

# Self-triggered Model Predictive Control for Continuous-Time Systems: A Multiple Discretizations Approach

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**Abstract**—In this paper, we propose a new self-triggered formulation of Model Predictive Control for continuous-time linear networked control systems. Our control approach, which aims at reducing the number of transmitting control samples to the plant, is derived by parallelly solving optimal control problems with different sampling time intervals. The controller then picks up one sampling pattern as a transmission decision, such that a reduction of communication load and the stability will be obtained. The proposed strategy is illustrated through comparative simulation examples.

## I. INTRODUCTION

Event-triggered and self-triggered control have been active areas of research in the community of Networked Control Systems (NCSs), due to their potential advantages over the typical time-triggered controllers [1]–[3]. In contrast to the time-triggered case where the control signals are executed periodically, event-triggered and self-triggered strategies require the executions based on the violation of prescribed control performances, such as Input-to-State Stability (ISS) [2] and  $\mathcal{L}_\infty$  gain stability [3].

In another line of research, Model Predictive Control (MPC) has been one of the most successful control strategies applied in a wide variety of applications, such as process industries, robotics, autonomous systems, and moreover, recent research also includes NCSs [4]. The basic idea of MPC is to obtain the current control action by solving the Optimal Control Problem (OCP) online, based on the knowledge about current state and predictive behaviors of the dynamics.

The application of the event-triggered or self-triggered framework to MPC is in particular of importance as the possible way to alleviate communication resources for NCSs. Combining these strategies has been a relatively recent research topic; see, e.g., [5]–[7] for discrete-time case and [8]–[11] for continuous-time case. In this paper, we are interested in designing a self-triggered MPC for continuous-time linear systems. In [9], [11], the self-triggered condition was derived based on the condition that the optimal cost as a Lyapunov function is guaranteed to decrease. However, the self-triggered strategy may lead to a conservative result in the following sense; the obtained self-triggered condition includes several parameters, such as Lipschitz constant of stage and terminal costs, which are characterized by the maximum size of state regions. Depending on the problem formulation, therefore, these parameters are sometimes over-approximated and the

corresponding self-triggered condition may then become conservative. A related work is also reported in [8], where the authors proposed an event-triggered scheme for continuous-time systems. In their approach, the OCP is solved only when the error between the actual and predictive state exceeds a certain threshold.

The self-triggered strategy proposed in this paper takes a different problem formulation from previous works in the literature. The basic idea is to parallelly solve OCPs, which provides different transmission time intervals under a piecewise constant control policy. Based on the different solutions, the controller then selects one solution providing the largest transmission time interval while at the same time guaranteeing the control performance. The new formulation of the proposed self-triggered strategy leads to the following main contributions of this paper with respect to the earlier approaches:

- (i) The proposed self-triggered strategy does not include parameters (such as Lipschitz constant parameters) that may be the potential source of conservativeness. The simulation result also illustrates that less conservative results can be obtained than the previous framework.
- (ii) The optimal costs can be compared under various transmission time intervals. This allows us to obtain the desired control performance by evaluating how much this becomes better or worse according to different values of transmission time intervals.

The approach presented in this paper is relevant to roll-out strategy [12], where the authors proposed a way to determine the transmission time interval providing a better control performance than the conventional periodic optimal control in terms of the reduced value function. In contrast to the research presented in [12], we will provide a discretizing sampling pattern such that a trade-off between the control performance and the transmission time intervals can be evaluated.

This paper is organized as follows. In Section II, the OCP is formulated. In Section III, the self-triggered strategy is provided. In Section IV, the feasibility of our proposed algorithm and the stability are investigated. In Section V, the proposed scheme is validated through a numerical example. We finally conclude in Section VI.

The notations used in the sequel are as follows:  $\mathbb{R}$ ,  $\mathbb{R}_{\geq 0}$ ,  $\mathbb{N}_{\geq 0}$ ,  $\mathbb{N}_{\geq 1}$  are the real, non-negative real, non-negative integers and positive integers, respectively. For a matrix  $Q$ , we use  $Q \succ 0$  to denote that  $Q$  is positive definite. Denote  $\|x\|$  as the Euclidean norm of vector  $x$ .

