Consensus Control for Leader-follower Multi-agent Systems under Prescribed Performance Guarantees

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***Abstract*— This paper addresses the problem of distributed control for leader-follower multi-agent systems under prescribed performance guarantees. Leader-follower is meant in the sense that a group of agents with external inputs are selected as leaders in order to drive the group of followers in a way that the entire system can achieve consensus within certain prescribed performance transient bounds. Under the assumption of tree graphs, a distributed control law is proposed when the decay rate of the performance functions is within a sufficient bound. Then, two classes of tree graphs that can have additional followers are investigated. Finally, several simulation examples are given to illustrate the results.**

I. INTRODUCTION

The consensus problem has attracted great interest due to its wide applications in cooperative control and formation control. Consensus is achieved when a group of agents converge to a common value. The first order consensus protocol was first introduced in [10], while the second order consensus protocol has been investigated in [13].

In this work, we study the consensus problem in a leader-follower framework, that is, one or more agents are selected as *leaders* with external inputs in addition to the first order consensus protocol. The remaining agents are *followers* only obeying the first order consensus protocol. Recent research that has been done in the leader-follower framework can be divided into two parts. The first part deals with the controllability of leader-follower multi-agent systems [14], [3], [12], [5]. The second part targets leader selection [11], [4], which involves the problems of how to choose the leaders among the agents such that the leader-follower system satisfies the requirements like controllability, optimal performance or formation maintenance.

Prescribed performance control (PPC) was proposed in [1] to prescribe the evolution of system output or tracking error within some predefined region. For example, an agreement protocol that can additionally achieve prescribed performance for a combined error of positions and velocities is designed in [8] for multi-agent systems with double integrator dynamics, while PPC for multi-agent average consensus with single integrator dynamics is presented in [6].

In this work, we are interested in how to design control strategies for the leaders such that the leader-follower multi-agent system achieves consensus within certain performance bounds. Compared with existing work of PPC for multi-agent systems [8], we apply a PPC law only to the leaders while most of the related work applies PPC to all the agents to achieve consensus. Unlike other leader-follower consensus approaches using PPC [7], in which the multi-agent system only has one leader and the leader is treated as a reference for the followers, we focus on a more general framework in the sense that we can have more than one leader and the leaders are designed in order to steer the entire system achieving consensus within the prescribed performance bounds. The difficulties in this work are due to the combination of uncertain topologies, leader amount and leader positions. In addition, the leader can only communicate with its neighbouring agents. The contributions of the paper can be summarized as: i) within this general leader-follower framework, under the assumption of tree graphs, a distributed control law is proposed when the decay rate of the performance functions is within a sufficient bound; ii) the specific classes of chain and star graphs that can have additional followers are investigated.

The rest of the paper is organized as follows. In Section II, preliminary knowledge is introduced and the problem is formulated, while Section III presents the main results, which are further verified by simulation examples in Section IV. Section V closes with concluding remarks and future work.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Graph Theory

An undirected graph [9] $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ comprises of the vertices set $\mathcal{V} = \{1, 2, \dots, n\}$ and the edges set $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid j \in \mathcal{N}_i\}$ indexed by $e_1, e_2, \dots, e_m$. Here, $m = |\mathcal{E}|$ is the number of edges and $\mathcal{N}_i$ denotes the agents in the neighbourhood of agent i that can communicate with i. The *adjacency matrix* $\mathbb{A}$ of $\mathcal{G}$ is the $n \times n$ symmetric matrix whose elements a_{ij} are given by $a_{ij} = 1$, if $(i,j) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. The degree of vertex i is defined as $d_i = \sum_{j \in \mathcal{N}_i} a_{ij}$. Then the *degree matrix* is $\Delta = \text{diag}(d_1, d_2, \dots, d_n)$. The *graph Laplacian* of $\mathcal{G}$ is $L = \Delta - \mathbb{A}$. A *path* is a sequence of edges connecting two distinct vertices. A graph is *connected* if there exists a path between any pair of vertices. By assigning an orientation to each edge of $\mathcal{G}$ we can define the *incidence matrix* $D = D(\mathcal{G}) = [d_{ij}] \in \mathbb{R}^{n \times m}$. The rows of D are indexed by the vertices and the columns are indexed by the edges with $d_{ij} = 1$ if the vertex i is the head of the edge (i,j), $d_{ij} = -1$ if the vertex i is the tail of the edge (i,j) and $d_{ij} = 0$ otherwise. Based on the incidence matrix, the graph Laplacian of $\mathcal{G}$ can be described as $L = DD^T$. In addition,

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$L_e = D^T D$ is the so called *edge Laplacian* [9] and c_{ij} denotes the elemnts of L_e.

B. System Description

In this work, we consider a multi-agent system with vertices $\mathcal{V} = \{1, 2, \dots, n\}$. Without loss of generality, we suppose that the first n_f agents are selected as followers while the last n_l agents are selected as leaders with respective vertices set $\mathcal{V}_F = \{1, 2, \dots, n_f\}$, $\mathcal{V}_L = \{n_f + 1, n_f + 2, \dots, n_f + n_l\}$ and $n = n_f + n_l$.

