Abstractions of Varying Decentralization Degree for Coupled Multi-Agent Systems

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***Abstract*— In this paper, we aim at the development of a decentralized abstraction framework for multi-agent systems under coupled constraints, with the possibility for a varying degree of decentralization. The methodology is based on the analysis employed in our recent work, where decentralized abstractions based exclusively on the information of each agent's neighbors were derived. In the first part of this paper, we define the notion each agent's m-neighbor set, which constitutes a measure for the employed degree of decentralization. Then, sufficient conditions are provided on the space and time discretization that provides the abstract system's model, which guarantee the extraction of a meaningful transition system with quantifiable transition possibilities.**

I. Introduction

The analysis and control of multi-agent systems constitutes an active area of research with numerous applications, ranging from the analysis of power networks to the automatic deployment of robotic teams. Of central interest in this field is the problem of high level task planning by exploiting tools from formal verification [8]. In order to follow this approach for dynamic systems it is required to provide a suitable discrete representation of the system which allows the automatic synthesis of discrete plans that guarantee satisfaction of the high level tasks. Then, under appropriate relations between the continuous system and its discrete analogue, these plans can be converted to low level primitives such as sequences of feedback controllers, and hence, enable the continuous system to implement the corresponding tasks.

The need for a formal approach to the aforementioned control synthesis problem has lead to a considerable research effort towards the extraction of discrete state symbolic models, also called abstractions. Results in this direction for the nonlinear single plant case have been obtained in the papers [20] and [26], which exploit approximate simulation and bisimulation relations. Symbolic models for piecewise affine systems on simplices and rectangles were introduced in [14] and have been further studied in [6]. Closer related to the control framework that we adopt for the abstraction, are the papers [15], [16] which build on the notion of In-Block Controllability [7]. Other abstraction techniques for nonlinear systems include [23], where discrete time systems are studied in a behavioral framework and [1], where box abstractions are studied for polynomial and other classes of systems. It is also noted that certain of the aforementioned approaches have been extended to switched systems [11], [12]. Furthermore, abstractions for interconnected systems have been recently developed in [25], [22], [21], [24], [19], [9] and rely mainly on compositional approaches based on small gain arguments. Finally, in [18], a compositional approach with a varying selection of subsystems for the abstraction is exploited, providing a tunable tradeoff between complexity reduction and model accuracy.

In this framework, we focus on multi-agent systems and assume that the agents' dynamics consist of feedback interconnection terms and additional bounded input terms, which we call free inputs and provide the ability for motion planning under the coupled constraints. In this paper, we generalize the corresponding results of our recent work [4], where each agent's abstract model has been based on the knowledge of its neighbors' discrete positions, by allowing the agent to have this information for all members of the network up to a certain distance in the communication graph. The latter provides an improved estimate of its neighbors' potential evolution and allows for more accurate discrete agent models, due to the reduction of the part of the available control which is required for the manipulation of the coupling terms. In addition, the derived abstractions are coarser than the ones in [4] and can reduce the computational complexity of high level task verification.

The rest of the paper is organized as follows. Basic notation and preliminaries are introduced in Section II. In Section III, we define well posed abstractions for single integrator multi-agent systems and prove that the latter provide solutions consistent with the design requirement on the systems' free inputs. Section IV is devoted to the study of the control laws that realize the transitions of the proposed discrete system's model. In Section V we quantify space and time discretizations which guarantee well posed transitions with motion planning capabilities. The framework is illustrated through an example with simulation results in Section VI, and we conclude in Section VII. Due to space constraints, the proofs of the results are provided in [3].

II. Preliminaries and Notation

We use the notation $|x|$ for the Euclidean norm of a vector $x \in \mathbb{R}^n$ and $\mathrm{int}(S)$ for the interior of a set $S \subset \mathbb{R}^n$. Given $R > 0$ and $x \in \mathbb{R}^n$, we denote $B(x;R) := \{y \in \mathbb{R}^n : |x-y| \le R\}$ and $B(R) := B(0;R)$.

Consider a multi-agent system with N agents. For each agent $i \in \mathcal{N} := \{1,\ldots,N\}$ we use the notation $\mathcal{N}_i \subset \mathcal{N} \setminus \{i\}$ for the set of its neighbors and N_i for its cardinality.

