Controller Synthesis of Collaborative Signal Temporal Logic Tasks for Multi-Agent Systems via Assume-Guarantee Contracts

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***Abstract*— This paper considers the problem of controller synthesis of a fragment of signal temporal logic (STL) specifications for large-scale multi-agent systems, where the agents are dynamically coupled and subject to collaborative tasks. A compositional framework based on continuous-time assume-guarantee contracts is developed to break the complex and large synthesis problem into subproblems of manageable sizes. We first show how to formulate the collaborative STL tasks as assume-guarantee contracts by leveraging the idea of prescribed performance control. The concept of contracts is used to establish our compositionality result, which allows us to guarantee the satisfaction of a global contract by the multi-agent system when all agents satisfy their local contracts. Then, a closed-form continuous-time feedback controller is designed to enforce local contracts over the agents in a distributed manner, which further guarantees global task satisfaction based on the compositionality result. The effectiveness of our results is demonstrated by two numerical examples.**

***Index Terms*— Multi-agent systems, signal temporal logics, formal methods, assume-guarantee contracts, distributed control, prescribed performance control.**

I. Introduction

Over the last few decades, multi-agent systems have received increasing attention with a wide range of applications in various areas, including multi-robot systems, social networks, autonomous driving, and smart grids. Revolving around the control of multi-agent systems, extensive research works have been established in the past two decades that deal with tasks such as formation control [1], consensus [2], [3], and coverage [4]; see [5] for an overview.

In recent years, there has been a growing trend in the planning and control of single- and multi-agent systems under more complex high-level specifications expressed as temporal logics. Temporal logic, including linear temporal logics (LTL) and signal temporal logics (STL), resembles natural languages and offers a powerful framework for rigorously reasoning about the temporal behaviors of systems. Thus, they can be used to express complex high-level specifications that extend beyond standard control objectives [6]. One potential solution that appeared more recently is the *correct-by-construction synthesis scheme* [6], [7], which is built upon formal methods-based approaches to formally verify or synthesize certifiable controllers against rich specifications given by temporal logic formulae. Planning and control of temporal logic formulae are addressed in [8], [9] for single agent and in [10]–[12] for multi-agent systems. However, when encountering large-scale systems, most of the above-mentioned approaches suffer severely from the *curse of dimensionality* due to their high computational complexity, which limits their applications to systems of moderate size. To tackle this complexity issue, different compositional approaches were developed for the analysis and control of large-scale and interconnected systems. The most common types of compositional approaches are based on input-output properties (e.g., small-gain or dissipativity properties) [13]–[17] and assume-guarantee contracts [14], [15], [18]–[22], respectively. Specifically, the notion of assume-guarantee contracts (AGCs) prescribes properties that a component must guarantee under assumptions on the behavior of its environment (or its neighboring agents) [23].

The main goal of this paper is to develop a compositional framework for the controller synthesis of signal temporal logic (STL) formulae on large-scale multi-agent systems. The synthesis of STL properties for control systems has attracted a lot of attention in the last few years. Compared to LTL, which is a propositional temporal logic that deals only with discrete-time signals, STL [24] is a predicate logic interpreted over continuous-time signals that allows to formulate more expressive tasks with real-time and real-valued constraints. Moreover, STL naturally entails space robustness [25] which enables one to assess the robustness of satisfaction. Despite the strong expressivity of STL formulae, the synthesis of control systems under STL specifications is known to be challenging due to its nonlinear, nonconvex, noncausal, and nonsmooth semantics. In [26], the problem of synthesizing STL tasks on discrete-time systems is handled using model predictive control (MPC) where space robustness is encoded as mixed-integer linear programs. The results in [27] established a connection between prescribed performance control (PPC) and the robust semantics of STL specifications, based on which continuous-time feedback control laws are derived for multi-agent systems [28]. Barrier function-based approaches are proposed for the collaborative control of STL tasks for

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multi-agent systems [29]. The recent results in [30] developed efficient decentralized and distributed control frameworks to enforce a prescribed-time stability property for multi-agent systems, which allows dynamical interactions with limited communications among agents.

In this paper, we consider continuous-time and interconnected multi-agent systems under a fragment of STL specifications. In this setting, each agent is subject to a local STL task that may depend on the behavior of other agents, whereas the interconnection of agents is induced by the dynamical couplings between each other. In order to provide a compositional framework to synthesize distributed controllers for a multi-agent system, we first formulate the desired STL formulae as prescribed performance control problems, and then introduce assume-guarantee contracts for the agents by leveraging the derived prescribed performance functions for the STL tasks. Two concepts of contract satisfaction, i.e., *weak satisfaction* and *uniform strong satisfaction* (cf. Definition 8) are introduced to establish our compositionality results using assume-guarantee reasoning, i.e., if all agents satisfy their local contracts, then the global contract is satisfied by the multi-agent system. In particular, we show that *weak satisfaction* of the local contracts is sufficient to deal with multi-agent systems with acyclic interconnection topologies, while *uniform strong satisfaction* is needed to reason about general interconnection topologies containing cycles. Based on the compositional reasoning, we then present a controller synthesis approach in a distributed manner. In particular, continuous-time closed-form feedback controllers are derived for the agents that ensure the satisfaction of local contracts, thus leading to the satisfaction of global contract by the multi-agent system. To the best of our knowledge, this paper is the first to handle STL specifications on multi-agent systems using assume-guarantee contracts. Thanks to the derived closed-form control strategy and the distributed framework, our approach requires very low computational complexity compared to existing results in the literature, which mostly rely on discretizations in state space or time [6], [7].

Related work: While AGCs have been extensively used in the computer science community [23], [31], new frameworks of AGCs for dynamical systems with continuous state-variables have been proposed recently in [18], [20] for continuous-time systems, and [15], [32, Chapter 2] for discrete-time systems. In this paper, we follow the same behavioral framework of AGCs for continuous-time systems proposed in [18]. In the following, we provide a comparison with the approach proposed in [18], [20]. A detailed comparison between the framework in [18], the one in [15] and existing approaches from the computer science community [23], [31] can be found in [18, Section 1].

The contribution of the paper is threefold:

• At the level of compositionality rules: The authors in [18] rely on a notion of *strong contract satisfaction* to provide a compositionality result (i.e., how to go from the satisfaction of local contracts at the component's level to the satisfaction of the global specification for the interconnected system) under the condition of the set of guarantees (of the contracts) being closed. In this paper, we are dealing with STL specifications, which are encoded as AGCs made of open sets of assumptions and guarantees. The non-closedness of the set of guarantees makes the concept of contract satisfaction proposed in [18] not sufficient to establish a compositionality result. For this reason, in this paper, we introduce the concept of *uniform strong contract satisfaction* and show how the proposed concept makes it possible to go from the local satisfaction of the contracts at the component's level to the satisfaction of the global STL specification at the interconnected system's level.

• At the level of the considered control objectives: When the objective is to synthesize controllers to enforce the satisfaction of AGCs for continuous-time systems, to the best of our knowledge, existing approaches in the literature can only deal with the particular class of invariance AGCs[1] in [20], where the authors used symbolic control techniques to synthesize controllers. In this paper, we present a new approach to synthesize controllers for a more general class of AGCs, where the set of assumptions and guarantees are described by STL formulae, by leveraging tools in the spirit of prescribed performance-based control.

• At the level of distributed control of STL tasks: Compared to the results in [28] which use similar PPC-based strategies for the control of STL tasks for multi-agent systems, our proposed control law allows for distinct collaborative STL tasks over the agents, whereas [28] is restricted to identical STL tasks shared by all neighboring agents. A preliminary investigation of our results appeared in [33]. Our results here improve and extend those in [33] in the following directions. First, the compositional approach developed in [33] is tailored to interconnected systems without communication or collaboration among the components, and thus can only deal with non-collaborative STL tasks. Here, we deal with general multi-agent systems with three types of interactions among agents, including dynamical couplings, communication/sensing, and task dependencies, which facilitate the handling of collaborative STL tasks. Second, different from the result in [33], we present two different compositionality results which show that weak satisfaction of contracts is sufficient to deal with acyclic interconnections (in terms of dynamical couplings) while strong satisfaction is needed to reason about general interconnections containing cycles.

II. Preliminaries and Problem Formulation

Notation: We denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real and non-negative integers, respectively. These symbols are annotated with subscripts to restrict them in the usual way, e.g., $\mathbb{R}_{>0}$ denotes the positive real numbers. We denote by $\mathbb{R}^n$ an n-dimensional Euclidean space and by $\mathbb{R}^{n\times m}$ a space of real matrices with n rows and m columns. We denote by I_n the identity matrix of size n, and by $\mathbf{1}_n = [1,\dots,1]^\mathsf{T}$ the vector of all ones of size n. We denote by $\mathrm{diag}(a_1,\dots,a_n)$ the diagonal matrix with diagonal elements being $a_1,\dots,a_n$. Consider sets $S_1, S_2, \dots, S_n$. We denote by $\prod_{i=1}^n S_i$ the Cartesian product of $S_1, S_2, \dots, S_n$. For a set K, the cardinality of K is denoted $|K|$. For each $j \in \{1,2,\dots,n\}$, the jth projection on $S = \prod_{i=1}^n S_i$ is the mapping $\mathrm{proj}^j : S \to S_j$ defined by:

[1] where the set of assumptions and guarantees of the contract are described by invariants.

$\text{proj}^j(s_1, s_2, \ldots, s_j, \ldots, s_n) = s_j$ for all $(s_1, s_2, \ldots, s_n) \in S$. Moreover, we further define $\overline{\mathbf{Proj}}^j(S) = S_j$ for all $j \in \{1, 2, \ldots, n\}$.

A. Signal Temporal Logic (STL)

Signal temporal logic (STL) is a predicate logic based on continuous-time signals, which consists of predicates μ that are obtained by evaluating a continuously differentiable predicate function $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}$. Specifically, $\mathcal{P}$ is the predicate function associated with μ which assigns the respective true or false boolean value of μ as: $\mu := \begin{cases} \top & \text{if } \mathcal{P}(x) \geq 0 \\ \bot & \text{if } \mathcal{P}(x) < 0, \end{cases}$ for $x \in \mathbb{R}^n$, where $\top$ and $\bot$ denotes true and false, respectively. We consider in this paper an STL fragment that is defined as

$$\psi ::= \top \mid \mu \mid \neg\mu \mid \psi_1 \wedge \psi_2, \tag{1}$$

$$\phi ::= G_{[a,b]}\psi \mid F_{[a,b]}\psi \mid F_{[\underline{a},\underline{b}]}G_{[\bar{a},\bar{b}]}\psi, \tag{2}$$

where μ is the predicate, ψ in (2) and ψ_1, ψ_2 in (1) are formulae of class ψ given in (1). The operators $\neg, \wedge, G_{[a,b]}, F_{[a,b]}$ denote the negation, conjunction, always, and eventually operators, respectively, with $a, b \in \mathbb{R}_{\geq 0}$ and $a \leq b$. The meaning of these operators will be specified later by the definition of STL semantics. We refer to ψ given in (1) as non-temporal formulae, i.e., boolean formulae, while ϕ is referred to as temporal formulae due to the use of always- and eventually-operators. We use $(\mathbf{x}, t) \models \phi$ to denote that the state trajectory $\mathbf{x} : \mathbb{R}_{\geq 0} \to X \subseteq \mathbb{R}^n$ satisfies ϕ at time t, where $\models$ denotes the satisfaction relation. The trajectory $\mathbf{x} : \mathbb{R}_{\geq 0} \to X \subseteq \mathbb{R}^n$ satisfying formula ϕ is denoted by $(\mathbf{x}, 0) \models \phi$. The STL semantics [24, Definition 1] of the fragment in (1)-(2) can be recursively given by:

$$\begin{aligned} (\mathbf{x}, t) \models \mu &\iff \mathcal{P}(\mathbf{x}(t)) \geq 0 \\ (\mathbf{x}, t) \models \neg\phi &\iff \neg((\mathbf{x}, t) \models \phi) \\ (\mathbf{x}, t) \models \phi_1 \wedge \phi_2 &\iff (\mathbf{x}, t) \models \phi_1 \wedge (\mathbf{x}, t) \models \phi_2 \\ (\mathbf{x}, t) \models F_{[a,b]}\phi &\iff \exists \bar{t} \in [t+a, t+b] \text{ s.t. } (\mathbf{x}, \bar{t}) \models \phi \end{aligned}$$

The always-operator can be derived as $G_{[a,b]}\phi = \neg F_{[a,b]}\neg\phi$.

Next, we recall that STL is naturally equipped with a quantitative semantics, called robust semantics [25, Def. 3] (also referred to as space robustness). This quantitative semantics can be interpreted as "how much (robustly) a signal $\mathbf{x}$ satisfies or violates a STL formula ϕ". Formally, space robustness for the STL operators considered in this work is defined as:

$$\begin{aligned} \rho^{\mu}(\mathbf{x}, t) &:= \mathcal{P}(\mathbf{x}(t)) \\ \rho^{\neg\phi}(\mathbf{x}, t) &:= -\rho^{\phi}(\mathbf{x}, t) \\ \rho^{\phi_1 \wedge \phi_2}(\mathbf{x}, t) &:= \min(\rho^{\phi_1}(\mathbf{x}, t), \rho^{\phi_2}(\mathbf{x}, t)) \\ \rho^{F_{[a,b]}\phi}(\mathbf{x}, t) &:= \max_{t_1 \in [t+a, t+b]} \rho^{\phi}(\mathbf{x}, t_1) \\ \rho^{G_{[a,b]}\phi}(\mathbf{x}, t) &:= \min_{t_1 \in [t+a, t+b]} \rho^{\phi}(\mathbf{x}, t_1) \\ \rho^{F_{[\underline{a},\underline{b}]}G_{[\bar{a},\bar{b}]}\phi}(\mathbf{x}, t) &:= \max_{t_1 \in [t+\underline{a}, t+\underline{b}]} \min_{t_2 \in [t_1+\bar{a}, t_1+\bar{b}]} \rho^{\phi}(\mathbf{x}, t_2) \end{aligned}$$

where $\rho^{\phi}(\mathbf{x}, t)$ are real-valued functions mapping from $X \times \mathbb{R}_{\geq 0}$ to $\mathbb{R}$ and are referred to as *robustness functions*. Note that every STL formula ϕ is equipped with a robustness function $\rho^{\phi}(\mathbf{x}, t)$ for which it holds that: $(\mathbf{x}, t) \models \phi$ if $\rho^{\phi}(\mathbf{x}, t) > 0$ [34, Proposition 16]. The robustness functions will be employed in the controller design process to enforce STL satisfaction. Similarly to [35], the non-smooth conjunction can be approximated by smooth functions as $\rho^{\phi_1 \wedge \phi_2}(\mathbf{x}, t) \approx -\frac{1}{\eta}\ln(\exp(-\eta\rho^{\phi_1}(\mathbf{x}, t)) + \exp(-\eta\rho^{\phi_2}(\mathbf{x}, t)))$. Note that $-\frac{1}{\eta}\ln(\exp(-\eta\rho^{\phi_1}(\mathbf{x}, t)) + \exp(-\eta\rho^{\phi_2}(\mathbf{x}, t))) \leq \min(\rho^{\phi_1}(\mathbf{x}, t), \rho^{\phi_2}(\mathbf{x}, t))$ holds for any choice of $\eta > 0$, and the equality holds as $\eta \to \infty$. Thus, we have $(\mathbf{x}, t) \models \phi_1 \wedge \phi_2$ as long as $-\frac{1}{\eta}\ln(\exp(-\eta\rho^{\phi_1}(\mathbf{x}, t)) + \exp(-\eta\rho^{\phi_2}(\mathbf{x}, t))) > 0$.

Note that the considered STL fragment allows us to encode concave temporal tasks, which is a necessary assumption used later for the design of closed-form, continuous feedback controllers (cf. Assumption 18). It should be mentioned that by leveraging the results in e.g., [36], it is possible to expand our results to full STL semantics.

B. Multi-agent systems and interconnection topologies

Consider a team of $N \in \mathbb{N}$ agents Σ_i, $i \in I = \{1, \ldots, N\}$. Each agent Σ_i is a tuple $\Sigma_i = (X_i, U_i, W_i, f_i, g_i, h_i)$, where

- $X_i = \mathbb{R}^{n_i}$, $U_i = \mathbb{R}^{m_i}$ and $W_i = \mathbb{R}^{p_i}$ are the state, external input, and internal input spaces, respectively;
- $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is the flow drift, $g_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times m_i}$ is the external input matrix, and $h_i : \mathbb{R}^{p_i} \to \mathbb{R}^{n_i}$ is the internal input map.