Let $x_i \in \mathbb{R}$ be the position of agent i, where we only consider the one dimensional case, without loss of generality. Specifically, the results can be extended to higher dimensions with appropriate use of the Kronecker product. The state evolution of follower $i \in \mathcal{V}_F$ is governed by the first order consensus protocol:

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i), \tag{1}$$

while the state evolution of leader $i \in \mathcal{V}_L$ is governed by the first order consensus protocol with an external input $u_i \in \mathbb{R}$:

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i) + u_i. \tag{2}$$

Let $x = [x_1, \dots, x_{n_f}, \dots, x_n]^T \in \mathbb{R}^n$ be the stack vector of absolute positions of all the agents and $u = [u_{n_f+1}, \dots, u_{n_f+n_l}]^T \in \mathbb{R}^{n_l}$ be the control input vector . Denote $\bar{x} = [\bar{x}_1, \dots, \bar{x}_m]^T$ as the stack vector of relative positions between the pair of communicating agents $(i,j) \in \mathcal{E}$, where $\bar{x}_k \triangleq x_{ij} = x_i - x_j, k = 1, 2, \dots, m$. It can be verified that $Lx = D\bar{x}$ and $\bar{x} = D^T x$. Moreover, if $\bar{x} = 0$, we have that $Lx = 0$. Stacking (1) and (2), the dynamics of the leader-follower multi-agent system is rewritten as:

$$\Sigma : \dot{x} = -Lx + Bu, \tag{3}$$

where L is the graph Laplacian and $B = \begin{bmatrix} 0_{n_f \times n_l} \\ I_{n_l} \end{bmatrix}$.

C. Prescribed Performance Control

The aim of PPC is to prescribe the evolution of the system output or the tracking error within some predefined region described as follows:

$$-M_{ij}\rho_{ij}(t) < x_{ij}(t) < \rho_{ij}(t) \qquad \text{if } x_{ij}(0) > 0 \tag{4}$$

$$-\rho_{ij}(t) < x_{ij}(t) < M_{ij}\rho_{ij}(t) \qquad \text{if } x_{ij}(0) < 0 \tag{5}$$

$\rho_{ij}(t) : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ are positive, smooth and stirctly decreasing performance functions that introduce the predefined bounds for the target system outputs or the tracking errors. One example choice is $\rho_{ij}(t) = (\rho_{ij0} - \rho_{ij\infty})e^{-l_{ij}t} + \rho_{ij\infty}$ with $\rho_{ij0}, \rho_{ij\infty}$ and l_{ij} positive parameters and $\rho_{ij\infty} = \lim_{t\to\infty}\rho_{ij}(t)$ represents the maximum allowable tracking error at the steady state; M_{ij} represents the maximum allowed overshot.

Normalizing $x_{ij}(t)$ with respect to the performance function $\rho_{ij}(t)$, we define the modulated error as $\hat{x}_{ij}(t) = \frac{x_{ij}(t)}{\rho_{ij}(t)}$ and the corresponding prescribed performance region $\mathcal{D}_{ij}$:

$$\mathcal{D}_{ij} \triangleq \{\hat{x}_{ij} : \hat{x}_{ij} \in (-M_{ij}, 1)\} \qquad \text{if } x_{ij}(0) > 0 \tag{6}$$

$$\mathcal{D}_{ij} \triangleq \{\hat{x}_{ij} : \hat{x}_{ij} \in (-1, M_{ij})\} \qquad \text{if } x_{ij}(0) < 0 \tag{7}$$

Then the modulated error is transformed through the transformed function T_{ij} that defines the smooth and strictly increasing mapping $T_{ij} : \mathcal{D}_{ij} \to \mathbb{R}$ and $T_{ij}(0) = 0$. One example choice is $T_{ij}(\hat{x}_{ij}) = \ln\left(-M_{ij}\frac{\hat{x}_{ij}+1}{\hat{x}_{ij}-M_{ij}}\right)$. Hence the transformed error is defined as $\varepsilon_{ij}(\hat{x}_{ij}) = T_{ij}(\hat{x}_{ij})$. It can be verified that if the transformed error $\varepsilon_{ij}(\hat{x}_{ij})$ is bounded, then the modulated error $\hat{x}_{ij}$ is constrained within the regions (6), (7). This also implies the error x_{ij} evolves within the predefined performance bounds (4) and (5), respectively. Differentiating $\varepsilon_{ij}(\hat{x}_{ij})$ with respect to time, we derive

$$\dot{\varepsilon}_{ij}(\hat{x}_{ij}) = \mathcal{J}_{T_{ij}}(\hat{x}_{ij}, t)[\dot{x}_{ij} + \alpha_{ij}(t)x_{ij}] \tag{8}$$

where

$$\mathcal{J}_{T_{ij}}(\hat{x}_{ij}, t) \triangleq \frac{\partial T_{ij}(\hat{x}_{ij})}{\partial \hat{x}_{ij}} \frac{1}{\rho_{ij}(t)} > 0 \tag{9}$$

$$\alpha_{ij}(t) \triangleq -\frac{\dot{\rho}_{ij}(t)}{\rho_{ij}(t)} > 0 \tag{10}$$

are the normalized Jacobian of T_{ij} and the normalized derivative of the performance function, respectively.

