Corridor MPC: Towards Optimal and Safe Trajectory Tracking

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***Abstract*— We present a framework for safe and optimal trajectory tracking by combining Model Predictive Control and Sampled-Data Control Barrier functions. This framework, which we call Corridor MPC, safely and robustly keeps the state of the system within a corridor that is defined as a permissible error around a reference trajectory. By incorporating Sampled-Data Control Barrier functions into an MPC framework, we guarantee safety for the continuous-time system in the sense of staying within the corridor and practical stability in the sense of converging to the reference trajectory. The proposed framework is evaluated with a free-flyer kinematics system.**

I. Introduction

The safe control of autonomous systems has received growing attention from the academic community over the past few years. As of recently, the International Space Station became inhabited by autonomous free-flyers that will autonomously perform tasks that can range from inspection to human interaction, where safety - of the autonomus agent, but also of the humans and structures involved - is key. Safety is defined as navigation on a bounded error set in the neighborhood of a possibly time-varying reference. Although safe control is crucial for all autonomous systems that operate in a shared workspace with humans and obstacles, the optimality of such control inputs plays an important long-term role. For instance, optimality affects the operation time of a space-craft, that is linked with the fuel usage or actuator wear and tear. Therefore, safety-critical systems should be endowed with both safe and optimality-aware controllers that respect control input limits and safety constraints, while minimizing control effort.

Model Predictive Control (MPC) and its nonlinear extension (NMPC) [1]–[4] are commonly used to address constraint satisfaction and optimality. In short, NMPC is a finite-horizon optimal controller (FHOC) that optimizes a cost function - usually a function of the error and control effort - along a finite time horizon, while enforcing state and control constraints. Extensions of MPC to handle unknown disturbances have been suggested in the literature, namely constraint-tightening MPC [5] and tube-based MPC [6]. On the latter we observe a combined approach of online and offline controllers that might run at different frequencies, where the online component is an ancillary controller. Ancillary controllers rely on associated Robust Control Invariant (RCI) sets [1, Def. 3.6]. In [6] a Monte-Carlo approach is suggested to estimate RCI sets for nonlinear systems, and in [7] the use of a sliding mode controller as the ancillary controller is proposed to exploit the associated RCI set. Another approach to estimate RCI sets has been proposed in [8] where a deep neural network uses Gaussian Regression over several state-space trajectories of the system. With respect to tracking of time-varying references, trajectory tracking methods using linear MPC have been proposed under Linear Time-Varying [9] and Linear Parameter-Varying formulations [10]. With respect to the nonlinear case (NMPC), recent works [11], [12] propose the calculation of a stabilizing terminal ingredient to show exponential asymptotic stability towards the reference being tracked, even in the case of dynamic, non-compliant targets. Safety-focused control methods have gained traction in the literature. For instance, Zeroing Control Barrier Functions (ZCBFs) [13] allow a system to operate within safe bounds even when a nominal controller fails, acting as a safety filter that forces the system to evolve inside a safe set that is similar to an RCI set. The robustness properties of such control methods have been inspected in [14], [15]. The authors in [16] have further proposed sampled-data control barrier functions. A combination of ZCBFs with MPC has been proposed in [17] for setpoint stabilization, that however does not provide continuous-time guarantees, as opposed to what we present in this paper.

In this work, we aim at deriving safe control laws with the goal to stay within a desired bound to a given time-varying trajectory while minimizing a cost function over a specified planning horizon. To this end, we take advantage of the recent developments in ZCBFs and NMPC to derive an alternative to existing trajectory tracking controllers, hereafter referred to as Corridor MPC (CMPC). This approach allows for safe trajectory tracking under the optimal control inputs derived using NMPC, while providing safety guarantees for continuous-time systems. Moreover, we present stability and recursive feasibility guarantees for CMPC. In summary, the motivation and advantages of this alternative approach over the state-of-the-art are (i) a sampled-data ZCBF formulation that ensures robust safety of a time-varying set that can be encoded into an MPC framework; (ii) continuous-time safety guarantees in the presence of uncertainties; (iii) an alternative formulation paving the way for further investigation of similarities and differences to existing RCI tube methods.

The rest of the paper is organized as follows: Section II introduces preliminary work; Section III introduces the problem we aim at solving; Section IV presents the proposed Corridor MPC framework as the solution to the proposed

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problem; Section V provides a practical stability and recursive feasibility analysis; Section VI shows an example for a free-flyer system, with numerical results in Section VII; and Section VIII concludes this work and presents future directions.

Notation: In this paper, we use $\mathbb{N}$ for natural numbers, $\mathbb{N}_0$ for natural numbers including 0, $\mathbb{R}$ for real numbers, $\mathbb{R}_{>0}$ for all positive real numbers and $\mathbb{R}_{\geq 0}$ for the positive real numbers, including 0. All natural numbers, including zero, until N, are represented as $\mathbb{N}_{[0,N]}$. We denote $\mathbb{A} \oplus \mathbb{B}$ as the Minkowski sum of sets $\mathbb{A}$ and $\mathbb{B}$. The composition operator $\alpha \circ \beta$ represents $\alpha(\beta)$. The weighted norm of a vector $a \in \mathbb{R}^n$ by a positive definite matrix $A \in \mathbb{R}^{n \times n}$ is represented as $\|a\|_A = \sqrt{a^T A a}$. The 2-norm of a vector $a \in \mathbb{R}^n$ is defined as $\|a\|_2$. The predicted value of b for time $(k+n)\Delta t$, predicted at time $k\Delta t$, is written as $b(n\Delta t|k\Delta t), k, n \in \mathbb{N}_0$. The j-th component of the vector a is represented as a_j. A continuous function, $\alpha : \mathbb{R} \to \mathbb{R}$ is a *extended class-$\mathcal{K}$ function* if it is strictly increasing and $\alpha(0) = 0$. A continuous function, $\beta : \mathbb{R} \to \mathbb{R}$ is an *class-$\mathcal{K}_\infty$ function* if it is strictly increasing, $\beta(0) = 0$ and $\lim_{r \to \infty} \beta(r) = \infty$. A continuous function, $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an *class-$\mathcal{KL}$ function* if, for each fixed r, $\gamma(\cdot, r)$ if of class $\mathcal{K}$, and for each fixed s, $\gamma(s, \cdot)$ is decreasing and $\gamma(s, r) \to 0$ as $r \to \infty$.

II. Preliminaries

Let us first introduce the nonlinear control affine system

$$\dot{\tilde{x}} = f_{c,\tilde{x}}(\tilde{x}(t), u(t), w(t)) = f_c(\tilde{x}(t)) + g_c(\tilde{x}(t))u(t) + w(t), \tag{1}$$

where $\tilde{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $w(t) \in \mathbb{R}^n$ correspond to the state, the control input and piece-wise continuous disturbance, which evolve in their respective domains:

$$\tilde{x}(t) \in \tilde{\mathbb{X}}, \quad u(t) \in \mathbb{U}, \text{ and } w(t) \in \mathbb{W}. \tag{2}$$

The sets $\tilde{\mathbb{X}}$, $\mathbb{W}$ and $\mathbb{U}$ are assumed to be closed sets containing the origin. We further assume $f_c : \mathbb{R}^n \to \mathbb{R}^n$ and $g_c : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally Lipschitz continuous functions on $\tilde{\mathbb{X}}$ with respect to $\tilde{x}$ and u, and piece-wise continuous in t. The disturbance w is unknown but bounded, with $w(t) \in \mathbb{W} := \{w \in \mathbb{R}^n : \|w\| \leq \overline{w}\}, \forall t \geq 0$. The discrete-time dynamics of (1) are represented as

$$\begin{aligned} \tilde{x}((k+1)\Delta t) &= f_{\tilde{x}}(\tilde{x}, u, w) \\ &= f(\tilde{x}(k\Delta t)) + g(\tilde{x}(k\Delta t))u(k\Delta t) + w(k\Delta t), \end{aligned} \tag{3}$$

where f and g are the discrete-time dynamics of f_c and g_c in (1) obtained through, for instance, Runge-Kutta discretization with sampling-time Δt. In the same way, the nominal dynamics for $x(k\Delta t) \in \mathbb{X}$ and considering $w(k\Delta t) = 0$, are given by

$$\begin{aligned} x((k+1)\Delta t) &= f_x(x, u) \\ &= f(x(k\Delta t)) + g(x(k\Delta t))u(k\Delta t). \end{aligned} \tag{4}$$

where $f_x(\tilde{x}, u) \triangleq f_{\tilde{x}}(\tilde{x}, u, 0)$.