A trajectory of Σ_i is a uniformly continuous map $(\mathbf{x}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times W_i$ such that for all $t \geq 0$

$$\dot{\mathbf{x}}_i(t) = f_i(\mathbf{x}_i(t)) + g_i(\mathbf{x}_i(t))\mathbf{u}_i(t) + h_i(\mathbf{w}_i(t)), \tag{3}$$

where $\mathbf{u}_i : \mathbb{R}_{\geq 0} \to U_i$ is the external input trajectory, and $\mathbf{w}_i : \mathbb{R}_{\geq 0} \to W_i$ is the internal input trajectory. Note that $u_i \in U_i$ are "external" inputs served as interfaces for controllers, and $w_i \in W_i$ are termed as "internal" inputs describing the physical interaction between agents which may be unknown to agent Σ_i. Note that the considered nonlinear control affine systems as in (3) have been extensively studied in nonlinear control theory [37], [38]. Control affine systems with drift terms are very general, since they can model a wide range of real-world physical systems, including various types of vehicle dynamics and control systems, e.g., dynamical systems subject to underactuation or nonholonomic motion constraints.

Assumption 1: Consider agent Σ_i as in (3). The functions $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $g_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times m_i}$, and $h_i : \mathbb{R}^{p_i} \to \mathbb{R}^{n_i}$ are locally Lipschitz continuous, and $g_i(\mathbf{x}_i)g_i(\mathbf{x}_i)^\top$ is positive definite for all $\mathbf{x}_i \in \mathbb{R}^{n_i}$.

Remark 2: We assume that the functions that appeared in the system dynamics (3) are locally Lipschitz continuous, which will be used later in Theorem 22 to ensure the boundedness of $f_i(\mathbf{x}_i), g_i(\mathbf{x}_i), h_i(\mathbf{x}_i)$ in bounded domains. Note that any continuously differentiable function is locally Lipschitz. Remark that $g_i(\mathbf{x}_i)g_i(\mathbf{x}_i)^\top$ is positive definite if and only if $g_i(\mathbf{x}_i)$ has full row rank, which implies that $m \geq n$. Note that in the case where $m = n$, it implies that the system is fully actuated. This assumption captures for instance the dynamics of omnidirectional robots or room temperature regulation as in Sec. V, and other practical control applications including robotic manipulators with one actuator per degree of freedom.

A multi-agent system consisting of $N \in \mathbb{N}$ agents Σ_i is formally defined as follows.

Definition 3: (Multi-agent system) Given $N \in \mathbb{N}$ agents Σ_i, $i \in \{1, \ldots, N\}$, as described in (3). A multi-agent system denoted by $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ is a tuple $\Sigma = (X, U, f, g)$ where

- $X = \prod_{i \in I} X_i$ and $U = \prod_{i \in I} U_i$ are the state and external input spaces, respectively;
- $f : \mathbb{R}^n \to \mathbb{R}^n$ is the flow drift and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is the external input matrix defined as : $f(\mathbf{x}(t)) = [f_1(\mathbf{x}_1(t)) + h_1(\mathbf{w}_1(t)); \ldots; f_N(\mathbf{x}_N(t)) + h_N(\mathbf{w}_N(t))]$, $g(\mathbf{x}(t)) = \mathrm{diag}(g_1(\mathbf{x}_1(t)), \ldots, g_N(\mathbf{x}_N(t)))$, where $n = \sum_{i \in I} n_i$, $m = \sum_{i \in I} m_i$, $\mathbf{x} = [\mathbf{x}_1; \ldots; \mathbf{x}_N]$, $\mathbf{w}_i(t) = [\mathbf{x}_{j_1}(t); \ldots; \mathbf{x}_{j_{|\mathcal{N}_i^a|}}(t)]$, for all $i \in I$, where $\mathcal{N}_i^a$ denotes the set of agents providing internal inputs to Σ_i.

A trajectory of Σ is then a uniformly continuous map $\mathbf{x} : \mathbb{R}_{\geq 0} \to X$ such that for all $t \geq 0$, $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + g(\mathbf{x}(t))\mathbf{u}(t)$, where $\mathbf{u} = [\mathbf{u}_1; \ldots; \mathbf{u}_N]$ is the external input trajectory.

C. Interaction topologies among agents

Note that the interaction topology plays an important role in the analysis and synthesis of networked multi-agent systems [5]. In this paper, we consider a general setup in the synthesis of multi-agent systems by considering three types of interactions among agents, which are determined by the *dynamical interconnection topology*, *communication topology*, and *task dependency topology*, respectively. Throughout the paper, we will use the graph-based representations of these network topologies, as described in the following.

Dynamical interconnection graph $\mathcal{G}^a = (I, E^a)$**:** In this paper, we consider the existence of physical interactions in terms of unknown dynamical couplings between agents, captured by $h_i(\mathbf{w}_i)$ as in (3). We denote by a directed graph $\mathcal{G}^a = (I, E^a)$ as the *interconnection graph* capturing the dynamical couplings among the N agents, where $I = \{1, \ldots, N\}$, and $(j, i) \in E^a$, indicate that agent j provides internal inputs to agent i through dynamical couplings in (3). Thus, we formally have $\mathcal{N}_i^a = \{j \in I | (j, i) \in E^a\}$. Given an interconnection graph $\mathcal{G}^a$, we define $I^{Init} = \{i \in I | \mathcal{N}_i^a = \emptyset\}$ as the set of agents who do not have any adversarial neighbors that impose unknown dynamical couplings on them. Note that in Definition 3, the interconnection structure indicates that the internal input $\mathbf{w}_i$ of an agent Σ_i is the stacked state $[\mathbf{x}_{j_1}; \ldots; \mathbf{x}_{j_{|\mathcal{N}_i^a|}}]$ of its adversarial neighbors Σ_j, $j \in \mathcal{N}_i^a$.

Communication graph $\mathcal{G}^c = (I, E^c)$**:** We denote by an undirected graph $\mathcal{G}^c = (I, E^c)$ the *communication graph* among the N agents, where $I = \{1, \ldots, N\}$, $(i, j) \in E^c$, $\forall i, j \in I$ are unordered pairs indicating communication links between agents i and j. The existence of a communication link between agents i and j reflects the fact that these two agents can exchange information directly through communication channels or active sensing, and thus the control input of agent i can depend on the state of agent j and vice versa.

Task dependancy graph $\mathcal{G}^t = (I, E^t)$**:** We consider a multi-agent system subject to a global STL specification $\bar{\phi}$, which is decomposed into local ones ϕ_i, $i = 1, \ldots, N$ [2]. Note that the satisfaction of task ϕ_i do not only depend on the behavior of Σ_i, but may also depend on the behavior of other agents $I \setminus \{i\}$ [3]. If the satisfaction of ϕ_i depends on the behavior of more than one agent, ϕ_i is referred to as a *collaborative task*. If ϕ_i only depends on one agent Σ_i, ϕ_i is called a *non-collaborative task*. To characterize the task dependencies in the multi-agent system, we define the *task dependency graph* as $\mathcal{G}^t = (I, E^t)$, which is a directed graph with $I = \{1, \ldots, N\}$, and $(i, j) \in E^t$ if and only if the formula ϕ_i depends on Σ_j.

Assumption 4: For each ϕ_i depending on agent Σ_j, $j \in I$, agent Σ_i can communicate with Σ_j, i.e., we have $\mathcal{G}^t \subseteq \mathcal{G}^c$ with $(i, j) \in E^t \implies (i, j) \in E^c$.

Assumption 4 implies that the task dependencies are compatible with the communication graph topology of the multi-agent system. If this assumption does not hold, one can leverage communication-aware task decomposition techniques, such as the one proposed in our recent work [39], to decompose the global STL task such that the resulting task dependency graph is compatible with the given communication topology.

Note that a path is a sequence $v_1 v_2 \ldots v_m$ of nodes in the graph, such that $(v_i, v_{i+1}) \in E$, $\forall i \in \{1, \ldots, m-1\}$, i.e., every consecutive pair of nodes is an edge in the graph. A cycle is a path $v_1 v_2 \ldots v_m$ where $v_m = v_1$ and the vertices $v_1, \ldots, v_{m-1}$ are distinct. A directed acyclic graph (DAG) is a directed graph with no directed cycles [40].

The following assumption will be used only in Theorem 22 for the design of the distributed control law.

Assumption 5: The task dependency graph $\mathcal{G}^t = (I, E^t)$ of the multi-agent system is a directed acyclic graph.

Note that although Assumption 5 imposes the task dependencies to follow a tree structure, one can leverage existing task decomposition or assignment approaches to rewrite cyclic task graphs into acyclic ones under mild assumptions [39].

D. Problem statement

We now have all the ingredients to provide a formal statement of the problem considered in the paper:

Problem 6: Consider a multi-agent system $\Sigma = (X, U, f, g)$ as in Definition 3, consisting of N agents Σ_i, $i \in \{1, \ldots, N\}$, and given a global STL specification $\bar{\phi}$, where $\bar{\phi} = \wedge_{i=1}^N \phi_i$ and ϕ_i are local STL tasks of the form in (2). Synthesize local controllers $\mathbf{u}_i$ for agents Σ_i such that Σ satisfies the specification $\bar{\phi}$.

In the remainder of the paper, we first introduce the concepts of assume-guarantee contracts and contract satisfaction in Section III-B. Based on these notions of contracts, our main compositionality results are presented in Sections III-C and III-D, which are used to tackle acyclic and cyclic interconnection graphs, respectively. Then, we show that STL tasks can be casted as prescribed performance functions in Section III-F, and then formulated as assume-guarantee contracts in Section

[2] By "global" and "local" STL specifications, we mean the tasks defined for the entire network and individual agents, respectively.

[3] By saying that the satisfaction of a task ϕ_i "depends" on the behavior of an agent Σ_j, we mean that the predicate function $\mathcal{P}_i$ of ϕ_i (as defined in Sec. II-A before (1)) is a function of $\mathbf{x}_j$.

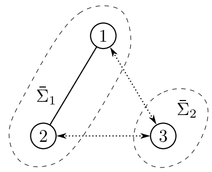


Fig. 1: A multi-agent system with two clusters $I_1 = \{1,2\}$ and $I_2 = \{3\}$.The solid lines indicate the communication links between agents, and the dotted lines with arrows indicate the dynamical couplings between clusters.

III-G. A closed-form continuous-time control law will be derived in Section IV to enforce local contracts over agents in a distributed manner.

III. Assume-Guarantee Contracts and Compositional Reasoning

In this section, we present a compositional approach based on assume-guarantee contracts which enables us to reason about the properties of a multi-agent system based on the those of its components. We will first introduce the new notions of assume-guarantee contracts and contract satisfactions for multi-agent systems. Then, two compositionality results will be presented which are tailored to multi-agent systems with acyclic and cyclic interconnection topologies, respectively.

In the next subsection, we first split the agents into clusters induced by the task dependency topology among the agents. Clusters will be leveraged to define assume-guarantee contracts representing the desired collaborative STL tasks.

A. Clusters induced by the task dependencies

Consider a multi-agent system with task dependency graph $\mathcal{G}^t = (I, E^t)$. We say that $I' \subseteq I$ is a *maximal dependency cluster* [41] if I' is a *weakly connected component* [5] in $\mathcal{G}^t$, i.e., $\forall i,j \in I'$, i and j are connected[4] $\mathcal{G}^t$ and $\nexists i \in I', i' \in I \setminus I'$ such that i and i' are connected. Thus, a multi-agent system under tasks $\{\phi_1, \ldots, \phi_N\}$ induces $K \leq N$ maximal dependency clusters I_k, $k \in \bar{I} = \{1, \ldots, K\}$. Note that these clusters are maximal, i.e., there are no task dependencies between different clusters.

Example 7: Consider a multi-agent system of three agents Σ_i, $i \in \{1,2,3\}$, as shown in Fig. 1. Agent Σ_1 is subject to a collaborative STL task $\phi_1 := G_{[0,10]}(\|\mathbf{x}_1 - \mathbf{x}_2\| \leq 3)$, i.e., agents 1 and 2 should always stay close to each other by 3 within time interval $[0,10]$; agent Σ_2 is subject to a non-collaborative STL task $\phi_2 := F_{[5,10]}(\|\mathbf{x}_2 - [50,20]^\top\| \leq 5)$, and agent Σ_3 is subject to a non-collaborative STL task $\phi_3 := G_{[0,10]}(\|\mathbf{x}_3 - [20,80]^\top\| \leq 10)$. The task dependency graph of the multi-agent system thus induces two maximal dependency clusters $I_1 = \{1,2\}$ and $I_2 = \{3\}$.

Let us denote a cluster by $\bar{\Sigma}_k = \mathcal{I}(\Sigma_{k_1}, \ldots, \Sigma_{k_{|I_k|}}) = (\bar{X}_k, \bar{U}_k, \bar{W}_k, \bar{f}_k, \bar{g}_k, \bar{h}_k)$, which results from the interconnection of agents Σ_i, $i \in I_k = \{k_1, \ldots, k_{|I_k|}\}$, $k \in \{1, \ldots, K\}$, induced by the task dependency graph. The formal definition of maximal dependency clusters is provided in Definition 25 in the appendix for completeness. A cluster's dynamics capture the collaborative behavior of all the agents involved within the same group whose tasks depend on each other. We define by $\mathcal{N}_i^c = I_k \setminus \{i\}$ the set of *cooperative neighbors* of each agent Σ_i, $i \in I_k$, and the *cooperative internal input* by the stacked state $\mathbf{z}_i(t) = [\mathbf{x}_{j_1}(t); \ldots; \mathbf{x}_{j_{|I_k|-1}}(t)]$, with $Z_i = \prod_{j \in \mathcal{N}_i^c} X_j$. For the sake of simplicity, we assume that $\mathcal{N}_i^a \cap \mathcal{N}_i^c = \emptyset$, which implies that there are no dynamical couplings among agents in the same cluster. However, the proposed results can be readily extended to handle the case that $\mathcal{N}_i^a \cap \mathcal{N}_i^c \neq \emptyset$ by deriving one more layer of compositional reasoning for each cluster; see detailed discussions at the end of Sec. III-B. Note that, since an agent can only belong to one cluster, it can be shown easily that the interconnection of all the clusters forms the same multi-agent system as the interconnection of all the single agents as in Definition 3, i.e., $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N) = \mathcal{I}(\bar{\Sigma}_1, \ldots, \bar{\Sigma}_K)$ where $\bar{\Sigma}_k = \mathcal{I}(\Sigma_{k_1}, \ldots, \Sigma_{k_{|I_k|}})$ are the clusters.

We further define for each cluster $\bar{\Sigma}_k$ the STL formula as $\bar{\phi}_k = \wedge_{i=1}^{I_k} \phi_i$. If the satisfaction of $\bar{\phi}_k$ for each cluster I_k is guaranteed, it holds by definition that the satisfaction of all the individual formulae $\phi_i, i \in I_k$ is guaranteed as well. As defined earlier, the satisfaction of $\bar{\phi}_k$, $k \in \{1, \ldots, K\}$, depends on the set of agents $I_k \subseteq I$. Note that we have by the definition of maximal dependency cluster that $I_1 \cup \cdots \cup I_K = I$ and $I_k \cap I_{k'} = \emptyset$ for all $k, k' \in \{1, \ldots, K\}$ with $k \neq k'$. Induced by the task dependency graph, the global STL specification can be written as $\bar{\phi} = \wedge_{k=1}^{K} \bar{\phi}_k$. Although there are no task dependencies between agents in different clusters I_k and $I_{k'}$, these agents might be dynamically coupled as shown in each agent's dynamics (3) based on the interconnection graph $\mathcal{G}^a$.

As mentioned in Subsection II-B, we defined the *interconnection graph* $\mathcal{G}^a = (I, E^a)$ for the multi-agent system with each agent Σ_i being a vertex in the graph. Fur future use, we further denote by a directed graph $\bar{\mathcal{G}}^a = (\bar{I}, \bar{E}^a)$ as the *cluster interconnection graph* capturing the dynamical couplings among the K clusters in the multi-agent system, where $\bar{I} = \{1, \ldots, K\}$, and $(j,i) \in \bar{E}^a$ indicate that cluster I_j provides internal inputs to cluster I_i, i.e., $\exists j_l \in I_j, i_{l'} \in I_i$ such that $(j_l, i_{l'}) \in E^a$.

B. Assume-guarantee contracts for multi-agent systems

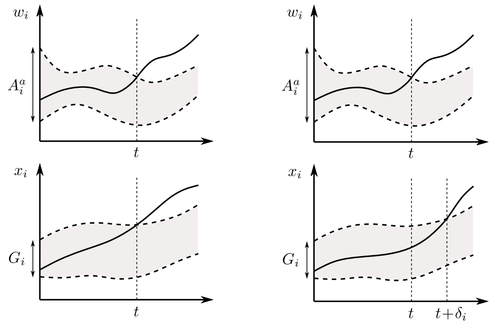
In this subsection, we introduce a notion of continuous-time assume-guarantee contracts to establish our compositional framework. A new concept of contract satisfaction is defined which is tailored to the PPC-based formulation of STL specifications as discussed in Subsection III-F.