D. Problem Statement

In this work, we are interested in how to design a control strategy for the leader-follower multi-agent system given by (3) such that the controlled system can achieve consensus while satisfying (4), (5). The control strategy is only applied to the leaders and these drive the followers to guarantee the entire multi-agent system meet the requirements. Formally,

Problem 1. *Let the leader-follower multi-agent system* Σ *defined by* (3) *with the communication graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *and the prescribed performance functions* $\rho_{ij}, (i,j) \in \mathcal{E}$. *Derive a control strategy such that the controlled leader-follower multi-agent system achieves consensus and satisfies* (4)*,* (5).

III. MAIN RESULTS

In this section, we design the control for the leader-follower multi-agent system (3) such that the system can achieve consensus within the prescribed performance bounds

$$\rho_{ij}(t) = (\rho_{ij0} - \rho_{ij\infty})e^{-l_{ij}t} + \rho_{ij\infty}. \tag{11}$$

Here the performance functions are chosen as (11) without loss of generality and the communication agents share information about their performance functions and transformation functions, that is, $\rho_{ij}(t) = \rho_{ji}(t), M_{ij} = M_{ji}$ and $T_{ij}(\hat{x}_{ij}) = -T_{ji}(\hat{x}_{ji})$. This means the communication between the neighbouring agents are bidirectional and the graph $\mathcal{G}$ is assumed undirected.

Consensus is achieved in the sense that the stack vector $\bar{x}$ of relative positions converges to zero as $t \to \infty$. We first rewrite the dynamics of the leader-follower multi-agent system (3) into the edge space in order to characterise the dynamics of the relative positions. We first rewrite (3) into the dynamics corresponding to followers and leaders, respectively. The corresponding incidence matrix is denoted as $D = \begin{bmatrix} D_F^T & D_L^T \end{bmatrix}^T$ with D_F, D_L denoting the incidence matrices that characterise how followers and leaders are connected with other agents. Then (3) is reorganised as

$$\Sigma : \begin{bmatrix}\dot{x}_f \\ \dot{x}_l\end{bmatrix} = \begin{bmatrix}A_F & B_F \\ B_F^T & A_L\end{bmatrix}\begin{bmatrix}x_f \\ x_l\end{bmatrix} + \begin{bmatrix}0_{n_f \times n_l} \\ I_{n_l}\end{bmatrix} u, \tag{12}$$

where x_f, x_l represents respectively $\begin{bmatrix}x_1 & x_2 & \cdots & x_{n_f}\end{bmatrix}^T$, $\begin{bmatrix}x_{n_f+1} & \cdots & x_{n_f+n_l}\end{bmatrix}^T$ and $A_F = D_F D_F^T, B_F = D_F D_L^T, A_L = D_L D_L^T$. Multiplying with D^T on both sides of (12), we obtain the dynamics on the edge space as

$$\Sigma_e : \dot{\bar{x}} = -L_e \bar{x} + D_L^T u, \tag{13}$$

with the edge Laplacian L_e. L_e is positive definite if the graph is a tree [2]. We thus here assume the following

Assumption 1. *The leader-follower multi-agent system* (3) *described by the graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is a connected tree.*

We consider tree graphs as a starting point since the positive definiteness of L_e is used in the analysis, and motivated by the fact that they require less communication load (less edges) for their implementation and show less couplings between agents. Note however that further results for a general graph could be built based on the results of tree graphs, for example, through graph decompositions [15]. For the leader-follower multi-agent system (13), the proposed controller applied to the leader agents is the composition of the term based on prescribed performance of the positions of the neighbours:

$$u_i = -\sum_{j \in \mathcal{N}_i} g_{ij} \mathcal{J}_{T_{ij}}(\hat{x}_{ij}, t) \varepsilon_{ij}(\hat{x}_{ij}), \qquad i \in \mathcal{V}_L, \tag{14}$$

where $g_{ij} = g_{ji}$ is a positive scalar gain to be appropriately tuned. Then the stack input vector is

$$u = -D_L \mathcal{J}_T(\hat{\bar{x}}, t) G \varepsilon(\hat{\bar{x}}), \tag{15}$$

where $\hat{\bar{x}}$ is the stack vector of transformed errors $\hat{x}_{ij}$, $G \in \mathbb{R}^{m \times m}$ is a positive definite diagonal gain matrix with entries g_{ij}. $\mathcal{J}_T(\hat{\bar{x}}, t) \in \mathbb{R}^{m \times m}$ is a time varying diagonal matrix with diagonal entries $\mathcal{J}_{T_{ij}}(\hat{x}_{ij}, t)$, $\varepsilon(\hat{\bar{x}}) \in \mathbb{R}^m$ is a stack vector with entries $\varepsilon_{ij}(\hat{x}_{ij})$. In the sequel, we develop the following result and will use Lyapunov-like methods to prove that.