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We also consider an ordering of the agent's neighbors which is denoted by $j_1, \dots, j_{N_i}$ and define the N_i-tuple $j(i) = (j_1, \dots, j_{N_i})$. The agents' network is represented by a directed graph $\mathcal{G} := (\mathcal{N}, \mathcal{E})$, with vertex set $\mathcal{N}$ the agents' index set and edge set $\mathcal{E}$ the ordered pairs (ℓ, i) with $i, \ell \in \mathcal{N}$ and $\ell \in \mathcal{N}_i$. The sequence $i_0 i_1 \cdots i_m$ with $(i_{\kappa-1}, i_\kappa) \in \mathcal{E}$, $\kappa = 1, \dots, m$, namely, consisting of m consecutive edges in $\mathcal{G}$, forms *a path of length* m in $\mathcal{G}$. We will use the notation $\mathcal{N}_i^m$ to denote for each $m \geq 1$ the set of agents from which i is reachable through a path of length m and not by a shorter one, excluding also the possibility to reach itself through a cycle. We also define $\mathcal{N}_i^0 := \{i\}$ and for each $m \geq 1$ the set $\bar{\mathcal{N}}_i^m := \bigcup_{\ell=0}^m \mathcal{N}_i^\ell$, namely, the set of all agents from which i is reachable by a path of length at most m, including i. We will use the terminology *m-neighbor set* of agent i for the set $\bar{\mathcal{N}}_i^m$ which always contains the agent itself. Finally, we denote by $\bar{N}_i^m$ and N_i^m the cardinality of the sets $\bar{\mathcal{N}}_i^m$ and $\mathcal{N}_i^m$, respectively. For each agent $i \in \mathcal{N}$, we consider a strict total order $\prec$ on each set $\bar{\mathcal{N}}_i^m$ that satisfies $i' \prec i''$ for each $i' \in \mathcal{N}_i^{m'}$, $i'' \in \mathcal{N}_i^{m''}$, with $0 \leq m' < m'' \leq m$. Given an index set $\mathcal{I}$, an agent $i \in \mathcal{N}$, its m-neighbor set $\bar{\mathcal{N}}_i^m$, and a strict total ordering of $\bar{\mathcal{N}}_i^m$ as above, we define the mapping $\mathrm{pr}_i : \mathcal{I}^N \to \mathcal{I}^{\bar{N}_i^m}$ which assigns to each N-tuple $(l_1, \dots, l_N) \in \mathcal{I}^N$ the $\bar{N}_i^m$-tuple $(l_i, l_{j_1^1}, \dots, l_{j_{N_i}^1}, l_{j_1^2}, \dots, l_{j_{N_i^2}^2}, \dots, l_{j_1^m}, \dots, l_{j_{N_i^m}^m}) \in \mathcal{I}^{\bar{N}_i^m}$, i.e., the indices of agent i and its m-neighbor set in accordance to the ordering. Finally, we define a transition system (see e.g., [20]) as a tuple $TS := (Q, Act, \longrightarrow)$, where: Q is a set of states; Act is a set of actions; $\longrightarrow$ is a transition relation with $\longrightarrow \subset Q \times Act \times Q$. The transition system is said to be finite, if Q and Act are finite sets. We also use the (standard) notation $q \xrightarrow{a} q'$ to denote an element $(q, a, q') \in \longrightarrow$. For every $q \in Q$ and $a \in Act$ we use the notation $\mathrm{Post}(q; a) := \{q' \in Q : (q, a, q') \in \longrightarrow\}$.

III. Abstractions for Multi-Agent Systems

We consider multi-agent systems with single integrator dynamics

$$\dot{x}_i = f_i(x_i, \mathbf{x}_j) + v_i, i \in \mathcal{N}, \tag{1}$$

that are governed by decentralized control laws consisting of two terms, a feedback term $f_i(\cdot)$ which depends on the states of i and its neighbors, which we compactly denote by $\mathbf{x}_j (= \mathbf{x}_{j(i)}) := (x_{j_1}, \dots, x_{j_{N_i}}) \in \mathbb{R}^{N_i n}$ (see Section II for the notation $j(i)$), and an extra input term v_i, which we call free input. The dynamics (1) are encountered in a popular set of multi-agent protocols [17], including consensus, connectivity maintenance, collision avoidance and formation control. In addition they may represent internal dynamics of the system as for instance in the case of smart buildings (see e.g., [2]).

In what follows, we consider a cell decomposition of the state space $\mathbb{R}^n$ and adopt a modification of the corresponding definition from [13, p 129-called cell covering]. In particular, we define a *cell decomposition* $\mathcal{S} = \{S_l\}_{l \in \mathcal{I}}$ of a domain $D \subset \mathbb{R}^n$ as a finite or countable family of uniformly bounded and connected sets S_l, $l \in \mathcal{I}$, such that $\mathrm{int}(S_l) \cap \mathrm{int}(S_{\hat{l}}) = \emptyset$ for all $l \neq \hat{l}$ and $\cup_{l \in \mathcal{I}} S_l = D$. Throughout the paper, we consider a fixed $m \in \mathbb{N}$ which specifies the m-neighbor set of each agent and will refer to it as the *degree of decentralization*. Also, given a cell decomposition $\{S_l\}_{l \in \mathcal{I}}$ of $\mathbb{R}^n$, we denote by $\mathbf{l}_i = (l_i, l_{j_1^1}, \dots, l_{j_{N_i}^1}, l_{j_1^2}, \dots, l_{j_{N_i^2}^2}, \dots, l_{j_1^m}, \dots, l_{j_{N_i^m}^m}) \in \mathcal{I}^{\bar{N}_i^m}$ the indices of the cells where agent i and its m-neighbors belong at a certain time instant and call it the *m-cell configuration* of agent i. We will also use the shorter notation $\mathbf{l}_i = (l_i, l_{j_1^1}, \dots, l_{j_{N_i^m}^m})$, or just $\mathbf{l}_i$. Similarly, we denote by $\mathbf{l} = (l_1, \dots, l_N) \in \mathcal{I}^N$ the indices of the cells where all the N agents belong at a given time instant and call it the cell configuration (of all agents). Thus, given a cell configuration $\mathbf{l}$, it is possible to determine the cell configuration of agent i as $\mathbf{l}_i = \mathrm{pr}_i(\mathbf{l})$ (see Section II for the definition of $\mathrm{pr}_i(\cdot)$).

Our aim is to derive finite or countable abstractions for each individual agent in the coupled system (1), through the selection of a cell decomposition and a time discretization step $\delta t > 0$. These will be based on the knowledge of each agent's neighbors discrete positions up to a certain distance in the network graph, namely, the degree of decentralization. Informally, we consider for each agent i the transition system with states the possible cells of the state partition, actions all the possible cells of the agents in its m-neighbor set and transition relation defined as follows. A final cell is reachable from an initial one, *if for all states in the initial cell there is a free input such that the trajectory of i will reach the final cell at time δt for all possible initial states of its m-neighbors in their cells and their corresponding free inputs*. For planning purposes we will require each corresponding system to be well posed (meaningful), in the sense that for each initial cell it is possible to perform a transition to at least one final cell. In order to provide meaningful decentralized abstractions we design appropriate hybrid feedback laws in place of the v_i's in order to guarantee well posed transitions.