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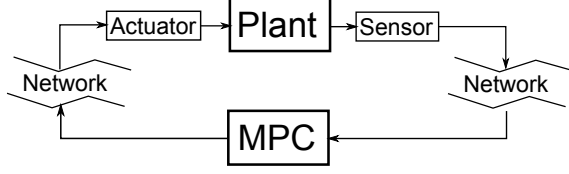


Fig. 1. Networked Control System

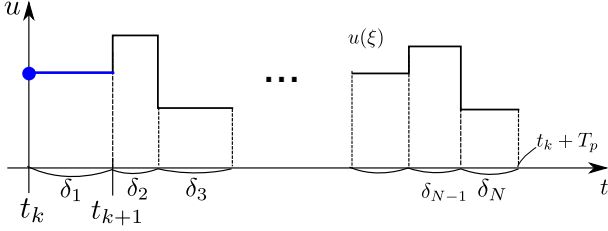


Fig. 2. The piece-wise constant control policy considered in this paper. The controller solves an OCP under above piece-wise constant control policy. Once the OCP is solved, the controller transmits the optimal control sample at  $t_k$  to the plant (Blue circle). The plant then applies it as sample-and-hold fashion until the next transmission time  $t_{k+1} = t_k + \delta_1$  (Blue line), and sends back the new state measurement to the controller.

## II. PROBLEM FORMULATION

### A. Dynamics and Cost

We consider a networked control system depicted in Fig. 1. The dynamics of the plant are assumed to be given by the following linear continuous-time invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state and  $u(t) \in \mathbb{R}^m$  is the control variable. We assume that the pair  $(A, B)$  is stabilizable, and  $u(t)$  is subject to the constraint  $u(t) \in \mathcal{U}$ , where  $\mathcal{U} \subset \mathbb{R}^m$  is a compact subset containing the origin in the interior. The control objective for the MPC is to drive the state to the origin, i.e.,  $x(t) \rightarrow 0$ , as  $t \rightarrow \infty$ .

Let  $t_k, k \in \mathbb{N}_{\geq 0}$  be the time instants when OCPs are solved; at  $t_k$ , the controller solves an OCP based on the knowledge about the state  $x(t_k)$ , and the dynamics given by (1). In this paper, we consider the following cost function to be minimized:

$$\begin{aligned} J(x(t_k), u(\cdot)) &= \int_{t_k}^{t_k+T_p} x^\top(\xi)Qx(\xi) + u^\top(\xi)Ru(\xi)d\xi \\ &\quad + x^\top(t_k + T_p)P_f x(t_k + T_p) \end{aligned} \quad (2)$$

where  $Q \succ 0, R \succ 0$  are the matrices for the stage cost,  $P_f^\top = P_f \succ 0$  is the matrix for the terminal cost, and  $T_p$  is the prediction horizon. More detailed characterization of  $P_f$  will be discussed in later sections.

In order to derive a self-triggered strategy, we first consider that the control input  $u(\xi), \xi \in [t_k, t_k + T_p]$  is constrained to be piece-wise constant with different sampling intervals, e.g.,  $\delta_1, \delta_2, \dots, \delta_N$ , as shown in Fig. 2. This discretizing scheme is motivated as follows: The solution of the OCP by minimizing

the cost (2) is in general given by a continuous trajectory of the optimal control input, say  $u^*(\xi)$ , for all  $\xi \in [t_k, t_k + T_p]$ . If the optimal control input *could* be applied until  $t_{k+1}$ , i.e.,  $u^*(\xi), \xi \in [t_k, t_{k+1}]$ , then we could utilize the classic MPC result to guarantee the asymptotic stability of the origin, see [13]. However, applying the continuous trajectory of the control input is not suited for practical NCSs applications in terms of the two aspects. Firstly, transmitting continuous control trajectory over the network requires an infinite-transmission bandwidth, which is un-realizable. Secondly, implementing the exact continuous control input is difficult for embedded control system architectures, since they only deal with samples as a discrete time domain, resulting in applying the control input eventually as a sampled-and-hold implementation at a high frequency. As the actual control trajectory for this case possibly differs from the optimal control trajectory, it fails to guarantee the asymptotic stability of the origin.

The OCP under the piece-wise constant control policy considered in this paper thus provides the optimal control sequence at discrete sampling intervals, i.e.,  $\{u^*(t_k), u^*(t_k + \delta_1), \dots, u^*(t_k + \sum_{j=1}^N \delta_j)\}$  rather than the whole control trajectory  $u^*(\xi), \xi \in [t_k, t_k + T_p]$ . As the procedure of transmitting control samples, we consider the following steps; (i) the controller transmits the optimal control sample  $u^*(t_k)$  to the plant; (ii) the plant then applies  $u^*(t_k)$  at constant until  $t_{k+1} = t_k + \delta_1$ ; (iii) the plant sends back a new state measurement  $x(t_{k+1})$  to the controller to solve the next OCP at  $t_{k+1}$ . Under this procedure, the transmission time interval is then given by  $t_{k+1} - t_k = \delta_1$ .

Applying the above transmission procedure not only allows the controller to transmit control command as a sample, but also allows us to formulate the OCP as the discrete time domain. The main difference of the problem formulation with respect to the periodic (or event-triggered) MPC for general discrete time systems is, however, that we are now free to select the sampling time intervals  $\delta_1, \dots, \delta_N$  in an appropriate way. Although there is a flexibility to select  $\delta_1, \dots, \delta_N$ , these intervals must be carefully determined such that:

- (i) The asymptotic stability of the origin is guaranteed under MPC with the piece-wise constant control policy.
- (ii) The reduction of communication load is achieved through the self-triggered formulation.