We define the error of the state $\tilde{x}(t)$ with respect to a possibly time-varying trajectory $x_r(t)$ as

$$\tilde{e}(t) = \tilde{x}(t) - x_r(t), \tag{5}$$

with the dynamics of the error $\tilde{e}(t)$

$$\dot{\tilde{e}} = f_{c,\tilde{x}}(\tilde{e}(t) + x_r(t), u(t), w(t)) - \dot{x}_r(t). \tag{6}$$

Assumption 1. *The reference trajectory $x_r(t)$ satisfies $x_r((k+1)\Delta t) = f(x_r(k\Delta t)) + g(x_r(k\Delta t))u_r(k\Delta t)$ at the sampling times $k\Delta t$, for $x_r(k\Delta t) \in \mathbb{X}$ and $u_r(k\Delta t) \in \mathbb{U}_r \subset \mathbb{U}$, and is continuously differentiable for all $t \in [k\Delta t, (k+1)\Delta t]$.*

Remark 1. *The upper bound on $\mathbb{U}_r$ is related to the upper bound on $\|\dot{x}_r(t)\|$ and $\|w(t)\|, \forall t \in \mathbb{R}_{\geq 0}$. We elaborate on this relation in Section VI.*

In discrete-time, the disturbed and nominal system errors are represented as

$$\tilde{e}(k\Delta t) = \tilde{x}(k\Delta t) - x_r(k\Delta t) \text{ and} \tag{7}$$

$$e(k\Delta t) = x(k\Delta t) - x_r(k\Delta t), \tag{8}$$

whereas their respective dynamics satisfy

$$\begin{aligned} \tilde{e}((k+1)\Delta t) &= f_{\tilde{e}}(\tilde{e}, u, w) = \\ &= f_{\tilde{x}}(\tilde{e}(k\Delta t) + x_r(k\Delta t), u(k\Delta t), w(k\Delta t)) - \dot{x}_r(k\Delta t), \end{aligned} \tag{9a}$$

$$\begin{aligned} e((k+1)\Delta t) &= f_e(\tilde{e}, u) = \\ &= f_x(e(k\Delta t) + x_r(k\Delta t), u(k\Delta t), 0) - \dot{x}_r(k\Delta t), \end{aligned} \tag{9b}$$

and $f_e(\tilde{e}, u) \triangleq f_{\tilde{e}}(\tilde{e}, u, 0)$. We denote by $\Psi_{\tilde{e}}(k\Delta t, \tilde{e}(0), \mathbf{w}_k)$ the solution of $f_{\tilde{e}}(\tilde{e}, w)$ at time $k\Delta t$, with initial condition $\tilde{e}(0)$, given a sequence of disturbances $\mathbf{w}_k = \{w(0), w(\Delta t), ..., w(k\Delta t)\}, \forall k \in \mathbb{N}_0$ and a feedback controller $u(k\Delta t) = K(\tilde{e}(k\Delta t))$. We denote the solution of the nominal system as $\Psi_e(k\Delta t, \tilde{e}(0)) \triangleq \Psi_{\tilde{e}}(k\Delta t, \tilde{e}(0), \mathbf{0}_k)$.

A. Model Predictive Control

Dealing with safety-critical systems imposes several requirements on the control synthesis. In particular, three requirements must be met at all times: (i) ensure the system evolves along a safe set $\mathbb{X}_S \subset \mathbb{X}$; (ii) ensure that the actuation limits $\mathbb{U}$ are respected; (iii) safely handle external disturbances $w \in \mathbb{W}$.

A common control approach to handle state and control constraints while minimizing the energy consumption is NMPC. NMPC is a FHOC that minimizes a cost function $J(x, u)$ along a receding horizon of length N under state and control constraints. The optimization problem is constrained by $x \in \mathbb{X}_S, u \in \mathbb{U}$ and the dynamics in (4). The optimization problem results in N predicted states and N control inputs for the system, of the form $\mathbf{x}_k^* = \{x^*(\Delta t|k\Delta t), \ldots, x^*(N\Delta t|k\Delta t)\}$, and $\mathbf{u}_k^* = \{u^*(0|k\Delta t), \ldots, u^*((N-1)\Delta t|k\Delta t)\}$ for a given initial state $x(0|k\Delta t) = \tilde{x}(k\Delta t)$, and an associated optimal cost value $J^*(\tilde{x}(k\Delta t))$. Each discrete control input is applied to the system (1) in a Zero Order Hold (ZOH) fashion - a piece-wise constant input between sampling instances, that is, $u(t) = u^*(k\Delta t)\ \forall t \in [k\Delta t, (k+1)\Delta t)$. Noting that the predicted value of x for time $(k+n)\Delta t$, predicted at time $k\Delta t$, is $x(n\Delta t|k\Delta t), k, n \in \mathbb{N}_0$, the NMPC problem is therefore defined as

$$J^*(\tilde{x}(k\Delta t)) = \min_{\mathbf{u}_k^*} \; J(x(n\Delta t|k\Delta t), u(n\Delta t|k\Delta t)) \quad \text{(10a)}$$

$$\text{s.t.:} \;\; x((m+1)\Delta t|k\Delta t) =$$

$$= f_x(x(m\Delta t|k\Delta t), u(m\Delta t|k\Delta t)) \quad \text{(10b)}$$

$$x(m\Delta t|k\Delta t) \in \mathbb{X}_S \quad \text{(10c)}$$

$$u(m\Delta t|k\Delta t) \in \mathbb{U}, \; \forall m \in \mathbb{N}_{[0,N-1]} \quad \text{(10d)}$$

$$x(N\Delta t|k\Delta t) \in \mathbb{X}_F \subset \mathbb{X}_S \quad \text{(10e)}$$

$$x(0|k\Delta t) = \tilde{x}(k\Delta t), \forall n \in \mathbb{N}_{[0,N]} \quad \text{(10f)}$$

When solved at each sampling time $k\Delta t$, we obtain a feedback controller $K_N(\tilde{x}(k\Delta t)) = u^*(0|k\Delta t)$. The set $\mathbb{X}_F$ is a terminal control invariant set under a state feedback controller of the form $u_K(t) = Kx(t)$. It is common [4] to use Linear Quadratic Regulators (LQR) and associated control invariant sets as terminal sets in NMPC.

B. Zeroing Control Barrier Functions

Recently, Zeroing Control Barrier Functions (ZCBF) [18] have emerged as an alternative way to ensure safety of a continuous-time system (1). This formulation ensures forward invariance of a safe set where we wish the system to evolve, defining an RCI set. Let $h(\tilde{x}, t) : \mathscr{D} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, where $\mathscr{D} \subset \mathbb{R}^n$ is a compact set, be a continuously differentiable function, and let us associate with $h(\tilde{x}, t)$ the following sets

$$\mathscr{C}(t) = \{\tilde{x} \in \mathbb{R}^n : h(\tilde{x}, t) \geq 0\}, \quad \text{(11a)}$$

$$\partial\mathscr{C}(t) = \{\tilde{x} \in \mathbb{R}^n : h(\tilde{x}, t) = 0\}, \quad \text{(11b)}$$

where $\partial\mathscr{C}(t)$ is the boundary of $\mathscr{C}(t)$.

Definition 1 ([18]). *Given a set $\mathscr{C}(t)$ defined by (11) for a continuously differentiable function $h(\tilde{x}, t)$, then $h(\tilde{x}, t)$ is called a zeroing control barrier function defined on a set $\mathscr{D}$ with $\mathscr{C}(t) \subset \mathscr{D} \subset \mathbb{R}^n$ $\forall t \geq 0$, if there exists a Lipschitz continuous extended class-$\mathcal{K}$ function α such that*

$$\sup_{u \in \mathbb{U}} [L_{f_c} h(\tilde{x}, t) + L_{g_c} h(\tilde{x}, t) u + \frac{\partial h(\tilde{x}, t)}{\partial t} + \alpha(h(\tilde{x}, t))] \geq 0,$$

$$\forall \tilde{x} \in \mathscr{D}, \forall t \in \mathbb{R}_{\geq 0}, \quad \text{(12)}$$

where L_{f_c} and L_{g_c} are the Lie derivatives of h along f_c and g_c, respectively.

