

Optimal Leader-Follower Control for Crowd Evacuation

Yuecheng Yang, Dimos V. Dimarogonas and Xiaoming Hu

Abstract—Crowd evacuation has become a primary safety issue in many public places in a metropolitan area. Experts from different fields have worked on modeling and designing evacuating policies by using different tools and methods. In this paper, an optimal control approach is used to derive guiding strategies for the rescue agents under different circumstances. Various optimal control problems are formulated to handle different assumptions of the scenario. Both the analytic solutions and the numerical simulation confirm the efficiency of this approach, which in turn can be potentially used as decision-making support in practical applications.

I. INTRODUCTION

Due to the rapid urbanization in many parts of the world and other factors, our societies today are facing increasingly more potential threats in public places with high density human crowds. Systematic crowd evacuation has thus become a very critical issue, and also an important topic for research. Since the past two decades [1], researchers from different fields have contributed to modeling and design of evacuation scenarios from different point of views. J.D. Sime [2] started from the psychology aspect to analyze the individual behaviors. In [3], [4] established a cellular automaton model with behavior inspired by insects in nature was established. With the development of multi-agent systems theory, some agent-based models came out in the past decade, e.g., [5], [6], [14], [18].

So far there seem to be two different methodologies for modeling the crowd behavior. The first is the so-called microscopic approach that treats individuals in the group as separate objects with certain influence from both the rest of the crowd and the environment. Most results that use tools from psychology, social forces, multi-agent systems theory belong to this category, e.g., [2], [9], [12], [20]. The other is the macroscopic approach that considers the whole crowd as one entity with certain parameters describing the density of the people, which applies only for large crowds. Tools from fluid mechanics and partial differential equations are used, e.g., [8], [21]. When safety is of concern, density based approximations of individuals may not be adequate. For example, the density being equal to 0.01 does not implies absence of agents in the area. We thus argue that microscopic approaches are more suitable when safety is prioritized and use this approach in this work.

The study on multi-agent systems has experienced a significant development over the past decade. Many agent-based models were established based on the assumptions given by C.W. Reynolds [7] when modeling animal flock behavior. Especially, the social force model [15] has been widely considered. The simulation of these models can illustrate the human crowd motion in realistic scenarios.

A question that is still largely open is how we can implement these models in order to control crowd evacuation in case of panic situation. Since efficiency is critical in such an evacuation, an optimal control approach seems to be suitable. If we can set some reasonable objectives for evaluating the evacuation progress, we can then design an optimal moving strategy for the crowd accordingly. Unfortunately, even if the optimal behavior can be determined, agents in panic may not be able to perform in that way. Therefore, in both practice and theory, a “leader-follower” approach would be more efficient, since even in panic people would still tend to follow a rescue worker. In the literature, several agent-based leader-follower models are established [10], [11]. Optimal control has also been used for such models, e.g., [16], [17]. In [16] an optimal control problem is formulated to control the system from one quasi-static equilibrium to another one. In [17] it is shown that optimal energy control for consensus implies that the interaction topology is the complete graph.

In this paper, we will focus on designing optimal control strategies for the leaders so that the followers can move from some given initial positions to a final destination in a most efficient way. The underlying assumption here is that one can not expect individuals in a big crowd to perform a global optimal behavior when they are in panic, as is already mentioned above. More realistically, we can assume that they will still follow some behavior primitives such as following a guiding leader, avoiding collision with each other, or avoiding walls and obstacles. On the other hand, skilled and calm leaders could do their best to guide the crowd to certain places considered to be safe enough. By choosing different objective functions and models, the optimal rescuing strategy for the leaders may vary in different situations. In order to get analytic solutions, we will focus on some relatively simple models. However, we argue that they still capture the essence of the problem.

The rest of the paper is organized as follows: In Section II, the optimal control problem is formulated. Situations with single follower are discussed in Section III, and the analytic solutions are provided. In Section IV, we investigate the scenario with multiple followers and derive the optimal solution to the problem. Numerical simulation is shown in Section V, and conclusions are given in Section VI.

