An inverse agreement control strategy with application to swarm dispersion

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***Abstract*— We propose an inverse agreement control strategy for multiple kinematic agents that forces the team members to disperse in the workspace in a distributed manner. Both the cases of an unbounded and a cyclic, bounded workspace are considered. In the first case, we show that the closed loop system reaches a configuration in which the minimum distance between any pair of agents is larger than a specific lower bound. It is proved that this lower bound coincides with the agents' sensing radius. In the case of a bounded cyclic workspace, the control law is redefined in order to force the agents to remain within the workspace boundary throughout the closed loop system evolution. Moreover the proposed control design guarantees collision avoidance between the team members in both cases. The results are supported through relevant computer simulations.**

I. Introduction

The emerging use of large-scale multi-robot and multi-vehicle systems in various modern applications has raised recently the need for the design of control laws that force a team of multiple vehicles/robots (from now on called "agents") to achieve various goals. As the number of agents increases, centralized control designs fail to guarantee robustness and are harder to implement than decentralized approaches, which also provide a reduce in the computational complexity of the overall feedback scheme.

Among the various objectives that the control design aims to impose on the multi-agent system, convergence of the agents to a common configuration, also known as the agreement problem, is a design specification that has been extensively pursued recently. Many feedback control schemes that achieve stabilization of the multi-agent team to an agreement point in a distributed manner have been presented recently, see for example [1],[4],[19],[14],[10],[7],[17],[13],[20],[16], for some recent results. Furthermore, the application of motion models of large populations of animals/insects (swarms) in nature to multi-vehicle/robot systems is also a field of extensive research activity in the last few years. Relative results include among others algorithms for swarm aggregation [9] and flocking motion [18],[22],[6].

In this paper we propose a control methodology for swarm dispersion which can be considered as an inverse agreement problem. Each agent follows a flow, whose inverse would lead the multi-agent team to agreement. The proposed control design is distributed, in the sense that each agent has only knowledge of the relative positions of agents located within its sensing zone at each time instant. The sensing zone in this paper is assumed to be a cyclic area around each agent whose radius is common for all agents. The application of this inverse agreement strategy is dispersion of the team members in the workspace, i.e. convergence to a configuration where the minimum distance between the swarm members is bounded from below by a *controllable* lower bound. It is shown that this lower bound coincides with the radius of the sensing zone of the agents in the case of an unbounded workspace. Furthermore, the results are extended in order to take into account the workspace boundary for the case of a cyclic bounded workspace.

Possible applications of the proposed dispersion algorithm include coverage control [5], and optimal placement of a large-scale multi-robot team in a relatively small area [8],[12],[21],[2]. However, in this paper it is also shown that inverse consensus/agreement algorithms can be used to provide solutions to various problems in multi-agent control. This is a topic of probable future research directions.

The rest of the paper is organized as follows: Section II presents the system and describes the problems treated in this paper. The swarm dispersion methodology is presented in Section III. The case of a bounded workspace is treated in Section IV, while simulations that support the presented algorithms are included in Section V. Section VI summarizes the results of this paper and indicates current research efforts.

II. System and Problem Description

Consider a system of N point agents operating in the same workspace $W \subset \mathbb{R}^2$. Let $q_i \in \mathbb{R}^2$ denote the position of agent i. The configuration space is spanned by $q = [q_1^T, \ldots, q_N^T]^T$. The motion of each agent is described by the single integrator:

$$\dot{q}_i = u_i, i \in \mathcal{N} = [1, \ldots, N] \tag{1}$$

where u_i denotes the velocity (control input) for each agent.

We assume that each agent has sense of agents that are found within a circle of radius d around the agent. This circle is called the *sensing zone* of each agent i and the parameter d its *sensing radius*. The sensing radius d is assumed common for all agents. We denote by N_i the subset of $\mathcal{N}$ that includes the agents that agent i can sense at each time instant, i.e.

$$N_i = \{j \in \mathcal{N}, j \neq i : \|q_i - q_j\| \leq d\}$$

The objective of this paper is dispersion of the team members in a distributed manner. We assume that a large number of agents is gathered in a workspace close to each other.

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The goal is to design control laws that force the agents to converge to sufficiently large distances between them, i.e. disperse in the workspace. Specifically, we equip each agent with a repulsive potential with respect to each other agent within its sensing zone. No global knowledge is imposed to any of the team members. This paper's main result states that the closed loop system converges to a configuration where the sensing zone of each agent is empty, i.e. every agent is located at a distance no less than d from every other agent. Moreover, the control design guarantees collision avoidance between the agents. The stability analysis is first performed assuming an unbounded workspace. We then obtain similar results for the case of a bounded workspace.

