Self-Triggered Control under Actuator Delays

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***Abstract*— In this paper we address the problem of self-triggered control of nonlinear systems under actuator delays. In particular, for globally asymptotically stabilizable systems we exploit the Lipschitz properties of the system's dynamics, and present a self-triggered strategy that guarantees the stability of the sampled closed-loop system with bounded actuator delays.**

I. INTRODUCTION

During the last decade, the study of event-triggered and self-triggered control for linear and nonlinear systems has attracted considerable attention, see for instance [1]-[3], [5], [7]-[9], [11]-[18], [21]-[30] and references therein. A property that characterizes event-triggered feedback techniques is that the controller is updated at sampling instants which are generated by a certain mechanism that monitors the state of the system in real-time. On the other hand, in self-triggered control, constant monitoring of the state is not required and the next controller update time is generated based on the last measurement of the system's state. Both techniques above overcome the drawback of periodic implementations which may generate unnecessary controller updates increasing the computational load of system devices.

In event-triggered control, the input-to-state stability (ISS) framework was utilized in [21], for the design of a mechanism which yields asymptotic stability of the closed-loop system avoiding infinitely fast sampling. Specifically, the controller is updated, only when the sampling error crosses a certain state-dependent threshold. Analogous techniques have also been used for a variety of problems, see for instance [3], [9], [13], [14], [17], [18], and [30]. To avoid the constant monitoring of the state, self-triggered techniques have been utilized for linear systems in [15], [27], [28], and [29], and for nonlinear systems in [1], [2], [11], [13], [16], [24], [25]. In particular, homogeneity and ISS assumptions were exploited in [1], small-gain conditions in [13] and [25] and high-gain designs in [16]. Finally, the works [2] and [24] propose self-triggered control schemes for locally stabilizable nonlinear systems.

Several of the aforementioned works require some ISS with respect to measurement errors assumption. For linear systems, this assumption is inherent from the stabilizability of the system. However, for nonlinear systems, this property holds for some special classes, see for instance [4] and [6]. In particular, in those works, there are counterexamples that show that the design of a feedback law that renders the system ISS with respect to measurement errors is not always possible. For event triggered and self-triggered control, techniques which do not require this assumption can be found for instance in [2], [14], [16], [18], [24].

In this paper we deal with the problem of self-triggered control for the following general class of nonlinear systems

$$\dot{x} = f(x, u),\ x \in \mathbb{R}^n,\, u \in \mathbb{R}^m, \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz with $f(0, 0) = 0$. In particular, a self-triggered control strategy for globally asymptotically stabilizable systems (1) is presented that guarantees the convergence of the system's state to an arbitrarily small neighborhood of the equilibrium under the presence of actuator delays. It should be noted that analogous self-triggered policies were also proposed in [2] and [24], yet for locally stabilizable nonlinear systems. Our methodology generalizes those results to the global case under weaker regularity assumptions on the systems' dynamics. Specifically, the proposed self-triggered condition, drives the state of the system from any initial condition $x_0 \in \mathbb{R}^n$ to a preassigned set containing the origin in finite time. Finally, a maximum bound on the actuator delays is derived in order for practical stability to be feasible.

It should be noted that this work extends one of the results in [23], where only the delay-free case was considered. Specifically, in [23], a (switching) self-triggered scheme was also proposed ensuring the boundedness of the system's state to an a priori selected small neighborhood of the equilibrium reducing the number of controller updates. In addition, event-triggered strategies for exponentially stabilizable systems were also developed with an updating threshold strategy, through which the controller is updated to preserve the stability properties of the system while the triggering condition is also updated to reduce future controller updates.

The rest of the paper is organized as follows. Section II includes the notation, certain definitions and the problem formulation. In Section III we present our main result and finally, in Section IV, examples and simulations are included to illustrate the proposed method.

II. PRELIMINARIES AND PROBLEM FORMULATION

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$, if it is continuous and strictly increasing with $\alpha(0) = 0$. If in addition $\lim_{s\to\infty} \alpha(s) = \infty$, then α is said to be of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed t the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed s it is decreasing to zero as $t \to \infty$. By $|x|$ we denote the Euclidean norm of a vector $x \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be Lipschitz continuous on compact

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sets if for every compact $S \subset \mathbb{R}^n$ there exists a constant $L > 0$ such that $|f(x)-f(y)| \leq L|x-y|$, for every $x, y \in S$. Next we recall some known definitions.