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II. PROBLEM FORMULATION

An individual agent in the crowd will be called *follower*, while a rescue agent will be called *leader*. Our objective is to plan the leaders' rescue guiding route in an optimal way according to a given leader-follower interaction model. In this paper, the considered models will be of first order, which indicates that we can control the velocity of a leader directly, and the velocity of followers can be modeled as a function of the leaders' and the followers' positions. Denoting the position of all followers as $x = (x_1^T \ x_2^T \ \dots \ x_n^T)^T$ and of all leaders as $x_l = (x_{l_1}^T \ x_{l_2}^T \ \dots \ x_{l_m}^T)^T$, where $x_i \in \mathbb{R}^2$ and $x_{l_j} \in \mathbb{R}^2$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, then we can write

$$\dot{x}_i(t) = f_i(x_1(t), \dots, x_n(t), x_{l_1}(t), \dots, x_{l_m}(t)), \quad (1)$$

for $i = 1, \dots, n$, where n is the number of followers and m is the number of leaders, or in stack vector form

$$\dot{x}(t) = f(x(t), x_l(t)). \quad (2)$$

The task of the leader is to guide the follower to a certain area in a given time. We assume that the leaders know the initial positions of the followers, the information about the target area, and the follower dynamics (2). It may happen that the leader cannot obtain all the initial positions of the followers, then leaders could only perform as good as they can based on the information they have. Hence, the assumption is reasonable. The evaluation of the guiding action involves both the terminal status of the followers and the rescuing cost generated during the procedure. The following optimal control problem (OCP) gives a standard form of this type of problems

$$\begin{aligned} \min_u \quad & \Phi(x(t_f)) + \int_0^{t_f} \mathcal{L}(t, x(t), u(t)) dt \quad (3) \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), x_l(t)), \\ & \dot{x}_l(t) = u(t), \\ & x(0) = x^0, \quad x_l(0) = x_l^0, \\ & x(t_f) \in E, \quad x_l(t_f) \in E_l, \\ & u(t) \in U, \end{aligned}$$

where $\Phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ and $\mathcal{L} : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}^{2m} \rightarrow \mathbb{R}$ are the cost functions of the final states and during the process, respectively, and $x^0 \in \mathbb{R}^{2n}$ and $x_l^0 \in \mathbb{R}^{2m}$ are the initial conditions of leaders and followers. The sets $E \subseteq \mathbb{R}^{2n}$ and $E_l \subseteq \mathbb{R}^{2m}$ describe the terminal constraints for the states while $U \subseteq \mathbb{R}^{2m}$ is the constraint for the control.

We can formulate many different problems in the form of (3) by choosing different functions Φ , \mathcal{L} and f , and different sets E , E_l and U . For example, one may want to guide the followers as close as possible, in terms of the sum of the distance squares, to a point with position x_e in a limited amount of time while the leaders have to be in a given area E_l at the terminal time. If there is no other constraint for the control, then we can set $\Phi(x(t_f)) = \sum_{i=1}^n \|x_i(t_f) - x_e\|^2$, $\mathcal{L} = 0$, $E = \mathbb{R}^{2n}$, $E_l = E_l$ and $U = \mathbb{R}^{2m}$.

In Sections III and IV, we will discuss cases with one leader but with different values for these functions and explore the analytical optimal solutions to the respective problems by using Pontryagin's minimum principle (PMP). Since nonlinear function f may make it difficult to obtain a analytical solution, we will focus on a linear interaction model among the leaders and the followers, which is reminiscent of standard consensus models in literature.

III. SINGLE-FOLLOWER CASE

Let us first consider the case where there is only one leader and one follower. This assumption is nevertheless illustrative of the general ideas. In this section, we assume that the follower follows the leader with velocity $f(x, x_l) = c(x_l - x)$ for some parameter $c > 0$, and we discuss two cases of the objective function and derive the analytic solution for each case.