The dispersion potential function between agents i and j is given by

$$\gamma_{ij}\left(\beta_{ij}\right) = \begin{cases} \frac{1}{2}\beta_{ij},\ 0 \le \beta_{ij} \le c^2 \\ \phi(\beta_{ij}),\ c^2 \le \beta_{ij} \le d^2 \\ h,\ d^2 \le \beta_{ij} \end{cases}$$

where $\beta_{ij} = \|q_i - q_j\|^2$ is the distance between agents i and j. The positive constant scalar parameters c, d, h and the function ϕ are chosen in such a way so that γ_{ij} is everywhere continuously differentiable. In this paper, we choose the following polynomial function: $\phi(x) = a_2 x^2 + a_1 x + a_0$.

The parameters of this function satisfy the differentiability requirement for γ_{ij}, provided that the coefficients satisfy the relations $a_2 = \frac{1}{4(c^2-d^2)}, a_1 = \frac{d^2}{2(d^2-c^2)},, a_0 = \frac{c^4}{4(c^2-d^2)}, h = \frac{d^2+c^2}{4}$. Figure 1 shows a plot of the function γ_{ij} with respect to β_{ij} for $c^2 = 0.56$ and $d^2 = 0.96$. The gradient and the

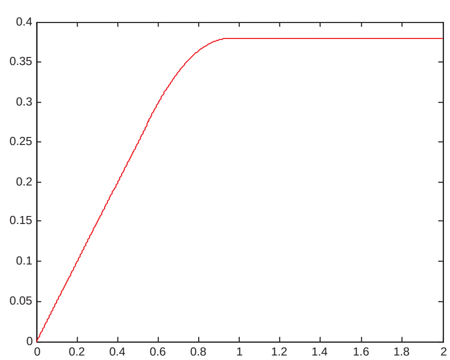

Fig. 1. The function γ_{ij} for $c^2 = 0.56$ and $d^2 = 0.96$.

partial derivative of γ_{ij} are computed by $\nabla\gamma_{ij} = 2\rho_{ij} D_{ij} q$ and $\frac{\partial \gamma_{ij}}{\partial q_i} = 2\rho_{ij}\left(D_{ij}\right)_i q$ where

$$\rho_{ij} \triangleq \frac{\partial \gamma_{ij}}{\partial \beta_{ij}}$$

and the matrices D_{ij},$(D_{ij})_i$, for $i < j$, are given by

$$D_{ij} = \left[\begin{array}{ccccc} & & O_{(i-1)\times N} & & \\ O_{1\times(i-1)} & 1 & O_{1\times(j-i-1)} & -1 & O_{1\times(N-j)} \\ & & O_{(j-i-1)\times N} & & \\ O_{1\times(i-1)} & -1 & O_{1\times(j-i-1)} & 1 & O_{1\times(N-j)} \\ & & O_{(N-j)\times N} & & \end{array}\right] \otimes I_2$$

and $\left(D_{ij}\right)_i =$

$$\left[\begin{array}{ccccc} O_{1\times(i-1)} & 1 & O_{1\times(j-i-1)} & -1 & O_{1\times(N-j)} \end{array}\right] \otimes I_2$$

where $\otimes$ denotes the standard Kronecker product between two matrices [11]. The definition of D_{ij},$(D_{ij})_i$, for $i > j$ is straightforward.

It can easily be shown that $\rho_{ij} > 0$ for $0 < \beta_{ij} < d^2$ and $\rho_{ij} = 0$ for $\beta_{ij} \ge d^2$.

III. Swarm Dispersion control design

A. Tools from Matrix Theory

In this subsection we review some tools from graph theory [3] and matrix analysis [11],[15] that we shall use in the stability analysis of the proposed control framework.

For an undirected graph $\mathcal{G} = (V, E)$ with n vertices we denote by V its set of vertices and by E its set of edges. If there is an edge connecting two vertices i, j, i.e. $(i, j) \in E$, then i, j are called *adjacent*. A *path* of length r from a vertex i to a vertex j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of the graph $\mathcal{G}$, then $\mathcal{G}$ is called *connected*.

The *undirected graph $\mathcal{G} = (V, E)$ corresponding to a real symmetric $n\times n$ matrix* M is a graph with n vertices indexed by $1, \ldots, n$ such that there is an edge between vertices $i, j \in V$ if and only if $M_{ij} \neq 0$, i.e. $(i, j) \in E \Leftrightarrow M_{ij} \neq 0$.