Definition 8: (Assume-guarantee contracts) Consider an agent $\Sigma_i = (X_i, U_i, W_i, f_i, g_i, h_i)$, and the cluster $\bar{\Sigma}_k$ that Σ_i belongs to as in Definition 25. An assume-guarantee contract for Σ_i is a tuple $\mathcal{C}_i = (A_i^a, G_i)$ where

- $A_i^a : \mathbb{R}_{\geq 0} \to W_i = \prod_{j \in \mathcal{N}_i^a} X_j$ is a set of assumptions on the adversarial internal input trajectories, i.e., the state trajectories of its adversarial neighboring agents;
- $G_i : \mathbb{R}_{\geq 0} \to \bar{X}_k = X_{k_1} \times \ldots \times X_i \times \ldots \times X_{k_{|I_k|}}$ is a set of guarantees on the state trajectories and cooperative internal input trajectories.

We say that Σ_i (*weakly*) satisfies $\mathcal{C}_i$, denoted by $\Sigma_i \models \mathcal{C}_i$, if for any trajectory $(\mathbf{x}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times W_i$ of Σ_i, the following

[4] Here, i and j are said to be connected if there is an *undirected path* $i, k_1, \ldots, k_m, j$ such that: $(i, k_1) \in E^t$ and/or $(k_1, i) \in E^t$ hold; $(k_1, k_2) \in E^t$ and/or $(k_1, k_2) \in E^t$ hold;...; $(k_m, j) \in E^t$ and/or $(k_m, j) \in E^t$ hold.



(a) Weak satisfaction. (b) Strong satisfaction.

Fig. 2: An illustration of weak and strong satisfaction of a contract. Top: trajectories of the internal inputs w_i. Bottom: trajectories of the state x_i. Intuitively, the weak (or strong) satisfaction states that if the assumptions A_i^a are satisfied up to time $t \in \mathbb{R}_{\geq 0}$, the system's state belongs to G_i at least until time t ($t + \delta_i$ with $\delta_i > 0$ in the case of strong satisfaction).

holds: for all $t \in \mathbb{R}_{\geq 0}$, if $\mathbf{w}_i(s) \in A_i^a(s)$ for all $s \in [0, t]$, then we have $(\mathbf{x}_i(s), \mathbf{z}_i(s)) \in G_i(s)$[5] for all $s \in [0, t]$.

We say that Σ_i *uniformly strongly* satisfies $\mathcal{C}_i$, denoted by $\Sigma_i \models_{us} \mathcal{C}_i$, if for any trajectory $(\mathbf{x}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \rightarrow X_i \times W_i$ of Σ_i, the following holds: there exists $\delta_i > 0$ such that for all $t \in \mathbb{R}_{\geq 0}$, if $\mathbf{w}_i(s) \in A_i^a(s)$ for all $s \in [0, t]$, then we have $(\mathbf{x}_i(s), \mathbf{z}_i(s)) \in G_i(s)$ for all $s \in [0, t + \delta_i]$.

Intuitively, an assume-guarantee contract in Definition 8 states that if the restriction on the adversarial internal input trajectories to the agent belongs to A_i^a, then the restriction of the state trajectories of the agent belongs to G_i. Note that $\Sigma_i \models_{us} \mathcal{C}_i$ implies $\Sigma_i \models \mathcal{C}_i$. An illustration of weak and strong satisfaction of a contract is depicted in Fig. 2. We say that the assumption set A_i^a (or guarantee set G_i) of a contract $\mathcal{C}_i = (A_i^a, G_i)$ is *closed* if for all $t \in \mathbb{R}_{\geq 0}$, $A_i^a(t)$ (or $G_i(t)$) is closed, i.e., each $A_i^a(t)$ (or $G_i(t)$) contains all of its limit points; otherwise, it is *open*. As it can be seen in the above definition, the set of guarantees G_i for each agent Σ_i is on both the state trajectories of Σ_i and the cooperative internal input trajectories, i.e., the state trajectories of its cooperative neighbors Σ_j, $j \in \mathcal{N}_i^c$. For later use, let us define $\overline{\textbf{Proj}}^i(G_i(t)) := \{x_i \in X_i | \exists x_{k_j} \in X_{k_j}, \forall k_j \in I_k, \text{s.t. } [x_{k_1}; \ldots; x_i; \ldots; x_{k_{|I_k|}}] \in G_i(t)\}$.

It should be mentioned that multi-agent systems have no assumptions on internal inputs since they have trivial null internal input sets as in Definition 3. Hence, an AGC for a multi-agent system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ will be denoted by $\mathcal{C} = (\emptyset, G)$ with $G \subseteq X$. The concept of contract satisfaction by a multi-agent system Σ is similar to those for the single agents by removing the conditions on internal inputs: We say that Σ (*weakly*) satisfies $\mathcal{C}$, denoted by $\Sigma \models \mathcal{C}$, if for any trajectory $\mathbf{x} : \mathbb{R}_{\geq 0} \rightarrow X$ of Σ, and for all $s \in \mathbb{R}_{\geq 0}$, we have $\mathbf{x}(s) \in G(s)$. Similarly, we can define assume-guarantee contracts for clusters of agents in the multi-agent system.

Definition 9: Consider a cluster of agents $\bar{\Sigma}_k = (\bar{X}_k, \bar{U}_k, \bar{W}_k, \bar{f}_k, \bar{g}_k, \bar{h}_k)$ as in Definition 25. An assume-guarantee contract for the cluster can be defined as $\bar{\mathcal{C}}_k = (\bar{A}_k^a, \bar{G}_k)$, where $\bar{A}_k^a : \mathbb{R}_{\geq 0} \rightarrow \bar{W}_k = \prod_{i \in I_k} W_i$ and $\bar{G}_k : \mathbb{R}_{\geq 0} \rightarrow \bar{X}_k$.

The concept of contract satisfaction by the clusters can be defined similarly as in Definition 8 and is provided in the appendix in Definition 26. The following proposition will be used to show the main compositionality result of this section.

Proposition 10: Consider a cluster of agents $\bar{\Sigma}_k = \mathcal{I}(\Sigma_{k_1}, \ldots, \Sigma_{k_{|I_k|}})$, where each agent is associated with a local assume-guarantee contract $C_i = (A_i^a, G_i)$, $i \in I_k$. Consider the assume-guarantee contract $\bar{\mathcal{C}}_k = (\bar{A}_k^a, \bar{G}_k)$ for the cluster $\bar{\Sigma}_k$, as in Definition 9 constructed from the local assume-guarantee contracts $\mathcal{C}_i$, $i \in I_k$ as follows: $\bar{A}_k^a = \prod_{i \in I_k} A_i^a$ and $\bar{G}_k = \cap_{i \in I_k} G_i$. Then, we have $\bar{\Sigma}_k \models \bar{\mathcal{C}}_k$ if $\Sigma_i \models \mathcal{C}_i$ for all $i \in I_k$; $\bar{\Sigma}_k \models_{us} \bar{\mathcal{C}}_k$ if $\Sigma_i \models_{us} \mathcal{C}_i$ for all $i \in I_k$.

Proof: The proof can be found in the appendix. ∎

Let us remark that Proposition 10 holds under the assumption that $\mathcal{N}_i^a \cap \mathcal{N}_i^c = \emptyset$. In the case that $\mathcal{N}_i^a \cap \mathcal{N}_i^c \neq \emptyset$, one can derive similar compositionality results for each cluster with acyclic or cyclic dynamical interconnections among agents in the same cluster by following the same reasoning as in Theorems 11 and 12. In the following subsections, we present our main compositionality results by providing conditions under which one can go from the satisfaction of local contracts at the agent's level to the satisfaction of a global contract for the multi-agent system.

C. Compositional reasoning for acyclic interconnections

In this subsection, we first present the compositionality result for multi-agent systems with *acyclic interconnections* among the clusters, i.e., the cluster interconnection graph is a directed acyclic graph (DAG).

Theorem 11: Consider a multi-agent system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ as in Definition 3 consisting of K clusters induced by the task dependency graph $\mathcal{G}^t$. Assume that the cluster interconnection graph $\bar{\mathcal{G}}^a = (\bar{I}, \bar{E}^a)$ is a directed acyclic graph. Suppose each agent Σ_i is associated with a contract $\mathcal{C}_i = (A_i^a, G_i)$. Let $\mathcal{C} = (\emptyset, G) = (\emptyset, \prod_{k \in \bar{I}} \bar{G}_k)$ be the corresponding contract for Σ, with $\bar{G}_k = \cap_{i \in I_k} G_i$, $k \in \bar{I}$. Assume the following conditions hold:

i) for all $i \in I$, $\Sigma_i \models \mathcal{C}_i$;

ii) for all $i \in I$, $\prod_{j \in \mathcal{N}_i^a} \overline{\textbf{Proj}}^j(G_j(t)) \subseteq A_i^a(t)$ for all $t \in \mathbb{R}_{\geq 0}$.

Then, $\Sigma \models \mathcal{C}$.

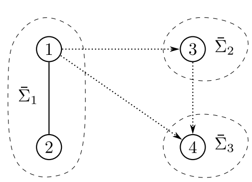
Proof: The proof can be found in the appendix. ∎

The above theorem provides us a compositionality result for multi-agent systems with acyclic cluster interconnection graphs via assume-guarantee contracts. However, we should mention that weak satisfaction is generally insufficient to reason about general networks with an interconnection graph containing cycles. Interested readers are referred to [18, Example 6] for a counter-example. A more general result is presented in the next subsection which allows us to deal with cyclic interconnections. An illustration of multi-agent systems with acyclic and cyclic cluster interconnection graphs is depicted in Fig. 3.

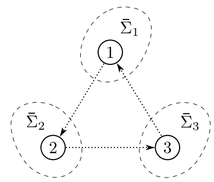
D. Compositional reasoning for cyclic interconnections

Next, we present a compositionality result on multi-agent systems with possibly *cyclic interconnections* between clusters.

[5]From here on, we slightly abuse the notation by denoting $(\mathbf{x}_i(s), \mathbf{z}_i(s))$ as the stacked vector $[\mathbf{x}_{j_1}(s); \ldots; \mathbf{x}_{j_{|I_k|}}(s)] \in \bar{X}_k$ for the sake of brevity.



(a) Acyclic cluster interconnection graph.



(b) Cyclic cluster interconnection graph.

Fig. 3: An illustration of a multi-agent system with acyclic or cyclic cluster interconnection graphs. The solid lines indicate the communication links between agents, and the dotted lines with arrows indicate the dynamical couplings between clusters.

Theorem 12: Consider a multi-agent system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ as in Definition 3 consisting of K clusters induced by the task dependency graph $\mathcal{G}^t$. Suppose each agent Σ_i is associated with a contract $\mathcal{C}_i = (A_i^a, G_i)$. Let $\mathcal{C} = (\emptyset, G) = (\emptyset, \prod_{k \in \bar{I}} \bar{G}_k)$ be the corresponding contract for Σ, with $\bar{G}_k = \cap_{i \in I_k} G_i$, $k \in \bar{I}$. Assume the following conditions hold:

i) for all $i \in I$, for any trajectory $(\mathbf{x}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times W_i$ of Σ_i, $\mathbf{x}_i(0) \in \overline{\mathbf{Proj}}^i(G_i(0))$;

ii) for all $i \in I$, $\Sigma_i \models_{us} \mathcal{C}_i$;

iii) for all $i \in I$, $\prod_{j \in \mathcal{N}_i^a} \overline{\mathbf{Proj}}^j(G_j(t)) \subseteq A_i^a(t)$ for all $t \in \mathbb{R}_{\geq 0}$.

Then, $\Sigma \models \mathcal{C}$.

Proof: The proof can be found in the appendix. ■

Note that condition (ii) in Theorem 11 and condition (iii) in Theorem 12 impose a compatibility condition between the local contracts of neighboring agents, which is essential for the compositional reasoning. Intuitively, this condition requires that for each agent i, the guarantee set of its adversarial neighboring agents $j \in \mathcal{N}_i^a$ is a subset of the assumption set of agent i.

Remark 13: It is important to note that while in the definition of the strong contract satisfaction in [18] the parameter δ may depend on time, our definition of assume-guarantee contracts requires a uniform δ for all time. The reason for this choice is that the uniformity of δ is critical in our compositional reasoning, since we do not require the set of guarantees to be closed as in [18]. See [18, Example 9] for an example, showing that the compositionality result does not hold using the concept of strong satisfaction when the set of guarantees of the contract is open. Indeed, as it will be shown in the next section, the set of guarantees of the considered contracts are open and one will fail to provide a compositionality result based on the classical (non-uniform) notion of strong satisfaction in [18].

E. From weak to strong satisfaction of contacts

Now, we provide an important result in Proposition 15 to be used to prove the main theorem in the next section, which shows how to go from *weak* to *uniform strong satisfaction* of AGCs by relaxing the assumptions. The following notion of ε-closeness of trajectories is needed to measure the distance between continuous-time trajectories.

Definition 14: (ε-closeness of trajectories) Let $Z \subseteq \mathbb{R}^n$. Consider $\varepsilon > 0$ and two continuous-time trajectories $z_1 : \mathbb{R}_{\geq 0} \to Z$ and $z_2 : \mathbb{R}_{\geq 0} \to Z$. Trajectory z_2 is said to be ε-close to z_1, if for all $t \in \mathbb{R}_{\geq 0}$, $\|z_1(t) - z_2(t)\| \leq \varepsilon$. We define the ε-expansion of z_1 by : $\mathcal{B}_\varepsilon(z_1) = \{z' : \mathbb{R}_{\geq 0} \to Z \mid z' \text{ is } \varepsilon\text{-close to } z_1\}$. For set $A = \{z : \mathbb{R}_{\geq 0} \to Z\}$, $\mathcal{B}_\varepsilon(A) = \cup_{z \in A} \mathcal{B}_\varepsilon(z)$.

Now, we introduce the following proposition which will be used later to prove our main theorem.

Proposition 15: (From weak to uniformly strong satisfaction of AGCs) Consider an agent $\Sigma_i = (X_i, U_i, W_i, f_i, g_i, h_i)$ associated with a local AGC $\mathcal{C}_i = (A_i^a, G_i)$. If trajectories of Σ_i are uniformly continuous and if there exists an $\varepsilon > 0$ such that $\Sigma_i \models \mathcal{C}_i^\varepsilon$ with $\mathcal{C}_i^\varepsilon = (\mathcal{B}_\varepsilon(A_i^a), G_i)$, then $\Sigma_i \models_{us} \mathcal{C}_i$.

Proof: The proof can be found in the appendix. ■

In the next subsection, we show how to cast STL formulae into time-varying prescribed performance functions which will be leveraged later to design continuous-time assume-guarantee contracts.

F. Casting STL as prescribed performance functions

As mentioned earlier, we consider that each agent in the multi-agent system is subject to an STL task ϕ_i which does not only depend on the behavior of agent Σ_i, but also on other agents Σ_j, $j \in I$. We denote by $I_{\phi_i} = \{i_1, \ldots, i_{P_i}\}$ the set of agents that are involved in ϕ_i, where P_i indicates the total number of agents that are involved in ϕ_i. We further define $\mathbf{x}_{\phi_i}(t) = [\mathbf{x}_{i_1}(t); \ldots; \mathbf{x}_{i_{P_i}}(t)] \in \mathbb{R}^{n_{P_i}}$, where $n_{P_i} := n_{i_1} + \ldots + n_{i_{P_i}}$.

In order to formulate the STL tasks as assume-guarantee contracts, we first show in this subsection how to cast STL formulae into prescribed performance functions. Note that the idea of casting STL as prescribed performance functions was originally proposed in [27].

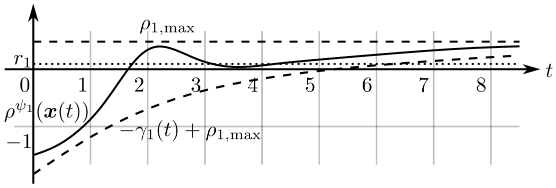
First, let us define a prescribed performance function $\gamma_i(t) = (\gamma_i^0 - \gamma_i^\infty)\exp(-l_i t) + \gamma_i^\infty$, where $l_i, t \in \mathbb{R}_{\geq 0}$, $\gamma_i^0, \gamma_i^\infty \in \mathbb{R}_{>0}$ with $\gamma_i^0 \geq \gamma_i^\infty$. Consider the robust semantics of STL introduced in Subsection II-A. For each agent with STL specification ϕ_i in (2) with the corresponding non-temporal formula ψ_i inside the F, G, FG operators as in (2), we can achieve $0 < r_i \leq \rho_i^{\phi_i}(\mathbf{x}_{\phi_i}, 0) \leq \rho_i^{max}$ by prescribing a temporal behavior to $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t))$ through a properly designed function γ_i and parameter ρ_i^{max}, and the prescribed region

$$\begin{aligned} & -\gamma_i(t) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t)) < \rho_i^{max} \\ \iff & -\gamma_i(t) < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t)) - \rho_i^{max} < 0. \end{aligned} \tag{4}$$

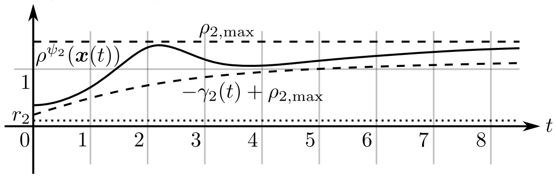
Note that functions $\gamma_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, $i \in \{1, \ldots, N\}$, are positive, continuously differentiable, bounded, and non-increasing [42]. The design of γ_i and ρ_i^{max} that leads to the satisfaction of $0 < r_i \leq \rho_i^{\phi_i}(\mathbf{x}_{\phi_i}, 0) \leq \rho_i^{max}$ through (4) will be discussed in Remark 23 in Section IV-A.