For the susequent analysis, we assume that the feedback terms $f_i(\cdot)$ are globally bounded, namely, that

$$|f_i(x_i, \mathbf{x}_j)| \leq M, \forall (x_i, \mathbf{x}_j) \in \mathbb{R}^{(N_i+1)n}, i \in \mathcal{N} \tag{2}$$

for certain $M > 0$ and that they are globally Lipschitz. Hence, we can choose $L_1 > 0$ and $L_2 > 0$, such that

$$|f_i(x_i, \mathbf{x}_j) - f_i(x_i, \mathbf{y}_j)| \leq L_1 |(x_i, \mathbf{x}_j) - (y_i, \mathbf{x}_j)| \tag{3}$$

$$|f_i(x_i, \mathbf{x}_j) - f_i(x_i, \mathbf{y}_j)| \leq L_2 |(x_i, \mathbf{x}_j) - (y_i, \mathbf{x}_j)|, \tag{4}$$

$$\forall x_i, y_i \in \mathbb{R}^n, \mathbf{x}_j, \mathbf{y}_j \in \mathbb{R}^{N_i n}, i \in \mathcal{N}.$$

Finally, we assume that each input v_i, $i \in \mathcal{N}$ is piecewise continuous and satisfies the bound

$$|v_i(t)| \leq v_{\max} (< M), \forall t \geq 0. \tag{5}$$

Based on the uniform bound on the diameters of the cells in the decomposition of the workspace, we can define the diameter $d_{\max}$ of the cell decomposition as $d_{\max} := \inf\{R > 0 : \forall l \in \mathcal{I}, \exists x \in S_l \text{ with } S_l \subset B(x; \frac{R}{2})\}$. Given a cell decomposition we fix a reference point $x_{l,G} \in S_l$ for each cell S_l, $l \in \mathcal{I}$. For each agent i and m-cell configuration of i, the corresponding reference points of the cells in the configuration will provide a trajectory which is indicative of the agent's reachability capabilities over the interval of the time discretization. Furthermore, they provide an estimate of the agent's neighbors' corresponding trajectories, for m-cell configurations of its neighbors that are consistent with the cell configuration of i, namely for which the "common agents" belong to the same cells. In particular, given an agent $i \in \mathcal{N}$, a neighbor $\ell \in \mathcal{N}_i$ of i and m-cell configurations $\mathbf{l}_i = (l_i, l_{j_1^1}, \ldots, l_{j^m_{N_i^m}})$ and $\mathbf{l}_\ell = (\bar{l}_\ell, \bar{l}_{j(\ell)_1^1}, \ldots, \bar{l}_{j(\ell)^m_{N_\ell^m}})$ of i and ℓ, respectively, we say that $\mathbf{l}_\ell$ *is consistent with* $\mathbf{l}_i$ if for all $\kappa \in \bar{\mathcal{N}}_\ell^{m-1} (\subset \bar{\mathcal{N}}_i^m)$ it holds $l_\kappa = \bar{l}_\kappa$. The following definition provides for each agent i its reference trajectory and the estimates of its neighbors' reference trajectories, based on i's m-cell configuration.

Definition 3.1: Given a cell decomposition $\mathcal{S} = \{S_l\}_{l \in \mathcal{I}}$ of $\mathbb{R}^n$, a reference point $x_{l,G} \in S_l$ for each $l \in \mathcal{I}$, a time step δt and a nonempty subset W of $\mathbb{R}^n$, consider an agent $i \in \mathcal{N}$, its m-neighbor set $\bar{\mathcal{N}}_i^m$ and an m-cell configuration $\mathbf{l}_i = (l_i, l_{j_1^1}, \ldots, l_{j^m_{N_i^m}})$ of i. We define the functions $\chi_i(t)$, $\boldsymbol{\chi}_j(t) := (\chi_{j_1}(t), \ldots, \chi_{j_{N_i}}(t))$, $t \geq 0$, through the solution of the *initial value problem*, specified by the following Cases (i) and (ii): **Case (i).** It holds $\mathcal{N}_i^{m+1} = \emptyset$. Then we have the initial value problem $\dot{\chi}_\ell(t) = f_\ell(\chi_\ell(t), \chi_{j(\ell)_1}(t), \ldots, \chi_{j(\ell)_{N_\ell}}(t)), t \geq 0, \ell \in \bar{\mathcal{N}}_i^m$, $\chi_\ell(0) = x_{l_\ell,G}, \forall \ell \in \bar{\mathcal{N}}_i^m$, where $j(\ell)_1, \ldots, j(\ell)_{N_\ell}$ denote the corresponding neighbors of each agent $\ell \in \bar{\mathcal{N}}_i^m$. **Case (ii).** It holds $\mathcal{N}_i^{m+1} \neq \emptyset$. Then we have the initial value problem $\dot{\chi}_\ell(t) = f_\ell(\chi_\ell(t), \chi_{j(\ell)_1}(t), \ldots, \chi_{j(\ell)_{N_i}}(t)), t \geq 0, \ell \in \bar{\mathcal{N}}_i^{m-1}$, with the terms $\chi_\ell(\cdot)$, $\ell \in \mathcal{N}_i^m$ defined as $\chi_\ell(t) := x_{l_\ell,G}, \forall t \geq 0, \ell \in \mathcal{N}_i^m$. ◁

Remark 3.2: Apart from the notation $\chi_i(\cdot)$ and $\boldsymbol{\chi}_j(\cdot)$ above, we will use the notation $\chi_\ell^{[i]}(\cdot)$ for the trajectory of each agent $\ell \in \bar{\mathcal{N}}_i^m$, as specified by the initial value problem corresponding to the m-cell configuration of i in Definition 3.1. We will refer to $\chi_i(\cdot) \equiv \chi_i^{[i]}(\cdot)$ as the *reference trajectory* of agent i .