A $n \times n$ real symmetric matrix with non-positive off-diagonal elements and zero row sums is called a *symmetric Metzler* matrix. It is shown in [15] that all the eigenvalues of a symmetric Metzler matrix are non-negative and zero is a trivial eigenvalue. The multiplicity of zero as an eigenvalue of a symmetric Metzler matrix is one (i.e. it is a simple eigenvalue) if and only if the corresponding undirected graph is connected. The trivial corresponding eigenvector is the vector of ones, $\overrightarrow{\mathbf{1}}$. This result has been used in the proof of the consensus algorithm for single integrator kinematic agents presented in [17]. Its usefulness in the present framework is verified in the sequel.

B. Swarm Dispersion with collision avoidance

We propose the following control law

$$u_i = -\sum_{j\in N_i} \frac{\partial\left(1/\gamma_{ij}\right)}{\partial q_i} \Rightarrow$$

$$u_i = -\sum_{j\in N_i}\left(-\frac{1}{\gamma_{ij}^2}\right)\frac{\partial \gamma_{ij}}{\partial q_i} = \sum_{j\in N_i}\frac{2\rho_{ij}}{\gamma_{ij}^2}\left(D_{ij}\right)_i q$$

which can be rewritten as

$$u_i = \sum_{j\neq i}\frac{2\rho_{ij}}{\gamma_{ij}^2}\left(D_{ij}\right)_i q \qquad (2)$$

since $\rho_{ij} = 0$ for $\beta_{ij} > d^2$. We should note that each agent takes into account only the agents within its sensing zone at each time instant. We then have $\dot{q} = 2\left(R_2 \otimes I_2\right) q$, where

$$\left(R_2\right)_{ij} = \begin{cases} \sum\limits_{j\neq i}\frac{\rho_{ij}}{\gamma_{ij}^2}, i = j \\ -\frac{\rho_{ij}}{\gamma_{ij}^2}, i \neq j \end{cases}$$

We consider $V = \sum_i \sum_{j\neq i} \frac{1}{\gamma_{ij}}$ as a candidate Lyapunov function. Its gradient is computed by

$$\nabla V = \sum_i \sum_{j\neq i} \left(-\frac{1}{\gamma_{ij}^2}\right) \nabla \gamma_{ij} = \\ = -\sum_i \sum_{j\neq i} \frac{2\rho_{ij}}{\gamma_{ij}^2} D_{ij} q = -2\left(R_1 \otimes I_2\right) q$$

where the matrix R_1 is given by

$$(R_1)_{ij} = \begin{cases} \sum_{j\neq i} \frac{\rho_{ij}}{\gamma_{ij}^2} + \sum_{j\neq i} \frac{\rho_{ji}}{\gamma_{ji}^2}, i = j \\ -\frac{\rho_{ij}}{\gamma_{ij}^2} - \frac{\rho_{ji}}{\gamma_{ji}^2}, i \neq j \end{cases}$$

We now have $\frac{\rho_{ij}}{\gamma_{ij}^2} = \frac{\rho_{ji}}{\gamma_{ji}^2} \Rightarrow R_1 = 2R_2$.

The time derivative of the candidate Lyapunov function is now calculated as follows

$$\dot{V} = (\nabla V)^T \cdot \dot{q} = \left(-2\left(R_1 \otimes I_2\right) q\right)^T \cdot 2\left(R_2 \otimes I_2\right) q \\ \overset{R_1=2R_2}{\Rightarrow} \dot{V} = -8\left\|\left(R_2 \otimes I_2\right) q\right\|^2 \leq 0 \tag{3}$$

The first result of this section establishes collision avoidance between the team members, as shown in the next Lemma:

Lemma 1: Consider the system of multiple kinematic agents (1) driven by the control law (2) and starting from a feasible set of initial conditions $\mathcal{I}(q) = \{q | \|q_i - q_j\| > 0, \forall i, j \in \mathcal{N}, i \neq j\}$. Then the set $\mathcal{I}(q)$ is invariant for the trajectories of the closed loop system.

Proof: For every initial condition $q(0) \in \mathcal{I}(q)$, the time derivative of V remains non-positive for all $t \geq 0$, by virtue of (3). Hence $V(q(t)) \leq V(q(0)) < \infty$ for all $t \geq 0$. Since $V \to \infty$ if and only if $\|q_i - q_j\| \to 0$ for at least one pair $i, j \in \mathcal{N}$, we conclude that $q(t) \in \mathcal{I}(q)$, for all $t \geq 0$. ♢

Next, we show that the "swarm center" remains constant:

Lemma 2: Consider the system of multiple kinematic agents (1) driven by the control law (2). Define the "swarm center" $\bar{q} \triangleq \frac{1}{N}\sum_{i=1}^{N} q_i$. Then $\bar{q}(t) = \bar{q}(0)$ for all $t \geq 0$.