To better illustrate the satisfaction of STL tasks using PPC-based strategy, we provide the next example.

Example 16: Consider STL formulae $\phi_1 := F_{[0,8]}\psi_1$ and $\phi_2 := G_{[0,8]}\psi_2$ with $\psi_1 = \mu_1$ and $\psi_2 = \mu_2$, where μ_1 and μ_2 are associated with predicate functions $\mathcal{P}_1(\mathbf{x}) = \mathcal{P}_2(\mathbf{x}) = \mathbf{x}$. Figs. 4a and 4b show the defined region in (4) prescribing a desired temporal behavior to satisfy ϕ_1 and ϕ_2, respectively. Specifically, it can be seen that $\rho_1^{\psi_1}(\mathbf{x}(t)) \in (-\gamma_1(t) + \rho_1^{max}, \rho_1^{max})$ and $\rho_2^{\psi_1}(\mathbf{x}(t)) \in (-\gamma_2(t) + \rho_2^{max}, \rho_2^{max})$ for all



(a) Prescribed region $(-\gamma_1(t) + \rho_1^{max}, \rho_1^{max})$ (dashed lines) for $\phi_1 := F_{[0,8]}\psi_1$, s.t. $\rho_1^{\phi_1}(\mathbf{x}, 0) \geq r_1$ with $r_1 = 0.1$ (dotted line).



(b) Prescribed region $(-\gamma_2(t) + \rho_2^{max}, \rho_2^{max})$ (dashed lines) for $\phi_2 := G_{[0,8]}\psi_2$, s.t. $\rho_2^{\phi_2}(\mathbf{x}, 0) \geq r_2$ with $r_2 = 0.1$ (dotted line).

Fig. 4: Prescribed regions for STL formulae.

$t \in \mathbb{R}_{\geq 0}$ as in Fig. 4. This shows that (4) is satisfied for all $t \in \mathbb{R}_{\geq 0}$. Then, the connection between atomic formulae $\rho_i^{\psi_i}(\mathbf{x}(t))$ and temporal formulae $\rho_i^{\phi_i}(\mathbf{x}, 0)$, is made by the choice of γ_1, γ_2, ρ_1^{max}, and ρ_2^{max}. For example, the lower bound $-\gamma_1(t) + \rho_1^{max}$ in Fig. 4a ensures that $\rho_1^{\psi_1}(\mathbf{x}(t)) \geq r_1 = 0.1$ for all $t \geq 6$, which guarantees that the STL task ϕ_1 is robustly satisfied by $\rho_1^{\phi_1}(\mathbf{x}, 0) \geq r_1$.

In the sequel, STL tasks will be formulated as contracts by leveraging the above-presented PPC-based framework. We will then design local controllers enforcing the local contracts over the agents (cf. Section IV, Theorem 22).

G. From STL tasks to assume-guarantee contracts

The objective of the paper is to synthesize local controllers $\mathbf{u}_i$, for agents Σ_i to achieve the global STL specification ϕ, where $\phi = \wedge_{i=1}^{N} \phi_i$ and ϕ_i is the STL task of Σ_i. Hence, in view of the interconnection between the agents and the distributed nature of the local controllers, one has to make some assumptions on the behaviour of the neighbouring components while synthesizing the local controllers, formalized in terms of assume-guarantee contracts. In this context, and using the concept of prescribed performance function to cast the STL tasks as presented in Section III-F, a natural assignment of the local assume-guarantee contract $\mathcal{C}_i = (A_i^a, G_i)$ for the agents Σ_i, can be defined formally as follows:

- $A_i^a = \prod_{j \in \mathcal{N}_i^a} \{\mathbf{x}_j : \mathbb{R}_{\geq 0} \to X_j \mid -\gamma_j(t) + \rho_j^{max} < \rho_j^{\psi_j}(\mathbf{x}_{\phi_j}(t)) < \rho_j^{max}, \forall t \in \mathbb{R}_{\geq 0}\}$;
- $G_i = \{(\mathbf{x}_i, \mathbf{z}_i) : \mathbb{R}_{\geq 0} \to X_i \times Z_i \mid -\gamma_i(t) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t)) < \rho_i^{max}, \forall t \in \mathbb{R}_{\geq 0}\}$;

where $-\gamma, \rho^{\psi}, \rho^{max}$ are the functions discussed in Subsection III-F corresponding to their STL tasks. Notice that the above definition of the local contracts $\mathcal{C}_i = (A_i^a, G_i)$ readily satisfies condition (ii) in Theorem 11 and (iii) in Theorem 12.

Once the specification ϕ is decomposed into local contracts[6] and in view of Theorems 11 and 12, Problem 6 can be solved by considering local control problems for each agent Σ_i. These control problems can be solved in a distributed manner and are formally defined as follows:

Problem 17: Given an agent $\Sigma_i = (X_i, U_i, W_i, f_i, g_i, h_i)$ associated with an assume-guarantee contract $\mathcal{C}_i = (A_i^a, G_i)$, where A_i^a, and G_i are given by STL formulae by means of prescribed performance functions, synthesize a local controller $\mathbf{u}_i : \bar{X}_k \times \mathbb{R}_{\geq 0} \to U_i$ such that $\Sigma_i \models_{us} \mathcal{C}_i$.

IV. Distributed Controller Design

In this section, we first provide a solution to Problem 17 by designing controllers ensuring that local contracts for agents are uniformly strongly satisfied. Then, we show that based on our compositionality result proposed in the last section, the global STL task for the network is satisfied by applying the derived local controllers to agents individually. First, we utilize the idea of prescribed performance control (PPC) [42] in order to enforce the satisfaction of local contracts by prescribing certain transient behavior of the prescribed regions that constrain the closed-loop trajectories of the agents.

A. Local controller design

As discussed in Subsection III-F, one can enforce STL tasks via PPC-based strategy by prescribing the transient behavior of $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t))$ within the predefined region in

$$-\gamma_i(t) < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t)) - \rho_i^{max} < 0, \tag{5}$$

where ψ_i is the corresponding non-temporal formula inside the F, G, FG operators as in (2). In order to leverage the idea of prescribed performance control to achieve this, an error term is first defined for each $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t))$ as $e_i(\mathbf{x}_{\phi_i}(t)) = \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t)) - \rho_i^{max}$. We can then define the modulated error as $\hat{e}_i(\mathbf{x}_{\phi_i}, t) = \frac{e_i(\mathbf{x}_{\phi_i}(t))}{\gamma_i(t)}$ and the corresponding prescribed performance region $\hat{\mathcal{D}}_i := (-1, 0)$. Notice that the regions $\hat{\mathcal{D}}_i := (-1, 0)$ defined for $\hat{e}_i(\mathbf{x}_{\phi_i}, t)$ are equivalent to the desired prescribed performance bounds for $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t))$ in (5). The transformed error is then defined as

$$\epsilon_i(\mathbf{x}_{\phi_i}, t) := T_i(\hat{e}_i(\mathbf{x}_{\phi_i}, t)) = \ln\Big(-\frac{\hat{e}_i(\mathbf{x}_{\phi_i}, t) + 1}{\hat{e}_i(\mathbf{x}_{\phi_i}, t)}\Big). \tag{6}$$

Let us also define

$$\mathcal{J}_i(\hat{e}_i, t) = \frac{\partial T_i(\hat{e}_i)}{\partial \hat{e}_i} \frac{1}{\gamma_i(t)} = -\frac{1}{\gamma_i(t)\hat{e}_i(1 + \hat{e}_i)}, \tag{7}$$

which is the normalized Jacobian of the transformation function. The basic idea of PPC is to derive a control policy such that $\epsilon_i(\mathbf{x}_{\phi_i}, t)$ is rendered bounded, which implies the satisfaction of (5). A more detailed description of the PPC-based strategy is provided in the appendix.

We make the following two assumptions on functions $\rho_i^{\psi_i}$ for formulae ψ_i, which are required for the local controller design in the main result of this section.

Assumption 18: Each formula within class ψ as in (1) has the following properties: (i) $\rho_i^{\psi_i} : \mathbb{R}^{n_i} \to \mathbb{R}$ is concave and (ii) the formula is well-posed in the sense that for all $C \in \mathbb{R}$ there exists $\bar{C} \geq 0$ such that for all $\mathbf{x}_{\phi_i} \in \mathbb{R}^{n_{P_i}}$ with $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}) \geq C$, one has $\|\mathbf{x}_{\phi_i}\| \leq \bar{C} < \infty$.

[6] Note that the decomposition of a global STL formula is out of the scope of this paper. In this paper, we use a natural decomposition of the specification, where the assumptions of a component coincide with the guarantees of its neighbours. However, given a global STL for an interconnected system, one can utilize existing methods provided in recent literature, e.g., [43], to decompose the global STL task into local ones.

Remark 19: Part (i) of Assumption 18 imposes that $\rho_i^{\psi_i}$ is concave. Concave functions contain the class of linear functions as well as functions which express, e.g., reachability tasks using predicates such as $\|\mathbf{x}_i - [50,20]^\top\| \leq 5$ as in Example 7, for which $\rho_i^{\psi_i}(\mathbf{x}_i) = 5 - \|\mathbf{x}_i - [50,20]^\top\|$. This assumption is required since the controller that will be proposed in Theorem 22 uses the gradient information of $\frac{\partial \rho^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial \mathbf{x}_i}$. Hence, local optima or saddle points may lead the system to get stuck at them. Part (ii) of Assumption 18 ensures bounded solutions of $\mathbf{x}_{\phi_i}(t)$, and thus the well definedness of $\mathbf{x}_{\phi_i}(t)$ for all $t \geq 0$. This assumption is not restrictive in practice since one can combine a new formula $\psi_i^{Asm} := (\|\mathbf{x}_{\phi_i}\| < \bar{C})$ with ψ_i for a sufficiently large $\bar{C}$ so that $\psi_i^{Asm} \wedge \psi_i$ is well-posed.

Define the global maximum of $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})$ as $\rho_i^{opt} = \sup_{\mathbf{x}_{\phi_i} \in \mathbb{R}^{nP_i}} \rho_i^{\psi_i}(\mathbf{x}_{\phi_i})$.

Assumption 20: (i) The global maximum of $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})$ is positive, i.e. $\rho_i^{opt} > 0$ and (ii) $\frac{\partial \rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial \mathbf{x}_i}$ is a non-zero vector.

Remark 21: The global maximum of $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})$ being positive guarantees that the local STL formula ψ_i is satisfiable, since $\rho_i^{opt} > 0$ implies that $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}, 0) > 0$ is possible. Note that ρ_i^{opt} is easy to compute since $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})$ is concave. The assumption on $\frac{\partial \rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial \mathbf{x}_i}$ being a non-zero vector is used to avoid local optima which can cause infeasibility issues. Note that since $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})$ is concave under Assumption 18, one has $\frac{\partial \rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial \mathbf{x}_i} = 0$ if and only if $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}) = \rho_i^{opt}$, thus (ii) of Assumption 20 can be guaranteed by choosing $\rho_i^{\max}$ such that $0 < \rho_i^{\max} < \rho_i^{opt} < 0$ holds as in (10), under the assumption that $\rho_i^{opt} > 0$ as in part (i) of Assumption 20.

Now, we are ready to present the main result of this section solving Problem 17 for the local controller design.

Theorem 22: Consider an agent Σ_i as in (3) belonging to cluster $\bar{\Sigma}_k$ and satisfying Assumption 1. Given that Σ_i is subject to STL task ϕ_i, and is associated with its local assume-guarantee contract $\mathcal{C}_i = (A_i^a, G_i)$, where

- $A_i^a = \prod_{j \in \mathcal{N}_i^a} \{\mathbf{x}_j : \mathbb{R}_{\geq 0} \to X_j \mid -\gamma_j(t) + \rho_j^{max} < \rho_j^{\psi_j}(\mathbf{x}_{\phi_j}(t)) < \rho_j^{max}, \forall t \in \mathbb{R}_{\geq 0}\}$;
- $G_i = \{(\mathbf{x}_i, \mathbf{z}_i) : \mathbb{R}_{\geq 0} \to X_i \times Z_i \mid -\gamma_i(t) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t)) < \rho_i^{max}, \forall t \in \mathbb{R}_{\geq 0}\}$;

where ψ_i is an atomic formula as in (1) satisfying Assumptions 18 and 20. Assume $-\gamma_i(0) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(0)) < \rho_i^{max} < \rho_i^{opt}$, for all $i \in I_k$. If the task dependency graph $\mathcal{G}^t = (I, E^t)$ is a directed acyclic graph as in Assumption 5, and all agents Σ_i in the cluster apply the controller

$$\mathbf{u}_i(\bar{\mathbf{x}}_k, t) = -g_i^\top(\mathbf{x}_i) \sum_{j \in I_k} \left(\frac{\partial \rho_j^{\psi_j}(\mathbf{x}_{\phi_j})}{\partial \mathbf{x}_i} \mathcal{J}_j(\hat{e}_j, t) \epsilon_j(\mathbf{x}_{\phi_j}, t) \right), \quad (8)$$

where $\mathcal{J}_i(\hat{e}_i, t), \epsilon_j(\mathbf{x}_{\phi_j}, t)$ are defined in (7) and (6), respectively, then we have $\Sigma_i \models_{us} \mathcal{C}_i$ for all $i \in I_k$.

Proof: The proof can be found in the appendix. ■

As can be seen from (8), the presented control strategy is distributed, but not fully decentralized, as it requires the information from its cooperative neighbors w.r.t. their states and prescribed performance functions. From an implementation perspective, such a distributed approach is realistic since communication between neighboring agents is feasible and efficient in many practical applications, and is often necessary to achieve coordinated behavior.

Remark 23: (Parameter design for the prescribed regions) Remark that the connection between atomic formulae $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t))$ and temporal formulae $\rho_i^{\phi_i}(\mathbf{x}_{\phi_i}, 0)$ is made by γ_i and ρ_i^{max} as in (4), which need to be designed as instructed in [27]. Specifically, suppose $\rho_i^{opt} > 0$ holds as in part (i) of Assumption 20, and select

$$t_i^* \in \begin{cases} a_i & \text{if } \phi_i = G_{[a_i,b_i]}\psi_i \\ [a_i, b_i] & \text{if } \phi_i = F_{[a_i,b_i]}\psi_i \\ [\underline{a}_i + \bar{a}_i, \underline{b}_i + \bar{a}_i] & \text{if } \phi_i = F_{[\underline{a}_i,\underline{b}_i]} G_{[\bar{a}_i,\bar{b}_i]}\psi_i \end{cases} \quad (9)$$

$$\rho_i^{\max} \in \left(\max(0, \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(0))), \rho_i^{\text{opt}} \right) \quad (10)$$

$$r_i \in (0, \rho_i^{\max}) \quad (11)$$

$$\gamma_i^0 \in \begin{cases} (\rho_i^{\max} - \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(0)), \infty) & \text{if } t_i^* > 0 \\ (\rho_i^{\max} - \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(0)), \rho_i^{\max} - r_i] & \text{else} \end{cases} \quad (12)$$

$$\gamma_i^\infty \in \left(0, \min(\gamma_i^0, \rho_i^{\max} - r_i)\right] \quad (13)$$

$$l_i \in \begin{cases} \mathbb{R}_{\geq 0} & \text{if } -\gamma_i^0 + \rho_i^{\max} \geq r_i \\ \frac{-\ln\left(\frac{r_i + \gamma_i^\infty - \rho_i^{\max}}{-\gamma_i^0 + \gamma_i^\infty}\right)}{t_i^*} & \text{else.} \end{cases} \quad (14)$$

Note the choice of γ_i^0 ensures the satisfaction of the assumption on initial conditions $-\gamma_i(0) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(0)) < \rho_i^{max} < \rho_i^{opt}$ as stated in Theorem 22. With γ_i and ρ_i^{max} chosen properly (as shown above), one can achieve $0 < r_i \leq \rho_i^{\phi_i}(\mathbf{x}_{\phi_i}, 0) \leq \rho_i^{max}$ by prescribing a temporal behavior to $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t))$ as in the set of guarantees G_i in Theorem 22, i.e., $-\gamma_i(t) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t)) < \rho_i^{max}$ for all $t \geq 0$.

B. Global task satisfaction

Here, we show that by applying the local controllers to the agents, the global STL task for the multi-agent system is also satisfied based on our compositionality result.