Theorem 1. *Consider the leader-follower multi-agent system* Σ *under Assumption 1 with dynamics* (3)*, the predefined performance functions* ρ_{ij} *as in* (11) *and the transformation function s.t.* $T_{ij}(0) = 0, \forall (i,j) \in \mathcal{E}$*, and assume that the initial conditions* $x_{ij}(0)$ *are within the performance bounds* (4) *or* (5)*. If the following condition holds:*

$$\bar{\gamma} \geq l = \max_{(i,j) \in \mathcal{E}} (l_{ij}), \tag{16}$$

where l *is the largest decay rate of* $\rho_{ij}(t)$ *and* $\bar{\gamma}$ *is the maximum value of* γ *that ensures:*

$$\Gamma = \begin{bmatrix} D_L^T D_L & \frac{1}{2}\big(L_e - \gamma\big(I_m - D_L^T D_L\big)\big) \\ \frac{1}{2}\big(L_e - \gamma\big(I_m - D_L^T D_L\big)\big) & \gamma L_e \end{bmatrix} \geq 0. \tag{17}$$

Then, the controlled system achieves consensus and satisfies (4) *or* (5) *when applying the control* (15).

Proof. Consider the Lyapunov-like function $V(\varepsilon_{\hat{\bar{x}}}, \bar{x}) = \frac{1}{2}\varepsilon_{\hat{\bar{x}}}^T G \varepsilon_{\hat{\bar{x}}} + \frac{\gamma}{2}\bar{x}^T \bar{x}$, with $\varepsilon_{\hat{\bar{x}}}$ denoting $\varepsilon(\hat{\bar{x}})$ and $\mathcal{J}_{T_{\hat{\bar{x}}}}$ denoting $\mathcal{J}_T(\hat{\bar{x}}, t)$. Then, $\dot{V} = \varepsilon_{\hat{\bar{x}}}^T G \dot{\varepsilon}_{\hat{\bar{x}}} + \gamma \bar{x}^T \dot{\bar{x}}$. Replacing $\dot{\varepsilon}_{\hat{\bar{x}}}$ according to (8), we obtain $\dot{V} = \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}}(\dot{\bar{x}} + \alpha(t)\bar{x}) + \gamma \bar{x}^T \dot{\bar{x}}$, where $\alpha(t)$ is the diagonal matrix with diagonal entries $\alpha_{ij}(t)$. According to (10) and (11), we know that $\alpha_{ij}(t) < l_{ij}, \forall t$. Substituting (13), (15), we can further derive that

$$\begin{aligned}\dot{V} =& \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}}(-L_e \bar{x} - D_L^T D_L \mathcal{J}_{T_{\hat{\bar{x}}}} G \varepsilon_{\hat{\bar{x}}} + \alpha(t)\bar{x}) \\ &+ \gamma \bar{x}^T(-L_e \bar{x} - D_L^T D_L \mathcal{J}_{T_{\hat{\bar{x}}}} G \varepsilon_{\hat{\bar{x}}}) \\ =& -\varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}} L_e \bar{x} + \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}} \alpha(t)\bar{x} \\ &- \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}} D_L^T D_L \mathcal{J}_{T_{\hat{\bar{x}}}} G \varepsilon_{\hat{\bar{x}}} - \gamma \bar{x}^T L_e \bar{x} \\ &- \gamma \bar{x}^T D_L^T D_L \mathcal{J}_{T_{\hat{\bar{x}}}} G \varepsilon_{\hat{\bar{x}}}\end{aligned} \tag{18}$$

Adding and subtracting $\gamma \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}} \bar{x}$ on the right hand side of (18), we obtain

$$\begin{aligned}\dot{V} =& -\varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}}(\gamma I_m - \alpha(t))\bar{x} - \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}} D_L^T D_L \mathcal{J}_{T_{\hat{\bar{x}}}} G \varepsilon_{\hat{\bar{x}}} \\ &- \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}} L_e \bar{x} - \gamma \bar{x}^T L_e \bar{x} + \gamma \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}}(I_m - D_L^T D_L)\bar{x} \\ =& -\varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}}(\gamma I_m - \alpha(t))\bar{x} \\ &- y^T \begin{bmatrix} D_L^T D_L & \frac{1}{2}\big(L_e - \gamma\big(I_m - D_L^T D_L\big)\big) \\ \frac{1}{2}\big(L_e - \gamma\big(I_m - D_L^T D_L\big)\big) & \gamma L_e \end{bmatrix} y \\ =& -\varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}}(\gamma I_m - \alpha(t))\bar{x} - y^T \Gamma y\end{aligned} \tag{19}$$