Proof: We have $\dot{\bar{q}} = \frac{1}{N}\sum_{i=1}^{N} \dot{q}_i = \frac{2}{N}\sum_{i=1}^{N}\sum_{j\in N_i} \frac{2\rho_{ij}}{\gamma_{ij}^2}\left(D_{ij}\right)_i q =$
$= \frac{2}{N}\sum_{i=1}^{N}\sum_{j\in N_i} \frac{2\rho_{ij}}{\gamma_{ij}^2}\left(q_i - q_j\right) = 0$ and the result follows. ♢

By virtue of Lemma 1, collision avoidance is guaranteed. The control design however is also directly related to the final configurations of the swarm members. The main result of this section is summarized in the following Theorem:

Theorem 3: Consider the multi-agent system (1) driven by the control (2) and starting from a set of initial conditions $\mathcal{I}(q) \cup \mathcal{F}(q)$ where $\mathcal{I}(q) = \{q | \|q_i - q_j\| > 0, \forall i, j \in \mathcal{N}, i \neq j\}$ was defined in Lemma 1 and

$$\mathcal{F}(q) = \{q | \|q_i - q_j\| < (N-1) d^*, \forall i, j \in N, i \neq j,\}$$

where $d^* > d$ is chosen arbitrarily. Then the agents reach a static configuration (i.e. all agents eventually stop) which satisfies

$$\|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j.$$

Proof: Since the set of initial conditions coincides with $\mathcal{I}(q)$, we have $q_i(t) \neq q_j(t)$, for all $i, j \in \mathcal{N}, i \neq j$, and for all $t \geq 0$, by virtue of Lemma 1. We take V as a candidate Lyapunov function. V is continuously differentiable within $\mathcal{I}$. Its time derivative is given by (3): $\dot{V} = -8\left\|\left(R_2 \otimes I_2\right) q\right\|^2 \leq 0$. Since by virtue of Lemma 2, the swarm center remains constant, the boundedness of the solutions of the closed loop system can be checked based on the relative positions of the swarm members. Pick $d^* > d$. It is easy to see that since $\rho_{ij} = 0$ whenever $\beta_{ij} > d$, the set $\|q_i - q_j\| \leq (N-1) d^*$ for all $i, j \in \mathcal{N}$ is positively invariant for the trajectories of the closed loop system. By virtue of Lemma 1, $\mathcal{I}(q) \cup \mathcal{F}(q)$ is also positively invariant. Since this set is closed and bounded, we can apply LaSalle's Invariance principle.

By LaSalle's Principle, the trajectories of the closed loop system converge to the largest invariant subset of the set

$$S = \left\{q | \dot{V} = 0\right\} = \{q | \left(R_2 \otimes I_2\right) q = 0\}$$

Note that within S, we have $\dot{q} = u = 2\left(R_2 \otimes I_2\right) q = 0 \Rightarrow u_i = 0$, for all $i \in \mathcal{N}$, i.e. all agents eventually stop.

We show next that the largest invariant subset of S is the set $S_0 = \{q | \rho_{ij} = 0, \forall i, j \in \mathcal{N}, i \neq j\}$. Clearly, S_0 is a subset of S which is invariant for the trajectories of the closed loop system. Suppose now that $\rho_{ij} > 0$ for some pairs of the team members. We denote the undirected graph corresponding to the matrix R_2 by $\mathcal{G}(R_2)$. The assumption that $\rho_{ij} > 0$ for some pairs of agents guarantees that $\mathcal{G}(R_2)$ has at least one edge. The graph $\mathcal{G}(R_2)$ can now be decomposed into its connected components. Note that since the graph is undirected, no vertex can belong to two different components simultaneously. Ignoring the connected components containing only one vertex (i.e. vertices k for which $\rho_{kj} = 0$ for all $j \neq k$), and rearranging the agent indices accordingly, equation $\left(R_2 \otimes I_2\right) q = 0$ can be decomposed into m different equations, each of which corresponds to a different connected component of $\mathcal{G}(R_2)$. Specifically for the connected component containing agents/vertices $\{i_1, i_2, \ldots, i_l\}, i_j \in \mathcal{N}, j = 1, \ldots l$ with $l \leq n$ we have