In the next subsection, we provide one possible way to determine the sampling time intervals  $\delta_1, \dots, \delta_N$ , such that the above problems can be tackled.

### B. Determining sampling time intervals

Under the piece-wise constant control policy outlined in Fig. 2, the sampling time intervals are determined in this subsection. By making use of the flexibility of selecting the sampling time intervals, consider at first that we have *multiple patterns* of sampling time intervals, i.e., we have  $M$  ( $M \in \mathbb{N}_{\geq 1}$ ) different sampling patterns in total, where each  $i$ -th ( $i \in \{1, 2, \dots, M\}$ ) sampling pattern has  $N_i$  sampling intervals,  $\delta_1^{(i)}, \delta_2^{(i)}, \dots, \delta_{N_i}^{(i)}$ . More specifically, in this paper

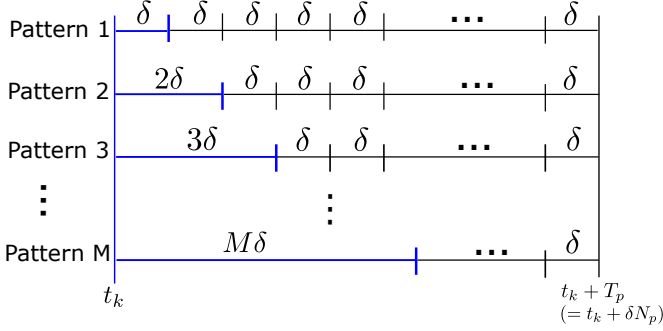


Fig. 3. Sampling patterns considered in this paper. Blue lines represent the transmission time intervals.

we consider the sampling patterns as shown in Fig. 3. Stated formally, for given  $M, N_p \in \mathbb{N}_{\geq 1}$ , where  $M < N_p$  and  $N_p$  represents the maximum number of sampling intervals among all patterns, and  $\delta = T_p/N_p$ , the sampling time intervals for the  $i$ -th ( $i \in \{1, 2, \dots, M\}$ ) pattern are given by

$$\delta_1^{(i)} = i\delta, \quad \delta_j^{(i)} = \delta \quad (j = 2, 3, \dots, N_i), \quad (3)$$

with  $N_i = N_p - i + 1$ . That is, the 1<sup>st</sup> pattern has the same interval:  $\delta_1^{(1)} = \dots = \delta_{N_p}^{(1)} = \delta$ . The 2<sup>nd</sup> pattern is the same as the 1<sup>st</sup> pattern only except the first sampling interval:  $\delta_1^{(2)} = 2\delta, \delta_2^{(2)} = \dots = \delta_{N_p-1}^{(2)} = \delta$ . Similarly, for the general  $i$ -th pattern we have  $\delta_1^{(i)} = i\delta$ , and  $\delta$  for the remaining intervals. The controller solves the corresponding OCPs under all sampling patterns above, and then selects one sampling pattern according to the self-triggered strategy proposed in the next section.

The main motivation of using the sampling patterns shown in Fig. 3, is that it allows to evaluate the trade-off between the transmission interval and the control performance quantitatively. According to the transmission procedure given in the previous subsection, the transmission time interval is given by  $\delta_1^{(i)} = i\delta$ . Thus, using larger patterns leads to longer transmission intervals. From the self-triggered point of view, it is desirable to have larger patterns. However, as we will see in the analysis that follows, the control performance instead becomes worse; this will be proved by the fact that the optimal cost becomes larger as larger patterns are selected. In later sections, we will provide a framework of selecting one sampling pattern, such that the trade-off between the transmission time interval and the control performance can be taken into account.

### C. Optimal Control Problem

In this subsection the OCP under each sampling pattern is formulated. For the  $i$ -th sampling pattern, we denote

$$\mathbf{u}_i(t_k) = \{u_i(t_k), u_i(t_k + i\delta), u_i(t_k + (i+1)\delta), \dots, u_i(t_k + (N_p - 1)\delta)\} \quad (4)$$

as the control input sequence to be applied. Note that  $u_i(t_k + i\delta)$  is used after  $u_i(t_k)$ , as  $u_i(t_k)$  is applied for the time

interval  $i\delta$ . The cost given by (2) under the  $i$ -th sampling pattern can be re-written as