If $h(\tilde{x}, t)$ is a zeroing control barrier function and if we have a locally Lipschitz continuous control law $u(\tilde{x}, t)$ that satisfies the constraint $L_{f_c} h(\tilde{x}, t) + L_{g_c} h(\tilde{x}, t) u(\tilde{x}, t) + \frac{\partial h(\tilde{x}, t)}{\partial t} \geq -\alpha(h(\tilde{x}, t))$, which is as in (12), then the condition $\dot{h}(\tilde{x}(t), t) \geq -\alpha(h(\tilde{x}(t), t))$ is enforced for all $t \geq 0$ for the nominal system $\dot{\tilde{x}} = f_c(\tilde{x}(t)) + g_c(\tilde{x}(t))u(t)$. By [18], it then follows that $\tilde{x}(t) \in \mathscr{C}(t)$ for all $t \geq 0$ when $\mathscr{C}(t)$ is compact.

It is worth remarking that the set $\mathscr{C}(t)$ can be seen as the set $\mathbb{X}_S$ in (10), with respect to the safety properties, and $\tilde{\mathbb{X}}$ as the set $\mathscr{D}$.

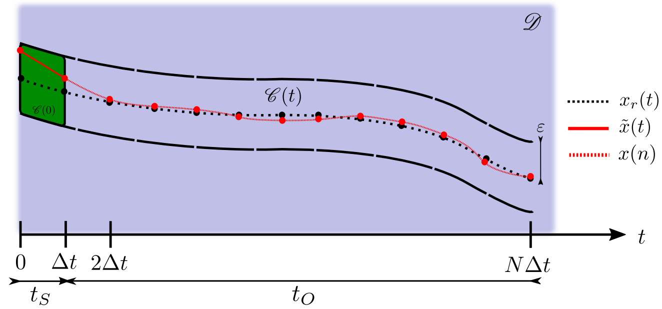

Fig. 1: Corridor MPC prediction horizon detail. From time $t_s \in [0, \Delta t]$, using an optimally safe control input, the state $\tilde{x}$ is ensured to evolve within the safe set $\mathscr{C}(t)$, defined with ZCBFs h, and ε maximum deviation from $x_r(t)$. The safe control input applied at $t = 0$ is optimized taking into account the optimality of the solution for the entire receding horizon planning, $t \in \{t_S \cup t_O\}$. The procedure is repeated at all sampling times, ensuring that $\tilde{x} \in \mathscr{C}(t)$ through the whole operation.

III. Problem Statement

In this work, we will focus on the reference tracking problem for Nonlinear Model Predictive Control. In particular, we are interested in robustly following a time-varying reference, that is, to stay within an ε-small neighborhood of the desired reference, even in the presence of an additive noise $w(t) \in \mathbb{W}$. Moreover, as we plan to apply our solution in safety-critical systems, it is required that the state $\tilde{x}$ remains in the predefined safe set $\mathscr{C}(t)$, ε-close to the desired reference trajectory $x_r(t)$, and that the control input $u(k\Delta t)$ satisfies the actuation limits set by $u(k\Delta t) \in \mathbb{U}$.

Formally, we define the problem as follows:

Problem 1. *Let the dynamics for $\tilde{x}(t)$ be defined by (1) and let $\tilde{e}(t)$ be as in (5). Let the admissible safe set $\mathscr{C}(t)$ be defined by (11) for $h(\tilde{x}, t) = \varepsilon^2 - \|\tilde{x} - x_r(t)\|_2^2$, where $\varepsilon \in \mathbb{R}_{>0}$ is the maximum allowed error to the reference and $x_r(t)$ is a time-varying reference, as in Assumption 1. The goal is to design a feedback control law $K_N(\tilde{e}(k\Delta t)) \in \mathbb{U}, k \in \mathbb{N}_0$ through (10), such that $\tilde{x}(t)$ remains in $\mathscr{C}(t)$ – and therefore $\|\tilde{e}(t)\| \leq \varepsilon, \forall t \geq 0$, – while minimizing a cost function $J(e(k\Delta t), u(k\Delta t))$, where $e(k\Delta t)$ is as in (8).*

IV. Corridor MPC

To solve Problem 1 we propose CMPC, a novel approach that takes advantage of the robust and safety properties of ZCBFs and the constrained optimality of NMPC.

A. ZCBF for Sampled-Data Systems

ZCBFs guarantees, as presented in the literature (e.g. in [18]), are targeted at continuous-time systems and, as argued before, require locally Lipschitz continuous control laws $u(x, t)$. As in this work we integrate ZCBFs into a NMPC that operates at discrete time steps $k\Delta t, k \in \mathbb{N}_0$, we instead use sampled-data control barrier functions as proposed in [16]. To this end, robustness margins can be incorporated in the ZCBF framework (12), ensuring that the system remains safe in-between sample times. The formulation proposed in [16] ensures continuous-time safety using a sampled-data control barrier function, assuming that the constraint set $\mathscr{C}$ is compact and the system dynamics are bounded. Lemma 1 extends the result in [16, Lem. 4] for a time-varying safe set $\mathscr{C}(t)$.

Definition 2 (SD-ZCBF). *Consider the system* (1) *and the compact sets $\mathscr{C}(t)$ and $\mathscr{D}$, where $\mathscr{C}(t)$ is defined by* (11) *for a continuously differentiable function $h : \mathscr{D} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, and $\mathscr{D} \supset \mathscr{C}(t)$ for all $t \geq 0$. The function h is a* sampled-data zeroing control barrier function *(SD-ZCBF) if for a given $\Delta t > 0$ there exists an extended class-$\mathcal{K}$ function α, where $\alpha \circ h : \mathscr{D} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is Lipschitz continuous on $\mathscr{D} \times \mathbb{R}_{\geq 0}$, such that for any point $\tilde{x} \in \mathscr{D}$ and $k \in \mathbb{N}_0$ there is a constant feedback input $u(\tilde{x}, k\Delta t) \in \mathbb{U}$, shortened to $u(k\Delta t)$, satisfying the following condition:*

$$L_{f_c}h(\tilde{x}, k\Delta t) + L_{g_c}h(\tilde{x}, k\Delta t)u(k\Delta t) + \alpha(h(\tilde{x}, k\Delta t)) \geq \bar{\nu} + c_6 + c_7\bar{w}, \tag{13}$$

with $\bar{\nu} \geq \nu(\Delta t)$, where

$$\nu(\Delta t) := \big((\bar{c}_1 + \bar{c}_2 + \bar{c}_3 c_4)(c_5 + \bar{w}) + (\underline{c}_1 + \underline{c}_2 + \underline{c}_3 c_4)\big)\,\Delta t, \tag{14}$$

and where $\bar{c}_1, \bar{c}_2, \bar{c}_3 \in \mathbb{R}_{>0}$ are the respective Lipschitz constants of $\alpha \circ h$, $L_{f_c}h$, $L_{g_c}h$ with respect to $\tilde{x}$, $\underline{c}_1, \underline{c}_2, \underline{c}_3 \in \mathbb{R}_{>0}$ are the respective Lipschitz constants of $\alpha \circ h$, $L_{f_c}h$, $L_{g_c}h$ with respect to t, and $c_4, c_5, c_6, c_7, \bar{w} \in \mathbb{R}_{>0}$ are the respective uniform bounds of $\|u\|$, the dynamics $\|f_c(\tilde{x}) + g_c(\tilde{x})u\|$, $\|\frac{\partial h(\tilde{x}(k),t)}{\partial t}\|$, $\|\frac{\partial h(\tilde{x}(k),t)}{\partial \tilde{x}}\|$, and $\|w\|$ on $\mathscr{D} \times \mathbb{R}_{\geq 0}$.