Definition 2.1: [10] The system $\dot{x} = f(x)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ being locally Lipschitz and $f(0) = 0$, is *globally asymptotically stable* (GAS) if there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that for any initial state $x(0)$ the solution exists for all $t \geq 0$ and satisfies $|x(t)| \leq \beta(|x(0)|, t)$, $x(0) \in \mathbb{R}^n$.

Lemma 2.1: [10, Gronwall-Bellman Lemma] Let $\lambda : [a, b] \to \mathbb{R}$ be continuous and χ, $\psi : [a, b] \to \mathbb{R}$ be non-negative continuous functions. If

$$\chi(t) \leq \lambda(t) + \int_a^t \psi(s)\chi(s)ds, \; a \leq t \leq b$$

then,

$$\chi(t) \leq \lambda(t) + \int_a^t \lambda(s)\psi(s) \exp\left[\int_s^t \psi(\tau)d\tau\right] ds.$$

Definition 2.2: [21] A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov function for the closed system $\dot{x} = f(x, k(x + e_r))$ if there exist α_1, α_2, α_3 and $\gamma \in \mathcal{K}_\infty$, such that:

$$\begin{aligned} \bar{\alpha}_1(|x|) \leq V(x) \leq \bar{\alpha}_2(|x|) \\ \nabla V(x) f(x,k(x + e_r)) \leq -\bar{\alpha}_3(|x|) + \gamma(|e_r|). \end{aligned}$$

The closed loop system $\dot{x} = f(x, k(x+e_r))$ is said to be ISS with respect to measurement errors $e_r \in \mathbb{R}^n$ if there exists an ISS Lyapunov function.

Consider the general nonlinear system (1) where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz with $f(0, 0) = 0$, and assume that a locally Lipschitz state feedback law $h : \mathbb{R}^n \to \mathbb{R}^m$, $h(0) = 0$, has been designed such that the closed loop system

$$\dot{x} = f(x, h(x)) \tag{2}$$

is globally asymptotically stable. Then, standard Converse Lyapunov Theorems guarantee the existence of a smooth Lyapunov function such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{3}$$

$$\nabla V(x) f(x, h(x)) \leq -\alpha(|x|) \tag{4}$$

where α_1, α_2, $\alpha \in \mathcal{K}_\infty$.

Suppose now that the feedback law $u = h(x)$ is applied to system (1) in discrete time under sample and hold, i.e.,

$$u(t) = h(x(t_k)), \; t \in [t_k, t_{k+1}),$$

where $\{t_k\}_{k\in\mathbb{N}}$ is an increasing sequence of sampling instants. Then, the resulting closed-loop system is,

$$\dot{x}(t) = f(x(t), h(x(t_k))), \; t \in [t_k, t_{k+1}), \tag{5}$$

where the time t_{k+1} is generated by the sampler

$$t_{k+1} = t_k + \Gamma(x(t_k)), \tag{6}$$

where $\Gamma : \mathbb{R}^n \to [\eta, \infty)$, for some $\eta > 0$. However, in practice, there also exists a sequence $\{\Delta_k\}_{k\in\mathbb{N}}$, $\Delta_k > 0$, $k \in \mathbb{N}$, of actuation delays, namely, the time required for the sensors to recompute the control law and update the actuators. Thus, in this case, the input applied to the system is

$$u(t) = \begin{cases} 0, & t \in [t_0, t_0 + \Delta_0) \\ h(x(t_{k-1})), & t \in [t_k, t_k + \Delta_k) \\ h(x(t_k)), & t \in [t_k + \Delta_k, t_{k+1}), \end{cases} \tag{7}$$

as shown in Figure 1.

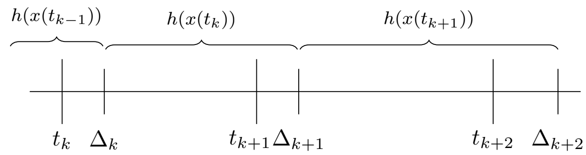

Fig. 1. Controller implementation with delay.