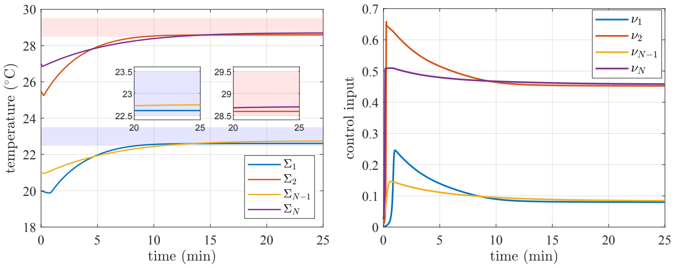
Corollary 24: Consider a multi-agent system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ as in Definition 3 consisting of K clusters induced by the task dependency graph $\mathcal{G}^t = (I, E^t)$. Suppose Assumption 5 holds. Suppose each agent Σ_i is associated with a contract $\mathcal{C}_i = (A_i^a, G_i)$, and each cluster of agents is associated with a contract $\bar{\mathcal{C}}_k = (\bar{A}_k^a, \bar{G}_k)$ as in Definition 9. If we apply the controllers as in (8) to agents Σ_i, then we get $\Sigma \models \mathcal{C} = (\emptyset, \prod_{k \in \bar{I}} \bar{G}_k)$. This means that the control objective in Problem 6 is achieved, i.e., system Σ satisfies signal temporal logic task $\bar{\phi}$.

Proof: The proof can be found in the appendix. ■

According to the above corollary, we remark that the global task satisfaction holds for multi-agent systems with acyclic task dependency graphs, irrespective of whether the dynamical interconnection graph is acyclic or cyclic.

V. Case Study

We demonstrate the effectiveness of the proposed results on two case studies: a room temperature regulation and a mobile multi-robot control problem.



(a) Room temperature (b) Control inputs

Fig. 5: Trajectories of the temperature (a) and control input (b) of the closed-loop agents Σ_1, Σ_2, Σ_{N-1} and Σ_N, where $N = 1000$, under control policy in (8).

A. Room Temperature Regulation

Here, we apply our results to the temperature regulation of a circular building with $N = 1000$ rooms each equipped with a heater. The evolution of the temperature of the interconnected model is described by the differential equation:

$$\Sigma : \dot{\mathbf{T}}(t) = A\mathbf{T}(t) + \alpha_h T_h \nu(t) + \alpha_e T_e, \quad (15)$$

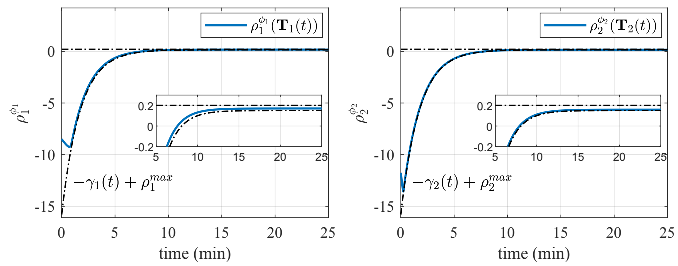
adapted from [44], where $A \in \mathbb{R}^{N\times N}$ is a matrix with elements $\{A\}_{ii} = (-2\alpha - \alpha_e - \alpha_h \nu_i)$, $\{A\}_{i,i+1} = \{A\}_{i+1,i} = \{A\}_{1,N} = \{A\}_{N,1} = \alpha$, $\forall i \in \{1, \dots, N-1\}$, and all other elements are identically zero, $\mathbf{T}(k) = [\mathbf{T}_1(k); \dots; \mathbf{T}_N(k)]$, $T_e = [T_{e1}; \dots; T_{eN}]$, $\nu(k) = [\nu_1(k); \dots; \nu_N(k)]$, where $\nu_i(k) \in [0, 1]$, $\forall i \in \{1, \dots, N\}$, represents the ratio of the heater valve being open in room i. Parameters $\alpha = 0.05$, $\alpha_e = 0.008$, and $\alpha_h = 0.12$ are heat exchange coefficients, $T_{ei} = -20\,°C$ is the external environment temperature, and $T_h = 55\,°C$ is the heater temperature. Now, by introducing Σ_i described by

$$\Sigma_i : \dot{\mathbf{T}}_i(t) = a\mathbf{T}_i(t) + d\mathbf{w}_i(t) + \alpha_h T_h \nu_i(t) + \alpha_e T_{ei},$$

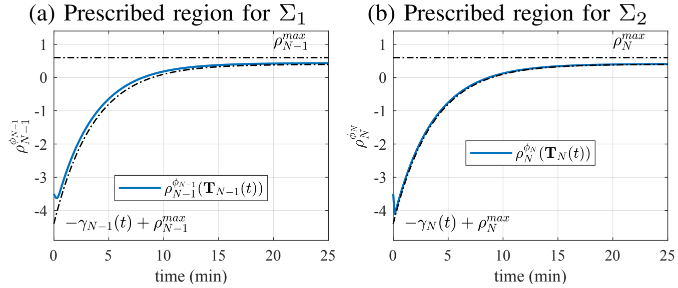
where $a = -2\alpha - \alpha_e - \alpha_h \nu_i$, $d = 0.8\alpha$, and $\mathbf{w}_i(t) = [\mathbf{T}_{i-1}(t); \mathbf{T}_{i+1}(t)]$ (with $\mathbf{T}_0 = \mathbf{T}_N$ and $\mathbf{T}_{N+1} = \mathbf{T}_1$), one can readily verify that $\Sigma = \mathcal{I}(\Sigma_1, \dots, \Sigma_N)$ as in Definition 3. The initial temperatures of these rooms are, respectively, $\mathbf{T}_1(0) = 20\,°C$, $\mathbf{T}_2(0) = 25.5\,°C$, $\mathbf{T}_i(0) = 21\,°C$ if $i \in \{i \text{ is odd } | i \in \{3, \dots, N\}\}$, and $\mathbf{T}_i(0) = 27\,°C$ if $i \in \{i \text{ is even } | i \in \{4, \dots, N\}\}$.

The room temperatures are subject to the following STL tasks ϕ_1: $F_{[0,5]}G_{[10,20]}(\|\mathbf{T}_1 - \mathbf{T}_{999}\| \le 2) \wedge (\|\mathbf{T}_1 - \mathbf{T}_3\| \le 2) \wedge (\|\mathbf{T}_1 - 23\| \le 0.5)$, ϕ_2: $F_{[0,5]}G_{[10,20]}(\|\mathbf{T}_2 - \mathbf{T}_{1000}\| \le 2) \wedge (\|\mathbf{T}_2 - \mathbf{T}_4\| \le 2) \wedge (\|\mathbf{T}_2 - 29\| \le 0.5)$, ϕ_i: $F_{[0,5]}G_{[10,20]}(\|\mathbf{T}_i - \mathbf{T}_{i+2}\| \le 2)$, for $i \in \{3, \dots, N-2\}$, and ϕ_{N-1}: $F_{[0,25]}(\|\mathbf{T}_{N-1} - 23\| \le 0.5)$, and ϕ_N: $F_{[0,25]}(\|\mathbf{T}_N - 29\| \le 0.5)$. Intuitively, the STL tasks require the controllers (heaters) to be synthesized such that the temperature of every other rooms should eventually get close to each other, and the temperature of the rooms reach the specified region ($[22.5, 23.5]$ for rooms Σ_1 and Σ_{N-1}, or $[28.5, 29.5]$ for rooms Σ_2 and Σ_N) in the desired time slots. In this setting, the circular building consists of two clusters: $I_1 = \{i \text{ is odd } | i \in \{1, \dots, N\}\}$ and $I_2 = \{i \text{ is even } | i \in \{1, \dots, N\}\}$. The cluster interconnection graph is cyclic due to the dynamical couplings between the neighboring rooms.

To enforce the STL tasks on this large-scale multi-agent system, we apply the proposed compositional framework by leveraging the results in Theorem 12 and applying the proposed PPC-based feedback controllers as in (8). Note that Assumptions 18-20 and 1 on the STL formulae and the system dynamics are satisfied, and the task dependency graph also satisfies Assumption 5. We can thus apply the feedback controller as in (8) on the agents to enforce STL tasks in a distributed manner. Numerical implementations were performed using MATLAB on a computer with a processor Intel Core i7 3.6 GHz CPU. Note that the computation of local controllers took on average less than 0.1 ms, which is negligible. The computation cost is very cheap since the local controller $\boldsymbol{u}_i$ is given by a closed-form expression and computed in a distributed manner. The simulation results for agents Σ_1, Σ_2, Σ_{N-1} and Σ_N are shown in Figs. 5-6. The state and input trajectories of the closed-loop agents are depicted in Fig. 5. The shaded areas represent the desired temperature regions to be reached by the systems. As can be seen from Figs. 5, all these agents satisfy their desired STL tasks. In Fig. 6, we present the temporal behaviors of $\rho_i^{\psi_i}(\mathbf{T}_1(t))$ for the four rooms Σ_1, Σ_2, Σ_{N-1} and Σ_N. It can be readily seen that the prescribed performances of $\rho_i^{\psi_i}(\mathbf{T}_i(t))$ are satisfied, which indicates that the time bounds are also respected. Remark that the design parameters of the prescribed regions are chosen according to the instructions listed in (9)-(14), which guarantees the satisfaction of temporal formulae $\rho_i^{\phi_i}(\mathbf{T}_i, 0)$ by prescribing temporal behaviors of atomic formulae $\rho_i^{\psi_i}(\mathbf{T}_i(t))$ as in Fig. 6. We can conclude that all STL tasks are satisfied within the desired time interval. Note that the proposed compositional framework allows us to deal with multi-agent systems in a distributed manner, thus rendering the controller synthesis problem of large-scale multi-agent systems tractable. Moreover, the proposed local controller allows us to deal with STL tasks that not only depends on single agents but may also depend on multiple agents. Unlike the methods



(a) Prescribed region for Σ_1 (b) Prescribed region for Σ_2



(c) Prescribed region for Σ_{N-1} (d) Prescribed region for Σ_N

Fig. 6: Prescribed regions for the STL tasks for agents Σ_1, Σ_2, Σ_{N-1} and Σ_N, where $N = 1000$. Performance bounds are indicated by dashed lines. Evolution of the prescribed performance of $\rho_i^{\phi_i}(\mathbf{T}_i(t))$ are depicted using solid lines.

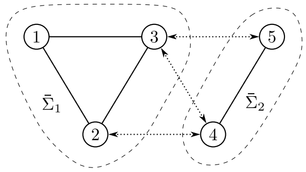


Fig. 7: The multi-robot system with two clusters $\bar{\Sigma}_1 = \{\Sigma_1, \Sigma_2, \Sigma_3\}$ and $\bar{\Sigma}_2 = \{\Sigma_4, \Sigma_5\}$. The solid lines indicate the communication links between agents and the dotted lines indicate the dynamical couplings between clusters.

in [28] which require all agents in a cluster to be subject to the same STL task, our designed local controllers can handle different STL tasks in the same cluster. Therefore, the methods in [28], cannot be applied to deal with the considered problem in this paper.

B. Mobile Robot Control

In this subsection, we demonstrate the effectiveness of the proposed results on a network of $N = 5$ mobile robots, where the dynamic of each robot is adapted from [45] with induced dynamical couplings. Each mobile robot has three omni-directional wheels. The dynamics of each robot Σ_i, $i \in \{1, 2, \ldots, 5\}$ can be described by

$$\Sigma_i : \dot{\boldsymbol{x}}_i = A_i \left(B_i^\top\right)^{-1} R_i \boldsymbol{u}_i - f_i(\boldsymbol{x}),$$

where the state variable of each robot is defined as $\boldsymbol{x}_i := [x_{i,1}; x_{i,2}; x_{i,3}]$ with $\boldsymbol{p}_i := [x_{i,1}; x_{i,2}]$ indicating the robot position and state $x_{i,3}$ indicating the robot orientation with respect to the $x_{i,1}$-axis; $R_i := 0.02$ m is the wheel radius of each robot; $A_i := \begin{bmatrix} \cos(x_{i,3}) & -\sin(x_{i,3}) & 0 \\ \sin(x_{i,3}) & \cos(x_{i,3}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $B_i := \begin{bmatrix} 0 & \cos(\pi/6) & -\cos(\pi/6) \\ -1 & \sin(\pi/6) & -\sin(\pi/6) \\ L_i & L_i & L_i \end{bmatrix}$ describes geometrical constraints with $L_i := 0.2$ m being the radius of the robot body; the term $f_i(\boldsymbol{x}) = \sum_{j \in \mathcal{N}_i^a} k_i \frac{[\boldsymbol{p}_i - \boldsymbol{p}_j; 0]}{\|\boldsymbol{p}_i - \boldsymbol{p}_j\| + 0.00001}$ is the induced dynamical coupling between agents that is used for the sake of collision avoidance, where $\boldsymbol{x} = [\boldsymbol{x}_1; \ldots; \boldsymbol{x}_N]$, $k_i = 0.1$, and $\boldsymbol{p}_j$, $j \in \mathcal{N}_i^a$, are the positions of Σ_i's adversarial neighboring agents with $\mathcal{N}_2^a = \{4\}$, $\mathcal{N}_3^a = \{4, 5\}$, $\mathcal{N}_4^a = \{2, 3\}$, $\mathcal{N}_5^a = \{3\}$. Each element of the input vector $\boldsymbol{u}_i$ corresponds to the angular rate of one wheel. The initial states of the robots are, respectively, $\boldsymbol{x}_1(0) = [2; 2; -\pi/4]$, $\boldsymbol{x}_2(0) = [4; 0; 0]$, $\boldsymbol{x}_3(0) = [6; 2; \pi/4]$, $\boldsymbol{x}_4(0) = [8; 0; 0]$, $\boldsymbol{x}_5(0) = [10; 2; -\pi/4]$.

The robots are subject to the following STL tasks:

$$\phi_1 := F_{[0,40]} G_{[0,20]} ((\|\boldsymbol{p}_1 - \boldsymbol{p}_2\| \le 1) \wedge (\|\boldsymbol{p}_1 - \boldsymbol{p}_3\| \le 1)$$
$$\phi_2 := F_{[0,40]} G_{[0,20]} ((\|\boldsymbol{p}_2 - \boldsymbol{p}_3\| \le 1) \wedge (|\deg(x_{2,3}) - \deg(x_{3,3})| \le 7.5)$$
$$\phi_3 := F_{[0,40]} G_{[0,20]} ((\|\boldsymbol{p}_3 - [14; 7]\| \le 0.1) \wedge (|\deg(x_{3,3})| \le 7.5)$$
$$\phi_4 := F_{[0,40]} G_{[0,20]} ((\|\boldsymbol{p}_4 - [14; 7.5]\| \le 0.1) \wedge (|\deg(x_{4,3})| \le 7.5)$$
$$\phi_5 := F_{[0,40]} G_{[0,20]} ((\|\boldsymbol{p}_5 - \boldsymbol{p}_4\| \le 1)$$

where $\deg(\cdot)$ converts angle units from radians to degrees. Intuitively, robots 3 and 4 are assigned to move to their predefined goal points $[14; 7]$ and $[14; 7.5]$, respectively, and stay there within the desired time interval, while satisfying

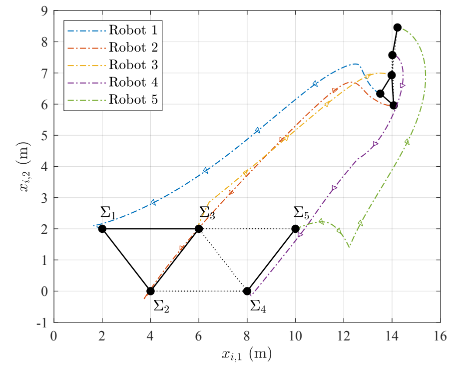
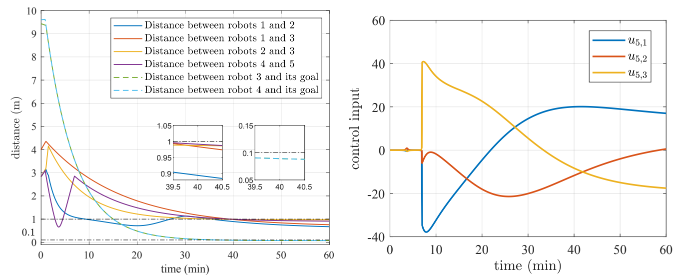


Fig. 8: State trajectories of the closed-loop robot systems on the position plane. Triangles indicate the orientation of robots.

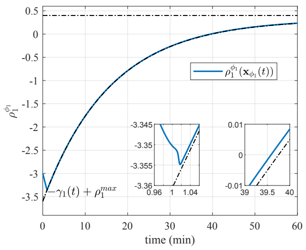


(a) Relative distances (b) Control input for robot 5

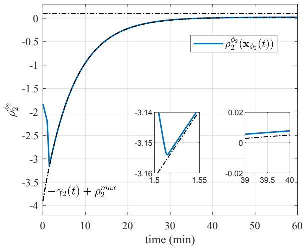
Fig. 9: The evolution of the relative distances between the agents (or between an agent and its goal position) (a) and control input for robot 5 (b) as time progresses.

the additional requirements on the robots' orientation; robot 2 is required to chase and follow robot 3 within the desired time interval, in the meanwhile keeping a similar orientation as robot 3; robot 2 is required to chase and follow robots 2 and 3 within the desired time interval; and robot 5 is required to chase and follow robot 4. Induced by the task dependency graph, we obtain two clusters from the multi-agent system: $\bar{\Sigma}_1 = \{\Sigma_1, \Sigma_2, \Sigma_3\}$ and $\bar{\Sigma}_2 = \{\Sigma_4, \Sigma_5\}$, as depicted in Fig. 7. The cluster interconnection graph in Fig. 7 is cyclic due to the dynamical couplings between the agents in different clusters.