$$\left(\tilde{R}_2 \otimes I_2\right) \tilde{q} = 0$$

where $\tilde{q} = \left[\begin{array}{ccc} q_{i_1}^T & \ldots & q_{i_l}^T \end{array}\right]^T$ and the $l \times l$ matrix $\tilde{R}_2$ has the same form as R_2 taking into account the set of agents $\{i_1, i_2, \ldots, i_l\}$. By denoting $\tilde{x}, \tilde{y}$ the stack vectors of $\tilde{q}$ in the x, y directions, we have $\left(\tilde{R}_2 \otimes I_2\right) \tilde{q} = 0 \Rightarrow \tilde{R}_2 \tilde{x} = \tilde{R}_2 \tilde{y} = 0$. The symmetric matrix $\tilde{R}_2$ has zero row sums and nonpositive off-diagonal elements, i.e. it is a symmetric Metzler matrix. As mentioned in Section IIIA, the eigenvalues of $\tilde{R}_2$ are nonnegative and zero is the smallest eigenvalue. However, since $\tilde{R}_2$ corresponds to a connected graph (a connected component of $\mathcal{G}(R_2)$), zero is a simple eigenvalue of $\tilde{R}_2$ with trivial corresponding eigenvector the vector of ones, $\overrightarrow{\mathbf{1}}$. Hence equations $\tilde{R}_2 \tilde{x} = \tilde{R}_2 \tilde{y} = 0$ guarantee that both $\tilde{x}, \tilde{y}$ are eigenvectors of $\tilde{R}_2$ belonging to span$\{\overrightarrow{\mathbf{1}}\}$. Thus all elements of $\tilde{q}$ attain the same value, implying that all agents converge to a common point at steady state. However this is impossible, since, due to the invariance of $\mathcal{I}(q)$, no trajectory of the closed loop system starting from $\mathcal{I}(q)$ can ever leave this set, i.e. $q_i(t) \neq q_j(t)$ for all $t \geq 0$. We conclude that the largest invariant subset of S is S_0. Since $\rho_{ij} = 0$ only for $\|q_i - q_j\| \geq d$, the proof is complete. $\diamondsuit$

Hence at steady state, the closed loop system converges to a configuration in which each agent is located at a distance no less than d from every other agent in the group. This reveals an important geometric property of the system at steady state. Since any pair of agents is located at least at a distance d from each other, each agent occupies a disc of radius $d/2$ in which no other agent is present. In other words, the agents are dispersed to n disjoint circular regions of radius $d/2$.

IV. The Bounded Workspace Case

The previous case proposed a dispersion algorithm for multiple kinematic agents in an unbounded workspace. In practical applications such as coverage control and sensor deployment the problem is to redefine the algorithm in order to take into account the workspace boundary. In this paper, we consider the case of a cyclic boundary of radius R_W. However, the proposed design is applicable to any convex workspace. The purpose is to construct an inverse agreement control law that forces the dispersing agents to remain within the workspace limits.

A similar potential field to the one for the inter-agent dispersion potential is used for the agent-boundary repulsion potential. Copying with the limited sensing capabilities of the agents, the repulsive potential of each agent with respect to the boundary of the workspace is given by

$$\gamma_{ib}\left(\beta_{ib}\right) = \begin{cases} \frac{1}{2}\beta_{ib},\ 0 \leq \beta_{ib} \leq c_b^2 \\ \varphi_b\left(\beta_{ib}\right),\ c_b^2 \leq \beta_{ib} \leq d_b^2 \\ h_b,\ d_b^2 \leq \beta_{ib} \end{cases}$$

where $\beta_{ib} = \|q_i - q_{i,\min}\|^2$, $d_b < d$ and $q_{i,\min} = \arg\min_{q \in \partial W} \|q_i - q\|^2$. Note that $q_{i,\min}$ is continuous for all i due to the convexity of W. The positive scalars h_b, c_b and the function φ_b are defined in such a way so that γ_{ib} is rendered everywhere continuously differentiable. Each agent has to have knowledge of the workspace boundary only when located at a distance smaller than d_b from it.

The control law for agent i is now redefined as

$$u_i = -\sum_{j \in N_i} \frac{\partial\left(1/\gamma_{ij}\right)}{\partial q_i} - \frac{\partial\left(1/\gamma_{ib}\right)}{\partial q_i}$$

Using the notation $\rho_{ib} = \frac{\partial \gamma_{ib}}{\partial \beta_{ib}}$ the control law can be rewritten as

$$u_i = \sum_{j \neq i} \frac{2\rho_{ij}}{\gamma_{ij}^2}\left(D_{ij}\right)_i q + 2\frac{\rho_{ib}}{\gamma_{ib}}\left(q_i - q_{i,\min}\right) \tag{4}$$

since

$$\frac{\partial\left(1/\gamma_{ib}\right)}{\partial q_i} = -\frac{1}{\gamma_{ib}^2}\frac{\partial \gamma_{ib}}{\partial q_i} = -2\frac{\rho_{ib}}{\gamma_{ib}}\left(q_i - q_{i,\min}\right)$$