$$\begin{aligned} J_i(x(t_k), \mathbf{u}_i(t_k)) &= \int_0^{i\delta} \{x^\top(t_k + \xi)Qx(t_k + \xi) + u_i^\top(t_k)Ru_i(t_k)\} d\xi \\ &+ \sum_{n=i}^{N_p-1} \int_0^\delta \{x^\top(t_k + n\delta + \xi)Qx(t_k + n\delta + \xi) \\ &\quad + u_i^\top(t_k + n\delta)Ru_i(t_k + n\delta)\} d\xi \\ &+ x^\top(t_k + N_p\delta)P_f x(t_k + N_p\delta), \end{aligned}$$

where the total cost is separated by each component of the control sequence  $\mathbf{u}_i(t_k)$ . Here we denoted  $J_i$  instead of  $J$  to emphasize that the piece-wise constant control policy under the  $i$ -th sampling pattern is used. By computing each integral in the above equation, the total cost for the  $i$ -th sampling pattern can be translated into a summation of costs:

$$\begin{aligned} J_i(x(t_k), \mathbf{u}_i(t_k)) &= F(x(t_k), u_i(t_k), i\delta) \\ &+ \sum_{n=i}^{N_p-1} \{F(x(t_k + n\delta), u_i(t_k + n\delta), \delta)\} \\ &+ x^\top(t_k + N_p\delta)P_f x(t_k + N_p\delta), \end{aligned}$$

where  $F(x(t), u(t), i\delta)$  denotes a new stage cost given by

$$\begin{aligned} F(x(t), u(t), i\delta) &= \int_0^{i\delta} x^\top(t + \xi)Qx(t + \xi) + u^\top(t)Ru(t) d\xi \\ &= \tilde{x}^\top(t)\Gamma(i\delta)\tilde{x}(t), \end{aligned} \quad (5)$$

where  $\tilde{x}(t) = [x^\top(t) \quad u^\top(t)]^\top$  and

$$\Gamma(i\delta) = \begin{bmatrix} \int_0^{i\delta} A_s^\top Q A_s ds & \int_0^{i\delta} B_s^\top Q A_s ds \\ \int_0^{i\delta} A_s^\top Q B_s ds & \int_0^{i\delta} (B_s^\top Q B_s + R) ds \end{bmatrix}$$

with  $A_s = e^{As}$ ,  $B_s = \int_0^s e^{A\tau} d\tau B$ . The OCP for the  $i$ -th sampling pattern is now formulated as follows.

(Problem 1) : Given  $x(t_k)$ , the OCP at  $t_k$  for the  $i$ -th pattern is to minimize  $J_i(x(t_k), \mathbf{u}_i(t_k))$ , subject to

$$\begin{cases} x(t_k + i\delta) = A_{i\delta}x(t_k) + B_{i\delta}u_i(t_k) & (6) \\ x(t_k + (n+1)\delta) & (n = i, \dots, N_p - 1) \\ \quad = A_\delta x(t_k + n\delta) + B_\delta u_i(t_k + n\delta) & (7) \\ u_i(t_k + n\delta) \in \mathcal{U}, \quad n = 0, i, i+1, \dots, N_p - 1 & (8) \\ x(t_k + N_p\delta) \in \Phi & (9) \end{cases}$$

The constraints (6) and (7) represent the dynamics by applying the control sequence  $\mathbf{u}_i(t_k)$ , and (8) represents the constraint for the control input. The last constraint (9) represents the terminal state penalty, where  $\Phi = \{x \in \mathbb{R}^n : x^\top P_f x \leq \varepsilon\}$  for a given  $\varepsilon > 0$ . We let

$$\begin{aligned} \mathbf{u}_i^*(t_k) &= \{u_i^*(t_k), u_i^*(t_k + i\delta), \dots, u_i^*(t_k + (N_p - 1)\delta)\} \\ \mathbf{x}_i^*(t_k) &= \{x_i^*(t_k), x_i^*(t_k + i\delta), \dots, x_i^*(t_k + N_p\delta)\} \end{aligned}$$

be the optimal control and the corresponding state sequence with  $x_i^*(t_k) = x(t_k)$ , obtained by solving Problem 1. We further denote  $J_i^*(x(t_k)) = J_i(x(t_k), \mathbf{u}_i^*(t_k))$  as the optimal cost.

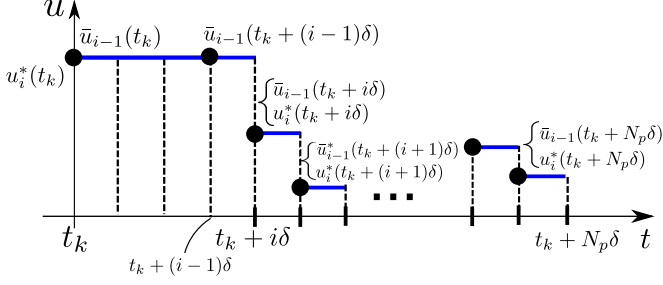


Fig. 4. Optimal piece-wise constant control policy for the  $i$ -th sampling pattern (blue line). The control policy in the figure also provides a feasible solution to Problem 1 under the  $(i-1)$ -th sampling pattern, since  $u_i^*(t_k)$  is applied for  $t \in [t_k, t_k + (i-1)\delta) \in [t_k, t_k + i\delta)$ . The black circles represent the admissible control sequence for the  $(i-1)$ -th pattern  $\bar{\mathbf{u}}_{i-1}(t_k)$ .