The intuition behind Definition 2 is given by Lemma 1.

Lemma 1. *For a given $\Delta t > 0$, suppose $h : \mathscr{D} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a SD-ZCBF for the system* (1)*. For any $k \in \mathbb{N}_0$, let $t_0 = k\Delta t$ and $\tilde{x}(t_0) = \tilde{x}(k\Delta t) \in \mathscr{C}(t_0) \subset \mathscr{D}$. Consider* (1) *in closed-loop with $u(t) = u(\tilde{x}(k\Delta t), k\Delta t) \in \mathbb{U}$ $\forall t \in [k\Delta t, (k+1)\Delta t]$, for which* (13) *holds, on $[k\Delta t, (k+1)\Delta t]$. Then $\tilde{x}(t)$ is uniquely defined on $[k\Delta t, (k+1)\Delta t]$, and $\tilde{x}(t) \in \mathscr{C}(t)$ for all $t \in [k\Delta t, (k+1)\Delta t]$.*

Proof. Since the dynamics of (1) are locally Lipschitz continuous on $\mathbb{R}_{\geq 0} \times \tilde{\mathbb{X}}$, piece-wise continuous in t, and u is constant on $[k\Delta t, (k+1)\Delta t]$, then $\tilde{x}(t)$ is uniquely defined on $[k\Delta t, \tau_1]$ for some $\tau_1 > k\Delta t$ [19, Thm. 54]. We note that if $\tau_1 \geq (k+1)\Delta t$, then of course $\tilde{x}(t)$ is uniquely defined on $[k\Delta t, (k+1)\Delta t]$. Suppose then that $\tau_1 < (k+1)\Delta t$. Furthermore, since $\tilde{x}(k\Delta t) \in \mathscr{C}(k\Delta t) \subset \mathscr{D}$, then by continuity of $\tilde{x}$ there exists a $\tau_0 \in (k\Delta t, \tau_1]$ for which $\tilde{x}(t) \in \mathscr{D}$ for all $t \in [k\Delta t, \tau_0]$. We define the following function on $[k\Delta t, \tau_0]$:

$$\begin{aligned} m_k(t) = &\Big(L_{f_c}h(\tilde{x}(t), t) - L_{f_c}h(\tilde{x}(k\Delta t), k\Delta t)\Big) \\ &+ \Big(\alpha(h(\tilde{x}(t)), t) - \alpha(h(\tilde{x}(k\Delta t), k\Delta t))\Big) \\ &+ \Big(L_{g_c}h(\tilde{x}(t), t) - L_{g_c}h(\tilde{x}(k\Delta t), k\Delta t)\Big)u(k\Delta t). \end{aligned} \tag{15}$$

By the Lipschitz properties of $f_c(\tilde{x})$, $g_c(\tilde{x})$, $h(\tilde{x}, t)$, $\alpha(h(\tilde{x}, t))$, and the fact that $k\Delta t < \tau_0 < (k+1)\Delta t \Rightarrow \tau_0 - k\Delta t < \Delta t$, it follows that:

$$\begin{aligned} ||m_k(t)|| \leq{} &(\bar{c}_1 + \bar{c}_2 + \bar{c}_3||u(k\Delta t)||)||\tilde{x}(t) - \tilde{x}(k\Delta t)|| \\ &+ (\underline{c}_1 + \underline{c}_2 + \underline{c}_3||u(k\Delta t)||)\Delta t. \end{aligned} \tag{16}$$

Next we introduce the bounds on the system dynamics $\|f_c(\tilde{x}) + g_c(\tilde{x})u\|$, denoted $c_5 \in \mathbb{R}_{>0}$, and on the input u, denoted $c_4 \in \mathbb{R}$. Note that c_5 is known to exist since the system dynamics are continuous on the compact set $\mathscr{D}$. Furthermore, since $\mathbb{U}$ is compact there exists a c_4 such that $\mathbb{U} \subset \{u \in \mathbb{R}^m : ||u|| \leq c_4\}$. Now, we get $\|\tilde{x}(t) - \tilde{x}(k\Delta t)\| = \int_{k\Delta t}^{\tau_0} \|f_c(\tilde{x}(t)) + g_c(\tilde{x}(t))u(k\Delta t) + w(t)\| dt \leq (c_5 + \bar{w})\Delta t$. Substitution into (16) yields: $||m_k(t)|| \leq \nu(\Delta t)$.

Now we have derived a bound on $\|m_k(t)\|$ for $t \in [k\Delta t, \tau_0]$. We add $L_{f_c}h(\tilde{x}, t) + L_{g_c}h(\tilde{x}, t)u(k\Delta t) + L_w h(\tilde{x}, t) + \frac{\partial h(\tilde{x}, t)}{\partial t} + \alpha(h(\tilde{x}, t))$ to both sides of (13) and substitute (15) which yields:

$$\begin{aligned} &L_{f_c}h(\tilde{x}, t) + L_{g_c}h(\tilde{x}, t)u(k\Delta t) + L_w h(\tilde{x}, t) + \frac{\partial h(\tilde{x}, t)}{\partial t} \\ &+\alpha(h(\tilde{x}, t)) \geq L_w h(\tilde{x}, t) + \frac{\partial h(\tilde{x}, t)}{\partial t} + \bar{\nu} + m_k(t) \\ &+ c_7\bar{w} + c_6, \end{aligned}$$

Since $0 \leq \|m_k(t)\| \leq \nu(\Delta t) \leq \bar{\nu}$, $\|L_w h\| = \|\frac{\partial h(\tilde{x}, t)}{\partial \tilde{x}} w(t)\| \leq c_7\bar{w}$, and $\|\frac{\partial h(\tilde{x}, t)}{\partial t}\| \leq c_6$ it follows that $L_{f_c}h(\tilde{x}, t) + L_{g_c}h(\tilde{x}, t)u(k\Delta t) + L_w h(\tilde{x}, t) + \frac{\partial h(\tilde{x}, t)}{\partial t} + \alpha(h(\tilde{x}, t)) \geq 0$ for all $t \in [k\Delta t, \tau_0]$. It is clear then that $\dot{h}(\tilde{x}, t) \geq -\alpha(h(t))$, which by Lemma 2 of [20] ensures that $\tilde{x}(t) \in \mathscr{C}(t)$ for all $t \in [k\Delta t, \tau_0]$. Now we prove that $\tilde{x}(t) \in \mathscr{C}(t)$ for all $t \in [k\Delta t, \tau_1]$ by contradiction. Suppose instead that for some $\tau_a \in (\tau_0, \tau_1]$, $\tilde{x}(\tau_a) \in \mathscr{D} \setminus \mathscr{C}(t)$ and $\tilde{x}(t) \in \mathscr{D}$ for all $t \in [k\Delta t, \tau_a]$ (i.e., the solution has left $\mathscr{C}(t)$, but not $\mathscr{D}$). Then $\tilde{x}(t)$ must leave $\mathscr{C}(t)$ at some $t < \tau_a$. Furthermore, since the closed-loop dynamics are locally Lipschitz on $\mathscr{D}$, $\tilde{x}(t)$ is uniquely defined on $[k\Delta t, \tau_a]$ (this is shown by repeatedly applying [19, Thm. 54] since $\tilde{x}(t)$ remains in $\mathscr{D}$ over which local Lipschitz continuity of the closed-loop dynamics holds). To leave $\mathscr{C}(t)$, $\dot{h}(\tilde{x}, t) < 0$ must hold on $\partial\mathscr{C}(t)$ (11b). We recompute $m_k(t)$ by repeating the previous steps and replacing τ_0 with τ_a. We note that since all the previously defined bounds $c_1, ..., c_7, \bar{w}$ are independent of time and $\tau_a \leq (k+1)\Delta t$, that the repeated analysis yields the exact same bound on $m_k(t)$ as in (16). Therefore we see that $\dot{h}(\tilde{x}, t) \geq 0$ holds for any $\tilde{x}(t) \in \mathscr{C}(t), t \in [k\Delta t, \tau_a]$. We arrive at a contradiction, and so $\tilde{x}(t)$ can never leave $\mathscr{C}(t)$ (and $\mathscr{D}$) on $t \in [k\Delta t, \tau_1]$. If $\tau_1 < (k+1)\Delta t$, then the solution $\tilde{x}(t)$ must have left every compact subset over which local Lipschitz continuity of the closed-loop dynamics holds (see proof of [21, Thm. 3.3]). However we have just shown that $\tilde{x}(t) \in \mathscr{C}(t) \subset \mathscr{D}$, and the closed-loop system is locally Lipschitz continuous on $\mathscr{D}$. Thus in fact $\tau_1 \geq (k+1)\Delta t$ and so $\tilde{x}$(t) is uniquely defined on $[k\Delta t, (k+1)\Delta t]$. Finally, we once more compute $m_k(t)$ by repeating the previous steps and substituting τ_0 with $(k+1)\Delta t$. Once again we arrive at the same bound on $m_k(t)$ from (16) such that $\dot{h}(\tilde{x}, t) \geq 0$

for all $t \in [k\Delta t, (k+1)\Delta t]$, which ensures that $\tilde{x}(t) \in \mathscr{C}(t)$ for all $t \in [k\Delta t, (k+1)\Delta t]$. □