The objectives of this paper are the determination of a suitable sampler (6) and a maximum delay $\Delta > 0$ such that,

(i) $t_{k+1} - t_k > 0$ for all $k \in \mathbb{N}$,
(ii) $\Delta \in [0, t_{k+1} - t_k)$ for all $k \in \mathbb{N}$.
(iii) Drive the state of the sampled system (5)-(7) from any $x_0 \in \mathbb{R}^n$ to a predefined set containing zero for any delay $\Delta_k \in [0, \Delta)$, $k \in \mathbb{N}$.

Remark 1: Property (i) implies that the time elapsed between two sampling instants is strictly positive in order to avoid infinitely fast sampling, i.e. Zeno behavior. Property (ii) implies that the delay induced by the sensors is smaller than the inter-sampling time. The latter is necessary in order for the problem to be solvable. Property (iii) implies practical stability. Analogous results were also proposed in [2] and [24] for locally asymptotically stable systems. However, in this work, the set where practical stability is achieved is a priori selected and in addition, we provide explicit bounds for the maximum delay Δ so that both (ii) and (iii) hold.

III. Main Result

In this section we present our main result concerning the self-triggered control of globally asymptotically stabilizable systems (1). In particular, we assume a Lyapunov function for system (1) together with a continuous-time controller that guarantee that the equilibrium $x = 0$ is GAS. Thus, we make the following assumption:

A1. There exist a smooth, positive definite and proper Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a locally Lipschitz map $h : \mathbb{R}^n \to \mathbb{R}^m$ with $h(0) = 0$ and a function $\alpha \in \mathcal{K}_\infty$ such that

$$\nabla V(x) f(x, h(x)) \leq -\alpha(|x|), \; x \in \mathbb{R}^n.$$

Note that assumption A1 suggests that the control law u has been designed in continuous time so that the closed loop system is globally asymptotically stable. Assumption A1 is weaker than ISS, [20]. However, continuity of the feedback law provides some local robustness properties, [19].

We propose next a self-triggered control scheme that guarantees the convergence of the state to a sufficiently small neighborhood of the equilibrium when global asymptotic stability is considered for the continuous-time case.

Proposition 1: Consider the system (1) and assume that A1 holds. Then, for any $x_0 \in \mathbb{R}^n$, there exist a compact set $\mathcal{V}(x_0) \ni 0$ and constants $\tau > 0$, $\beta > 0$, $\gamma > 0$, and $\Delta > 0$, such that for any delay $\Delta_k \in [0, \Delta)$, the self-triggered condition

$$t_{k+1} = t_k + \frac{1}{L_f} \ln\left(\frac{\frac{\tau}{L_{f,h}\beta} + L_h|x_k|}{2L_{f,h}\gamma\Delta_k + L_h|x_k|} \right), \tag{8}$$

where L_f, L_h, $L_{f,h} = L_f L_h$ are the Lipschitz constants of f, h over $\mathcal{V}$, guarantees that the state of the sampled system (5)-(7) will be confined to any neighborhood S of zero, with $S \subset \mathcal{V}$, for all $k \geq k_0$, for some $k_0 \in \mathbb{N}$. Moreover, the inter-sampling times $t_{k+1} - t_k$ are strictly positive for all $k \in \mathbb{N}$ and the maximum delay Δ satisfies $\Delta \in [0, t_{k+1} - t_k)$, for all $k \in \mathbb{N}$.

Proof: Let $x_0 \in \mathbb{R}^n$ and consider the Lyapunov function of assumption A1. Also, let

$$\xi := V(x_0) \tag{9}$$

and consider a constant $c > 0$ such that $c < \xi$. Next, define the sets

$$\mathcal{V} := \{x \in \mathbb{R}^n : V(x) \leq \xi\}. \tag{10}$$
$$S := \{x \in \mathbb{R}^n : V(x) < c\}, \tag{11}$$

Since f and h are locally Lipschitz we have that, on the compact set $\mathcal{V}$,

$$|f(x, u) - f(\hat{x}, u)| \leq L_f|x - \hat{x}|,$$

$$|h(x) - h(\hat{x})| \leq L_h|x - \hat{x}|,$$

where L_f, $L_h > 0$ are the Lipschitz constants of f and h, respectively. Thus, from the previous Lipschitz conditions we obtain that

$$|f(x, h(\hat{x})) - f(x, h(x))| \leq L_{f,h}|\hat{x} - x|,$$

where $L_{f,h} := L_f L_h$ is the Lipschitz constant of f, h on the compact set $\mathcal{V}$.