Similarly to the classic strategy of MPC, we consider that the matrix  $P_f$  and  $\varepsilon$  are chosen such that the following condition on the terminal region  $\Phi$  is satisfied:

**Assumption 1.** *There exists a local state feed-back controller  $\kappa(x) = Kx \in \mathcal{U}$ , satisfying*

$$x(t_k + \delta)^\top P_f x(t_k + \delta) - x^\top(t_k) P_f x(t_k) \leq -F(x(t_k), Kx(t_k), \delta) \quad (10)$$

for all  $x(t_k) \in \Phi$ , where  $x(t_k + \delta) = (A_\delta + B_\delta K)x(t_k)$ .

Since the system (1) is assumed to be stabilizable, the local controller  $\kappa(x)$  and  $\Phi$  satisfying (10), can be found off-line by following the procedure presented in [13]. To arrive at the self-triggered strategy, we will in the following derive some useful properties for the optimal costs obtained under different sampling patterns. These properties are key ingredients to quantify the control performances for the self-triggered strategy, as well as for the asymptotic stability provided in later sections.

**Lemma 1.** *Suppose that Problem 1 admits a solution at  $t_k$  under each sampling pattern  $i \in \{1, 2, \dots, M\}$ , which provides the optimal costs  $J_i^*(x(t_k))$  for all  $i \in \{1, \dots, M\}$ . Then we have*

$$J_1^*(x(t_k)) \leq J_2^*(x(t_k)) \leq J_3^*(x(t_k)) \dots \leq J_M^*(x(t_k)) \quad (11)$$

*Proof.* Let  $\mathbf{u}_i^*(t_k)$ ,  $\mathbf{x}_i^*(t_k)$ ,  $i \in \{1, 2, \dots, M\}$  be the optimal control and the corresponding state sequence obtained by Problem 1 under the  $i$ -th sampling pattern. The illustration of the corresponding optimal piece-wise constant control policy is depicted in Fig. 4. Under the  $i$ -th ( $i \geq 2$ ) sampling pattern,  $u_i^*(t_k)$  is applied at constant for all  $t \in [t_k, t_k + i\delta)$  as shown in Fig. 4. The control policy for the  $i$ -th ( $i \geq 2$ ) sampling pattern is thus admissible also for the  $(i-1)$ -th sampling pattern, as  $u_i^*(t_k)$  is applied for  $t \in [t_k, t_k + (i-1)\delta) \in [t_k, t_k + i\delta)$ . More specifically, let

$$\bar{\mathbf{u}}_{i-1}(t_k) = \{\bar{u}_{i-1}(t_k), \bar{u}_{i-1}(t_k + (i-1)\delta) \dots \dots, \bar{u}_{i-1}(t_k + (N_p - 1)\delta)\},$$

where  $\bar{u}_{i-1}(t_k) = u_i^*(t_k)$ ,  $\bar{u}_{i-1}(t_k + (i-1)\delta) = u_i^*(t_k)$  and

$$\bar{u}_{i-1}(t_k + j\delta) = u_i^*(t_k + j\delta), \quad j = i, \dots, N_p - 1,$$

and  $\bar{\mathbf{x}}_{i-1}(t_k) = \{\bar{x}_{i-1}(t_k), \bar{x}_{i-1}(t_k + (i-1)\delta) \dots, \bar{x}_{i-1}(t_k + N_p\delta)\}$  be the corresponding state sequence with  $\bar{x}_{i-1}(t_k) = x(t_k)$  (see the illustration of  $\bar{\mathbf{u}}_{i-1}$  in Fig. 4). Then,  $\bar{\mathbf{u}}_{i-1}(t_k)$  provides a feasible solution to Problem 1 under the  $(i-1)$ -th pattern, satisfying all constraints (6), (7), (8) and (9). The last constraint (9) is obtained by the fact that  $\bar{x}_{i-1}(t_k + N_p\delta) = x_i^*(t_k + N_p\delta) \in \Phi$ . Since  $\bar{\mathbf{u}}_{i-1}$  is a feasible controller for the  $(i-1)$ -th pattern, we obtain

$$\begin{aligned} J_{i-1}^*(x(t_k)) &\leq J_{i-1}(x(t_k), \bar{\mathbf{u}}_{i-1}(t_k)) \\ &= J_i(x(t_k), \mathbf{u}_i^*(t_k)) \\ &= J_i^*(x(t_k)), \end{aligned} \quad (12)$$

and the above inequality holds for all  $i \in \{2, 3, \dots, M\}$ . The proof is thus complete.  $\square$