The following lemma establishes conditions on the network structure in a neighborhood of each agent i, which guarantee that i's neighbors' reference trajectories coincide with their estimates by i, for consistent cell configurations.

Lemma 3.3: Assume that for agent $i \in \mathcal{N}$ it holds $\mathcal{N}_i^{m+1} = \emptyset$ and let $\mathbf{l}_i$ be an m-cell configuration of i. Then, for each $\ell \in \mathcal{N}_i$ with $\mathcal{N}_\ell^{m+1} = \emptyset$ and m-cell configuration $\mathbf{l}_\ell$ of ℓ consistent with $\mathbf{l}_i$, it holds $\chi_\ell^{[\ell]}(t) = \chi_\ell^{[i]}(t)$, for all $t \geq 0$, with $\chi_\ell^{[\ell]}(\cdot)$ and $\chi_\ell^{[i]}(\cdot)$ as determined by Definition 3.1 for the m-cell configurations $\mathbf{l}_\ell$ and $\mathbf{l}_i$, respectively.

Despite the result of Lemma 3.3, in principle, the trajectory of each agent's neighbor and its estimate, based on the solution of the initial value problem for the reference trajectory of the specific agent do not coincide. Explicit bounds for this deviation are given in Proposition 3.4 below.

Proposition 3.4: Consider the agent $i \in \mathcal{N}$ and let $\mathbf{l}_i$ be an m-cell configuration of i. Also, pick $\ell \in \mathcal{N}_i$ and any m-cell configuration $\mathbf{l}_\ell$ of ℓ consistent with $\mathbf{l}_i$. Finally, let $\delta t \in (0, t^*]$, with t^* being the unique positive solution of the equation $e^{L_2 t^*} - \left(L_2 + \frac{L_2^2}{L_1\sqrt{N_{\max}}}\right)t^* - 1 = 0$, with $N_{\max} := \max\{N_i : i \in \mathcal{N}\}$. Then, the difference $|\chi_\ell^{[i]}(\cdot) - \chi_\ell^{[\ell]}(\cdot)|$ satisfies the bound $|\chi_\ell^{[i]}(t) - \chi_\ell^{[\ell]}(t)| \leq H_m(t), \forall t \in [0, \delta t]$, where the functions $H_\kappa(\cdot)$, $\kappa \geq 1$, are defined recursively as $H_1(t) := Mt, t \geq 0$, $H_\kappa(t) := \int_0^t e^{L_2(t-s)} L_1 \sqrt{N_{\max}} H_{\kappa-1}(s) ds, t \geq 0$ and $\chi_\ell^{[i]}(\cdot)$, $\chi_\ell^{[\ell]}(\cdot)$ are determined by the initial value problem of Definition 3.1 for the m-cell configurations $\mathbf{l}_i$ and $\mathbf{l}_\ell$, respectively.

In order to provide the definition of well posed transitions for the individual agents, we will exploit for each agent $i \in \mathcal{N}$ the following system with disturbances:

$$\dot{x}_i = f_i(x_i, \mathbf{d}_j) + v_i, \tag{6}$$

where $d_{j_1}, \ldots, d_{j_{N_i}} : [0, \infty) \to \mathbb{R}^n$ are continuous functions. Also, before defining the notion of a well posed space-time discretization we provide a class of hybrid feedback laws which are assigned to the free inputs v_i in order to obtain meaningful discrete transitions. These control laws are parameterized by each agent's initial conditions and a set of auxiliary parameters from a nonempty subset W of $\mathbb{R}^n$, which are exploited for motion planning. In particular, the choice of each vector $w_i \in W$ is in a one-to-one correspondence with the choice of a point inside a reachable ball for i, thus, providing the agent the possibility to perform transitions to multiple cells. In addition, for each agent i, the feedback laws in the following definition depend on the selection of the cells where i and its m-neighbors belong.

Definition 3.5: Given a cell decomposition $\{S_l\}_{l \in \mathcal{I}}$ of $\mathbb{R}^n$ and a nonempty subset W of $\mathbb{R}^n$, consider an agent $i \in \mathcal{N}$ and an initial cell configuration $\mathbf{l}_i$ of i. For each $x_{i0} \in S_{l_i}$ and $w_i \in W$, consider the mapping $k_{i,\mathbf{l}_i}(\cdot, \cdot, \cdot; x_{i0}, w_i) : [0, \infty) \times \mathbb{R}^{(N_i+1)n} \to \mathbb{R}^n$, parameterized by $x_{i0} \in S_{l_i}$ and $w_i \in W$. We say that $k_{i,\mathbf{l}_i}(\cdot)$ satisfies *Property* **(P)**, if the following conditions are satisfied. **(P1)** The mapping $k_{i,\mathbf{l}_i}(t, x_i, \mathbf{x}_j; x_{i0}, w_i)$ is continuous on $[0, \infty) \times \mathbb{R}^{(N_i+1)n} \times S_{l_i} \times W$. **(P2)** The mapping $k_{i,\mathbf{l}_i}(\cdot, \cdot, \cdot; x_{i0}, w_i)$ is globally Lipschitz on $(x_i, \mathbf{x}_j)$ for all $x_{i0} \in S_{l_i}$ and $w_i \in W$. ◁

The next definition characterizes the bounds on the deviation between the reference trajectory of each agent's neighbor and its estimate obtained from the solution of the initial value problem for the specific agent.