A. Optimizing the terminal position

For a leader, the goal may be to guide followers to an exit or a safe place as soon as possible. Mathematically this can be formulated as the problem of minimizing the distance between the follower and the exit at some given time instance t_f , i.e., $\Phi(x(t_f)) = \|x(t_f) - x_e\|^2$, where x_e is the position of the exit; $\mathcal{L}(t, x(t), u(t))$ is set to be zero in this case. Then we can rewrite (3) as

$$\begin{aligned} \min_u \quad & \|x(t_f) - x_e\|^2 \quad (4) \\ \text{s.t.} \quad & \dot{x} = c(x_l - x), \\ & \dot{x}_l = u, \\ & x(0) = x^0, \\ & x_l(0) = x_l^0 \\ & \|u(t)\| \leq v_{max}. \end{aligned}$$

The bound on $u(t)$ reflects the fact that both agents have limited motion capabilities. Pontryagin's minimum principle can be used to solve this problem. The Hamiltonian in this case is

$$H = \lambda^T (c(x_l - x)) + \lambda_l^T u, \quad (5)$$

where λ and λ_l are the Lagrange multipliers. If x , x_l , u^* are the optimal states and control, then the pointwise minimization gives

$$\begin{aligned} u^* &= \operatorname{argmin} H((x, x_l), u, (\lambda, \lambda_l)) = \operatorname{argmin} \lambda_l^T u \\ &= \begin{cases} -\frac{v_{max}}{\|\lambda_l\|} \lambda_l, & \lambda_l \neq 0 \\ \text{arbitrary}, & \lambda_l = 0 \end{cases} \quad (6) \end{aligned}$$

By plugging this control to the system and using the adjoint system to solve the Lagrange multipliers, we can get

$$x(t) = e^{-ct} x^0 - \frac{v_{max}}{c \|\alpha\|} (ct - 1 + e^{-ct}) \alpha + (1 - e^{-ct}) x_l^0, \quad (7)$$

where $\alpha := x(t_f) - x_e \neq 0$. If we let $t = t_f$ in (7) and subtract x_e on both sides of it, then we get $\alpha = k\beta$, where

$$k = (1 + \frac{v_{max}}{c \|\alpha\|} (ct_f - 1 + e^{-ct_f}))^{-1} > 0, \quad (8)$$

and

$$\beta = e^{-ct_f}x^0 + (1 - e^{-ct_f})x_l^0 - x_e. \quad (9)$$

Hence, the optimal control is given by

$$u^*(t) = -\frac{v_{max}}{\|k\beta\|}k\beta = -\frac{v_{max}}{\|\beta\|}\beta \quad (10)$$

If $\alpha = 0$, then both $\lambda(t)$ and $\lambda_l(t)$ will be constant and equal to zero. The solution of the pointwise minimization will be arbitrary values. On the other hand, $\alpha = 0$ indicates that $x(t_f) = x_e$. Therefore, any control $u(t)$ that will lead the follower to the exit in time t_f will then be optimal.

By plugging $\|\alpha\| = k\|\beta\|$ back to (8), we get

$$k = 1 - \frac{v_{max}}{c\|\beta\|}(ct_f - 1 + e^{-ct_f}). \quad (11)$$

Meanwhile, we know that $k > 0$ by (8), which implies

$$v_{max}(ct_f - 1 + e^{-ct_f}) \leq c\|\beta\|. \quad (12)$$

We can solve the inequality (12) to get an upper bound for t_f such that $\alpha \neq 0$. If t_f satisfies (12), then the solution (10) is the only solution satisfying the PMP conditions, which is also the unique optimal solution to OCP (4).

The above analysis is summarized as follows:

Theorem 3.1: If the terminal time t_f satisfies (12), then the control (10) solves the optimal control problem (4), where β is given by (9).

Although the optimal control (10) obtained by PMP is an open-loop solution, we can always write it as a state feedback control according to Bellman's Principle of Optimality, i.e., at time t , considering a similar OCP to (10) with the current states as the initial condition and the new time limit $t_f - t$. We can derive the optimal solution to the new problem according to Theorem 3.1, which can be written as

$$u^*(t) = -\frac{v_{max}}{\|\beta(t, x(t), x_l(t))\|}\beta(t, x(t), x_l(t)), \quad (13)$$