Lemma 1 states that the 1<sup>st</sup> pattern provides the best control performance in the sense that the optimal cost takes the minimum value among all patterns. The next lemma states that the optimal cost is guaranteed to decrease whenever the 1<sup>st</sup> pattern is used:

**Lemma 2.** *Suppose that the  $i$ -th pattern was used at  $t_{k-1}$  and the next time to solve the OCP is given by  $t_k = t_{k-1} + i\delta$ . Then, under Assumption 1, the optimal cost satisfies*

$$J_1^*(x(t_k)) - J_i^*(x(t_{k-1})) \leq -F(x(t_{k-1}), u_i^*(t_{k-1}), i\delta) \quad (13)$$

*Proof.* (Sketch) Let

$$\begin{aligned} \mathbf{u}_i^*(t_{k-1}) &= \{u_i^*(t_{k-1}), u_i^*(t_k), \dots, u_i^*(t_k + (N_p - i - 1)\delta)\} \\ \mathbf{x}_i^*(t_{k-1}) &= \{x_i^*(t_{k-1}), x_i^*(t_k), \dots, x_i^*(t_k + (N_p - i)\delta)\} \end{aligned}$$

be the optimal control input and the corresponding state sequence obtained at  $t_{k-1}$  under the  $i$ -th pattern. From the constraint (9), we have  $x_i^*(t_k + (N_p - i)\delta) \in \Phi$ . At  $t_k$ , we consider the following control and the corresponding state sequence for the 1<sup>st</sup> pattern;  $\bar{\mathbf{u}}_1(t_k) = \{\bar{u}_1(t_k), \bar{u}_1(t_k + \delta), \dots, \bar{u}_1(t_k + (N_p - 1)\delta)\}$ ,  $\bar{\mathbf{x}}_1(t_k) = \{\bar{x}_1(t_k), \bar{x}_1(t_k + \delta), \dots, \bar{x}_1(t_k + N_p\delta)\}$ , where each component of  $\bar{\mathbf{u}}_1(t_k)$  is given by

$$\bar{u}_1(t_k + j\delta) = \begin{cases} u_i^*(t_k + j\delta) & (\text{for } j = 0, \dots, N_p - i - 1) \\ \kappa(\bar{x}_1(t_k + j\delta)) & (\text{for } j = N - i, \dots, N_p - 1) \end{cases} \quad (14)$$

Applying the local controller  $\kappa$  from  $t_k + (N_p - i)\delta$  is admissible since we have  $\bar{x}_1(t_k + (N_p - i)\delta) = x_i^*(t_k + (N_p - i)\delta) \in \Phi$ . Thus  $\bar{\mathbf{u}}_1(t_k)$  is a feasible controller for Problem 1 under the 1<sup>st</sup> sampling pattern, and the upper bound of the difference between  $J_1^*(x(t_k))$  and  $J_i^*(x(t_{k-1}))$  is given by

$$\begin{aligned} J_1^*(x(t_k)) - J_i^*(x(t_{k-1})) &\leq J_1(x(t_k), \bar{\mathbf{u}}_1(t_k)) \\ &\quad - J_i(x(t_{k-1}), \mathbf{u}_i^*(t_{k-1})) \end{aligned} \quad (15)$$

Some calculations of the right hand side in (15) yield (13). The derivation of (13) from (15) is omitted here and described in the technical report<sup>1</sup>.  $\square$

<sup>1</sup><http://arxiv.org/abs/1609.02259>

### III. SELF-TRIGGERED STRATEGY

In this section we propose the self-triggered strategy as one of our main results. The key idea of the framework is to select the best pattern in the sense that it provides the largest possible transmission time interval, while satisfying some conditions to obtain the desired control performance. In the following proposed algorithm, we denote  $i_k$ ,  $k \in \mathbb{N}_{\geq 0}$  as the sampling pattern selected by the controller to transmit the corresponding optimal control sample  $u_{i_k}^*(t_k)$ .

#### Algorithm 1: (Self-triggered MPC strategy)

- (i) *Initialization* : At initial time  $t_0$ , the controller solves Problem 1 only for the 1<sup>st</sup> sampling pattern based on  $x(t_0)$ . The controller then transmits the optimal control sample  $u_1^*(t_0)$  to the plant, i.e.,  $i_0 = 1$ . The plant applies the constant controller  $u_1^*(t_0)$  until  $t_1 = t_0 + \delta$ , and sends back  $x(t_1)$  to the controller as a new state measurement.
- (ii) At  $t_k$ ,  $k \in \mathbb{N}_{\geq 1}$ , the controller solves Problem 1 for all patterns  $i = 1, \dots, M$  based on  $x(t_k)$ . This provides the optimal control sequences  $\mathbf{u}_1^*(t_k), \mathbf{u}_2^*(t_k), \dots, \mathbf{u}_M^*(t_k)$ , and the corresponding optimal costs  $J_1^*(x(t_k)), \dots, J_M^*(x(t_k))$ .
- (iii) The controller selects one pattern  $i_k \in \{1, \dots, M\}$  by solving the following problem;