with $y^T = \begin{bmatrix}\varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}} & \bar{x}^T\end{bmatrix}$. Since $G, \mathcal{J}_{T_{\hat{\bar{x}}}}$ are both diagonal and positive definite matrices, we have that $G\mathcal{J}_{T_{\hat{\bar{x}}}}$ is also a diagonal positive definite matrix. $(\gamma I_m - \alpha(t))$ is a diagonal positive definite matrix if $\gamma \geq l = \max(l_{ij}) > \bar{\alpha} = \sup \alpha_{ij}(t)$. Due to $T_{ij}(0) = 0$, we have $\varepsilon_{ij}(\hat{x}_{ij})\hat{x}_{ij} \geq 0$. Then, by by setting $\gamma := \theta + \bar{\alpha}$, with θ being a positive constant we get $-\varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}}(\gamma I_m - \alpha(t))\bar{x} \leq -\theta \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}} \bar{x}$. Then, according to (9), we further obtain $-\theta \varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}} \bar{x} = -\theta \varepsilon_{\hat{\bar{x}}}^T G \frac{\partial \varepsilon_{\hat{\bar{x}}}}{\partial \hat{\bar{x}}} \hat{\bar{x}} \leq 0$. This holds because the transformed function is smooth and strictly increasing and $\varepsilon_{ij}(\hat{x}_{ij})\hat{x}_{ij} \geq 0$. Therefore, in order for $\dot{V} \leq 0$ to hold, it suffices that $\gamma \geq l = \max(l_{ij}) > \sup \alpha_{ij}(t)$ and in addition, Γ should be semi-positive definite. Here, in order for $\Gamma \geq 0$ to be feasible, we need the assumption that the communication graph is a tree. Then, based on condition (16), and choosing $\gamma = \bar{\gamma}$, we obtain $-\varepsilon_{\hat{\bar{x}}}^T G \mathcal{J}_{T_{\hat{\bar{x}}}}(\bar{\gamma} I_m - \alpha(t))\bar{x} \leq 0$ and $\Gamma \geq 0$. Finally, we can conclude that $\dot{V} \leq 0$ when $\gamma = \bar{\gamma}$. This also implies $V(\varepsilon_{\hat{\bar{x}}}, \bar{x}) \leq V(\varepsilon_{\hat{\bar{x}}}(0), \bar{x}(0))$. Hence if $\bar{x}(0))$ is chosen within the region (6) or (7) then $V(\varepsilon_{\hat{\bar{x}}}(0), \bar{x}(0))$ is finite, which implies that $V(\varepsilon_{\hat{\bar{x}}}, \bar{x})$ is bounded $\forall t$. Therefore $\varepsilon_{\hat{\bar{x}}}, \bar{x}$ are bounded and the boundedness of the transformed error $\varepsilon_{\hat{\bar{x}}}$ implies that the relative position $\bar{x}(t)$ evolves within the prescribed performance bounds $\forall t$. Then we can prove the boundedness of $\ddot{V}(\varepsilon_{\hat{\bar{x}}}, \bar{x})$ based on the boundedness of $\varepsilon_{\hat{\bar{x}}}, \dot{\varepsilon}_{\hat{\bar{x}}}$. The boundedness of $\ddot{V}(\varepsilon_{\hat{\bar{x}}}, \bar{x})$ implies the uniform continuity of $\dot{V}(\varepsilon_{\hat{\bar{x}}}, \bar{x})$, which in turn implies that $\dot{V}(\varepsilon_{\hat{\bar{x}}}, \bar{x}) \to 0$ as $t \to \infty$ by applying Barbalat's Lemma. This implies $\bar{x} \to 0$ as $t \to \infty$ and consensus will be achieved. □

Remark 1. (16) *and* (17) *show the trade-off between the largest decay rate of the performance bounds and the number of leaders. We are interested in specifying the state of* *the multi-agent system at the equilibrium. Denote* $x_c = \frac{1}{n}\sum_{i=1}^{n} x_i$ *as the centroid. In most of the work regarding PPC [8],* $\lim_{t\to\infty} x_c(t) = x_c(0) = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$*. This is because a PPC input for every agent exists. In our work, the main difference is that when we choose some leaders, we can achieve a varying equilibrium state of each agent by tuning the gain matrix, which is quite useful in practical design as we can decide where all the agents should gather.*

In the sequel, we will discuss the results for two specific classes of tree graphs: chain and star graph. First we consider the chain graph, which is wildly used for instance in autonomous vehicle platooning.

Definition 1. *A chain* $\mathcal{G}^c = (\mathcal{V}^c, \mathcal{E}^c)$ *is a tree graph with vertices set* $\mathcal{V}^c = \{1, 2, \ldots, n\}, n \geq 2$ *and edges set* $\mathcal{E}^c = \{(i, i+1) \in \mathcal{V}^c \times \mathcal{V}^c \mid i \in \mathcal{V}^c \setminus \{n\}\}$ *indexed by* $e_i = (i, i+1), i = 1, 2, \ldots, n-1$.

Note that (16) in Theorem 1 is a sufficient but not necessary condition. For a chain graph, the matrix inequality (17) may be actually infeasible when the graph has 2 or more followers. This is reasonable because when there exist more followers and the relative positions between them are close to the performance boundary, these followers cannot perform quickly by only obeying a first-order consensus protocol to stay in the funnel due to lack of control. The following result for $\mathcal{G}^c$ is derived.

Proposition 1. *Consider the leader-follower multi-agent system* Σ *described by* (3) *with the communication chain graph* $\mathcal{G}^c = (\mathcal{V}^c, \mathcal{E}^c)$ *and the followers set* $\mathcal{V}_F^c = \{1, 2, \ldots, n_f\}$*, the predefined performance functions* ρ_{ij} *as in* (11) *and the transformation function s.t.* $T_{ij}(0) = 0, \forall (i,j) \in \mathcal{E}$*, and assume that the initial conditions* $x_{ij}(0)$ *are within the performance bounds* (4) *or* (5)*. Then, the chain can only have at most 3 followers* ($n_f \leq 3$) *in order to achieve consensus within the prescribed performance bounds* $\rho_{ij}(t)$ *when applying* (15)*. Specifically,*

$$\begin{aligned} \max_{(i,j)\in\mathcal{E}} (l_{ij}) = l \leq 2, \quad & n_f = 2; \\ \max_{(i,j)\in\mathcal{E}} (l_{ij}) = l \leq 1, \quad & n_f = 3 \end{aligned} \tag{20}$$

are the respective sufficient conditions under which the chain achieves consensus and satisfies (4)*,* (5) *when applying* (15)*.*