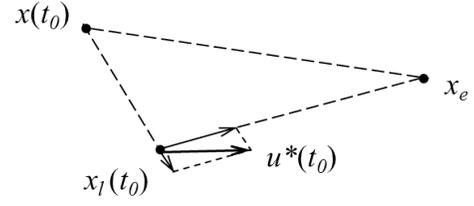
where

$$\begin{aligned} \beta(t, x(t), x_l(t)) &= e^{-c(t_f-t)}x(t) + (1 - e^{-c(t_f-t)})x_l(t) - x_e \\ &= e^{-c(t_f-t)}(x(t) - x_l(t)) + (x_l(t) - x_e). \end{aligned} \quad (14)$$

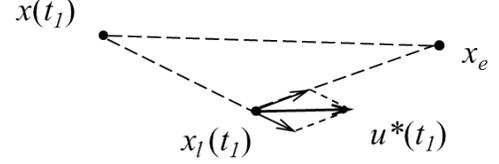
Hence, u^* can be considered as linear combination of the two directions $x_l(t) - x(t)$ and $x_e - x_l(t)$ with time varying weights. When t is small, the leader tends to move in the direction of $x_e - x_l(t)$, i.e., towards the exit, since $e^{-c(t_f-t)}$ is relatively small. When t is getting larger, the leader will focus more on guiding the follower, and the vector $u^*(t)$ will become parallel to $x_e - x(t)$. The phenomenon is depicted in Fig 1.

Remark: If we would like the follower to be close enough to the exit in the end, i.e., $\|x(t_f) - x_e\| \leq r$, for some $r > 0$, the lower bound of t_f that make the system state fulfill this constraint can also be given by solving the inequality

$$\|\beta\| - \frac{v_{max}}{c}(ct_f - 1 + e^{-ct_f}) \leq r, \quad (15)$$



(a) $t = t_0$



(b) $t = t_1$

Fig. 1. A Graphic illustration of the optimal moving directions of the leader as linear combinations of $x_l(t) - x(t)$ and $x_e - x_l(t)$. In (1(a)), $u^*(t_0)$ is dominated by the vector $x_e - x_l(t)$ because t_0 is small. In (1(b)), $u^*(t_1)$ is almost parallel to the vector $x_e - x(t)$ since the value of $e^{-c(t_f-t_1)}$ is approaching 1.

B. Optimizing the rescuing cost

There are scenarios where (semi-)autonomous robots are deployed, e.g., for the search and rescue missions. For such systems, energy is an issue. Thus it is reasonable to define the objective function of the process $\mathcal{L}(x(t), u(t), t)$ as $\|u(t)\|^2$. Meanwhile, we want the leader to guide followers to a certain region defined by a disc with the center x_e and the radius r . Then we obtain a new optimal control problem

$$\begin{aligned} \min_u \quad & \int_0^{t_f} \|u(s)\|^2 ds \\ \text{s.t.} \quad & \dot{x} = c(x_l - x), \\ & \dot{x}_l = u, \\ & x(0) = x^0, \quad x_l(0) = x_l^0 \\ & x(t_f) \in \mathbf{B}(x_e, r), \end{aligned} \quad (16)$$

where $\mathbf{B}(x_e, r) = \{x \in \mathbb{R}^2 : \|x - x_e\| \leq r\}$.

The Hamiltonian is

$$H = \|u(t)\|^2 + \lambda^T(c(x_l - x)) + \lambda_l^T u. \quad (17)$$

The pointwise minimization gives

$$u^* = \operatorname{argmin} H((x, x_l), u, (\lambda, \lambda_l)) = -\frac{\lambda_l}{2}. \quad (18)$$

For the single-follower case, we can replace the closed ball with a sphere in the boundary condition of (16), and obtain a boundary constraint for $\lambda(t)$, which is $\lambda(t_f) = k(x(t_f) - x_e) \triangleq k\alpha$ for some $k \in \mathbb{R}$. Then we get

$$\lambda_l = -\int_{t_f}^t ce^{c(s-t_f)}\lambda(t_f)ds + \lambda_l(t_f) = k(1 - e^{c(t-t_f)})\alpha, \quad (19)$$

and

$$u^* = -\frac{k}{2}(1 - e^{c(t-t_f)})\alpha. \quad (20)$$

We can again perform the integration and get

$$x(t) = -\frac{k}{2c} \left(ct - 1 + e^{-ct} + e^{-ct_f} - \frac{1}{2}(e^{c(t-t_f)} + e^{-c(t+t_f)}) \right) + e^{-ct}x^0 + (1 - e^{-ct})x_l^0, \quad (21)$$