$$i_k = \max_{i \in \{1, 2, \dots, M\}} i, \quad (16)$$

subject to

$$J_i^*(x(t_k)) \leq J_1^*(x(t_k)) + \beta \quad (17)$$

$$J_i^*(x(t_k)) \leq J_{i_{k-1}}^*(x(t_{k-1})) - \gamma F(x(t_{k-1}), u_{i_{k-1}}^*(t_k), i_{k-1}), \quad (18)$$

where  $\beta$  and  $\gamma$  are the constant parameters, satisfying  $0 \leq \beta, 0 < \gamma \leq 1$ .

- (iv) The controller transmits  $u_{i_k}^*(t_k)$ , and then the plant applies  $u_{i_k}^*(t_k)$  as sample-and-hold implementation until  $t_{k+1} = t_k + i_k \delta$ . The plant then sends back  $x(t_{k+1})$  to the controller as a new current state measurement.

The main point of our proposed algorithm is the way to select the pattern  $i_k$  given in the step (iii). From Lemma 1, the 1<sup>st</sup> pattern provides the minimum cost among all sampling patterns. Thus, the first condition (17) implies that larger patterns are allowed to be selected to obtain longer transmission intervals, but the optimal cost should not go far from the 1<sup>st</sup> pattern; the optimal cost is allowed to be larger only by  $\beta$  from  $J_1^*(x(t_k))$ , so that it does not degrade much the control performance. Thus, the parameter  $\beta$  plays a role to regulate the trade-off between the control performance and the transmission time intervals. That is, a smaller  $\beta$  leads to better control performance (but resulting in less transmissions), and larger  $\beta$  leads to less transmissions (but resulting in worse control performance). The second condition (18) takes into account the optimal cost obtained at the previous time  $t_{k-1}$ , and this aims at guaranteeing the asymptotic stability of the origin. Note that  $\gamma$  needs to satisfy  $0 < \gamma \leq 1$ . As we

will describe in the next section, this condition ensures that Algorithm 1 is always implementable. Since it is desirable to reduce the communication load as much as possible, the controller selects the pattern providing the largest transmission interval satisfying (17), (18), i.e.,  $\max i$  in (16).

The main advantage of using the proposed method is that the optimal cost  $J_i^*(t_k)$  can be compared not only with the previous one  $J_{i_{k-1}}^*(t_{k-1})$ , but also with the current ones obtained at  $t_k$  under different sampling patterns. This allows us not only to ensure stability, but also to evaluate how much the control performance becomes better or worse according to the transmission time intervals. Note that the control performance may also be regulated through the tuning of  $\gamma$  in (18). However, due to the condition  $0 < \gamma \leq 1$ , we cannot select  $\gamma$  large enough such that small patterns (good control performance) are ensured to be obtained. Thus the desired control performance can be suitably specified through the first condition (17), rather than (18).

### IV. ANALYSIS

One of the desirable properties of Algorithm 1 is to ensure that it is always implementable, i.e., we need to exclude the case when all the patterns do not satisfy both (17) and (18). Furthermore, the stability of the closed loop system under Algorithm 1 needs to be verified. In the following theorem, we deduce that both of these properties are satisfied.

**Theorem 1.** *Consider the networked control system in Fig. 1 where the plant follows the dynamics given by (1) and the proposed self-triggered strategy (Algorithm 1) is implemented. The followings are then satisfied:*

- (i) *The way to obtain the pattern  $i_k$  in step (iii) in Algorithm 1, is always feasible. That is, there exists at least one pattern  $i$ , satisfying both (17), (18) for all  $k \in \mathbb{N}_{\geq 0}$ .*
- (ii) *The closed loop system is asymptotically stabilized to the origin.*

*Proof.* The proof of (i) is obtained by showing that the 1<sup>st</sup> sampling pattern ( $i = 1$ ) always satisfies (17) and (18). The first condition is clearly satisfied when  $i = 1$  since  $\beta \geq 0$ . Furthermore, from Lemma 2, we obtain

$$\begin{aligned} J_1^*(x(t_k)) &\leq J_{i_{k-1}}^*(x(t_{k-1})) - F(x(t_{k-1}), u_{i_{k-1}}^*(t_k), i_{k-1}) \\ &\leq J_{i_{k-1}}^*(x(t_{k-1})) - \gamma F(x(t_{k-1}), u_{i_{k-1}}^*(t_k), i_{k-1}) \end{aligned}$$

Thus the second condition holds for  $i = 1$ . Thus the proof of (i) is complete.