The time-derivative $\dot{V}$ of V along system (5) is written

$$\begin{aligned} \dot{V}(x) =& \nabla V(x) f(x, h(x)) \\ &+ \nabla V(x)(f(x, h(x_k)) - f(x, h(x))). \end{aligned} \tag{12}$$

Notice now that the sets $\mathcal{V}$ and $\mathcal{V} \setminus S$ are compact. Thus, we can define

$$\ell := \min_{x \in \mathcal{V} \setminus S} \alpha(|x|), \tag{13}$$
$$\beta := \max_{x \in \mathcal{V}} |\nabla V(x)| \tag{14}$$
$$\gamma := \max_{x \in \mathcal{V}} \{|x|\} \tag{15}$$

By taking into account assumption A1, (13), and (14), we obtain from (12)

$$\dot{V}(x) \leq -\ell + \beta L_{f,h}|x - x_k|, \; x \in \mathcal{V} \setminus S. \tag{16}$$

Consider now the solution of (5)-(7) over the interval $[t_k, t_{\max})$, $x(t_k) = x_k \in \mathcal{V} \setminus S$, where $t_{\max} \in (t_k, \infty]$ is the maximal existence time of the solution,

$$\begin{aligned} x(t) = x_k &+ \int_{t_k}^{t_k+\Delta_k} f(x(s), h(x_{k-1}))ds \\ &+ \int_{t_k+\Delta_k}^{t} f(x(s), h(x_k))ds. \end{aligned} \tag{17}$$

By adding and subtracting terms and by taking into account (15), and that f, h are locally Lipschitz, we have

$$\begin{aligned} |x(t) - x_k| \leq& \int_{t_k}^{t} f(x_k, h(x_k))ds \\ &+ \int_{t_k}^{t} |f(x(s), h(x_k)) - f(x_k, h(x_k))|ds \\ &+ \int_{t_k}^{t_k+\Delta_k} |f(x(s), h(x_{k-1})) - f(x(s), h(x_k))|ds \\ \leq& L_{f,h}|x_k|(t - t_k) + L_{f,h}|x_{k-1} - x_k|\Delta_k \\ &+ L_f \int_{t_k}^{t} |x(s) - x_k|ds \\ \leq& L_{f,h}|x_k|(t - t_k) + 2L_{f,h}\gamma\Delta \\ &+ L_f \int_{t_k}^{t} |x(s) - x_k|ds. \end{aligned} \tag{18}$$

From (18) and the Gronwall-Bellman Lemma we obtain

$$\begin{aligned} |x(t) - x_k| \leq& L_h(|x_k| + 2L_f\Delta\gamma)e^{L_f(t-t_k)} \\ &- L_h|x_k|, \; t \in [t_k, t_{\max}) \end{aligned} \tag{19}$$

Notice now that, (16) and (19) imply that for all $t \in [t_k, t_{\max})$

$$\begin{aligned} \dot{V}(x(t)) \leq& -\ell + \beta L_{f,h}\Big(L_h(|x_k| + 2L_f\Delta\gamma)e^{L_f(t-t_k)} \\ &- L_h|x_k| \Big). \end{aligned} \tag{20}$$

We proceed by determining a maximum delay $\Delta > 0$ such that the following properties hold

(P1) $t_{k+1} - t_k > \Delta$ for all $k \in \mathbb{N}$,
(P2) $t_{k+1} + \Delta < t_{\max}$ for all $k \in \mathbb{N}$,
(P3) $\dot{V} < 0$ for $x \in \mathcal{V} \setminus S$.