Remark 2. *If $u = u(\tilde{x}(k\Delta t), k\Delta t)$ for almost all $t \in [k\Delta t, (k+1)\Delta t]$ and $u \in \mathbb{U}$, then the results of Lemma 1 still hold since conditions of Theorem 54 of [19] and Lemma 2 of [20] hold, and u is bounded. Thus Lemma 1 ensures forward invariance of $\mathscr{C}(t)$ for* (4) *in closed-loop with a zero-order hold control input u satisfying* (13) *at each sampling time.*

B. Corridor MPC Formulation

Formally, the Corridor MPC, solved at each sampling time $k\Delta t, k \in \mathbb{N}_0$, is formulated as

$$J_N^*(\tilde{e}(k\Delta t)) = \min_{\mathbf{u}_k^*} \quad J_N(e, u) \tag{17a}$$

$$\text{s.t.: } x((m+1)\Delta t|k\Delta t) = f_x(x, u) \tag{17b}$$

$$\begin{aligned} &L_{f_c}h(x(0|k\Delta t), k\Delta t) \\ &\quad + L_{g_c}h(x(0|k\Delta t), k\Delta t)u(0|k\Delta t)) \\ &\quad + \alpha(h(x(0|k\Delta t), k\Delta t)) \geq \bar{\nu} + c_6 + c_7\bar{w} \end{aligned} \tag{17c}$$

$$u(m\Delta t|k\Delta t) \in \mathbb{U}, \quad \forall m \in \mathbb{N}_{[0,N-1]} \tag{17d}$$

$$e(n\Delta t|k\Delta t) = x(n\Delta t|k\Delta t) - x_r(n\Delta t|k\Delta t) \tag{17e}$$

$$x(0|k\Delta t) = \tilde{x}(k\Delta t), \quad \forall n \in \mathbb{N}_{[0,N]} \tag{17f}$$

where

$$\begin{aligned} J_N(e, u) = &\sum_{n=0}^{N-1} l(e(n\Delta t|k\Delta t), u(n\Delta t|k\Delta t)) \\ &+ V(e(N\Delta t|k\Delta t)). \end{aligned} \tag{18}$$

The FHOC yields a feedback optimal control input $K_N(\tilde{e}) = u^*(0|k\Delta t)$, with respect to the cost function $J_N(e, u)$ and subject to the constraints (17b)-(17f), yielding predicted trajectories for the nominal state $\mathbf{x}_k^*$ - and with (17e), $\mathbf{e}_k^*$ -, and control inputs $\mathbf{u}_k^*$, similarly to (10).

In this approach, the optimization horizon of the MPC, and consequent control inputs, can be divided in two time phases: the set $t_S := \{t \in \mathbb{R}_{\geq 0} : k\Delta t \leq t \leq (k+1)\Delta t\}$, and the set $t_O := \{t \in \mathbb{R}_{\geq 0} : (k+1)\Delta t < t \leq (k+N)\Delta t\}$, as illustrated in Fig. 1. This virtual division is owed to condition (17c), which is imposed only for the first-step of the receding horizon. Condition (17c) implies (as later proved in Lemma 2) that $\tilde{x}(t) \in \mathscr{C}(t), \forall t \in [k\Delta t, (k+1)\Delta t]$, meeting the safety goals defined in Problem 1. On what concerns the second step of the receding horizon, associated with the prediction time t_O, the FHOC (17) is only constrained by the dynamics (17b) and control constraints (17d) where the goal is to minimize the error $e(k\Delta t)$ and control effort $u(k\Delta t), k \in \mathbb{N}$, subject to (17b). When recursively solving the FHOC (17) at each sampling time, safety of the continuous time system (1) through the whole operation is ensured. Consequently, it is ensured that all control inputs sent to the system are optimally safe.

V. Feasibility and Stability Analysis

In this section we present the feasibility and stability analysis derived for the CMPC problem (17).

A. Recursive Feasibility

Lemma 2 details the recursive feasibility of the proposed Corridor MPC defined in (17).

Lemma 2. *Let $h : \mathscr{D} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a SD-ZCBF for the system* (1) *and let $\tilde{x}(0) \in \mathscr{C}(0)$. Then the system* (1) *under the control $u^*(\tilde{x}, k\Delta t)$ ensures that the FHOC problem* (17) *is recursively feasible.*

Proof. Since h is a SD-ZCBF as per Definition 2, there exists a $u \in \mathbb{U}$ such that (17c) holds for any $\tilde{x}(0) \in \mathscr{C}(0)$. Let $t_0 = 0$ and $t_1 = \Delta t$. At t_0, and using Lemma 1, the control input $u(t_0)$ obtained by solving (17) ensures that $\tilde{x}(t) \in \mathscr{C}(t), \forall t \in [t_0, t_1]$. Repeating the same procedure for $t_0 = k\Delta t$ and $t_1 = (k+1)\Delta t, k = 1, \ldots, \infty$, concludes the proof. We note that at $t = (k+1)\Delta t$ a switching of the control input occurs from $u(k\Delta t)$ to $u((k+1)\Delta t)$. However, on the interval $[k\Delta t, (k+1)\Delta t]$, this switch occurs on a set of measure zero such that $u(k\Delta t)$ is applied for almost all $t \in [k\Delta t, (k+1)\Delta t]$. Note, from Remark 2, it is clear that the results of Lemma 1 still hold. □

B. Input-to-State Stability

Input-to-State Stability (ISS) serves as an adequate framework to obtain stability bounds when controlling the system in (3).

Assumption 2. *There exists a $h_e(\tilde{e}(t))$ such that $\forall t \geq 0, \tilde{x} \in \mathscr{D}, h_e(\tilde{e}(t)) = h(\tilde{x}, t)$.*

A common example that satisfies Assumption 2 is $h(\tilde{x}, t) = \varepsilon - \|\tilde{x} - \tilde{x}_r(t)\|_2 = \varepsilon - \|\tilde{e}(t)\|_2 = h_e(\tilde{e}(t))$. Analogously to $\tilde{x}$, let $\mathscr{C}_e(t)$ be defined according to

$$\mathscr{C}_e(t) = \{\tilde{e} \in \mathbb{R}^n : h_e(\tilde{e}(t)) \geq 0\}. \tag{19}$$

We are interested on showing that (9a) is ISS w.r.t $w(k\Delta t) \in \mathbb{W}$ for all $\tilde{e}(k\Delta t) \in \mathscr{C}_e, k \in \mathbb{N}_0$, as the reference trajectory $x_r(t)$ can be designed respecting this condition. The derivations here presented follow closely the results in [22, Sec. 4], adapted to the dynamics in (9). Let ISS be defined as follows.