It should be noted that $\rho_{ib} = 0$ for $\beta_{ib} > d_b^2$ and $\rho_{ib} > 0$ for $\beta_{ib} \leq d_b^2$. In stack vector form we then have

$$\dot{q} = 2\left(R_3 \otimes I_2\right) q - 2\left(R_4 \otimes I_2\right) q_{\min}$$

where

$$R_3 = R_2 + \mathrm{diag}\left\{\frac{\rho_{1b}}{\gamma_{1b}^2}, \ldots, \frac{\rho_{Nb}}{\gamma_{Nb}^2}\right\}$$

and

$$R_4 = \mathrm{diag}\left\{\frac{\rho_{1b}}{\gamma_{1b}^2}, \ldots, \frac{\rho_{Nb}}{\gamma_{Nb}^2}\right\}$$

We also denote by $q_{\min}$ the stack vector of all q_{min}^i. Similarly to the case of an unbounded workspace, using

$$V_b = \sum_i \sum_{j \neq i} \frac{1}{\gamma_{ij}} + \sum_i \frac{1}{\gamma_{ib}}$$

as a candidate Lyapunov function and computing its gradient with respect to q we get

$$\nabla V_b = -4\left(R_3 \otimes I_2\right) q + 4\left(R_4 \otimes I_2\right) q_{\min}$$

The time derivative of V_b is now given by

$$\dot{V}_b = \left(\nabla V_b\right)^T \cdot \dot{q} = -8\left\|\left(R_3 \otimes I_2\right) q - \left(R_4 \otimes I_2\right) q_{\min}\right\|^2 \leq 0 \tag{5}$$

We first show that the interior of the workspace is a positively invariant set for the trajectories of the closed loop system:

Lemma 4: Consider the multi-agent system (1) driven by the control (4) and starting from the set of initial conditions $\mathcal{I}(q) \cap \mathcal{J}(q)$ where $\mathcal{J}(q) = \left\{q | q \in \mathrm{int}(W) \triangleq W \backslash \partial W\right\}$ coincides with the interior of the workspace and $\mathcal{I}(q)$ was defined previously. Then $\mathcal{I}(q) \cap \mathcal{J}(q)$ is invariant for the trajectories of the closed loop system.

Proof: The invariance of $\mathcal{I}(q)$ was shown in Lemma 1. Similar arguments are used to show the invariance of $\mathcal{J}(q)$. For every initial condition $q(0) \in \mathcal{I}(q) \cap \mathcal{J}(q)$, the time derivative of V_b remains non-positive for all $t \geq 0$, by virtue of (5). Hence $V_b(q(t)) \leq V_b(q(0)) < \infty$ for all $t \geq 0$. Since $V_b \to \infty$ whenever $q_i \to q_{i,\min}$ for at least one agent $i \in \mathcal{N}$, and the latter implies $q \to \partial W$, we conclude that $q(t) \in \mathcal{J}(q)$, for all $t \geq 0$. $\diamondsuit$

Thus, if the agents start within the interior of the workspace, they are forced to remain within it. Furthermore, Lemma 1 still holds and hence collisions are avoided. Similar convergence results can now be derived from the stability analysis held in the previous sections. We first formally state that the agents reach a configuration where $u_i = 0$ for all i:

Corollary 5: Consider the system of multiple agents (1) driven by the control law (4) and starting from the set of initial conditions $\mathcal{I}(q) \cap \mathcal{J}(q)$. Then the system reaches a configuration in which $u = 0$, i.e. $u_i = 0$ for all $i \in \mathcal{N}$.

Proof: The set $\mathcal{J}(q)$ is closed and bounded for the trajectories of the closed loop system, by virtue of Lemma 4. From (5) we know that $\dot{V}_b$ is negative semidefinite. By LaSalle's Invariance Principle, the trajectories of the closed loop system reach the largest invariant subset of the set

$$S_b = \left\{q | \dot{V}_b = 0\right\} = \left\{q | \left(R_3 \otimes I_2\right) q - \left(R_4 \otimes I_2\right) q_{\min} = 0\right\}$$

Within S_b, we have $\dot{q} = u = 2\left(R_3 \otimes I_2\right) q - 2\left(R_4 \otimes I_2\right) q_{\min} = 0$, where u is the stack vector of u_i's. Hence $u_i = 0$ for all $i \in \mathcal{N}$. $\diamondsuit$