Claim 1: There exists $\Delta > 0$ such that

$$\frac{\frac{\tau}{L_{f,h}\beta} + L_h|x_k|}{2L_{f,h}\Delta\gamma + L_h|x_k|} > 1,$$

where $0 < \tau < \ell$ and ℓ is defined in (13).

Indeed, for Claim 1 to hold, it suffices to find $\Delta > 0$ such that

$$2L_{f,h}\Delta\gamma < \frac{\tau}{L_{f,h}\beta},$$

or equivalently, it suffices that Δ satisfies,

$$\Delta < \frac{\tau}{2L_{f,h}^2\beta\gamma}. \tag{21}$$

The previous claim guarantees that if Δ satisfies (21), then the argument in the logarithm is greater that one and thus, $t_{k+1} - t_k > 0$. In the following analysis, we determine a bound on the maximum delay Δ such that property (P1) holds.

Let

$$\Delta := \frac{\tau}{2aL_{f,h}^2\beta\gamma} < 1, \tag{22}$$

where $a > 1$.Then, due to (21), Claim 1 holds. We next determine $a > 1$ in (22) such that (P1) holds.

Claim 2: There exists $\Delta > 0$ such that

$$t_{k+1} > t_k + \Delta.$$

In order for this claim to hold, it suffices to select $a > 1$ in (22) so that

$$\frac{1}{L_f}\ln\left(\frac{\frac{\tau}{L_{f,h}\beta} + L_h|x_k|}{2L_{f,h}\Delta\gamma + L_h|x_k|}\right) > \Delta. \tag{23}$$

By taking into account the inequality

$$\ln(x) \geq \frac{x-1}{x},\ x > 0$$

and (22), we have that the left hand side of (23) is written

$$\ln\left(\frac{\frac{\tau}{L_{f,h}\beta} + L_h|x_k|}{2L_{f,h}\Delta\gamma + L_h|x_k|}\right) = \ln\left(\frac{a(L_{f,h}L_h|x_k|\beta + \tau)}{aL_{f,h}L_h|x_k|\beta + \tau}\right)$$

$$\geq \frac{\frac{a(L_{f,h}L_h|x_k|\beta+\tau)}{aL_{f,h}L_h|x_k|\beta+\tau} - 1}{\frac{a(L_{f,h}L_h|x_k|\beta+\tau)}{aL_{f,h}L_h|x_k|\beta+\tau}} = \frac{\tau(a-1)}{a(L_{f,h}L_h|x_k|\beta + \tau)}. \tag{24}$$

Since $a > 1$ and β, $L_{f,h}$, L_h, $\tau > 0$, we have from (15)

$$|x_k| \leq \gamma \Rightarrow \frac{\tau(a-1)}{a(L_{f,h}L_h|x_k|\beta + \tau)} \geq \frac{\tau(a-1)}{a(L_{f,h}L_h\gamma\beta + \tau)}$$

Thus, in order to prove Claim 2, it suffices to select $a > 1$ in (22) such that the following strict inequality holds

$$\frac{\tau(a-1)}{aL_f(L_{f,h}L_h\gamma\beta + \tau)} > \frac{\tau}{2aL_{f,h}^2\beta\gamma}. \tag{25}$$

For the latter to be true, it suffices to select

$$a > 1 + \frac{L_f(L_{f,h}L_h\beta\gamma + \tau)}{2L_{f,h}^2\beta\gamma}. \tag{26}$$

Therefore, for a as above, we have that (25) holds and consequently that (23) holds, which proves Claim 2.

Thus, from (22) and (26), we obtain a bound on the maximum delay Δ. Then, any delay $\Delta_k \in [0, \Delta)$ satisfies property (P1). We next show that for $\Delta_k \in [0, \Delta)$, property (P2) holds.

Claim 3: It holds that $t_{k+1} + \Delta < t_{\max}$.

Indeed, consider the time t_{k+1} as given by (8) and assume that $t_{k+1} + \Delta \geq t_{\max}$ and $t_{\max} < \infty$ which implies

$$\limsup_{t\to t_{\max}^-} |x(t)| = +\infty. \tag{27}$$

Then, it follows from (8) and (20) with $\Delta_k \leq \Delta$, that for $0 < \tau < \ell$,

$$\dot{V}(x(t)) \leq -(\ell - \tau) < 0,\ \ \forall t \in [t_k, t_{\max}).$$

The latter implies that $x(t) \in \mathcal{V}$ for all $t \in [t_k, t_{\max})$ which contradicts (27). Hence, property (P2) holds.