Definition 3 (Input-to-State Stability (ISS) [22])**.** *Consider that* (9a) *is controlled by a feedback control law $u(k\Delta t) = K_N(e(k\Delta t))$. Then,* (9a) *is ISS w.r.t. $w(k\Delta t) \in \mathbb{W}, k \in \mathbb{N}_0$, if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that for all initial states $\tilde{e}(0)$ and disturbances $w(k\Delta t) \in \mathbb{W}$,*

$$\|\Psi_{\tilde{e}}(k\Delta t, \tilde{e}(0), \mathbf{w}_k)\| \leq \beta(\|\tilde{e}(0)\|, k\Delta t) + \gamma(\|\mathbf{w}_{k-1}\|), k \in \mathbb{N}. \tag{20}$$

For simplicity, let $a \triangleq a(k\Delta t)$, for any function a in the reminder of this section, except where otherwise specified.

Assumption 3. *V(e) is a control Lyapunov function (CLF) for system* (9b) *such that for all $\tilde{e} \in \mathscr{C}_e$ there exist two $\mathcal{K}_\infty$-functions α_V and β_V satisfying $\alpha_V(\|\tilde{e}\|) \leq V(\|\tilde{e}\|) \leq \beta_V(\|\tilde{e}\|)$ and*

$$\min_u \{V(f_e(\tilde{e}, u)) - V(\tilde{e}) + l(\tilde{e}, u)\} \leq 0 \tag{21}$$

such that $\tilde{e} \in \mathscr{C}_e, u \in \mathbb{U}$ and $f_e(\tilde{e}, u) \in \mathscr{C}_e$. Moreover, the cost function $J_N(\tilde{e}, \mathbf{u}_k)$ is uniformly continuous such that

$$\|J_N(\tilde{e}_1, \mathbf{u}_k) - J_N(\tilde{e}_2, \mathbf{u}_k)\| \leq \alpha_{J_N}(\|\tilde{e}_1 - \tilde{e}_2\|) \tag{22}$$

for $\tilde{e}_1, \tilde{e}_2 \in \mathscr{C}_e$, α_{J_N} being a $\mathcal{K}_\infty$-function and for any control sequence $\mathbf{u}_k \in \mathbb{U}$.

It is known [22] that Assumption 3 is sufficient to prove that the optimal cost $J_N^*(\tilde{e})$ is a Lyapunov function of the closed-loop nominal system, for all $\tilde{e} \in \mathscr{C}_e$, yielding asymptotic stabilization of (9b). However, for the disturbed system, we can at best expect to converge to a neighborhood of the origin. Consider now the dynamic system in (9a) satisfying Assumption 1, controlled by $u = K_N(\tilde{e})$ which is solution to the FHOC problem (17) where the terminal cost $V(\tilde{e})$ respects Assumption 3. In this case, it is shown in [22, Thm. 4] that (9a) fulfils the ISS property in $\mathscr{C}_e$ if (i) $f_e(\tilde{e}, u)$ is uniformly continuous for all $\tilde{e} \in \mathscr{C}_e$, or (ii) the optimal cost $J_N^*(\tilde{e})$ is uniformly continuous in $\mathscr{C}_e$. In Theorem 1 it is proven that the latter case can be fulfilled by the CMPC.

Theorem 1. *Consider the system* (9a) *fulfilling Assumption 1, $h(\tilde{x}, t)$ a valid SD-ZCBF and let $h_e(\tilde{e}(t))$ be such that Assumption 2 holds. Let $K_N(\tilde{e})$ be the FHOC in* (17)*, where $V(\tilde{e})$ satisfies Assumption 3, for all $\tilde{e} \in \mathscr{C}_e$. Then, system* (9a) *fulfils the ISS property, defined in Definition 3, in the set $\mathscr{C}_e$ for a sufficiently small bound on the uncertainty w.*

Proof. From [22, Prop. 1-C2] we can show that the optimal cost $J_N^*(\tilde{e})$ is uniformly continuous in the set $\mathscr{C}_e$. For any $\tilde{e} \in \mathscr{C}_e$ and $\mathbf{e}_k^*$ (the optimal trajectory obtained by solving the FHOC in (17) with initial state $\tilde{e}$, and considering the nominal dynamics in (9b)), Bellman's optimality principle implies that $J_N^*(\tilde{e}(k\Delta t)) > J_{N-j}^*(e^*(j\Delta t|k\Delta t)), \forall j = 0, 1, ..., N, \forall k \in \mathbb{N}_0$, given that Assumptions 1 and 3 are respected. Then, for all $\tilde{e} \in \mathscr{C}_e$, all predicted optimal trajectories $\mathbf{e}_k^*$ remain in $\mathscr{C}_e$. Let $\tilde{\mathbf{u}}_k$ be a sequence of $N-1$ control inputs $\tilde{\mathbf{u}}_k = \{u(k\Delta t), .., u((k+N-1)\Delta t)\}$ obtained by solving the FHOC in (17) at the sampling times $t = k\Delta t, ..., (k+N-1)\Delta t$. In other words, $\tilde{\mathbf{u}}_k = \{u^*(0|k\Delta t), ..., u^*(0|(k+N-1)\Delta t)\}$. Such control input trajectory $\tilde{\mathbf{u}}_k$ is known to exist since for any $\tilde{e}(k\Delta t) \in \mathscr{C}_e$ there exists a control input $u \in \mathbb{U}$ such that $\tilde{e}((k+1)\Delta t) \in \mathscr{C}_e$, as long as Assumption 2 holds and $h(\tilde{x}, t)$ is a valid SD-ZCBF. Then, let $\tilde{e}_1, \tilde{e}_2 \in \mathscr{C}_e$ such that $J_N^*(\tilde{e}_1) \geq J_N^*(\tilde{e}_2)$, for which it holds

$$\begin{aligned}\|J_N^*(\tilde{e}_1) - J_N^*(\tilde{e}_2)\| &\leq \|J_N(\tilde{e}_1, \tilde{\mathbf{u}}_k) - J_N(\tilde{e}_2, \tilde{\mathbf{u}}_k)\| \\ &\overset{(22)}{\leq} \alpha_{J_N}(\|\tilde{e}_1 - \tilde{e}_2\|).\end{aligned} \tag{23}$$

Note that $\|J_N^*(\tilde{e}_1) - J_N^*(\tilde{e}_2)\| \leq \|J_N(\tilde{e}_1, \tilde{\mathbf{u}}_k) - J_N(\tilde{e}_2, \tilde{\mathbf{u}}_k)\|$ since $\|u^*(1|k\Delta t)\| \leq \|u^*(0|(k+1)\Delta t)\|$ due to the ZCBF condition in (17c). Therefore, the optimal cost is uniformly continuous in $\mathbb{R}^n$. From [22, Thm. 4], we have that the system (9a) is ISS w.r.t. the disturbance $w \in \mathbb{W}$, completing the proof. ∎

VI. Corridor MPC for Free-flyer Kinematics

In this section we apply the CMPC framework to a free-flyer kinematics model. We consider the free-flyer kinematics [23] given by

$$\dot{\tilde{p}} = u_1 + w_p, \text{ and} \tag{24a}$$
$$\dot{\tilde{\theta}} = \Psi(\tilde{\theta})u_2 + w_\theta, \tag{24b}$$

where $\tilde{p} \in \mathbb{R}^3$ is the position, $\tilde{\theta} \in \mathbb{R}^3$ the attitude euler angles and $\Psi(\theta)$ the attitude Jacobian that translates body rates to Euler angles [24],

$$\Psi(\theta) = \begin{bmatrix} 1 & s_\phi t_\varphi & c_\phi t_\varphi \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi / c_\varphi & c_\phi / c_\varphi \end{bmatrix} \tag{25}$$

where $\phi \in]-\pi, \pi[, \varphi \in]-\frac{\pi}{2}, \frac{\pi}{2}[, \psi \in [-\pi, \pi]$ correspond to the roll, pitch and yaw angles, respectively, and $s_i = \sin i, c_i = \cos i, t_i = \tan i, i = \phi, \varphi$. Then, $\theta := [\phi, \varphi, \psi]^T$. The control input to (24) is a linear velocity u_1 and an angular velocity u_2, which are concatenated in $u := [u_1, u_2]^T \in \mathbb{U}$. Without loss of generality, we assume that $\mathbb{U} = \{u \in \mathbb{R}^6 : \|u_1\| \leq \bar{u}_1 \wedge \|u_2\| \leq \bar{u}_2\}$. Disturbances w_p and w_θ are upper bounded on the sets $\mathbb{W}_p := \{w_p \in \mathbb{R}^3 : \|w_p\| \leq \overline{w}_p\}$ and $\mathbb{W}_\theta := \{w_\theta \in \mathbb{R}^3 : \|w_\theta\| \leq \overline{w}_\theta\}, \forall t \geq 0$, respectively.