Noticing the fact that $\alpha = x(t_f) - x_e$ and $\|\alpha\| = r$, we can solve α and k , and get

$$\alpha = \frac{r}{\|\beta\|} \beta, \quad (22)$$

and

$$k = \frac{2c \left(\frac{\|\beta\|}{r} - 1 \right)}{ct_f - 1 + 2e^{-ct_f} - \frac{1}{2}(1 + e^{-2ct_f})}, \quad (23)$$

where

$$\beta = e^{-ct_f}x^0 + (1 - e^{-ct_f})x_l^0 - x_e \quad (24)$$

Theorem 3.2: The control (20) solves the optimal control problem (16), where $\alpha = \frac{r}{\|\beta\|} \beta$, β is given by (24), and k is given by (23).

IV. MULTIPLE-FOLLOWER CASE

We will consider the case of one leader and multiple followers in this section, i.e., $m = 1$ and $n > 1$. Assuming that followers interact with each other linearly with respect to a fixed graph G , whose Laplacian is denoted by L , then the follower dynamics (2) can be written as

$$\dot{x} = Ax + Bx_l, \quad (25)$$

where $A = -L \otimes I_2 - \text{diag}\{\vec{b}\} \otimes I_2$, $B = \vec{b} \otimes I_2$, and the nonnegative vector \vec{b} indicates the power of the links between the leader and the followers. We also assume the joint undirected graph with leader and followers is connected, i.e., the leader is connected to at least one agent in each component of G . In this section, we will only extend the OCP (16) where we want to minimize the total energy cost. The extended version is given by

$$\begin{aligned} \min \quad & \int_0^{t_f} \|u(s)\|^2 ds \\ \text{s.t.} \quad & \dot{x} = Ax + Bx_l, \\ & \dot{x}_l = u, \\ & x_i(0) = x_i^0, \quad x_l(0) = x_l^0 \\ & x_i(t_f) \in \mathbf{B}(x_e, r). \end{aligned} \quad (26)$$

One can apply PMP to solve the optimal control problem by using a particular technique to deal with the boundary condition for $x_i(t_f)$. The Hamiltonian will be

$$H = \|u(t)\|^2 + \lambda^T (Ax + Bx_l) + \lambda_l^T u. \quad (27)$$

The point-wise minimization will be again

$$u^* = -\frac{\lambda_l}{2}. \quad (28)$$

Since we assumed that the joint undirected graph with leader and followers is connected, the matrix A is negative definite

[22]. From the adjoint system, we can do the standard integration and get

$$\lambda_l(t) = B^T A^{-1} (e^{-A(t-t_f)} - I_{2n}) \lambda(t_f). \quad (29)$$

As mentioned in [13], the condition for the boundary value of the Lagrange multiplier $\lambda(t)$ can be obtained in the following way. We can rewrite the terminal set as $E = \{x | \psi(t_f, x(t_f)) \leq 0\}$, where $\psi_i(t, x) = \|x_i - x_e\|^2 - r^2$. The value of the Lagrange multiplier λ_i at time t_f must be in the form of $\lambda_i(t_f) = \left(\frac{\partial \phi}{\partial x_i} + v_i \frac{\partial \psi}{\partial x_i} \right) \Big|_{x_i=x_i(t_f)}$, where $v = (v_1 \ v_2 \ \dots \ v_n)^T$ and the following conditions are fulfilled:

$$\begin{cases} \psi(t_f, x(t_f)) \leq 0 \\ v \geq 0 \\ v^T \psi(t_f, x(t_f)) = 0. \end{cases} \quad (30)$$