Definition 3.6: Consider an agent $i \in \mathcal{N}$. We say that a continuous function $\alpha_i : [0, \delta t] \to \mathbb{R}_{\geq 0}$ satisfies the *neighbor reference trajectory deviation bound*, if for each cell configuration $\mathbf{l}_i$ of i, neighbor j_κ, $\kappa = 1, \ldots, N_i$ of i and cell configuration of each j_κ consistent with $\mathbf{l}_i$ it holds:

$$|\chi_{j_\kappa}^{[i]}(t) - \chi_{j_\kappa}^{[j_\kappa]}(t)| \leq \alpha_i(t), \forall t \in [0, \delta t]. ◁ \tag{7}$$

We can now formalize our requirement on acceptable discrete transitions, based on the knowledge of each agent's m-cell configuration. The corresponding definition below includes certain bounds on the evolution of each agent, which we sharpen, by requiring the reference points of the decomposition to satisfy

$$|x_{l,G} - x| \leq \tfrac{d_{\max}}{2}, \forall x \in S_l, l \in \mathcal{I}. \tag{8}$$

The definition exploits the auxiliary system with disturbances (6) and is inspired by the approach adopted in [10].

Definition 3.7: Consider a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$, a time step δt, a nonempty subset W of $\mathbb{R}^n$, and a continuous function $\beta : [0,\delta t] \to \mathbb{R}_{\geq 0}$ satisfying

$$\tfrac{d_{\max}}{2} \leq \beta(0); \beta(\delta t) \leq v_{\max}\delta t. \tag{9}$$

Also, consider an agent $i \in \mathcal{N}$, a continuous function $\alpha_i : [0,\delta t] \to \mathbb{R}_{\geq 0}$ satisfying the neighbor reference trajectory deviation bound (7), an m-cell configuration $\mathbf{l}_i$ of i and the solution of the initial value problem of Definition 3.1. Then, given a control law

$$v_i = k_{i,\mathbf{l}_i}(t, x_i, \mathbf{x}_j; x_{i0}, w_i) \tag{10}$$

as in Definition 3.5, that satisfies Property (P), a vector $w_i \in W$, and a cell index $l' \in \mathcal{I}$, we say that the *Consistency Condition* is satisfied if the following hold. The set $\mathrm{int}(B(\chi_i(\delta t); \beta(\delta t))) \cap S_{l'_i}$ is nonempty, and there exists a point $x'_i \in \mathrm{int}(B(\chi_i(\delta t); \beta(\delta t))) \cap S_{l'_i}$, such that for each initial condition $x_{i0} \in \mathrm{int}(S_{l_i})$ and selection of continuous functions $d_{j_1}, \ldots, d_{j_{N_i}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the solution $x_i(\cdot)$ of the system with disturbances (6) with $v_i = k_{i,\mathbf{l}_i}(t, x_i, \mathbf{d}_j; x_{i0}, w_i)$, satisfies the following implication for each $\bar{t} \in [0,\delta t]$

$$\begin{aligned}|d_{j_\kappa}(t) - \chi^{[i]}_{j_\kappa}(t)| &\leq \alpha_i(t) + \beta(t), \forall t \in [0,\bar{t}], \kappa \in \{1,\ldots,N_i\}\\ &\Longrightarrow |x_i(\bar{t}) - \chi^{[i]}_i(\bar{t})| < \beta(\bar{t}),\end{aligned} \tag{11}$$

where $\chi^{[i]}_{j_\kappa}(\cdot)$, $\kappa = 1,\ldots,N_i$ correspond to the trajectories of i's neighbors in the solution of its reference trajectory. Furthermore, if the left hand side of the implication in (11) holds with $\bar{t} = \delta t$, then it follows that $x_i(\delta t) = x'_i \in S_{l'_i}$ and $|k_{i,\mathbf{l}_i}(t, x_i(t), \mathbf{d}_j(t); x_{i0}, w_i)| \leq v_{\max}, \forall t \in [0,\delta t]$. ◁

Notice, that when the Consistency Condition is satisfied, agent i can be driven to cell $S_{l'_i}$ precisely in time δt for all disturbances which satisfy the left hand side of the implication in (11). The latter capture the possibilities for the evolution of i's neighbors over the time interval $[0,\delta t]$, given the knowledge of i's m-cell configuration. Based on the Consistency Condition we next provide the definition of a well posed space-time discretization.

Definition 3.8: Consider a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$, a time step δt and a nonempty subset W of $\mathbb{R}^n$. **(i)** Given a continuous function $\beta : [0,\delta t] \to \mathbb{R}_{\geq 0}$ that satisfies (9), an agent $i \in \mathcal{N}$, a continuous function $\alpha_i : [0,\delta t] \to \mathbb{R}_{\geq 0}$ satisfying (7) an initial m-cell configuration $\mathbf{l}_i$ of i and a cell index $l'_i \in \mathcal{I}$ we say that *the transition $l_i \xrightarrow{\mathbf{l}_i} l'_i$ is well posed with respect to the space-time discretization $\mathcal{S} - \delta t$*, if there exist a feedback law $v_i = k_{i,\mathbf{l}_i}(\cdot,\cdot,\cdot; x_{i0}, w_i)$ as in Definition 3.5 that satisfies Property (P), and a vector $w_i \in W$, such that the Consistency Condition of Definition 3.7 is fulfilled. **(ii)** We say that the *space-time discretization $\mathcal{S} - \delta t$ is well posed*, if there exists a continuous function $\beta : [0,\delta t] \to \mathbb{R}_{\geq 0}$ that satisfies (9), such that for each agent $i \in \mathcal{N}$ there exists a continuous function $\alpha_i : [0,\delta t] \to \mathbb{R}_{\geq 0}$ satisfying (7), in a way that for each cell configuration $\mathbf{l}_i$ of i, there exists a cell index $l'_i \in \mathcal{I}$ such that the transition $l_i \xrightarrow{\mathbf{l}_i} l'_i$ is well posed with respect to $\mathcal{S} - \delta t$.

Given a space-time discretization $\mathcal{S} - \delta t$ and based on Definition 3.8(i), it is now possible to provide an exact definition of the discrete transition system which serves as an abstract model for the behaviour of each agent.