For the system at hand, it is convenient to separate the problem in two barriers, one for the translation and another for the attitude. Accordingly, these candidate barriers are written as

$$h_1(\tilde{p}, t) = \varepsilon_p^2 - \|\tilde{p} - p_r(t)\|_2^2, \tag{26}$$
$$h_2(\tilde{\theta}, t) = \varepsilon_\theta^2 - \|\tilde{\theta} - \theta_r(t)\|_2^2 \tag{27}$$

where $p_r : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ and $\theta_r : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ are bounded, continuously differentiable, time-varying references, with bounded derivatives. For simplicity, we drop the explicit time dependence for these variables. The barriers (26) and (27) represent a safe set around a time varying reference, where the maximum allowed error is given by $\varepsilon_p \in \mathbb{R}_{>0}$ and $\varepsilon_\theta \in \mathbb{R}_{>0}$, for the position and attitude, respectively. Proposition 1 establishes the validity of the candidate barrier h_1, while h_2 is handled in a similar way in a manner given after the proof.

Proposition 1. *Consider the kinematics in* (24a) *and the candidate barrier function h_1 in* (26)*. Let $\mathscr{C}_p(t)$ be defined in* (11) *for $h := h_1$. Given a compact set $\mathscr{D}_p$ such that $\mathscr{C}_p(t) \subset \mathscr{D}_p, \forall t \geq 0$, let $\bar{\epsilon}_p = |\min_{\tilde{p} \in \mathscr{D}_p, t \in [0, \infty)} h_1(\tilde{p}, t)|$, and given a $\delta_p \in (0, \varepsilon_p^2)$ suppose the following holds*

$$\delta_p \frac{2\bar{u}_1(\varepsilon_p^2 - \delta_p)}{c_8} - (c_6 + c_7 \bar{w})(\delta_p + \bar{\epsilon}_p) > 0. \tag{28}$$

Then, there exists a $\Delta t^* > 0$ such that $\forall \Delta t \in (0, \Delta t^*)$, h_1 is a SD-ZCBF.

Proof. Consider the system in (24a) and (13), (14). Let $\alpha_1(h_1) = \lambda_1 h_1$. For (24a), it holds that $L_{f_c} h_1(\tilde{p}, k) = 0$ and $L_{g_c} h_1(\tilde{p}, k) = -2(\tilde{p} - p_r(k))$, $\bar{c}_2 = \underline{c}_2 = 0, c_4 = c_5 = \bar{u}_1$, $\bar{w} = \max_{t \in [0, \infty)} \|w_p(t)\|$. Then, let $\bar{c}_1, \bar{c}_3 \in \mathbb{R}_{>0}$ be the respective Lipschitz constants of $\lambda_1 h_1(\tilde{p},t)$, $L_{g_c} h_1(\tilde{p},t)$ with respect to $\tilde{p}$, $\underline{c}_1, \underline{c}_3 \in \mathbb{R}_{>0}$ be the respective Lipschitz constants of $\lambda_1 h_1(\tilde{p},t)$, $L_{g_c} h_1(\tilde{p},t)$ with respect to t, $\forall \tilde{p} \in \mathscr{D}_p, t \in [0,\infty)$, $c_6 = \max_{\tilde{p}\in\mathscr{D}_p, t\in[0,\infty)} \|2(\tilde{p}-p_r(t))^T \dot{p}_r(t)\|$, $c_7 = \max_{\tilde{p}\in\mathscr{D}_p, t\in[0,\infty)} \|2(\tilde{p}-p_r(t))\|$, and let $\nu(\Delta t)$ be defined as in (14).

Let $u_1(k\Delta t)$ be defined as

$$u_1(k\Delta t) = \begin{cases} -\frac{1}{2}(\tilde{p} - p_r(k\Delta t))\kappa_p, & \text{for } h_1 \in [-\bar{\epsilon}_p, \delta_p], \\ 0, & \text{otherwise} \end{cases}, \tag{29}$$

$\forall k \in \mathbb{N}_0$, where $\bar{\epsilon}_p = |\min_{\tilde{p}\in\mathscr{D}_p, t\in[0,\infty)} h_1(\tilde{p},t)|$, $0 < \delta_p < \varepsilon_p^2$ and $\kappa_p \in \mathbb{R}_{>0}$. We note that $\bar{\epsilon}_p$ is well-defined due to $\mathscr{D}$ being compact and $p_r(t)$ bounded. Furthermore, u_1 is well defined on $\mathscr{D}$ for any $k \in \mathbb{N}_0$. We will show that with this choice of u_1 and α_1, (13) hold.

We will start by looking at the case when $h_1 \in [-\bar{\epsilon}_p, \delta_p]$. For this case, we substitute u_1 into the left hand-side of (13): $\|\tilde{p} - p_r(t))\|_2^2 \kappa_p + \lambda_1 h_1$. Considering the value of h_1 at the extremes of its interval, we can write $h_1 \le \delta_p \implies \|\tilde{p}-p_r(t))\|_2^2 \ge \varepsilon_p^2 - \delta_p$, which together with $h_1 \ge -\bar{\epsilon}_p$ yields $\|\tilde{p} - p_r(t))\|_2^2\kappa_p + \lambda_1 h_1 \ge (\varepsilon_p^2 - \delta_p)\kappa_p - \lambda_1\bar{\epsilon}_p$. Furthermore, we choose κ_p to satisfy the following condition

$$\kappa_p \ge \frac{\lambda_1\bar{\epsilon}_p + \nu(\Delta t) + c_6 + c_7\bar{w}}{\varepsilon_p^2 - \delta_p} \tag{30}$$

for which (13) holds.

Next, we investigate the case when $h_1 \in [\delta_p, \varepsilon_p^2]$. Selecting λ_1 such that

$$\lambda_1 \ge \frac{((\bar{c}_2 + \bar{c}_3 c_4)(c_5 + \bar{w}) + (\underline{c}_2 + \underline{c}_3 c_4))\Delta t + c_6 + c_7\bar{w}}{\delta_p - (c_7(c_5+\bar{w}) + c_6)\Delta t} \tag{31}$$

guarantees that (13) holds.

For notation brevity, let $d_1 = \big((\bar{c}_2 + \bar{c}_3 c_4)(c_5 + \bar{w}) + (\underline{c}_2 + \underline{c}_3 c_4)\big)$, $d_2 = c_7(c_5+\bar{w}) + c_6$, $d_3 = c_6 + c_7\bar{w}$, $d_4 = \varepsilon_p^2 - \delta_p$ and $d_5 = \frac{2\bar{u}_1 d_4}{c_8}$ and note that $d_1, ..., d_5 \in \mathbb{R}_{>0}$, $\bar{c}_1 = \lambda_1 c_7$ and $\underline{c}_1 = \lambda_1 c_6$. In this manner, (30) and (31) may be written as $\kappa_p \ge \frac{\lambda_1\bar{\epsilon}_p + \nu(\Delta t) + d_3}{d_4}$ and $\lambda_1 \ge \frac{d_1\Delta t + d_3}{\delta_p - d_2\Delta t}$, respectively. To ensure that $\|u_1\| \in \mathbb{U}$ and that (13) is satisfied, we include the actuation limit $\bar{u}_1 \ge \|-\frac{1}{2}c_8\kappa_p\|$, where $c_8 = \max_{\tilde{p}\in\mathscr{D}_p, t\in[0,\infty)}\|\tilde{p}-p_r(t)\|$. Then, we have $\bar{u}_1 \ge \kappa_p c_8 \frac{1}{2}$, which with (30) leads to $\frac{2\bar{u}_1}{c_8} \ge \frac{1}{d_4}(\lambda_1\bar{\epsilon}_p + \nu(\Delta t) + d_3)$. With $\bar{c}_1 = \lambda_1 c_7$ and $\underline{c}_1 = \lambda_1 c_6$, we get $\frac{2\bar{u}_1 d_4}{c_8} \ge \lambda_1\bar{\epsilon}_p + \lambda_1 d_2\Delta t + d_1\Delta t + d_3$, and with (31), yields $d_5 - d_3 \ge \frac{d_1\Delta t + d_3}{\delta_p - d_2\Delta t}(\bar{\epsilon}_p + d_2\Delta t) + d_1\Delta t$. At this point, let $\Delta t^* = \min\{\Delta t_1^*, \Delta t_2^*\} > 0$, where $\Delta t_1^* = \frac{\delta_p}{d_2}$ and $\Delta t_2^* = \frac{\delta_p d_5 - d_3(\delta_p + \bar{\epsilon}_p)}{\delta_p d_1 + d_2 d_5 + d_1\bar{\epsilon}_p}$. Then, for any $\Delta t \in (0, \Delta t^*)$, the following relation holds:

$$(\delta_p - d_2\Delta t)(d_5 - d_3 - d_1\Delta t) \ge (d_1\Delta t + d_3)(\bar{\epsilon}_p + d_2\Delta t). \tag{32}$$