Proof. When the chain graph has only one follower, that is $n_f = 1$, the result can be proved by using Theorem 1. Let $\bar{\gamma}$ be the maximum value of γ that ensures (17) holds. By further choosing the decay rate of the performance functions (11) to satisfy (16), we can conclude that the controlled system achieves consensus within the prescribed performance bounds by applying (15) based on Theorem 1. When the chain has additional followers, the condition in Theorem 1 may be infeasible since it is a sufficient but not necessary condition. But for this kind of special chain structure, we can resort to checking the edge dynamics (13) directly. It can be shown that $-L_e$ has elements given by $c_{ij} = -2$ when $i = j$, $c_{ij} = 1$ when $|i - j| = 1$ and $c_{ij} = 0$ otherwise when the graph is a chain. We then rewrite (13) as

$$\begin{bmatrix} \dot{\bar{x}}_f \\ \hline \dot{\bar{x}}_l \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B}^T & \mathbf{C} \end{array}\right] \begin{bmatrix} \bar{x}_f \\ \hline \bar{x}_l \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{D} \end{bmatrix} u, \tag{21}$$

where $\bar{x}_f \in \mathbb{R}^{(n_f-1)}$ represents the edges between followers, while $\bar{x}_l \in \mathbb{R}^{n_l}$ represents the edge that connects the leader node $\{n_f+1\}$ and the follower node $\{n_f\}$, and the edges between leaders. Both $\mathbf{A} \in \mathbb{R}^{(n_f-1)\times(n_f-1)}, \mathbf{C} \in \mathbb{R}^{n_l\times n_l}$ have the same structure as $-L_e$ but with different dimensions, $\mathbf{B}$ has an element 1 at row $(n_f - 1)$, column 1 (bottom left corner) that represents the connection between the follower node $\{n_f\}$ and the leader node $\{n_f+1\}$. $\mathbf{0}$ is a $(n_f-1)\times n_l$ zero matrix. $\mathbf{D} \in \mathbb{R}^{n_l\times n_l}$ has elements given by $\mathbf{d}_{ij} = 1$ when $i = j$, $\mathbf{d}_{ij} = -1$ when $i - j = 1$ and $\mathbf{d}_{ij} = 0$ otherwise. Then we can analyse the leader part $\bar{x}_l$ and the follower part $\bar{x}_f$ separately. For $\bar{x}_l$, it can be proved that $\bar{x}_l$ achieves consensus within the performance bounds based on the positive definiteness of $\mathbf{D}\mathbf{D}^T$ when applying control (15). We further rewrite the follower part as

$$\dot{\bar{x}}_f = \mathbf{A}\bar{x}_f + \mathbf{b}\bar{x}_\star, \tag{22}$$

where $\mathbf{b} \in \mathbb{R}^{(n_f-1)}$ is the first column of $\mathbf{B}$, i.e., with the last element equals to 1 and all other elements equal to 0. $\bar{x}_\star$ represents the edge between the follower node $\{n_f\}$ and the leader node $\{n_f + 1\}$. We can further solve the state evolution of (22) as follows:

$$\begin{aligned} \bar{x}_f(t) &= e^{\mathbf{A}t}\bar{x}_f(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}\bar{x}_\star(\tau)d\tau \\ &= M^T e^{\mathbf{\Lambda}t} M\bar{x}_f(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}\bar{x}_\star(\tau)d\tau, \\ &= \bar{x}_f^0(t) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}\bar{x}_\star(\tau)d\tau, \end{aligned} \tag{23}$$

where $\bar{x}_f^0(t) = \begin{bmatrix}\bar{x}_1^0(t) & \bar{x}_2^0(t) & \ldots & \bar{x}_{n_f-1}^0(t)\end{bmatrix}^T$ is zero input trajectories, that is when $\bar{x}_\star(t) = 0, \forall t$; $\mathbf{A} = M^T\mathbf{\Lambda}M$, where $\mathbf{\Lambda}$ is a diagonal matrix with diagonal entries negative and equal to the eigenvalues of $\mathbf{A}$, which is due to $\mathbf{A}$ having the same structure as $-L_e$, and M is the matrix composed with the corresponding eigenvectors. Without loss of generality, suppose all performance functions are the same and described by

$$\rho(t) = (\rho_0 - \rho_\infty)e^{-lt} + \rho_\infty. \tag{24}$$

When $n_f = 2$, $\bar{x}_f = \bar{x}_1$ and $\mathbf{A} = -2$, we have that

$$\bar{x}_1^0(t) = M^T e^{\mathbf{\Lambda}t} M\bar{x}_1(0) = e^{-2t}\bar{x}_1(0) < \rho_0 e^{-2t}. \tag{25}$$