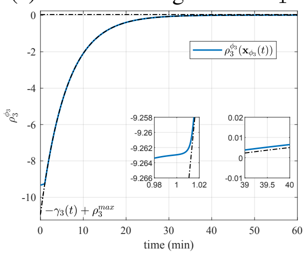
To enforce the STL tasks on this 5-robot system, we apply the proposed compositional framework by leveraging the results in Theorem 12 and the proposed feedback controllers as in (8). Note that Assumptions 18-20 and 1 on the STL formulae and the system dynamics are satisfied, and the task dependency graph also satisfies Assumption 5. We can thus apply the feedback controller as in (8) on the agents to enforce the STL tasks in a distributed manner. Numerical implementations were performed using MATLAB on a computer with a processor Intel Core i7 3.6 GHz CPU. Note that the computation of local controllers took on average 0.1 ms, which is negligible since $\boldsymbol{u}_i$ is given by a closed-form expression and computed in a distributed manner. Simulation results are shown in Figs. 8-10. The state trajectories of each robot are depicted as in Fig. 8 on the position plane. The initial positions and final positions of the agents are represented by solid circles, the solid lines between the circles indicate communication links, and the dotted lines between the circles indicate dynamical couplings between the agents. In Fig. 9, we show in subfigure (a) the evolution of the relative distances



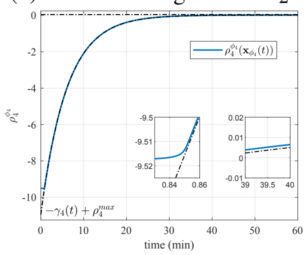
(a) Prescribed region for Σ_1



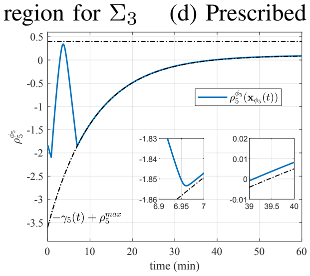
(b) Prescribed region for Σ_2



(c) Prescribed region for Σ_3



(d) Prescribed region for Σ_4



(e) Prescribed region for Σ_5

Fig. 10: The prescribed regions for the STL tasks. Performance bounds are indicated by dashed lines. Evolution of $\rho_i^{\phi_i}(\boldsymbol{x}_{\phi_i}(t))$ are depicted using solid lines.

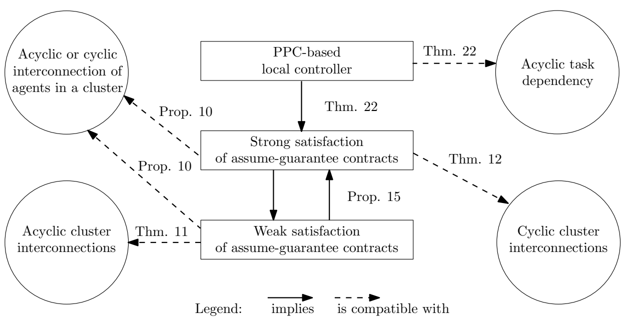


Fig. 11: Summary of the main results.

between the agents (or between an agent and its goal position) and in subfigure (b) the trajectory of control inputs for robot 5 as time progresses. As it can be seen from Figs. 8 and 9, all agents satisfy their desired STL tasks. In particular, agents 3 and 4 finally achieved their tasks to reach and stay around their goal point within a certain distance (0.1 meter) after 40 minutes; agents 1, 2, and 5 achieved their tasks to chase and stay close to their desired agents within a certain distance (1 meter). In Fig. 10, we further showcase the temporal behaviors of $\rho_i^{\phi_i}(\mathbf{x}_{\phi_i}(t))$ for the robots. It can be seen that the prescribed performances of $\rho_i^{\phi_i}(\mathbf{x}_{\phi_i}(t))$ are satisfied, which verifies that the time bounds are also respected. We can conclude that all STL tasks are satisfied within the desired time interval.

VI. Conclusions

We proposed a compositional approach for the synthesis of signal temporal logic tasks for large-scale multi-agent systems using assume-guarantee contracts. The notions and main results proposed in the paper and their relationships are sketched in Fig. 11. Each agent in the multi-agent system is subject to collaborative tasks in the sense that it does not only depend on the agent itself but may also depend on other agents. The STL tasks are first translated to assume-guarantee contracts so that the satisfaction of a contract guarantees the satisfaction of the signal temporal logic task. Two concepts of contract satisfaction were introduced to establish our compositionality results, where weak satisfaction was shown to be sufficient to deal with acyclic interconnections, and strong satisfaction was needed for cyclic interconnections. We then derived a continuous-time closed-form feedback controller to enforce the uniform strong satisfaction of local contracts in a distributed manner, thus guaranteeing the satisfaction of global STL task for the multi-agent system based on the proposed compositionality result. Finally, the theoretical results were validated via two numerical case studies.

Appendix

Definition 25: (Maximal dependency clusters) Consider a cluster of agents Σ_i, $i \in I_k = \{k_1, \ldots, k_{|I_k|}\}$, $k \in \{1, \ldots, K\}$, with each agent as described in (3). The product system for the cluster of agents denoted by $\bar{\Sigma}_k = \mathcal{I}(\Sigma_{k_1}, \ldots, \Sigma_{k_{|I_k|}})$ is a tuple $\bar{\Sigma}_k = (\bar{X}_k, \bar{U}_k, \bar{W}_k, \bar{f}_k, \bar{g}_k, \bar{h}_k)$ where

- $\bar{X}_k = \prod_{i \in I_k} X_i$, $\bar{U}_k = \prod_{i \in I_k} U_i$, and $\bar{W}_k = \prod_{i \in I_k} W_i$ are the state, external input spaces, and internal input spaces, respectively;
- $\bar{f}_k$ is the flow drift, $\bar{g}_k$ is the external input matrix, and $\bar{h}_k$ is the internal input matrix defined as: $\bar{f}_k(\bar{\mathbf{x}}_k(t)) = [f_{k_1}(\mathbf{x}_{k_1}(t)); \ldots; f_{k_{|I_k|}}(\mathbf{x}_{k_{|I_k|}}(t))]$, $\bar{g}_k(\bar{\mathbf{x}}_k(t)) = \mathrm{diag}(g_{k_1}(\mathbf{x}_{k_1}(t)), \ldots, g_{k_{|I_k|}}(\mathbf{x}_{k_{|I_k|}}(t)))$, $\bar{h}_k(\bar{\mathbf{w}}_k(t)) = [h_{k_1}(\mathbf{w}_{k_1}(t)); \ldots; h_{k_{|I_k|}}(\mathbf{w}_{k_{|I_k|}}(t))]$, where $\bar{\mathbf{x}}_k = [\mathbf{x}_{k_1}; \ldots; \mathbf{x}_{k_{|I_k|}}] \in \mathbb{R}^{\bar{n}_k}$ with $\bar{n}_k = n_{k_1} + \cdots + n_{k_{|I_k|}}$, $\bar{\mathbf{w}}_k = [\mathbf{w}_{k_1}; \ldots; \mathbf{w}_{k_{|I_k|}}] \in \mathbb{R}^{\bar{p}_k}$ with $\bar{p}_k = p_{k_1} + \cdots + p_{k_{|I_k|}}$.

A trajectory of $\bar{\Sigma}_k$ is a uniformly continuous map $(\bar{\mathbf{x}}_k, \bar{\mathbf{w}}_k): \mathbb{R}_{\geq 0} \to \bar{X}_k \times \bar{W}_k$ such that for all $t \geq 0$

$$\dot{\bar{\mathbf{x}}}_k(t) = \bar{f}_k(\bar{\mathbf{x}}_k(t)) + \bar{g}_k(\bar{\mathbf{x}}_k(t))\bar{\mathbf{u}}_k(t) + \bar{h}_k(\bar{\mathbf{w}}_k(t)), \quad (16)$$

where $\bar{\mathbf{u}}_k = [\mathbf{u}_{k_1}; \ldots; \mathbf{u}_{k_{|I_k|}}]$.

Definition 26: (Satisfaction of assume-guarantee contracts by the clusters) Consider a cluster of agents $\bar{\Sigma}_k$ an assume-guarantee contract $\bar{\mathcal{C}}_k = (\bar{A}_k^a, \bar{G}_k)$ as in Definition 9.

We say that $\bar{\Sigma}_k$ (*weakly*) satisfies $\bar{\mathcal{C}}_k$, denoted by $\bar{\Sigma}_k \models \bar{\mathcal{C}}_k$, if for any trajectory $(\bar{\mathbf{x}}_k, \bar{\mathbf{w}}_k): \mathbb{R}_{\geq 0} \to \bar{X}_k \times \bar{W}_k$ of $\bar{\Sigma}_k$, the following holds: for all $t \in \mathbb{R}_{\geq 0}$ such that $\bar{\mathbf{w}}_k(s) \in \bar{A}_k^a(s)$ for all $s \in [0, t]$, we have $\bar{\mathbf{x}}_k(s) \in \bar{G}_k(s)$ for all $s \in [0, t]$.

We say that $\bar{\Sigma}_k$ *uniformly strongly* satisfies $\bar{\mathcal{C}}_k$, denoted by $\bar{\Sigma}_k \models_{us} \bar{\mathcal{C}}_k$, if for any trajectory $(\bar{\mathbf{x}}_k, \bar{\mathbf{w}}_k): \mathbb{R}_{\geq 0} \to \bar{X}_k \times \bar{W}_k$ of $\bar{\Sigma}_k$, the following holds: there exists $\bar{\delta}_k > 0$ such that for all $t \in \mathbb{R}_{\geq 0}$, $\bar{\mathbf{w}}_k(s) \in \bar{A}_k^a(s)$ for all $s \in [0, t]$, we have $\bar{\mathbf{x}}_k(s) \in \bar{G}_k(s)$ for all $s \in [0, t + \bar{\delta}_k]$.

A. Supplementaries on prescribed performance control

In order to design feedback controllers to prescribe the transient behavior of $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t))$ within the predefined region:

$$-\gamma_i(t) < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t)) - \rho_i^{max} < 0, \quad (17)$$

one can translate the prescribed performance functions into notions of errors as follows. First, define a one-dimensional error as $e_i(\mathbf{x}_{\phi_i}(t)) = \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t)) - \rho_i^{max}$. Now, by normalizing the error $e_i(\mathbf{x}_{\phi_i}(t))$ with respect to the prescribed performance function γ_i, we define the modulated error as $\hat{e}_i(\mathbf{x}_{\phi_i}, t) = \frac{e_i(\mathbf{x}_{\phi_i}(t))}{\gamma_i(t)}$. Then, (17) can be rewritten as $-1 < \hat{e}_i(t) < 0$. We use $\hat{\mathcal{D}}_i := (-1, 0)$ to denote the performance region for $\hat{e}_i(t)$. Next, the modulated error is transformed through a transformation function $T_i : (-1, 0) \to \mathbb{R}$ defined as $T_i(\hat{e}_i(\mathbf{x}_{\phi_i}, t)) = \ln(-\frac{\hat{e}_i(\mathbf{x}_{\phi_i},t)+1}{\hat{e}_i(\mathbf{x}_{\phi_i},t)})$. Note that the transformation function $T_i : (-1, 0) \to \mathbb{R}$ is a strictly increasing function, bijective and hence admitting an inverse. By differentiating the transformed error $\epsilon_i := T_i(\hat{e}_i(\mathbf{x}_{\phi_i}, t))$ w.r.t time, we obtain

$$\dot{\epsilon}_i = \mathcal{J}_i(\hat{e}_i, t)[\dot{e}_i + \alpha_i(t)e_i], \tag{18}$$

where $\mathcal{J}_i(\hat{e}_i, t) = \frac{\partial T_i(\hat{e}_i)}{\partial \hat{e}_i}\frac{1}{\gamma_i(t)} = -\frac{1}{\gamma_i(t)\hat{e}_i(1+\hat{e}_i)} > 0$, for all $\hat{e}_i \in (-1, 0)$, is the normalized Jacobian of the transformation function, and $\alpha_i(t) = -\frac{\dot{\gamma}_i(t)}{\gamma_i(t)} > 0$ for all $t \in \mathbb{R}_{\geq 0}$ is the normalized derivative of the performance function γ_i.

It can be seen that, if the transformed error ϵ_i is bounded for all t, then the modulated error $\hat{e}_i$ is constrained within the performance region $\hat{\mathcal{D}}_i$, which further implies that the error e_i evolves within the prescribed performance bounds as in (17).

B. Proofs not contained in the main body

Proof of Proposition 10. We only provide the proof for the case of strong satisfaction of contracts. The case of weak satisfaction can be derived similarly. Suppose that $\Sigma_i \models_{us} \mathcal{C}_i$ holds for all $i \in I_k$. Consider an arbitrary trajectory $(\bar{\mathbf{x}}_k, \bar{\mathbf{w}}_k) : \mathbb{R}_{\geq 0} \to \bar{X}_k \times \bar{W}_k$ of the cluster $\bar{\Sigma}_k$. Let us show the existence of $\bar{\delta}_k > 0$ such that for all $t \in \mathbb{R}_{\geq 0}$ with $\bar{\mathbf{w}}_k(s) \in \bar{A}_k^a(s)$ for all $s \in [0, t]$, we have $\bar{\mathbf{x}}_k(s) \in \bar{G}_k(s)$ for all $s \in [0, t + \bar{\delta}_k]$.

First, given the trajectory $(\bar{\mathbf{x}}_k, \bar{\mathbf{w}}_k) : \mathbb{R}_{\geq 0} \to \bar{X}_k \times \bar{W}_k$, we have by Definition 25 that $\bar{\mathbf{x}}_k = [\mathbf{x}_{k_1}; \ldots; \mathbf{x}_{k_{|I_k|}}]$ and $\bar{\mathbf{w}}_k = [\mathbf{w}_{k_1}; \ldots; \mathbf{w}_{k_{|I_k|}}]$, and thus, $(\mathbf{x}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times W_i$ is a trajectory of Σ_i for all $i \in I_k$. Now, consider any arbitrary $t \in \mathbb{R}_{\geq 0}$, such that $\bar{\mathbf{w}}_k(s) \in \bar{A}_k^a(s)$ for all $s \in [0, t]$, it follows that $\mathbf{w}_i(s) \in A_i^a$ for all $s \in [0, t]$ for all $i \in I_k$ since $\bar{A}_k^a = \prod_{i \in I_k} A_i^a$. Using the fact that $\Sigma_i \models_{us} \mathcal{C}_i$ for all $i \in I_k$, we have by Definition 8 that for each $i \in I_k$ there exists δ_i such that $(\mathbf{x}_i(s), \mathbf{z}_i(s)) \in G_i(s)$ for all $s \in [0, t + \delta_i]$, where $\mathbf{z}_i(t) = [\mathbf{x}_{j_1}(t); \ldots; \mathbf{x}_{j_{|I_k|-1}}(t)]$ is the cooperative internal input (the stacked state of its cooperative agents that are involved in the same cluster). Let us define $\bar{\delta}_k = \min_{i \in I_k} \delta_i$. Then, using the fact that $\bar{G}_k = \cap_{i \in I_k} G_i$ we get that $\bar{\mathbf{x}}_k(s) = [\mathbf{x}_{k_1}(s); \ldots; \mathbf{x}_{k_{|I_k|}}(s)] \in \bar{G}_k(s)$ for all $s \in [0, t + \bar{\delta}_k]$, where $G_i(s) \subseteq \bar{X}_k = X_{k_1} \times \ldots \times X_i \times \ldots \times X_{k_{|I_k|}}$. Therefore, we obtain that $\bar{\Sigma}_k \models_{us} \bar{\mathcal{C}}_k$. ■

Proof of Theorem 11. Let $\mathbf{x} : \mathbb{R}_{\geq 0} \to X$ be a trajectory of the multi-agent system Σ. Then, from the definition of multi-agent systems, we have for all $i \in I$, $(\mathbf{x}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times W_i$ is a trajectory of Σ_i, where $\mathbf{x} = [\mathbf{x}_1; \ldots; \mathbf{x}_N]$, and $\mathbf{w}_i(t) = [\mathbf{x}_{j_1}(t); \ldots; \mathbf{x}_{j_{|\mathcal{N}_i^a|}}(t)]$. Since the cluster interconnection graph $\bar{\mathcal{G}}^a = (\bar{I}, \bar{E}^a)$ is a directed acyclic graph, there exists at least one initial cluster that does not have adversarial internal inputs. Now, consider the initial clusters $\{\bar{\Sigma}_k\}_{k \in \bar{I}^{Init}}$. First, we have by condition (i) that $\Sigma_i \models \mathcal{C}_i$ holds for all $i \in I$, and thus we obtain that $\bar{\Sigma}_k \models \bar{\mathcal{C}}_k$ by Proposition 10. This implies that for all $t \in \mathbb{R}_{\geq 0}$ with $\bar{\mathbf{w}}_k(s) \in \bar{A}_k^a(s)$ for all $s \in [0, t]$, we have $\bar{\mathbf{x}}_k(s) \in \bar{G}_k(s)$ for all $s \in [0, t]$. Now consider an arbitrary $t \in \mathbb{R}_{\geq 0}$ with $\bar{\mathbf{w}}_k(s) \in \bar{A}_k^a(s)$ for all $s \in [0, t]$. Since the initial clusters do not have adversarial internal inputs, this implies:

$$\bar{\mathbf{x}}_k(s) \in \bar{G}_k(s), \forall s \in [0, t], \forall k \in \bar{I}^{Init}. \tag{19}$$