We now proceed to show that the control law is related to the final relative positions of the agents in a manner similar to the unbounded case. From the proof of Corollary 5 we derive that the system converges to the largest invariant subset of the set S_b. Please note that the result of Lemma 4 holds for arbitrarily small c_b, d_b. For $c_b, d_b \to 0$, we have that either $q_i \to q_{i,\min}$, or $\rho_{ib} \to 0$, for those agents that do not satisfy the condition $q_i \to q_{i,\min}$. Thus, in this case

$$\begin{aligned}
&(R_3 \otimes I_2)\, q - (R_4 \otimes I_2)\, q_{\min} = \\
&= (R_2 \otimes I_2)\, q - (R_4 \otimes I_2)\,(q - q_{\min}) \\
&= (R_2 \otimes I_2)\, q - \left(\left(\text{diag}\left\{\frac{\rho_{1b}}{\gamma_{1b}^2}, \ldots, \frac{\rho_{Nb}}{\gamma_{Nb}^2}\right\}\right) \otimes I_2\right)(q - q_{\min}) \\
&= (R_2 \otimes I_2)\, q - \\
&- \left[\begin{array}{ccc} \frac{\rho_{1b}}{\gamma_{1b}^2}(q_1 - q_{1,\min})^T & \ldots & \frac{\rho_{Nb}}{\gamma_{Nb}^2}(q_N - q_{N,\min})^T \end{array}\right]^T \\
&= (R_2 \otimes I_2)\, q
\end{aligned}$$

since for each $i \in \mathcal{N}$, we have either $q_i \to q_{i,\min}$, or $\rho_{ib} \to 0$, for $c_b, d_b \to 0$ as discussed above.

Therefore the set S_b coincides with the set S of the proof of Theorem 3. As proved in that Theorem, the largest invariant subset within S is the set

$$\begin{aligned}
S_0 &= \{q | \rho_{ij} = 0, \forall i, j \in N, i \neq j\} = \\
&= \{q | \, \|q_i - q_j\| \geq d, \forall i, j \in N, i \neq j\}
\end{aligned}$$

Hence the system reaches a configuration in which all agents remain within the workspace bounds and each agent is located at a distance no less than d from every other agent, *provided that such configuration exists within the workspace bounds*. This result is formally stated in the next Theorem:

Theorem 6: Consider the multi-agent team (1) driven by the control law (4) and starting from the set of initial conditions $\mathcal{I}(q) \cap \mathcal{J}(q)$. Assume furthermore that the set

$$B(q) = \{q \in \text{int}(W) \,|\, \|q_i - q_j\| \geq d, \forall i, j \in N, i \neq j\}$$

is nonempty. Then the system reaches a configuration in which all agents remain in the interior of the workspace, and $\|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j$.

Remark 1: Similarly to the unbounded case, the non-emptiness of the set $B(q)$ corresponds to a situation where each agent occupies a disc of $d/2$ at steady state. Whenever the set $B(q)$ is empty, i.e. there does not exist a configuration in the interior of the workspace such that $\|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j$, the workspace is not large enough to fulfill the above geometric condition, and the system converges to a configuration that minimizes the cost function V_b, respecting the constraint imposed by Lemma 4 that the agents are forced to remain within the workspace boundary. In essence, some of the $d/2$-discs may overlap. This will be visualized via a specific example in the simulations section.

Remark 2: The results can be extended to the case where the workspace W is an arbitrary convex region.

V. Simulations

To support the results presented in the previous paragraphs, we provide a series of computer simulations.

In the first simulation, nine single integrator agents navigate under the control law (2). Screenshots I-III in Figure 2 show the evolution of the closed loop system in time. The agents are located at their initial positions in the screenshot I. Collision avoidance is fulfilled, due to the proposed control design. The agents disperse in the workspace and eventually stop in screenshot III. Screenshot IV depicts the final positions of the swarm members. Each agent occupies a disc of radius $d/2$. These discs are visualized in the last screenshot by the large discs whose center is the corresponding agent. By virtue of Theorem 3, the large discs are disjoint.

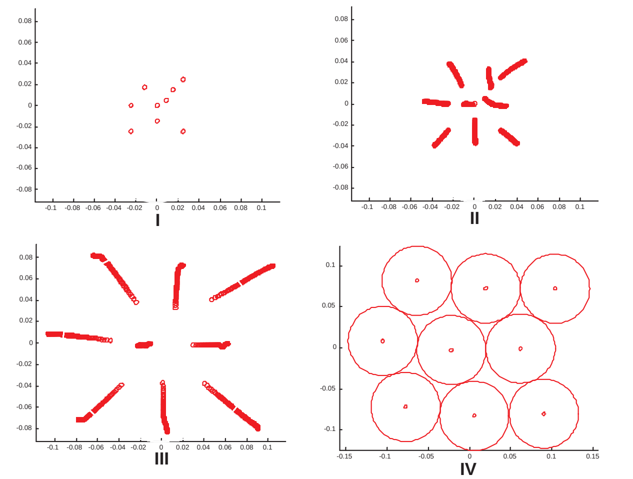

Fig. 2. Swarm dispersion for nine single integrator agents. The agents disperse in the workspace and eventually occupy nine disjoint discs of radius $d/2$, one for each agent.