The last equation of (30) is called the complementarity condition, which means that each agent will either finally reach the boundary of the set or not contribute to the optimal solution. For example, if we want to lead all the agents to reach boundary at the same time, $\psi(t_f, x(t_f))$ will be zero while v is free to choose according to (30). Then we have

$$\begin{aligned} \lambda(t_f) &= (\lambda_1(t_f)^T \ \dots \ \lambda_n(t_f)^T)^T \\ &= \left(v_1 \frac{\partial \psi}{\partial x_1} \Big|_{x_i=x_i(t_f)} \ \dots \ v_n \frac{\partial \psi}{\partial x_n} \Big|_{x_i=x_i(t_f)} \right)^T \\ &= (2v_1(x_1(t_f)^T - x_e^T) \ \dots \ 2v_n(x_n(t_f)^T - x_e^T))^T \triangleq \eta. \end{aligned}$$

If we can obtain η , then we know the optimal control will be

$$u^*(t) = -\frac{1}{2} B^T A^{-1} (e^{-A(t-t_f)} - I_{2n}) \eta. \quad (31)$$

By doing certain integrations similar to the ones we did in Section III.B, we get

$$\begin{aligned} x(t_f) &= e^{At_f} x^0 + (e^{At_f} - I) A^{-1} B x_l^0 + \frac{1}{2} \left((-t_f I_{2n} - A^{-1} \right. \\ &\quad \left. + A^{-1} e^{At_f}) A^{-1} B B^T A^{-1} + W(t_f) A^{-2} \right. \\ &\quad \left. + (I_{2n} - e^{At_f}) A^{-1} B B^T A^{-2} e^{At_f} \right) \eta, \end{aligned} \quad (32)$$

where $W(t)$ is the controllability Gramian of the system $\dot{x} = Ax + Bu$. If we define

$$\begin{aligned} \beta &= (\beta_1 \ \beta_2 \ \dots \ \beta_n)^T \\ &= e^{At_f} x^0 + (e^{At_f} - I_{2n}) A^{-1} B x_l^0 - x_e \otimes \vec{1}_n, \end{aligned} \quad (33)$$

and

$$\begin{aligned} \Gamma &= (\Gamma_1^T \ \Gamma_2^T \ \dots \ \Gamma_n^T)^T \\ &= -\frac{1}{2} \left((-t_f I_{2n} - A^{-1} + A^{-1} e^{At_f}) A^{-1} B B^T A^{-1} \right. \\ &\quad \left. + W(t_f) A^{-2} + (I_{2n} - e^{At_f}) A^{-1} B B^T A^{-2} e^{-At_f} \right), \end{aligned} \quad (34)$$

then

$$x(t_f) - x_e \otimes \vec{1}_n = \beta - \Gamma \eta. \quad (35)$$

$x_i(t_f) - x_e$ can be written as $\beta_i - \Gamma_i \eta$ with the constraint that

$$\|\beta_i - \Gamma_i \eta\| = r. \quad (36)$$

It is more probable that some of the agents will reach the interior of the set E finally, i.e., $\psi_i(t_f, x(t_f)) < 0$ for some i . Then we must have $v_i = 0$ according to (30). But we can still write $\lambda(t_f)$ in the form of $2V(x(t_f) - x_e \otimes \bar{I}_n) \triangleq \eta$, where $V = \text{diag}\{v\} \otimes I_2$ with $v_i = 0$ for those i 's such that $\psi_i(t_f, x(t_f)) < 0$. Finally, we will obtain that $x_i(t_f) - x_e = \beta_i + \Gamma_i \eta$. However, we can only get the inequality constraint

$$\|\beta_i - \Gamma_i \eta\| \leq r, \quad (37)$$

since we do not know for which i , $\psi_i(t_f, x(t_f)) < 0$, and $v_i = 0$. Furthermore, for any η satisfying (37), it is not always possible to find the vector v such that (30) holds. Fortunately, we are not searching for any arbitrary η that satisfies (37). Note that given the optimal control (31), the objective value will be

$$\begin{aligned} \int_0^{t_f} \|u(s)\|^2 ds &= \int_0^{t_f} \left\| -\frac{1}{2} B^T A^{-1} (e^{-A(s-t_f)} - I_{2n}) \eta \right\|^2 ds \\ &= \int_0^{t_f} \eta^T K(s)^T K(s) \eta ds \\ &= \eta^T \left(\int_0^{t_f} K(s)^T K(s) ds \right) \eta \triangleq \eta^T Q \eta, \end{aligned} \quad (38)$$