We can now expand the left and right hand-sides of (32) to $\delta_p d_5 - \delta_p d_3 - \delta_p d_1\Delta t - d_2 d_5\Delta t + d_2 d_3\Delta t + d_1 d_2(\Delta t)^2 \ge d_1\bar{\epsilon}_p\Delta t + d_1 d_2(\Delta t)^2 + d_3\bar{\epsilon}_p + d_2 d_3\Delta t$. Simplifying this expression and factoring out Δt yields

$$\frac{\delta_p d_5 - d_3(\delta_p + \bar{\epsilon}_p)}{\delta_p d_1 + d_2 d_5 + d_1\bar{\epsilon}_p} \ge \Delta t. \tag{33}$$

Thus, there exists a non-empty interval $(0, \Delta t^*)$, for which any $\Delta t \in (0, \Delta t^*)$ ensures that h_1 is a SD-ZCBF. Since there exists a u_1 such that $\|u_1\| \le \bar{u}_1$, the candidate barrier function (26) is a valid SD-ZCBF, completing the proof. □

Translation Dynamics - (24a)	**Attitude Dynamics** - (24b)
c_4 = 0.8660	c_4 = 0.3464
c_7 = 3.4641	c_7 = 1.0392
$\bar{c}_2 = \underline{c}_2$ = 0	$\bar{c}_2 = \underline{c}_2$ = 0
$\bar{c}_3$ = 2	$\bar{c}_3$ = 2
$\underline{c}_3 = 2(c_4 + 0.1)$	$\underline{c}_3 = 2(c_4 + 0.01)$
$c_5 = c_4$	$c_5 = c_4$
$c_6 = 0.1c_7$	$c_6 = 0.01c_7$
$c_8 = 0.5c_7$	$c_8 = 0.5c_7$
λ_1 = 1.0075	λ_2 = 1.6249
ε_p = 1.32	ε_θ = 0.4338
δ_p = 0.2378	δ_θ = 0.0399
Δt_p = 0.01	Δt_θ = 0.01

TABLE I: System constants for the dynamics in (24).

Repeating Proposition 1's proof for h_2 and considering the candidate control input

$$u_2(k) = \begin{cases} -\frac{1}{2}\Psi(\tilde{\theta})^{-1}(\tilde{\theta} - \theta_r(k))\kappa_\theta, & \text{for } h_2 \in [-\bar{\epsilon}_\theta, \delta_\theta] \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

leads to the existence of a similar upper bound for Δt, proving that h_2 is, also, a valid SD-ZCBF.

It is important to note that there exists a trade-off between the maximum control input allowed by the system, the maximum disturbance and reference rate-of-change that the system can handle, and the maximum allowed tracking error. This relation is evident from (28). For small $\bar{u}_1$, ε needs to be large to satisfy $\Delta t > 0$. The same relation holds for large disturbances or fast-changing references, which influence the constants c_6, c_7 and $\bar{w}$.

Remark 3. *It is important to note that the candidate controllers* (29) *and* (34) *are only examples of feasible solutions. With CMPC, an optimized control input is generated, respecting the constraint* (17c) *for both* (26) *and* (27).

VII. Numerical Results

In this section we show simulation results for the CMPC, when applied to the free-flyer kinematics in Section VI. We consider the barriers defined in (26) and (27). To derive a feasible sampling time Δt that respects the conditions in Proposition 1, while minimizing $\varepsilon_{p/\theta}$ and $\delta_{p/\theta}$, a non-linear optimization routine was implemented in Matlab, taking into account the constants in Table I for the dynamics in (24a) and (24b). Accordingly, the Corridor MPC was initialized with $l(e,u) = e^TQe + u^TRu$, $V(e) = e^TPe$, where $Q = \text{diag}([100,100,100,50,50,50])$, $R = \text{diag}([50,50,50,30,30,30])$, $P = 100\cdot Q$, a prediction horizon of $N = 30$ and $\Delta t = 0.01s$. We also consider $\bar{w}_p =$

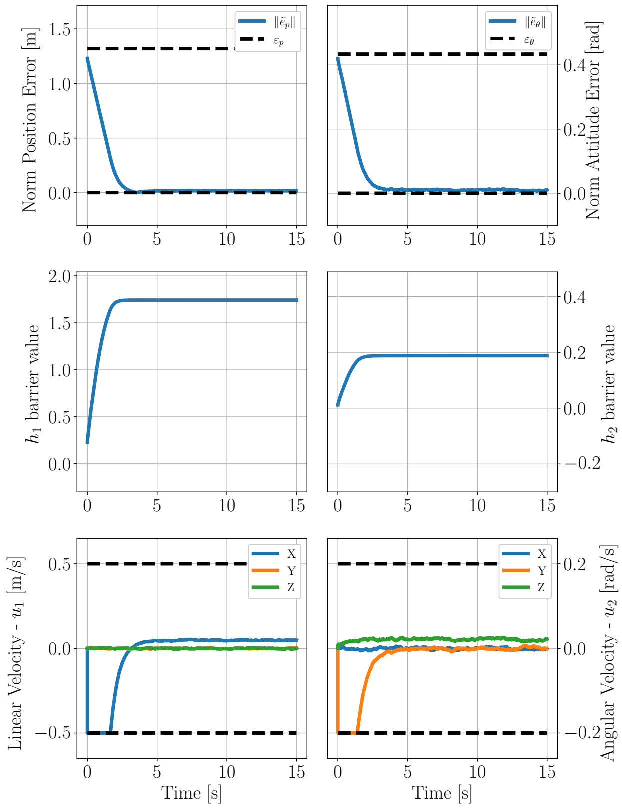

Fig. 2: Corridor MPC for a Free-flyer Kinematics. The system is robustly kept inside a safe corridor, and stabilized on a neighborhood of the origin through optimal constrained control inputs.

0.001 and $\bar{w}_\theta = 0.001$. Lastly, the reference trajectory was created according to Assumption 1 with $x_r(0) = 0_6$ and $u_r(t) = [0.1, 0, 0, 0, 0, 0.01]^T$, and the system initial state was set to $\tilde{x}(0) = [0.8, 0, 0, 0, 0.45, 0]$, where 0_6 is a zero vector in $\mathbb{R}^6$. The system was simulated for 15 seconds. The results for this simulation are presented in Fig. 2.

As it is possible to observe from Figure 2, the control limits are respected over the entire operation, while the system state is kept inside the safe region, delimited by the dashed lines in the control and error plots. It is also clear that the barrier values are always positive, showing that the system is kept within the prescribed safety bounds.

VIII. Conclusions and Future Work

In this paper we present a novel concept for combining safety and optimality for trajectory tracking controllers, named Corridor MPC. We provide recursive feasibility and practical stability results for all states starting inside a user-defined maximum error with respect to a time-varying trajectory. As future work, possible research directions span: introducing practical stability guarantees for the sampled-data system; introducing less conservative bounds for the continuous-time safety conditions; and lastly, exploring distributed frameworks for the Corridor MPC formulation, in a multi-agent setting.

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