Finally, by defining t_{k+1} as in (8), where $0 < \tau < \ell$, and where the delay Δ satisfies (22) with a as in (26), we get that

$$\dot{V}(x(t)) \leq -(\ell - \tau) < 0,\ \ x(t) \in \mathcal{V} \setminus S, \tag{28}$$

which due to (11), (10), and the fact that $\xi > c$, implies that the trajectory of the system will enter S in finite time. In fact, by integrating $\dot{V}(x(t)) \leq -(\ell - \tau)$, and by taking into account (11) and (10) we obtain that $x(t)$ will enter the set S within the time interval $[0, \frac{\xi - c}{\ell - \tau}]$.

Notice that, (8) generates the next sampling instant and does not require continuous monitoring of the state $x(t)$ since the constants τ, β and $L_{f,h}$ can be computed offline. In addition, Claim 1 and 2 imply that the inter-event times are lower bounded and the maximum delay $\Delta \in [0, t_{k+1} - t_k)$. Finally, as $|x_k|$ approaches the set S the inter-event times grow larger. ∎

Remark 2: It should be noted that a similar self-triggering mechanism to (8) was also proposed in [24] for locally asymptotically stabilizable systems. In particular, the approach in [24] was based on the Converse Lyapunov Theorem and the existence of a Lyapunov function that in addition to (3) and (4), also satisfies $|\nabla V(x)| \leq \alpha_3(|x|)$, $\alpha_3 \in \mathcal{K}$. However, for globally stabilizable systems the previous inequality is not always true. In this paper, our methodology requires only locally Lipschitz dynamics and the explicit form of a Lyapunov function. In addition, we have derived an explicit bound the delays must satisfy in order for practical stability to be achievable.

Remark 3: A self-triggered mechanism for practical stability was also proposed in [2], when the continuous-time system is locally asymptotically stabilizable. The suggested mechanism however, was based on a Taylor series expansion which required sufficiently smooth dynamics. It should also be noted, that under additional Lipschitz assumptions for the function $\alpha^{-1}(\cdot)$ in (4), the authors presented a self-triggered mechanism that guarantees asymptotic stability. This assumption however is not easy to verify and there are several examples of systems that do not satisfy it, for instance linear systems (see also [13]).

Notice that (22) and (26) provide a maximum bound for the delays Δ_k after each event, i.e., $\Delta_k < \Delta$. Hence, Proposition 1 shows that the state will reach and remain in the set (11) as long as $\Delta_k < \Delta$ for all $k \in \mathbb{N}$, where Δ satisfies (22) and (26). It should also be noted that (8) requires that the delays Δ_k are known. However, one can modify the self-triggering mechanism (8) by using the maximum delay bound Δ, instead of Δ_k. In particular, by taking into account (22) and (26) we could, instead of Δ_k, use

$$\Delta = \frac{L_h^2\tau}{L_{f,h}(L_{f,h} + 2(1+\varepsilon)L_{f,h})\beta\gamma + L_f\tau} \tag{29}$$

for any $\epsilon > 0$. Notice that due to (13)-(15), this estimate of the maximum delay Δ can be conservative. In particular, by selecting a sufficiently small $c > 0$ in (11), then also the maximum delay Δ must also be sufficiently small due to (22) and (26). Thus, to increase the value of Δ it suffices to increase the value of c which however, implies that we also enlarge the area where practical stability is achieved.

Next, we provide a semi-global version of Proposition 1 in the sense that a compact set of initial conditions is a priori given.

Proposition 2: Consider the system (1) and assume that A1 holds. Then, for any given compact set $0 \in E \subset \mathbb{R}^n$, and any $x_0 \in E$, the result of Proposition 1 holds.