The proof of (ii) is obtained by the fact that the optimal cost decreases along the time sequence. Since the optimal cost of the selected pattern satisfies (18), we have

$$\begin{aligned} J_{i_1}^*(x(t_1)) - J_{i_0}^*(x(t_0)) &\leq -\gamma F(x(t_0), u_{i_0}^*(t_0), i_0) \\ &< -\gamma \int_{t_0}^{t_1} x^\top(t) Q x(t) dt \\ J_{i_2}^*(x(t_2)) - J_{i_1}^*(x(t_1)) &< -\gamma \int_{t_1}^{t_2} x^\top(t) Q x(t) dt \\ &\vdots \end{aligned}$$

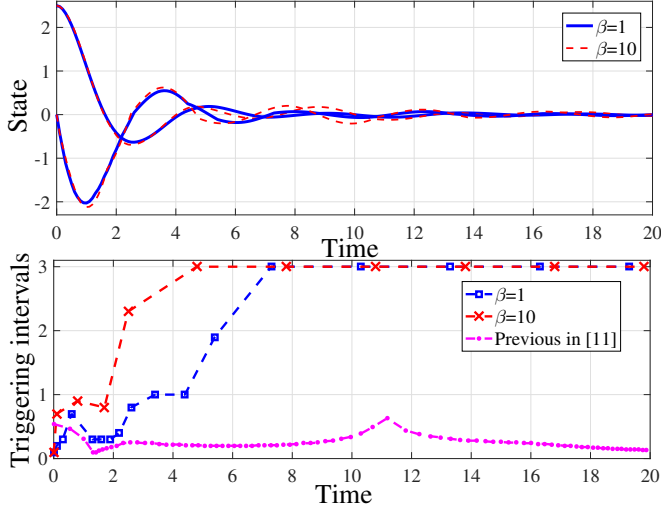


Fig. 5. State trajectories and transmission time intervals.

where the first to the second in-equality follows from (5). Summing over both sides of the above yields

$$\gamma \int_{t_0}^{\infty} x^T(t)Qx(t)dt < J_{i_0}^*(x(t_0)) - J_{i_\infty}^*(x(\infty)) < \infty$$

Since the function  $x^T(t)Qx(t)$  is uniformly continuous on  $t \in [0, \infty)$  and  $Q \succ 0$ , we obtain  $\|x(t)\| \rightarrow 0$  as  $t \rightarrow \infty$  from Barbalat's lemma [14]. This completes the proof.  $\square$

## V. ILLUSTRATIVE EXAMPLE

As an illustrative example, we consider the spring-mass system; the state vector  $x = [x_1; x_2]$  consists of the position  $x_1$  and the velocity  $x_2$ , and the dynamics are given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u, \quad (19)$$

where  $k = 2$  is the spring coefficient and  $m = 1$  is the mass. The matrices for the stage cost are  $Q = I_2$ ,  $R = 0.5$ , and the prediction horizon is  $T_p = 8$ . The terminal matrix  $P_f$  and the local controller  $\kappa$  are computed properly by following the procedure presented in [13]. We further assume that the control input  $u$  is constrained by  $\|u\| \leq 8$ . The total number of sampling patterns is given by  $M = 30$  with  $\delta = 0.1$ , i.e., the maximum transmission time interval is  $M\delta = 3$ .

Fig. 5 shows state trajectories of  $x_1$  and  $x_2$  (upper), with  $\gamma = 0.5$ ,  $\beta = 1$  and  $\beta = 10$  from the initial state  $x_0 = [2.5; 0]$ , and the transmission time intervals (lower). From the figure, the state achieves asymptotic stability of the origin, and larger patterns (i.e., longer transmission time intervals) are more likely to be obtained as the state gets closer to the origin. One can also see the trade-off between the control performance and the number of transmissions; faster convergence is achieved when  $\beta = 1$  than  $\beta = 10$  from the upper figure, while it requires more transmissions of control samples as shown in the lower figure.

To compare with the previous framework, we have also plotted the transmission time intervals in Fig. 5 obtained by the

methodology presented in [11]. Here we set  $\sigma = \gamma = 0.5$  in Eq.(19) in [11], to ensure the same rate of cost decrease. From Fig. 5, the proposed scheme attains much longer transmission time intervals than the previous method under the same performance guarantees.

## VI. CONCLUSION AND FUTURE WORK

In this paper, we propose a self-triggered control methodology for continuous-time linear networked control systems. Our proposed scheme was derived by solving multiple optimal control problems with different sampling time intervals, and the controller selects one sampling pattern resulting in the largest transmission time intervals while satisfying the desired control performances. Our proposed scheme was also validated by an illustrative example. Future work involves deriving the self-triggered strategies against random packet dropouts and extend the proposed result to the nonlinear case.

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