Definition 3.9: For each agent i, its individual transition system $TS_i := (Q_i, Act_i, \longrightarrow_i)$ is defined as follows: $Q_i := \mathcal{I}$ (the indices of the cell decomposition); $Act_i := \mathcal{I}^{\bar{N}^m_i}$ (all m-cell configurations of i); $l_i \xrightarrow{\mathbf{l}_i}_i l'_i$ iff $l_i \xrightarrow{\mathbf{l}_i} l'_i$ is well posed, for each $l_i, l'_i \in Q_i$ and $\mathbf{l}_i = (l_i, \ldots, l_{j^m_{N^m_i}}) \in \mathcal{I}^{\bar{N}^m_i}$. ◁

Remark 3.10: Given a well posed space-time discretization $\mathcal{S} - \delta t$ and an initial cell configuration $\mathbf{l} = (l_1, \ldots, l_N) \in \mathcal{I}^N$, it follows from Definitions 3.8 and 3.9 that for each agent $i \in \mathcal{N}$ it holds $\mathrm{Post}_i(l_i; \mathrm{pr}_i(\mathbf{l})) \neq \emptyset$ ($\mathrm{Post}_i(\cdot)$ refers to the transition system TS_i of each agent-see also Section II).

According to Definition 3.8, a well posed space-time discretization requires the existence of a well posed transition for each agent i and m-cell configuration of i, and the latter reduces to the selection of an appropriate feedback controller for i, which also satisfies Property (P) and guarantees that the auxiliary system with disturbances (6) satisfies the Consistency Condition. We next show, that given an initial cell configuration and a well posed transition for each agent, it is possible to choose a feedback law for each agent, so that the resulting closed-loop system will guarantee all these well posed transitions. At the same time, the magnitude of the hybrid feedback laws does not exceed the allowed magnitude $v_{\max}$ of the free inputs on $[0,\delta t]$, and hence, establishes consistency with the initial design requirement (5).

Proposition 3.11: Consider system (1), let $\mathbf{l} = (l_1, \ldots, l_N) \in \mathcal{I}^N$ be an initial cell configuration and assume that the space-time discretization $S - \delta t$ is well posed, which according to Remark 3.10 implies that for all $i \in \mathcal{N}$ it holds that $\mathrm{Post}_i(l_i; \mathrm{pr}_i(\mathbf{l})) \neq \emptyset$. Then, for every final cell configuration $\mathbf{l}' = (l'_1, \ldots, l'_N) \in \mathrm{Post}_1(l_1; \mathrm{pr}_1(\mathbf{l})) \times \cdots \times \mathrm{Post}_N(l_N; \mathrm{pr}_N(\mathbf{l}))$, there exist feedback laws $v_i = k_{i,\mathrm{pr}_i(\mathbf{l})}(t, x_i, \mathbf{x}_j; x_{i0}, w_i)$, $i \in \mathcal{N}$, satisfying Property (P), $w_1, \ldots, w_N \in W$ and a vector $x' = (x'_1, \ldots, x'_N) \in S_{l_1} \times \cdots \times S_{l_N}$, such that for each $i \in \mathcal{N}$, the solution of the closed-loop system (1) with $v_i = k_{i,\mathrm{pr}_i(\mathbf{l})}$, $i \in \mathcal{N}$, is well defined on $[0,\delta t]$, and its i-th component satisfies $x_i(\delta t, x(0)) = x'_i \in S_{l'_i}, \forall x(0) \in \mathbb{R}^{Nn} : x_\kappa(0) = x_{\kappa 0} \in S_{l_\kappa}, \kappa \in \mathcal{N}$. Furthermore, it follows that each control law $k_{i,\mathbf{l}_i}$ evaluated along the corresponding solution of the system satisfies $|k_{i,\mathrm{pr}_i(\mathbf{l})}(t, x_i(t), \mathbf{x}_j(t); x_{i0}, w_i)| \leq v_{\max}, \forall t \in [0,\delta t], i \in \mathcal{N}$.

IV. Design of the Hybrid Control Laws

According to Definition 3.8, the establishment of well posed space-time discretizations $\mathcal{S} - \delta t$ for system (1) relies on the design of appropriate feedback laws which guarantee well posed transitions for all agents and their possible cell configurations. We thus proceed by defining the control laws that are exploited in order to derive well posed discretizations. Consider a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$, a time step δt and a reference point $x_{l,G}$ for each cell as in (8). For each agent i and m-cell configuration $\mathbf{l}_i$ of i, we define the feedback laws $k_{i,\mathbf{l}_i} : [0,\infty) \times \mathbb{R}^{(N_i+1)n} \to \mathbb{R}^n$, parameterized by $x_{i0} \in S_{l_i}$ and $w_i \in W$ as $k_{i,\mathbf{l}_i}(t, x_i, \mathbf{x}_j; x_{i0}, w_i) := k_{i,\mathbf{l}_i,1}(t, x_i, \mathbf{x}_j) + k_{i,\mathbf{l}_i,2}(x_{i0}) + k_{i,\mathbf{l}_i,3}(t; w_i)$, where

$$W := B(v_{\max}) \subset \mathbb{R}^n, \tag{12}$$

$$k_{i,\mathbf{l}_i,1}(x_i, \mathbf{x}_j) := f_i(\chi_i(t), \boldsymbol{\chi}_j(t)) - f_i(x_i, \mathbf{x}_j), \tag{13}$$

$$k_{i,\mathbf{l}_i,2}(x_{i0}) := \frac{1}{\delta t}(x_{l_i,G} - x_{i0}), \tag{14}$$

$$k_{i,\mathbf{l}_i,3}(t; w_i) := \zeta(t) w_i, \tag{15}$$

$$t \in [0,\infty), (x_i, \mathbf{x}_j) \in \mathbb{R}^{(N_i+1)n}, x_{i0} \in S_{l_i}, w_i \in W,$$

$$\zeta_i : \mathbb{R}_{\geq 0} \to [\underline{\lambda}, \bar{\lambda}], 0 \leq \underline{\lambda} \leq \bar{\lambda} < 1. \tag{16}$$