Proof: Let $R > 0$ such that $|x| \leq R$ for all $x \in E$. Define $\xi := \alpha_2(R)$, where $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$ as in (3). Also, define

$$\mathcal{V}_R := \{x \in \mathbb{R}^n : V(x) \leq \xi\}. \tag{30}$$

$$S_R := \{x \in \mathbb{R}^n : V(x) < c\}, \tag{31}$$

for some constant $c > 0$ with $c < \xi$. Then, $E \subset B[0, R] \subset \mathcal{V}_R$, where $B[0, R]$ is the closed ball of radius R and center at $0 \in \mathbb{R}^n$. Next, if we define ℓ, β, and γ as in (13), (14), and (15), respectively, using V_R, S_R instead of $\mathcal{V}$ and S, and by using the same arguments as in the proof of Proposition 1, we again obtain

$$\dot{V}(x(t)) \leq -(\ell - \tau) < 0, \quad x(t) \in V_R \setminus S_R, \tag{32}$$

provided that the maximum delay Δ satisfies (22) and (26). We consider now the following two cases

(i) $x_0 \notin S_R$,
(ii) $x_0 \in S_R$.

For the first case, (32) implies that $x(t)$ will enter the set S_R in finite time and remain inside thereafter. For the second case, (32) implies that $x(t)$ will remain in S_R for all $t \geq 0$. ■

In Proposition 2, the sets E and S_R determine the number of controller updates and the maximum delay Δ. Shrinking the set E allows to increase the maximum delay Δ, while, enlarging the region S_R, where practical stability is achieved, decreases the number of controller updates. This is a direct consequence of the constant R in the proof of Proposition 2 and the definitions (13), (14), and (15), on the sets V_R, S_R and the bounds (22) and (26).

IV. Examples

Example 1: We first illustrate the proposed self-triggering mechanism on a system that was considered in [1]. Specifically, consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_1 x_2^2 \\ \dot{x}_2 &= x_1 x_2^2 - x_1^2 x_2 + u\end{aligned} \tag{33}$$

and the feedback law $u = -x_2^3 - x_1 x_2^2$ which asymptotically stabilizes the system with a Lyapunov function $V(x) = 1/2(x_1^2 + x_2^2)$ and $\alpha(|x|) = 1/2|x|^4$, $x = (x_1, x_2)^T$. Let $x_0 = (0.1, 0.4)$ and consider the constant $c = 0.015$. Then, we can calculate $\xi = 0.085$, $\beta = 0.41$, $\ell = 0.00045$, $L_f = 0.51$, $L_h = 0.58$. Finally, by taking into account (22) and (26) we obtain $\Delta = 0.0012$. Applying the self-triggered condition (8) we obtain the state evolution as shown in Fig.2. On the time interval $[0, 40]$ we have experienced a number of 7250 controller updates with an average inter-event period of $0.0055s$, Fig.3. Applying the self-triggered mechanism (8) with $\Delta = 0$, we have instead experienced 4979 controller updates with an average period of $0.008s$. Obviously, our approach is more conservative than the one proposed in [1]. However, it allows us to deal with systems that do not satisfy additional ISS assumptions as shown in the next example.

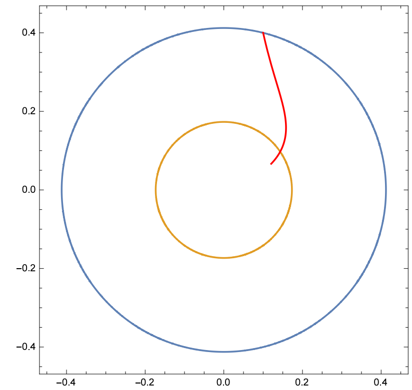

Fig. 2. The blue circle represents the set $\mathcal{V}$, the yellow circle represents the set S, and the red line the state of system (33).

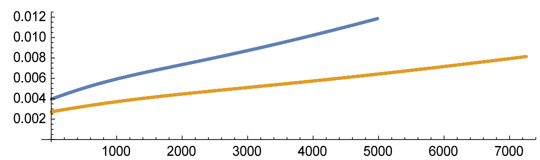

Fig. 3. Inter-sampling period with delays (yellow) and without delays (blue) for system (33).