Next, let us prove by contradiction that for all $k \in \bar{I}$, $\bar{\mathbf{x}}_k(s) \in \bar{G}_k(s)$, for all $s \in [0, t]$. Let us assume that there exists $k \in \bar{I} \setminus \bar{I}^{Init}$, such that $\bar{\mathbf{x}}_k(s) \notin \bar{G}_k(s)$, for some $s \in [0, t]$. Since by condition (i) we have $\Sigma_i \models \mathcal{C}_i$ for all $i \in I$, which implies that $\bar{\Sigma}_k \models \bar{\mathcal{C}}_k$ by Proposition 10, we have that $\bar{\mathbf{w}}_k(s) \notin \bar{A}_k^a(s)$ for some $s \in [0, t]$. Since $\bar{\mathbf{w}}_k = [\mathbf{w}_{k_1}; \ldots; \mathbf{w}_{k_{|I_k|}}]$, this means that for some $i \in I_k$, $\mathbf{w}_i(s) \notin A_i^a(s)$. Then using the fact that: $\mathbf{w}_i(s) = [\mathbf{x}_{j_1}(s); \ldots; \mathbf{x}_{j_{|\mathcal{N}_i^a|}}(s)]$ and $\prod_{j \in \mathcal{N}_i^a} \overline{\mathbf{Proj}}^j(G_j(s) \subseteq A_i^a(s)$, we deduce the existence of $j \in \mathcal{N}_i^a$ such that $\mathbf{x}_j(s) \notin \overline{\mathbf{Proj}}^j(G_j(s))$, which further implies that $\bar{\mathbf{x}}_{k'}(s) \notin \bar{G}_{k'}(s)$, for some $s \in [0, t]$ where $j \in I_{k'}$. Note that $k' \neq k$, since we assumed that the agents in the same cluster are not dynamically coupled via adversarial internal inputs, i.e., $\mathcal{N}_i^a \cap \mathcal{N}_i^c = \emptyset$. Consider the cluster $I_{k'}$. If $k' \in \bar{I}^{Init}$, this readily leads to a contradiction with (19); in the case that $k' \in \bar{I} \setminus \bar{I}^{Init}$, by leveraging the same argument as above, we get that there exists $k'' \in \bar{I} \setminus \{k, k'\}$ (due to the structure of a DAG and $\mathcal{N}_i^a \cap \mathcal{N}_i^c = \emptyset$) such that $\bar{\mathbf{x}}_{k''}(s) \notin \bar{G}_{k''}(s)$, for some $s \in [0, t]$. By further iterating this argument and the structure of DAG, there exists $l \in \bar{I}^{Init}$ such that $\bar{\mathbf{x}}_l(s) \notin \bar{G}_l(s)$, which contradicts (19). Hence, we have for all $k \in \bar{I}$, $\bar{\mathbf{x}}_k(s) \in \bar{G}_k(s)$, for all $s \in [0, t]$, and thus, $\mathbf{x}(s) = [\bar{\mathbf{x}}_1(s); \ldots; \bar{\mathbf{x}}_K(s)] \in \prod_{k \in \bar{I}} \bar{G}_k(s) = G(s)$ for all $s \in [0, t]$. Therefore, $\Sigma \models \mathcal{C}$. ■

Proof of Theorem 12. Let $\mathbf{x} : \mathbb{R}_{\geq 0} \to X$ be a trajectory of the multi-agent system Σ. Then, from the definition of multi-agent systems, we have for all $i \in I$, $(\mathbf{x}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times W_i$ is a trajectory of Σ_i, where $\mathbf{x} = [\mathbf{x}_1; \ldots; \mathbf{x}_N]$, and $\mathbf{w}_i(t) = [\mathbf{x}_{j_1}(t); \ldots; \mathbf{x}_{j_{|\mathcal{N}_i^a|}}(t)]$. We prove $\Sigma \models \mathcal{C}$ by inductively showing the existence of $\delta > 0$ such that $\mathbf{x}(s) \in G(s)$ for all $s \in [0, n\delta]$ for all $n \in \mathbb{N}$. First, we have from (i) that for all $i \in I$, $\mathbf{w}_i(0) = [\mathbf{x}_{j_1}(0); \ldots; \mathbf{x}_{j_{|\mathcal{N}_i^a|}}(0)] \in \prod_{j \in \mathcal{N}_i^a} \overline{\mathbf{Proj}}^j(G_j(0)) \subseteq A_i^a(0)$, where the set inclusion follows from (iii). Now, consider the clusters $\bar{\Sigma}_k$, $k \in \bar{I}$, in the multi-agent system. We have from (ii) that $\Sigma_i \models_{us} \mathcal{C}_i$ holds for all $i \in I$, and thus $\bar{\Sigma}_k \models_{us} \bar{\mathcal{C}}_k$ for all $k \in \bar{I}$ by Proposition 10. Note that by the definition of clusters, we have $\bar{\mathbf{w}}_k(0) = [\mathbf{w}_{k_1}(0); \ldots; \mathbf{w}_{k_{|I_k|}}(0)] \in \bar{A}_k^a(0) = \prod_{i \in I_k} A_i^a(0)$, since $\mathbf{w}_i(0) \in A_i^a(0)$ holds for all $i \in I$. Given that $\bar{\Sigma}_k \models_{us} \bar{\mathcal{C}}_k$, we thus have the existence of $\bar{\delta}_k > 0$ such that for $\bar{\mathbf{w}}_k(0) \in \bar{A}_k^a(0)$, we have $\bar{\mathbf{x}}_k(s) = [\mathbf{x}_{k_1}(s); \ldots; \mathbf{x}_{k_{|I_k|}}(s)] \in \bar{G}_k(s)$ for all $s \in [0, \bar{\delta}_k]$. Let us define $\delta > 0$ as $\delta := \min_{k \in \bar{I}} \bar{\delta}_k$. Then, it follows that $\mathbf{x}(s) = [\bar{\mathbf{x}}_1(s); \ldots; \bar{\mathbf{x}}_K(s)] \in \prod_{k \in \bar{I}} \bar{G}_k(s) = G(s)$ for all $s \in [0, \delta]$.

Next, let us show that if $\mathbf{x}(s) \in G(s)$ holds for all $s \in [n\delta, (n+1)\delta)]$, then it implies that $\mathbf{x}(s) \in G(s)$ holds for all $s \in [(n+1)\delta, (n+2)\delta)]$. First, let us assume that $\mathbf{x}(s) \in G(s)$ for all $s \in [n\delta, (n+1)\delta)]$ and show that $\mathbf{x}(s) \in G(s)$ for all $s \in [(n+1)\delta, (n+2)\delta)]$. By $\mathbf{x}(s) \in G(s)$ for all $s \in [n\delta, (n+1)\delta)]$, we obtain that $\bar{\mathbf{x}}_k(s) \in \bar{G}_k(s)$ for all $s \in [n\delta,(n+1)\delta)]$ for all $k \in \bar{I}$, since $\mathbf{x}(s) = [\bar{\mathbf{x}}_1(s);\ldots;\bar{\mathbf{x}}_K(s)] \in G(s) = \prod_{k\in\bar{I}} \bar{G}_k(s)$. This further implies that $(\mathbf{x}_i(s),\mathbf{z}_i(s)) \in G_i(s)$ for all $s \in [n\delta,(n+1)\delta)]$ for all $i \in I$, since $\bar{\mathbf{x}}_k(s) = [\mathbf{x}_{k_1}(s);\ldots;\mathbf{x}_{k_{|I_k|}}(s)] \in \bar{G}_k(s) = \cap_{i\in I_k} G_i(s)$, where $G_i(s) \subseteq \bar{X}_k = X_{k_1}\times\ldots\times X_i\times\ldots\times X_{k_{|I_k|}}$. Then, we obtain that for all $i \in I_k$, and for all $s \in [n\delta,(n+1)\delta]$, $\mathbf{w}_i(s) = [\mathbf{x}_{j_1}(s);\ldots;\mathbf{x}_{j_{|\mathcal{N}_i|}}(s)] \in \prod_{j\in\mathcal{N}_i^a} \overline{\mathbf{Proj}}^j(G_j(s)) \subseteq A_i^a(s)$, where the set inclusion follows from (iii). This implies that $\bar{\mathbf{w}}_k(s) = [\mathbf{w}_{k_1}(s);\ldots;\mathbf{w}_{k_{|I_k|}}(s)] \in \prod_{i\in I_k} A_i^a(s) = \bar{A}_k^a(s)$ for all $s \in [n\delta,(n+1)\delta]$. Again, since $\bar{\Sigma}_k \models_{us} \bar{\mathcal{C}}_k$, one gets for all $k \in \bar{I}$, $\bar{\mathbf{x}}_k(s) = [\mathbf{x}_{k_1}(s);\ldots;\mathbf{x}_{k_{|I_k|}}(s)] \in \bar{G}_k(s)$ for all $s \in [(n+1)\delta,(n+1)\delta+\bar{\delta}_k]$, which further implies that $\bar{\mathbf{x}}_k(s) \in \bar{G}_k(s)$ for all $s \in [(n+1)\delta,(n+2)\delta]$ since $\delta := \min_{k\in\bar{I}} \bar{\delta}_k$. Hence, $\mathbf{x}(s) = [\bar{\mathbf{x}}_1(s);\ldots;\bar{\mathbf{x}}_K(s)] \in \prod_{k\in\bar{I}} \bar{G}_k(s) = G(s)$ for all $s \in [(n+1)\delta,(n+2)\delta)]$. Therefore, by induction, one has that $\mathbf{x}(s) \in G(s)$ for all $s \in [0,n\delta]$ for all $n \in \mathbb{N}$, and thus for all $s \geq 0$, which concludes that $\Sigma \models \mathcal{C}$. ∎

Proof of Proposition 15. Consider $\varepsilon > 0$ such that $\Sigma_i \models \mathcal{C}_i^\varepsilon$. From uniform continuity of $\mathbf{w}_i : \mathbb{R}_{\geq 0} \to W_i$, and for $\varepsilon > 0$, we have the existence of $\delta_i > 0$ such that for all $t \geq 0$, if $\mathbf{w}_i(s) \in A_i^a(s)$, for all $s \in [0,t]$, then $\mathbf{w}_i(s) \in \mathcal{B}_\varepsilon(A_i^a)(s)$, for all $s \in [0,t+\delta_i]$. Let us now show the uniform strong satisfaction of contracts. Consider the $\delta_i > 0$ defined above, and consider any $t \geq 0$ such that $\mathbf{w}_i(s) \in A_i^a(s)$ for all $s \in [0,t]$. First we have by the uniform continuity of $\mathbf{w}_i$, $\mathbf{w}_i(s) \in \mathcal{B}_\varepsilon(A_i^a)(s)$, for all $s \in [0,t+\delta_i]$. Hence, from the weak satisfaction of $\mathcal{C}_i^\varepsilon$, one has that $(\mathbf{x}_i(s),\mathbf{z}_i(s)) \in G_i(s)$ for all $s \in [0,t+\delta_i]$, which in turn implies the uniform strong satisfaction of the contract $\mathcal{C}_i$ according to Definition 8. Hence, $\Sigma_i \models_{us} \mathcal{C}_i$. ∎

Proof of Theorem 22. We prove the uniform strong satisfaction of the contract using Proposition 15. Let $(\mathbf{x}_i,\mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times W_i$ be a trajectory of Σ_i. Since $-\gamma_i(0) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_i(0)) < \rho_i^{max}$ holds, we have $\mathbf{x}_i(0) \in G_i$. Now, consider $\varepsilon > 0$. Let us prove $\Sigma_i \models \mathcal{C}_i^\varepsilon$, where $\mathcal{C}_i^\varepsilon = (\mathcal{B}_\varepsilon(A_i^a), G_i)$. Consider any $\mathbf{w}_i = [\mathbf{x}_{j_1};\ldots;\mathbf{x}_{j_{|\mathcal{N}_i^a|}}]$ with $\mathbf{w}_i(t) \in \mathcal{B}_\varepsilon(A_i^a)(t)$ for all $t \in \mathbb{R}_{\geq 0}$. By Definition 14, $\mathbf{w}_i \in \mathcal{B}_\varepsilon(A_i^a)$ implies that $\exists \tilde{\mathbf{w}}_i = [\tilde{\mathbf{x}}_{j_1};\ldots;\tilde{\mathbf{x}}_{j_{|\mathcal{N}_i^a|}}] \in A_i^a$ such that $\forall t \geq 0$, $\|\mathbf{w}_i(t) - \tilde{\mathbf{w}}_i(t)\| \leq \varepsilon$, which further leads to $\|\mathbf{x}_j(t) - \tilde{\mathbf{x}}_j(t)\| \leq \varepsilon$, $\forall t \geq 0$, $\forall j \in \mathcal{N}_i^a$. Let us recall that the STL robustness functions $\rho_j^{\psi_j}$ are continuously differentiable in $\mathbf{x}$, thus there exists a real constant $D \in \mathbb{R}_{\geq 0}$ such that for any $\mathbf{x}_j(t)$ and $\tilde{\mathbf{x}}_j(t)$, it holds that $\left|\rho_j^{\psi_j}(\mathbf{x}_j(t)) - \rho_j^{\psi_j}(\tilde{\mathbf{x}}_j(t))\right| \leq D\|\mathbf{x}_j(t) - \tilde{\mathbf{x}}_j(t)\| \leq D\varepsilon$. Also note that by the definition of A_i^a, $\tilde{\mathbf{w}}_i(t) \in A_i^a$ gives us for all $j \in \mathcal{N}_i^a$, $-\gamma_j(t)+\rho_j^{max} < \rho_j^{\psi_j}(\tilde{\mathbf{x}}_j(t)) < \rho_j^{max}, \forall t \geq 0$. As a consequence, we get $-\gamma_j(t)+\rho_j^{max} - D\varepsilon < \rho_j^{\psi_j}(\mathbf{x}_j(t)) < \rho_j^{max} + D\varepsilon, \forall t \geq 0$.