The functions $\chi_i(\cdot)$ and in $\boldsymbol{\chi}_j(\cdot)$ in (13) are given through the solution of the initial value problem of Definition 3.1. In particular, $\chi_i(\cdot)$ constitutes a reference trajectory, whose endpoint agent i should reach at time δt, when the agent's initial condition lies in S_{l_i} and the feedback $k_{i,\mathbf{l}_i}(\cdot)$ is applied when $w_i = 0$ in (15). We also note that the feedaback laws $k_{i,\mathbf{l}_i}(\cdot)$ depend on the cell of agent i and specifically on its m-cell configuration $\mathbf{l}_i$, through the reference point $x_{l_i,G}$ in (14) and the trajectories $\chi_i(\cdot)$ and $\boldsymbol{\chi}_j(\cdot)$ in (13). The parameters $\underline{\lambda}$ and $\bar{\lambda}$ in (16) stand for the minimum and maximum portion of the free input, respectively, that can be exploited for motion planning. In particular, for each $w_i \in W$ in (12), the vector $\zeta_i(t) w_i$ provides the "velocity" of a motion that we superpose to the reference trajectory $\chi_i(\cdot)$ of agent i at time $t \in [0, \delta t]$. The latter allows the agent to reach all points inside a ball with center the position of the reference trajectory at time δt by following the curve $\bar{x}_i(t) := x_i(t) + w_i \int_0^t \zeta_i(s) ds$, as depicted in Fig. 1. Specifically, the feedback term $k_{i,\mathbf{l}_i,1}(\cdot)$ enforces the agent to move in parallel to its reference trajectory and the additional terms $k_{i,\mathbf{l}_i,2}(\cdot)$ and $k_{i,\mathbf{l}_i,3}(\cdot)$ navigate the agent to the point x inside the ball $B(\chi_i(\delta t); r_i)$ at time δt from any initial state $x_{i0} \in S_{l_i}$. In a similar way, it is possible to reach any point inside this ball by a different selection of w_i. This ball has radius $r_i := \int_0^{\delta t} \zeta_i(s) ds v_{\max} \geq \underline{\lambda} \delta t v_{\max}$, namely, the distance that the agent can cross in time δt by exploiting $k_{i,\mathbf{l}_i,3}(\cdot)$, which corresponds to the part of the free input that is available for planning. Hence, it is possible to perform a well posed transition to any cell which intersects $B(\chi_i(\delta t); r_i)$. Notice that due to the assumption $v_{\max} < M$ in (5), it is in principle not possible to cancel the interconnection terms. An example of a cooperative controller motivating this assumption can be found in our recent paper [5], where appropriate coupling terms ensure robust connectivity of the multi-agent network for bounded free inputs.

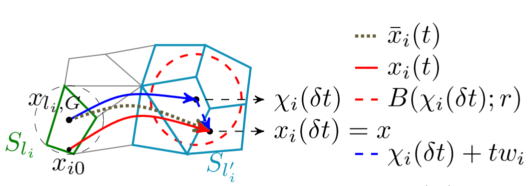

Fig. 1. For any point x inside the ball with center $\chi_i(\delta t)$ and each initial condition x_{i0} in S_{l_i}, the endpoint of i's trajectory $x_i(\cdot)$ coincides with the endpoint of the curve $\bar{x}_i(\cdot)$, which is precisely x, and lies in $S_{l'_i}$.

In order to verify the Consistency Condition for the derivation of well posed discretizations, we will select the function $\beta(\cdot)$ in Definition 3.7 as $\beta(t) := \frac{d_{\max}(\delta t - t)}{2\delta t} + \bar{\lambda} v_{\max} t, t \in [0, \delta t]$. Furthermore, we select a constant $\bar{c} \in (0,1)$, which for each agent, is a measure of the deviation between the reference trajectory of its neighbors and their estimates through the initial value problem for the specification of the reference trajectory of the agent (for corresponding consistent cell configurations). By defining $\bar{t} := \sup\left\{t > 0 : e^{L_2 t} - \left(L_2 + \bar{c}\frac{L_2^2}{L_1\sqrt{N_{\max}}}\right)t - 1 < 0\right\}$, it follows that $0 < \bar{t} < t^*$ (see Proposition 3.4 for the definition of t^* and $N_{\max}$), and that the function $H_m(\cdot)$ as given in Proposition 3.4 satisfies $H_m(t) \leq \bar{c}^{m-1} M t, \forall t \in [0, \bar{t}]$. Hence, it follows from Proposition 3.4 that if we select the functions $\alpha_i(\cdot) \equiv \alpha(\cdot), \forall i \in \mathcal{N}$, with $\alpha(t) := cMt, \forall t \in [0, \delta t]$; $c := \bar{c}^{m-1}$, then the neighbor reference trajectory deviation bound (7) is satisfied for all $0 < \delta t \leq \bar{t}$.

V. Well Posed Space-Time Discretizations with Motion Planning Capabilities

In this section, we exploit the feedback laws (13), (14), (15) introduced in Section IV in order to provide well posed space-time discretizations and their reachability properties for system (1). The desired sufficient conditions for well posed discretizations are given in the following theroem and rely on the selection of the functions $\alpha_i(\cdot)$ and $\beta(\cdot)$ in the previous section.