Then, $\bar{x}_1(t)$ is within the performance bound $\rho(t)$, i.e., $\bar{x}_1(t) < \rho(t), \forall t$, when $l \leq 2$ and in addition,

$$\int_0^t e^{-2(t-\tau)}\bar{x}_\star(\tau)d\tau < (\rho_0 - \bar{x}_1(0))e^{-2t} + \rho_\infty(1 - e^{-2t}), \tag{26}$$

which can be ensured by tuning a large enough gain g_{32} to the leader indexed by node 3. From (26), we know that when the relative position between the two followers is close to the boundary, we need to tune a larger gain for the leader that connects the followers. When $n_f = 3$, we can derive a similar result. In particular, we now have that

$$\begin{bmatrix} \bar{x}_1^0(t) \\ \bar{x}_2^0(t) \end{bmatrix} = M^T e^{\mathbf{\Lambda} t} M \begin{bmatrix} \bar{x}_1(0) \\ \bar{x}_2(0) \end{bmatrix} < k \begin{bmatrix} \rho_0 \\ \rho_0 \end{bmatrix} e^{-t}, \qquad (27)$$

with $k = 1$, which implies that $\bar{x}_i^0(t) < \rho_0 e^{-t}, i = \{1, 2\}$. Similarly, we can conclude that when $l \leq 1$, and in addition the tuning gain g_{43} for the leader indexed by node 4 is large enough, the controlled system achieves consensus within the prescribed performance bounds. When $n_f \geq 4$, it can be proved similarly that $\bar{x}_i^0(t) < k\rho_0 e^{\lambda_{\max}(\mathbf{A})t}, i = \{1, 2, \ldots, n_f - 1\}$, but with $k > 1$. This means that $\bar{x}_i^0(t)$ cannot be bounded by $\rho_0 e^{\lambda_{\max}(\mathbf{A})t}$ for any initial conditions within the performance bounds. Therefore, we can conclude that in order to achieve consensus within the performance bounds for all initial condition $x_{ij}(0)$ within the performance bounds (4) or (5), n_f should be less or equal to 3. □

Remark 2. *Proposition 1 indicates that for a chain graph, in order to achieve consensus within the prescribed performance bounds, we can only have at most 3 consecutive followers at the end of the graph. In addition, when the initial relative position between 2 followers is close to the prescribed performance boundary, we need to tune a large enough gain for the leader that connects the followers.*

Now we consider another specific class, in particular the star graph $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ which is defined as follows.

Definition 2. *A star $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ is a tree graph with vertices set $\mathcal{V}^s = \{1, 2, \ldots, n\}, n \geq 2$ where vertice n is called the centering node, and the edges set $\mathcal{E}^s = \{(i, n) \in \mathcal{V}^s \times \mathcal{V}^s \mid i \in \mathcal{V}^s \setminus \{n\}\}$ indexed by $e_i = (i, n), i = 1, 2, \ldots, n - 1$.*

Proposition 2. *Consider the leader-follower multi-agent system Σ described by* (3) *with the communication star graph $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ and the leader set $\mathcal{V}_L^s = \{n\}$, the predefined performance functions ρ_{ij} as in* (11) *and the transformation function s.t. $T_{ij}(0) = 0, \forall (i, j) \in \mathcal{E}$, and assume that the initial conditions $x_{ij}(0)$ are within the performance bounds* (4) *or* (5). *If*

$$\max_{(i,j)\in\mathcal{E}} (l_{ij}) = l \leq 1. \qquad (28)$$

Then, the controlled system achieves consensus and satisfies (4) *or* (5) *when applying the control* (15).

Proof. For a star graph defined as Definition 2 with the centering node n as the only leader, the edge Laplacian L_e and matrices $D_L^T D_L, D_F^T D_F$ have special structures. $D_L^T D_L$ has all elements equal to 1, while $D_F^T D_F = L_e - D_L^T D_L$ is an identity matrix. L_e has the elements given by $c_{ij} = 2$ when $i = j$, and $c_{ij} = 1$ otherwise. Under this special structure of star graphs and according to Theorem 1, it can be verified that (16) is always feasible with $\bar{\gamma} = 1$, and from (28), we know the condition $\bar{\gamma} \geq l = \max_{(i,j)\in\mathcal{E}} (l_{ij})$ holds. Finally, by applying Theorem 1, for a star graph, when the performance functions (11) are chosen such that (28) holds, then we can conclude that the controlled system achieves consensus and satisfies (4) or (5) when applying (15). □

We conclude this section with the following observations. A sufficient condition for a general tree graph was derived in Theorem 1, under which the leader-follower multi-agent system (3) achieves consensus and satisfies (4), (5). It can be seen that (16) may be infeasible when the decay rate of the performance functions is too large. This is reasonable since the followers only obey the first-order consensus protocol without any additional external input. And the decay rate constraint differs for different graph topologies, leader amount and leader positions.

IV. SIMULATIONS

In this section three simulation examples are presented in order to verify the results of the previous sections. The communication graphs are shown as Fig. 1, where the leaders and followers are represented by grey and white nodes, respectively. Regarding the prescribed performance functions, for all $(i, j) \in \mathcal{E}$, we choose $M_{ij} = 1$ and $T_{ij}(\hat{x}_{ij}) = \ln\left(-\frac{\hat{x}_{ij}+1}{\hat{x}_{ij}-1}\right)$. The prescribed performance bounds are chosen as $\rho_{ij}(t) = 4.9e^{-lt} + 0.1$ with different decay rate l for different simulation examples. In addition, prescribed performance bounds are depicted in black.