In the second simulation of Figure 3, we have again nine single integrator agents navigating under the control law (4). The workspace radius is given by $R_W = 18 * d$. The agents start from an initial condition where they are aggregated near the workspace center. Some agents approach the workspace boundary and are forced to remain within it due to the existence of the repulsive potential on the workspace boundary. Collision avoidance is fulfilled throughout the closed loop system evolution. The workspace is large enough to allow the agents to occupy nine disjoint discs of radius $d/2$ at steady state, i.e. the set B of Theorem 6 is nonempty. This is depicted in the last screenshot of Figure 3.

By reducing the workspace radius of the previous simulation, the set B of Theorem 6 is rendered empty, i.e. there does not exist a configuration in the interior of the workspace such that the condition $\|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j$ is fulfilled. This is the case in Figure 4, where we have set $R_W = 16 * d$. The agents disperse again within the limits of the workspace, avoiding collisions with each other. In the last screenshot, some of the big circles of radius $d/2$ surrounding the agents overlap, since the set B is now empty.

VI. Conclusions

We proposed an inverse agreement control strategy for multiple kinematic agents that forces the team members to disperse in the workspace in a distributed manner. Both the cases of an unbounded and a cyclic, bounded workspace

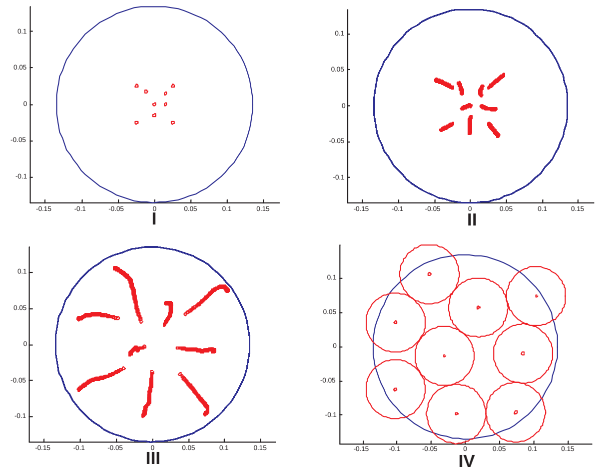

Fig. 3. Swarm dispersion for nine single integrator agents in a bounded workspace. The workspace is large enough to allow the agents to occupy nine disjoint discs of radius $d/2$ at steady state. Agents are forced to remain within the workspace boundary.

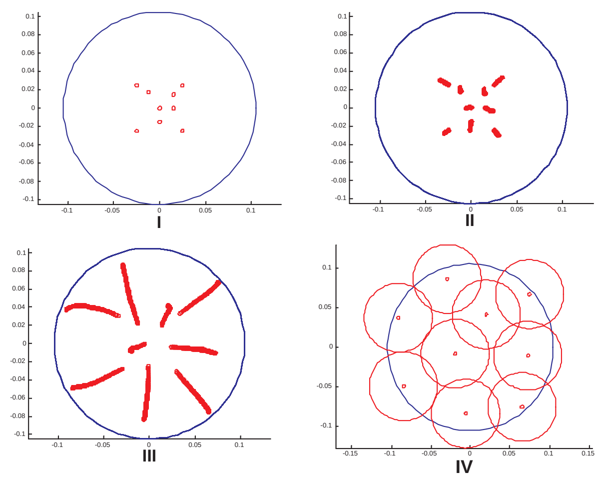

Fig. 4. Swarm dispersion for nine single integrator agents in a bounded workspace. The workspace is not large enough to allow the agents to occupy nine disjoint discs of radius $d/2$ at steady state. These discs are overlapping in screenshot IV.

were considered. In the first case, we showed that the closed loop system reaches a configuration in which the minimum distance between any pair of agents is larger than a specific lower bound. It was proven that this lower bound coincides with the agents' sensing radius. In the case of a bounded cyclic workspace, the control law was redefined in order to force the agents to remain within the workspace boundary throughout the closed loop system evolution. Moreover the proposed control design guaranteed collision avoidance between the team members in both cases. The results were supported through a series of computer simulations.

Current research involves exploring the relation of the sensing radius, the number of agents and the radius of the workspace with the emptiness of the set B of Theorem 6. Furthermore, the results should be extended to take into account second integrator and nonholonomic models of agents' motion. Finally, we aim to apply the results to a real experimental multi-robot testbed.

VII. Acknowledgements

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