Theorem 5.1: Consider a cell decomposition $\mathcal{S}$ of $\mathbb{R}^n$ with diameter $d_{\max}$, a time step δt, and the parameters $\underline{\lambda}$, $\bar{\lambda}$ in (16). We assume that $\delta t \in (0, \min\{\bar{t}, \frac{(1-\underline{\lambda})v_{\max}}{L_1\sqrt{N_{\max}}(cM + \bar{\lambda} v_{\max}) + \underline{\lambda} L_2 v_{\max}}\})$ and $d_{\max} \in (0, \min\{\frac{2(1-\underline{\lambda})v_{\max}\delta t}{1+(L_1\sqrt{N_{\max}}+L_2)\delta t}, 2(1-\underline{\lambda})v_{\max}\delta t - 2(L_1\sqrt{N_{\max}}(cM + \bar{\lambda} v_{\max}) + \underline{\lambda} L_2 v_{\max})\delta t^2\}]$, with L_1, L_2, M, and $v_{\max}$ as given in (3), (4), (2), and (5), respectively, and c, $\bar{t}$ as defined in the previous section. Then, the space-time discretization is well posed for the multi-agent system (1). In particular, for each agent $i \in \mathcal{N}$ and cell configuration $\mathbf{l}_i$ of i it holds $\mathrm{Post}_i(l_i; \mathbf{l}_i) \supset \{l \in \mathcal{I} : S_l \cap B(\chi_i(\delta t); r_i) \neq \emptyset\}$, where $\chi_i(\cdot)$ is the reference trajectory corrsponding to $\mathbf{l}_i$ and r_i is defined in the previous section with $\zeta_i(t) := \underline{\lambda}$.

We finally provide an improved version of Theorem 5.1, for the case where the conditions of Lemma 3.3 are satisfied for all agents.

Theorem 5.2: Assume that $\mathcal{N}_i^{m+1} = \emptyset$ for all $i \in \mathcal{N}$. Then, the result of Theorem 5.1 remains valid for any $\delta t \in (0, \frac{(1-\underline{\lambda})v_{\max}}{L_1\sqrt{N_{\max}}\bar{\lambda} v_{\max} + \underline{\lambda} L_2 v_{\max}})$ and $d_{\max} \in (0, \min\{\frac{2(1-\underline{\lambda})v_{\max}\delta t}{1+(L_1\sqrt{N_{\max}}+L_2)\delta t}, 2(1-\underline{\lambda})v_{\max}\delta t - 2(L_1\sqrt{N_{\max}}\bar{\lambda} v_{\max} + \underline{\lambda} L_2 v_{\max})\delta t^2\}]$.

VI. Example and Simulation Results

As an illustrative example we consider a system of four agents in $\mathbb{R}^2$. Their dynamics are given as $\dot{x}_1 = \mathrm{sat}_\rho(x_2 - x_1) + v_1, \dot{x}_2 = v_2, \dot{x}_3 = \mathrm{sat}_\rho(x_2 - x_3) + v_3, \dot{x}_4 = \mathrm{sat}_\rho(x_3 - x_4) + v_4$, where $\mathrm{sat}_\rho(x) := x$, if $|x| < \rho$ and $\mathrm{sat}_\rho(x) := \frac{\rho}{|x|}x$, if $|x| \geq \rho$. The agents' neighbors' sets in this example are $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_2 = \emptyset$, $\mathcal{N}_3 = \{2\}$, $\mathcal{N}_4 = \{3\}$ and specify the corresponding network topology. By selecting the degree of decentralization $m = 2$, it follows that the conditions of Theorem 5.2 are satisfied. For the simulation results we select $v_{\max} := \frac{\rho}{2}$, $\underline{\lambda} = 0.4$, $\bar{\lambda} = 1$, $\bar{r} = 10$ and pick the values of δt and $d_{\max}$ in accordance to Theorem 5.2. We assume that agent 2, which is unaffected by the coupled constraints has constant velocity $v_2 = (-1, -4)$ and study reachability properties of the system over the time interval $[0, 2]$. The sampled trajectory of agent 2 is visualized with the circles in the figure below, and the blue/magenta cells indicate the union of agents 3/1 reachable cells over the time interval (given the trajectory of 2). Finally, given the trajectory of 2 and selecting the discrete trajectory of 3 which is depicted by the red cells in the figure, we obtain with yellow the corresponding reachable cells of agent 4. The simulation results have been implemented in MATLAB with a running time of the order of a few seconds, on a PC with an Intel(R) Core(TM) i7-4600U CPU @ 2.10GHz processor.

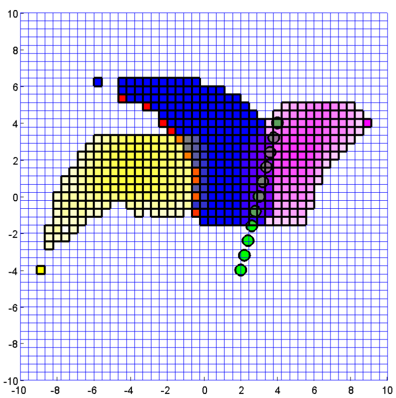

Fig. 2. Reachable cells of agents 1, 3 and 4.

VII. Conclusions

We have provided abstractions for multi-agent systems under a varying degree of decentralization and modeled well posed transitions by exploiting a system with disturbances in order to capture the possible evolution of each agent's neighbors. Sufficient conditions on the space and time discretization have been quantified in order to capture the reachability properties of the symbolic models through well defined transitions. The latter are realized by means of hybrid feedback control laws which take into account the coupled constraints and navigate the agents to their successor cells.

Ongoing work includes the improvement of the agents' reachabilty properties, based on their local dynamics bounds and Lipschitz constants which will enable the exploitation of a larger part of the free inputs for the transitions to successor cells. In addition we aim at the formulation of an online abstraction framework for heterogeneous agents with updated choices of the discretization and planning parameters.

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