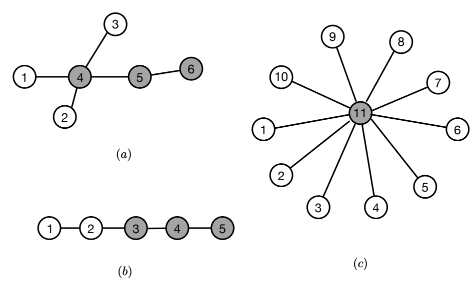

Fig. 1. Communication graphs with tree topologies.

In Fig. 1.(a), we first consider a tree graph with leaders set as $\mathcal{V}_L = \{4, 5, 6\}$, and the relative positions are initialised as $\begin{bmatrix}4.6 & 4.9 & 4.5 & 4.7 & 4.5\end{bmatrix}^T$. According to Theorem 1, the matrix inequality is feasible with $\bar{\gamma} = 1$, hence it suffices that $l \leq \bar{\gamma} = 1$. The simulation result when applying the PPC law (15) with a gain matrix G whose diagonal entries are all equal to 1 is shown on the right side of Fig. 2. As a comparison, the simulation result without PPC is shown on the left side of Fig. 2. We can see from Fig. 2 that the trajectories intersect the performance bound without extra control, which can be improved by applying the PPC law.

In Fig. 1.(b), we consider a chain graph with followers set as $\mathcal{V}_F = \{1, 2\}$, the relative positions are initialised as $\begin{bmatrix}4.8 & 3 & -2 & 1\end{bmatrix}^T$. The simulation results are shown in Fig. 3, where the left figure shows the simulation result without additional control. Here the decay rate of the prescribed performance function is 2. We can see that the trajectories intersect the performance bound, which is improved as shown in the middle figure by applying the PPC law (15) with gain matrix $G = diag(1, 10, 1, 1)$ and $g_{32} = 10$ is

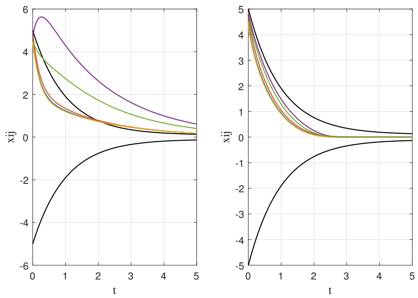

Fig. 2. The left figure shows the trajectories of relative positions without PPC, while the controlled system with PPC law is shown in the right figure under the communication graph as in Fig. 1.(a).

tuned for leader $\{3\}$ that connects the followers. However, it can be seen that the trajectories still intersect the performance bound. We then increase g_{32} to 200, and the simulation result is shown in the right figure. We can see that the controlled system achieves consensus within the performance bound.

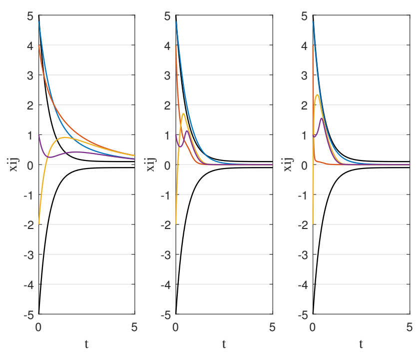

Fig. 3. The left figure shows the trajectories of relative positions without PPC, while the controlled system with PPC law but different gain matrix is shown in the middle and right figure, respectively under the communication graph as in Fig. 1.(b) with $\mathcal{V}_F = \{1, 2\}$.

In Fig. 1.(c), We consider a star graph with only one leader as $\mathcal{V}_L = \{11\}$, and the relative positions are initialised as $\begin{bmatrix}4 & 3 & -2 & -3 & 4.9 & 1 & 4.7 & -4 & 1 & 4.8\end{bmatrix}^T$. The simulation result when applying PPC law (15) with a gain matrix G whose diagonal entries are all equal to 1 is shown on the right side of Fig. 4. As a comparison, the simulation result without PPC is shown on the left side of Fig. 4.

V. CONCLUSIONS

In this paper, we have studied consensus problems of leader-follower multi-agent systems with prescribed performance bounds. Under the assumption of tree graphs, a distributed prescribed performance control law has been proposed for a group of selected leaders in order to drive the followers such that the entire system can achieve consensus under the prescribed performance guarantees. In addition, two specific classes of chain and star graphs that can have additional followers have been investigated.

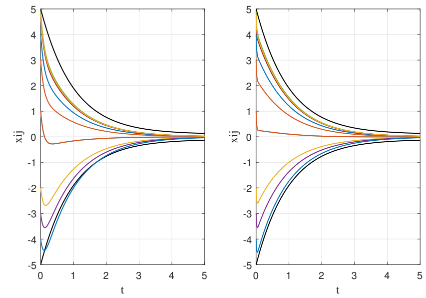

Fig. 4. The left figure shows the trajectories of relative positions without PPC, while the controlled system with PPC law is shown in the right figure under the communication graph as in Fig. 1.(c).

Future research directions include considering more general graphs with circles and applying other transient approaches to this leader-follower framework.

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