where $K(t) = -\frac{1}{2} B^T A^{-1} (e^{-A(t-t_f)} - I_{2n})$, and $Q = \int_0^{t_f} K(s)^T K(s) ds$. To minimize the objective value is therefore equivalent to minimizing $\eta^T Q \eta$, which implies that the η we are looking for is the optimal solution to the following quadratic-constraint quadratic programming (QCQP) problem

$$\begin{aligned} \min_{\eta} \quad & \eta^T Q \eta \\ \text{s.t.} \quad & \|\beta_i - \Gamma_i \eta\|^2 \leq r^2, \text{ for } i = 1, 2, \dots, n. \end{aligned} \quad (39)$$

The following lemma guarantees that for the optimal η solved from (39), we can always find v such that (30) is satisfied, and then derive the optimal control.

Lemma 4.1: If η is the optimal solution to the convex optimization problem (39), then there always exists a vector v such that the condition (30) is fulfilled for $x_i(t_f) = \beta_i - \Gamma_i \eta + x_e$.

Proof: We use the KKT optimality condition to prove this lemma. If η^* is the global optimal solution to (39), then there exist Lagrange multipliers $\mu = (\mu_1 \ \mu_2 \ \dots \ \mu_n)^T$ such that

$$\begin{cases} 2Q\eta^* + 2\sum_{i=1}^n \mu_i (\Gamma_i \eta^* - \beta_i) = 0 \\ \mu \geq 0 \\ \mu_i (\|\beta_i - \Gamma_i \eta^*\|^2 - r^2) = 0. \end{cases} \quad (40)$$

By noticing that $x_i(t_f) - x_e = \beta_i - \Gamma_i \eta$, we can directly choose $v = \mu$ to fulfill the conditions (30). ■

Theorem 4.2: The solution (31) with η as the optimal solution to the convex optimization problem (39) solves the optimal control problem (26).

Proof: The theorem can be proven by using Lemma 4.1 and the analysis before that. ■

Remark: The convex optimization problem (39) does not always have feasible solutions, especially for small r . For

given initial conditions x_{i_0}, x_{l_0}, x_e and given terminal time t_f , we need to first solve another QCQP problem:

$$\begin{aligned} \min_{r, \eta} \quad & r^2 \\ \text{s.t.} \quad & \|\beta_i - \Gamma_i \eta\|^2 \leq r^2, \text{ for } i = 1, 2, \dots, n. \end{aligned} \quad (41)$$

to obtain the lower bound for the feasible radius r . The objective function w.r.t r is strictly convex, so there will be only one global optimal r_{\min} , which can be found by using a standard convex programming solver. This specific problem can also be solved more efficiently by using the technique introduced in [19] if Γ_i 's are identical. If the given r in the OCP (26) is smaller than r_{\min} , then the OCP has no optimal solution. This lower bound also describes how well the leader can perform the evacuation during a given time.

On the other hand, for a given r , one can solve the following optimization problem

$$\begin{aligned} \min_{t, \eta} \quad & t \\ \text{s.t.} \quad & \|\beta_i(t) - \Gamma_i(t) \eta\|^2 \leq r^2, \text{ for } i = 1, 2, \dots, n. \end{aligned} \quad (42)$$

to obtain the lower bound for t_f , where $\beta_i(t)$ and $\Gamma_i(t)$ are nonlinear functions of t that are defined similarly to (33) and (34). The solution to this optimization problem answers the question that “what is the minimal time the leader needs to guide the follower into the safety zone with radius r ”.

We conclude the section by providing a summary of the algorithm for solving the OCP (26).

Algorithm 1: The algorithm consists of four steps

1. Compute the vectors/matrices β , Γ , and Q by the definition (33), (34), and (38), respectively.
2. Solve the QCQP problem (41) to get r_{\min} . If $r < r_{\min}$, then the OCP has no optimal solution.
3. Solve the QCQP problem (39) to derive the vector η .
4. The optimal control is given by $u^*(t) = -\frac{1}{2} B^T A^{-1} (e^{-A(t-t_f)} - I_{2n}) \eta$.