Next, we show that $\mathbf{x}_{\phi_i}(t) \in G_i$. The underlying idea of proving this is based on showing that the transformed error ϵ_i is bounded. This is proved by showing that the prescribed performance region $\hat{\mathcal{D}}_i = (-1,0)$ for $\hat{e}_i$ is forward invariant. Consider the cluster of agents $\bar{\Sigma}_k$ with $i \in I_k$ where each agent is subject to STL task ϕ_i. Let us define the stack vectors $\epsilon = [\epsilon_1;\ldots;\epsilon_{|I_k|}]$, $\hat{e} = [\hat{e}_1;\ldots;\hat{e}_{|I_k|}]$, and the prescribed performance region $\hat{\mathcal{D}} := \hat{\mathcal{D}}_1 \times \ldots \times \hat{\mathcal{D}}_{|I_k|}$ for $\hat{e}$, where $\hat{\mathcal{D}}_i := (-1,0)$. Consider a potential function $V : \hat{\mathcal{D}} \to \mathbb{R}_{\geq 0}$ defined as $V(\hat{e}) = \frac{1}{2}\epsilon(\hat{e})^\top\epsilon(\hat{e})$ with $\epsilon_i(\hat{e}_i)$ defined as in (6). By differentiating V with respect to time, we obtain $\dot{V} = \epsilon^\top\dot{\epsilon}$, where each $\dot{\epsilon}_i$ can be obtained by $\dot{\epsilon}_i \overset{(18)}{=} \mathcal{J}_i(\hat{e}_i,t)[\dot{e}_i + \alpha_i(t)e_i] = \mathcal{J}_i(\hat{e}_i,t)[\frac{\partial\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial\bar{\mathbf{x}}_k}^\top \dot{\bar{\mathbf{x}}}_k(t) - \dot{\gamma}_i(t)\hat{e}_i]$. Thus, we get

$$\dot{V} = \epsilon^\top\dot{\epsilon} = \epsilon^\top\mathcal{J}(\Gamma\dot{\bar{\mathbf{x}}}_k - \mathbf{p}), \tag{20}$$

where $\mathcal{J} \in \mathbb{R}^{|I_k|\times|I_k|}$ is a diagonal matrix with diagonal entries $\mathcal{J}_i(\hat{e}_i) = -\frac{1}{\gamma_i(t)\hat{e}_i(1+\hat{e}_i)}$, $i \in I_k$, $\Gamma \in \mathbb{R}^{|I_k|\times\bar{n}_k}$ is a matrix with row vectors $\frac{\partial\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial\bar{\mathbf{x}}_k}^\top$, and $\mathbf{p} \in \mathbb{R}^{|I_k|}$ is a column vector with entries $\dot{\gamma}_i(t)\hat{e}_i$. By inserting the dynamics of $\dot{\bar{\mathbf{x}}}_k$ (16) into (20), we obtain that

$$\begin{aligned}\dot{V} =& \epsilon^\top\mathcal{J}(\Gamma(\bar{f}_k(\bar{\mathbf{x}}_k) + \bar{g}_k(\bar{\mathbf{x}}_k)\bar{\mathbf{u}}_k + \bar{h}_k(\bar{\mathbf{w}}_k)) - \mathbf{p})\\ =& \epsilon^\top\mathcal{J}(\Gamma(\bar{f}_k(\bar{\mathbf{x}}_k) + \bar{h}_k(\bar{\mathbf{w}}_k)) - \mathbf{p}) + \epsilon^\top\mathcal{J}\Gamma\bar{g}_k(\bar{\mathbf{x}}_k)\bar{\mathbf{u}}_k, \end{aligned}\tag{21}$$

where $\bar{\mathbf{u}}_k = [\mathbf{u}_{k_1};\ldots;\mathbf{u}_{k_{|I_k|}}]$, with the local control law $\mathbf{u}_i = -g_i^\top(\mathbf{x}_i)\sum_{j\in I_k}(\frac{\partial\rho_j^{\psi_j}(\mathbf{x}_{\phi_j})}{\partial\mathbf{x}_i}\mathcal{J}_j(\hat{e}_j,t)\epsilon_j(\mathbf{x}_{\phi_j},t))$ as in (8), $i \in I_k = \{k_1,\ldots,k_{|I_k|}\}$. Then, we get

$$\dot{V} = \epsilon^\top\mathcal{J}(\Gamma(\bar{f}_k(\bar{\mathbf{x}}_k) + \bar{h}_k(\bar{\mathbf{w}}_k) - \Theta\mathbf{d}) - \mathbf{p}) - \epsilon^\top\mathcal{J}\Gamma\Theta\Lambda\mathcal{J}\epsilon, \tag{22}$$

where $\Theta := \text{diag}(g_{k_i}(\mathbf{x}_{k_i})g_{k_i}^\top(\mathbf{x}_{k_i})) \in \mathbb{R}^{\bar{n}_k\times\bar{n}_k}$, $\mathbf{d} \in \mathbb{R}^{\bar{n}_k}$ is a column vector with entries $h_i(d_i)$, and $\Lambda \in \mathbb{R}^{\bar{n}_k\times|I_k|}$ is a matrix with column vectors $\frac{\partial\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial\bar{\mathbf{x}}_k} = [\frac{\partial\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial\mathbf{x}_{i_1}};\ldots;\frac{\partial\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial\mathbf{x}_{i_{P_i}}}]$. Recall that $\Gamma \in \mathbb{R}^{|I_k|\times\bar{n}_k}$ is a matrix with row vectors $\frac{\partial\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial\bar{\mathbf{x}}_k}^\top$, we get that $\Gamma = \Lambda^\top$. Then by (22), and thus $-\epsilon^\top\mathcal{J}\Gamma\Theta\Lambda\mathcal{J}\epsilon = -\epsilon^\top\mathcal{J}\Lambda^\top\Theta\Lambda\mathcal{J}\epsilon$. Note that by Assumption 5, we get that Λ is a block matrix that can be converted to an upper triangular form. Moreover, the diagonal entries of the block matrix Λ are non-zero since $\frac{\partial\rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial\mathbf{x}_i}$ are non-zero. Therefore, we can obtain that the rank of matrix Λ is rank$(\Lambda) = |I_k|$. Note that according to Assumption 1, $g_i(\mathbf{x}_i)g_i(\mathbf{x}_i)^\top$ is positive definite. Therefore, the matrix $\Lambda^\top\Theta\Lambda$ is positive definite. Then, we can obtain that $\Lambda^\top\Theta\Lambda \geq \alpha_L I_{|I_k|}$ for some $\alpha_L > 0$. Recall that $\mathcal{J} \in \mathbb{R}^{|I_k|\times|I_k|}$ is a diagonal matrix with positive diagonal entries $\mathcal{J}_i(\hat{e}_i) = -\frac{1}{\gamma_i(t)\hat{e}_i(1+\hat{e}_i)}$, thus, $\mathcal{J}^2 \geq \alpha_J I_{|I_k|}$, where $\alpha_J = \min_{i\in I_k}\{\frac{1}{\sup_{t\in\mathbb{R}_{\geq 0}}\gamma_i(t)^2}\min_{\hat{e}_i\in\hat{\mathcal{D}}_i}(\frac{1}{\hat{e}_i(1+\hat{e}_i)})^2\} > 0$. Hence, we obtain that $-\epsilon^\top\mathcal{J}\Lambda^\top\Theta\Lambda\mathcal{J}\epsilon \leq -\alpha_L\alpha_J\epsilon^\top\epsilon \leq -2\alpha_L\alpha_J V$. Hence, we get the chain of inequality

$$\begin{aligned}\dot{V} =& \epsilon^\top\mathcal{J}(\Gamma(\bar{f}_k(\bar{\mathbf{x}}_k) + \bar{h}_k(\bar{\mathbf{w}}_k) - \Theta\mathbf{d}) - \mathbf{p}) - \epsilon^\top\mathcal{J}\Lambda^\top\Theta\Lambda\mathcal{J}\epsilon,\\ \leq& \epsilon^\top\mathcal{J}(\Gamma(\bar{f}_k(\bar{\mathbf{x}}_k) + \bar{h}_k(\bar{\mathbf{w}}_k) - \Theta\mathbf{d}) - \mathbf{p})\\ &- \epsilon^\top\mathcal{J}(\alpha_L - \xi)\mathcal{J}\epsilon - \xi\|\epsilon^\top\mathcal{J}\|^2\\ \leq& -\kappa V + \eta(t)\end{aligned}\tag{23}$$

for some ξ satisfying $0 < \xi < \alpha_L$, where $\kappa = 2(\alpha_L - \xi)\alpha_J$, $\eta(t) = \frac{1}{4\xi}(\|\Gamma\|(\|\bar{f}_k(\bar{\mathbf{x}}_k)\| + \|\bar{h}_k(\bar{\mathbf{w}}_k)\| + \|\Theta\mathbf{d}\|) + \|\mathbf{p}\|)^2$.

Now, we proceed with finding an upper bound $\bar{\eta}$ of $\eta(t)$ for all $\hat{e} \in \hat{\mathcal{D}}$ and all times $t \in \mathbb{R}_{\geq 0}$. Let us recall the definition of the modulated error $\hat{e}_i(\mathbf{x}_{\phi_i},t) := \frac{\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}) - \rho_i^{max}}{\gamma_i(t)}$ and its corresponding prescribed performance region $\hat{\mathcal{D}}_i := (-1,0)$ for each $i \in I_k$. Next, we define $\mathcal{X}_i(t) := \{\mathbf{x}_{\phi_i} \in \mathbb{R}^{n_{P_i}} | -1 <

$\hat{e}_i(\mathbf{x}_{\phi_i}, t) = \frac{\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}) - \rho_i^{max}}{\gamma_i(t)} < 0\}$ as the set of states $\mathbf{x}_{\phi_i}$ such that $\hat{e}_i(\mathbf{x}_{\phi_i}, t) \in \hat{\mathcal{D}}_i$ at time $t \in \mathbb{R}_{\geq 0}$. One can also observe that $\mathcal{X}_i(t)$ has the property that for $t_1 < t_2$, $\mathcal{X}_i(t_2) \subseteq \mathcal{X}_i(t_1)$ holds since $\gamma_i(t)$ is non-increasing in t. Thus, $\mathcal{X}_i(0)$ collects all states $\mathbf{x}_{\phi_i}$ such that $\hat{e}_i(\mathbf{x}_{\phi_i}, t) \in \hat{\mathcal{D}}_i$ at all times $t \in \mathbb{R}_{\geq 0}$. Note also that $\mathcal{X}_i(0)$ is bounded due to condition (ii) of Assumption 18 and γ_i is bounded by definition, for all $i \in I_k$. Thus, by the continuity of functions f_i, g_i and h_i, it holds that $\|f_i(\mathbf{x}_i)\|$ and $\|g_i(\mathbf{x}_i)g_i(\mathbf{x}_i)^\top h_i(d_i(t))\|$ are upper bounded, where $d_i(t) = [\gamma_{j_1}(t)\mathbf{1}_{n_{j_1}}; \ldots; \gamma_{j_{|\mathcal{N}_i|}}(t)\mathbf{1}_{n_{|\mathcal{N}_i|}}]$. Thus, $\|\bar{f}_k(\bar{\mathbf{x}}_k)\|$ and $\|\Theta\mathbf{d}\|$ are bounded. Note that $\mathbf{p}$ is a column vector with entries $\dot{\gamma}_i(t)\hat{e}_i$. Since $|\dot{\gamma}_i(t)\hat{e}_i| \leq |\dot{\gamma}_i(0)|$ and $|\dot{\gamma}_i(0)|$ is bounded by definition, $\|\mathbf{p}\|$ is bounded as well. Moreover, recall from the beginning of the proof that by the definition of A_i^a in the assume-guarante contract, we have $\mathbf{w}_i(t) = [\mathbf{x}_{j_1}(t); \ldots; \mathbf{x}_{j_{|\mathcal{N}_i^a|}}(t)] \in \mathcal{B}_\varepsilon(A_i^a)$, which implies $-\gamma_j(t) + \rho_j^{max} - D\varepsilon < \rho_j^{\psi_j}(\mathbf{x}_j(t)) < \rho_j^{max} + D\varepsilon, \forall t \geq 0$. By further combining this with Assumption 18, we get that $\mathbf{w}_i(t)$ is bounded, and thus it holds that $\|h_i(\mathbf{w}_i)\|$ and $\|\bar{h}_k(\bar{\mathbf{w}}_k)\|$ are also upper bounded. Additionally, note that $\frac{\partial \rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial \mathbf{x}_{\phi_i}} = 0$ if and only if $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}) = \rho_i^{opt}$ since $\rho_i^{\psi}(\mathbf{x}_{\phi_i})$ is concave under Assumption 18. However, since $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(0)) < \rho_i^{max} < \rho_i^{opt}$, and for all states $\mathbf{x}_{\phi_i} \in \mathcal{X}_i(0)$, $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}) < \rho_i^{max}$ holds, then, we have for all states $\mathbf{x}_{\phi_i} \in \mathcal{X}_i(0)$, $\frac{\partial \rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial \mathbf{x}_{\phi_i}} \neq 0_{n_i}$, and $\|\frac{\partial \rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial \mathbf{x}_{\phi_i}}\|^2 \geq k_\rho > 0$ holds for a positive constant k_ρ. Since Γ is a matrix with row vectors $\frac{\partial \rho_i^{\psi_i}(\mathbf{x}_{\phi_i})}{\partial \bar{\mathbf{x}}_k}^\top$, we get that $\|\Gamma\|$ is bounded. Consequently, we can define an upper bound $\bar{\eta}$ of $\eta(t)$, for all $\mathbf{x}_{\phi_i} \in \mathcal{X}_i(0)$ for all $i \in I_k$.

Next, we show that $\hat{\mathcal{D}}$ is forward invariant. To do this, we first introduce a function $\mathcal{S}(\hat{e}) = 1 - e^{-V(\hat{e})}$ for which $0 < \mathcal{S}(\hat{e}) < 1$, $\forall \hat{e}_i \in \hat{\mathcal{D}}_i$, $\forall i \in I_k$, and $\mathcal{S}(\hat{e}) \to 1$ as $\hat{e} \to \partial\hat{\mathcal{D}}$. By differentiating $\mathcal{S}(\hat{e})$ we get

$$\dot{\mathcal{S}}(t) = \dot{V}(\hat{e})\big(1 - \mathcal{S}(\hat{e})\big). \tag{24}$$

By substituting (23) and inserting $V(\hat{e}) = -\ln(1 - \mathcal{S}(\hat{e}))$ in (24), we get

$$\dot{\mathcal{S}}(t) \leq -\kappa\big(1 - \mathcal{S}(\hat{e})\big)\big(\ln(e^{-\frac{\eta(t)}{\kappa}}) - \ln(1 - \mathcal{S}(\hat{e}))\big). \tag{25}$$

Note that by definition, we have $\kappa > 0$ and $1 - \mathcal{S}(\hat{e}) > 0$. Now define the region $\Omega_{\hat{e}} = \{\hat{e} \in \hat{\mathcal{D}} | \mathcal{S}(\hat{e}) \leq 1 - e^{-\frac{\bar{\eta}}{\kappa}}\}$. Since $-\gamma_i(0) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(0)) < \rho_i^{max}$ holds $\forall i \in I_k$, then we obtain that $\hat{e}_i(\mathbf{x}_{\phi_i}(0)) \in \hat{\mathcal{D}}_i = (-1, 0)$ $\forall i \in I_k$, and consequently, $\mathcal{S}(\hat{e}(0)) < 1$ holds. Let us define $c = \mathcal{S}(\hat{e}(0))$ and the set $\Omega_c = \{\hat{e} \in \hat{\mathcal{D}} | \mathcal{S}(\hat{e}) \leq c\}$. Now, consider the case when $c < 1 - e^{-\frac{\bar{\eta}}{\kappa}}$. In this case, $\Omega_c \subset \Omega_{\hat{e}}$, and by (25), $\dot{\mathcal{S}}(t) \leq 0$ for all $\hat{e} \in \partial\Omega_{\hat{e}}$, therefore, $\hat{e}(t) \in \Omega_{\hat{e}}$, $\forall t \in \mathbb{R}_{\geq 0}$. Next, consider the other case when $c \geq 1 - e^{-\frac{\bar{\eta}}{\kappa}}$. In this case $\Omega_{\hat{e}} \subseteq \Omega_c$, and by (25), $\dot{\mathcal{S}}(t) < 0$ for all $\hat{e} \in \Omega_c \setminus \Omega_{\hat{e}}$, hence, $\mathcal{S}(\hat{e}) \to \Omega_{\hat{e}}$. Thus, starting from any point within the set Ω_c, $\mathcal{S}(\hat{e}(t))$ remains less than 1. Consequently, the modulated error $\hat{e}$ always evolves within a closed strict subset of $\hat{\mathcal{D}}$ (that is, set $\Omega_{\hat{e}}$ in the case that $c < 1 - e^{-\frac{\bar{\eta}}{\kappa}}$, or set Ω_c in the case that $c \geq 1 - e^{-\frac{\bar{\eta}}{\kappa}}$), which implies that $\hat{e}$ is not approaching the boundary $\partial\hat{\mathcal{D}}$. It follows that ϵ is bounded, and thus, ϵ_i is bounded for all $i \in I_k$. Thus, we can conclude that $\rho_i^{\psi_i}(\mathbf{x}_{\phi_i}(t))$ evolves within the predefined region (4), i.e., $\mathbf{x}_{\phi_i}(t) \in G_i$ for all $t \in \mathbb{R}_{\geq 0}$. Therefore, we have $\Sigma_i \models \mathcal{C}_i^\varepsilon$. By Proposition 15, it implies that $\Sigma_i \models_{us} \mathcal{C}_i$. ■

Part of the proof above was inspired by [46, Thm. 1], where similar Lyapunov arguments were used in the context of PPC-based control of multi-agent systems. However, the results there deal with consensus control of multi-agent systems only and cannot handle temporal logic properties.

Proof of Corollary 24. From Theorem 22, one can verify that the closed-loop agents under controller (8) satisfy: for all $i \in I$, $\Sigma_i \models_{us} \mathcal{C}_i$, and for all $i \in I$, $\prod_{j \in \mathcal{N}_i^a} \overline{\mathbf{Proj}}^j(G_j) \subseteq A_i^a$. Moreover, for all $i \in I$ and for any trajectory $(\mathbf{x}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times W_i$ of Σ_i, the choice of parameters of the prescribed regions as in (9)–(14) ensures that $\mathbf{x}_i(0) \in \overline{\mathbf{Proj}}^i(G_i)(0)$. Hence, all conditions required in Theorem 12 are satisfied. Note that since $\Sigma_i \models_{us} \mathcal{C}_i$ implies $\Sigma_i \models \mathcal{C}_i$, the conditions required in Theorem 11 hold as well. Thus, irrespective of whether the dynamical interconnection graph is acyclic or cyclic, we can conclude that $\Sigma \models \mathcal{C} = (\emptyset, \prod_{k \in \bar{I}} \bar{G}_k)$ as a consequence of Theorems 12 and 11. Therefore, the multi-agent system Σ satisfies the STL task $\bar{\phi} = \wedge_{i=1}^N \phi_i$. ■

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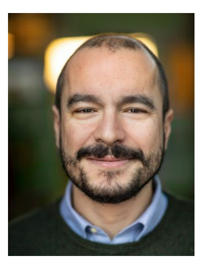

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