Example 2: Consider the following scalar system from [4]

$$\dot{x} = -x \sin^2(x^2) + u \cos(x^2) \tag{34}$$

The state feedback $u = h(x) := -x \cos(x^2)$ renders the closed-loop system exponentially stable with the Lyapunov function $V(x) = 1/2x^2$. However, this feedback does not render the system ISS with respect to measurement errors, see [4]. From the time-derivative of the Lyapunov function $V = 1/2x^2$ we get $\dot{V} = -x^2$ and thus Assumption A1 holds. Let $x_0 = 2$. According to (9), (14), and (15), we have $\xi = 2$ and $\beta = \gamma = 2$. Also, let $c = 0.5$. Then from (13) we obtain $\ell = 0.25$ and if we define $f(x) := -x \sin^2(x^2)$ and $g(x) := \cos(x^2)$, we obtain $L_f = 8.48$, and $L_h = 5.4$. Finally, according to (22) we select $\Delta = 0.0004$. We have added random delays Δ_k, after each event, satisfying $\Delta_k \leq \Delta$ and $\Delta_0 = \Delta$, Fig. 4. The state evolution is depicted in Fig. 7. In the time interval $[0, 10]$ we have experienced a number of 1608 controller updates under the presence of delays with the average inter-sampling period being $0.0061s$, Fig. 6 (left). Using the maximum delay Δ in (29) with $\epsilon = 0.1$, we have experienced a number of 2139 controller updates with an average period of $0.0046s$, Fig. 6 (right). Finally, for the delay-free case we have experienced 1284 controller updates. It should be noted that our result only guarantees convergence in the selected set (11) and this particular example can also be exponentially stabilized with the event-triggering mechanisms proposed in [23].

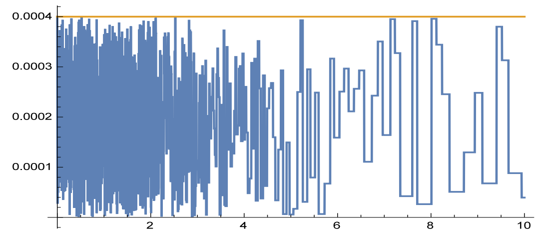

Fig. 4. Random delays Δ_k (blue) and maximum delay Δ (yellow).

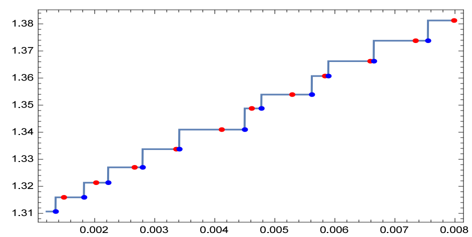

Fig. 5. The red dots represent the time t_{k+1} generated by the sampler (8) and the blue dots the time the controller is updated, i.e. $t_k + \Delta_k$.

V. CONCLUSION

In this work, we have extended one of the results of [23] by including actuator delays in the sample and hold implementation. In particular, a self-triggered mechanism was presented for the practical stabilization of nonlinear systems in a small neighborhood of the equilibrium. Explicit bounds were also derived for the magnitude of the applied delays.

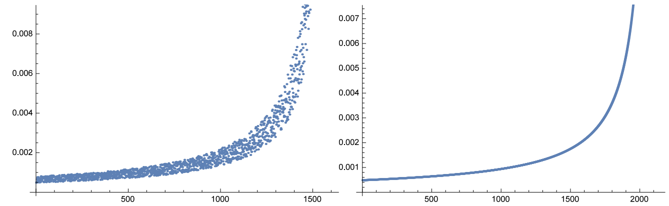

Fig. 6. Inter-sampling times with random delays (left) and with the maximum delay (right).

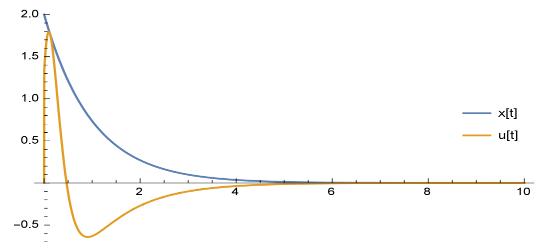

Fig. 7. State trajectory and control input of system (34).

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