V. SIMULATIONS

In this section, we provide a simulation to examine and demonstrate the performance of the theoretical optimal control strategy. Due to the page limitation, we only show one example with a slightly more advanced setting than problem (26). We consider the case where more than one exit is involved in the rescuing problem. In order to lead the followers to different places, multiple leaders are required. We assume leader j want to lead the followers to the exit j without knowing which leader the followers will follow. Meanwhile, a follower will choose the closest leader corresponding to the initial position and keep following that specific leader. Each leader will perform the rescuing strategy as if all the followers will follow it. By removing the follower-follower interaction, we can establish a model with multiple leaders as follows

$$\begin{aligned} \dot{x}_i &= c_i (x_{l_{k(i)}} - x_i) \\ \dot{x}_{l_j} &= u_j, \end{aligned} \quad (43)$$

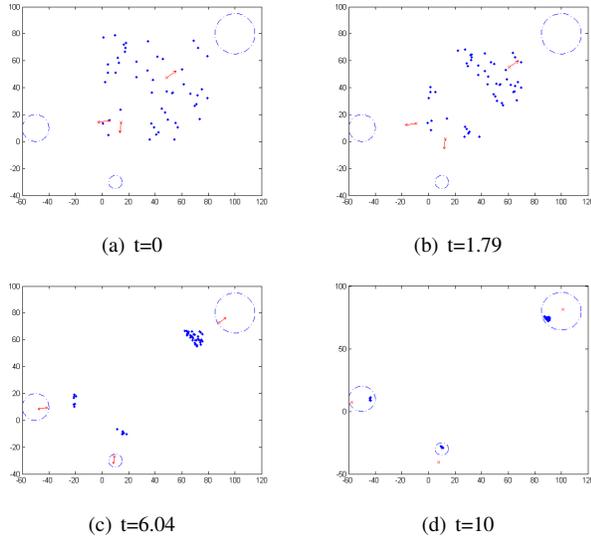


Fig. 2. Snapshots of the leader-follower movement with the model in the problem (43) with the parameter c chosen to be 0.3. The blue dots represent the follower's position, and the red crosses are the leaders' positions. The red arrow shows the heading direction of the leader, while the length of it indicates the speed. The followers are initially spread in the area of $[0, 80] \times [0, 80]$ while the position of the leaders is randomly generated. At $t = t_f = 10$, all the followers have entered the safety regions.

where $k(i) = 1, 2$ or 3 indicates which leader the follower i chose to follow. The set of followers who are following the leader j is a subset of the set of all the followers. Those followers will be able to reach the disk $B(x_{e_j}, r_j)$ since the optimal control (31) is a feasible solution for the problem (26) from Section IV.B, where x_{e_j} and r_j are the location and safety radius of the j^{th} exit, respectively. Here we run a simulation with three exits located at $(100, 80)$, $(-50, 10)$ and $(10, -30)$ with radius 15, 10 and 5, respectively. The 50 followers initially spread in the area of $[0, 80] \times [0, 80]$. The time limit is set to 10 seconds, and the constant c is chosen to be 0.3 again. Figure 2 shows how the final performance is. Since the set of the followers who follows leader j during the process is a subset of all the followers, those followers will finally reach the j -th exit.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we used optimal control as a tool to design the moving strategy for a leader agent to guide the follower agents in specific crowd evacuation scenarios. Different types of optimal control problems are formulated under different assumptions inspired from real world applications. Analytic solutions are obtained by using Pontryagin's Minimum Principle and related convex optimization problems. The solutions were also implemented in numerical experiments. Simulations indicated that the optimal control derived theoretically may still be applicable to more complicated models.

Future work will include both theoretical analysis of more complex scenarios and numerical experiments on more realistic settings. Using nonlinear follower dynamic equations or adding other social interactions in the corresponding optimal control problem will be considered. Developing methods

dealing with obstacles and building environment as well as more complex pedestrian behavior is another research direction.

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