Further results on formation infeasibility and velocity alignment

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***Abstract*— In previous work, we deduced that formation infeasibility results in velocity alignment in multi-agent systems with bidirectional communication topology and single integrator as well as nonholonomic kinematics. This paper contains additional results regarding the connection between formation infeasibility and velocity alignment. In particular, we obtain an analytic expression for the resulting common velocity vector in the case of formation infeasibility and extend the results to the case of unidirectional communication topology. The results are then extended to the case of a leader-follower architecture in which the followers are not aware of a global objective while the leaders are responsible for driving the team to the interior of a desired leader formation. When this formation is infeasible, we show that both leaders and followers attain a common velocity vector, of which an analytic expression is also provided. Computer simulations support the derived results.**

I. Introduction

Decentralized control of multi-agent systems is a field that is currently being rapidly explored in the robotics and control communities due to the fact that decentralized approaches respect the limited communication and sensing constraints of the agents and moreover provide a reduction in the computational complexity of the applied algorithms.

Among the various problems arising in the field of multi-agent systems, formation convergence and achievement of flocking behavior have been pursued extensively in the last few years. The main feature of formation control is the co-operative nature of the equilibria of the system. Agents must converge to a desired configuration encoded by the inter-agent relative positions. Many feedback control schemes that achieve formation stabilization to a desire formation in a distributed manner have been proposed recently (see for example [12],[14],[13],[11] for some recent results). The agreement problem, where agents must converge to the same point in the state space [15],[17],[1], [6] is also relevant. On the other hand, flocking behavior involves, among others, convergence of the velocity vectors and orientations of the agents to a common value at steady state; relevant contributions include [10], [19],[16].

In the authors' previous work [3],[5],[4] a connection between formation infeasibility and velocity alignment in kinematic multi-agent systems was established. Specifically, it was shown that if the formation encoded by the desired inter-agent relative positions was infeasible, then the agents would reach a common velocity vector under the initial formation control law. The results were first established for sphere world agents in [3],[5] while in [4] it was shown that formation infeasibility results in velocity alignment in the nonholonomic unicycle case as well. This paper contains additional results regarding this connection. We first obtain an analytic expression for the resulting common velocity vector in the case of formation infeasibility and extend the results to the case of unidirectional communication topology, for both the cases of single integrator and nonholonomic unicycle kinematic agents. The results are then extended to the case of a leader-follower architecture in which the followers are not aware of a global objective while the leaders' goal is to drive the team to the interior of a desired leader formation. When this formation is infeasible, we show that both leaders and followers attain a common velocity vector, of which an analytic expression is also provided.

The paper is organized as follows: in section II we present the problem formulation and review some algebraic graph theoretic tools used in the sequel. In Section III we consider the leaderless case and provide an analytic expression of the common velocity vector in the case of formation infeasibility for both the cases of nonholonomic and single integrator agents. The results are shown to hold for the case of directed graphs. Section IV contains the leader-follower case. In section V simulation results are presented while a summary of the results of this paper is given in section VI.

II. System and Mathematical Preliminaries

A. System and Problem Definition

Consider a system of N point agents operating in the same planar workspace $W \subset \mathbb{R}^2$. Let $q_i \in \mathbb{R}^2$ denote the position of agent i. We denote by $q = [q_1^T, \ldots, q_N^T]^T$ the stack vector of all agents positions. Let $q_i = [x_i, y_i]^T \in \mathbb{R}^2$ denote the position of agent i. Each of the N mobile agents has a specific orientation θ_i with respect to the global coordinate frame. The orientation vector of the agents is represented by $\theta = [\theta_1 \ldots \theta_N]^T$. The configuration of each agent is represented by $p_i = \begin{bmatrix} q_i^T & \theta_i^T \end{bmatrix}^T \in \mathbb{R}^2 \times (-\pi, \pi]$.

Each agent's objective is to converge to a desired relative configuration with respect to a certain subset of the rest of the team, in a manner that will lead the whole team to a desired formation. Specifically, each agent is assigned with a specific subset N_i of the rest of the team, called agent i's *communication set* with which it can communicate in order to achieve the desired formation. The desired formation can be encoded in terms of a directed graph, from now on called the *formation graph* $G = \{V, E\}$, whose set of vertices $V =$

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$\{1, ..., N\}$ are indexed by the team members, and whose set of edges $E = \{(i,j) \in V \times V | j \in N_i\}$ contains pairs of vertices that represent inter-agent formation specifications. A vector $c_{ij} \in \mathbb{R}^2$ is associated to each edge $(i,j) \in E$, in order to specify the desired inter-agent relative positions in the final formation configuration.

The objective of each agent i is to be stabilized in a desired relative position c_{ij} with respect to each member j of N_i. Each agent has only knowledge of the relative displacement of agents that belong to its neighboring set. Formation feasibility is defined in the following manner:

Definition 1: The formation configuration is called *feasible* if the set $\Phi \triangleq \{q \in W \, | q_i - q_j = c_{ij}, \, \forall (i,j) \in E\}$ of feasible formation configurations is nonempty. □

Whenever the latter does not hold, the formation configuration is called *infeasible*. For example, formation infeasibility occurs whenever there are *conflicting objectives* between any pair of agents that have an undirected edge between one another in the formation graph, i.e. whenever $c_{ij} \neq -c_{ji}, \forall i,j \in \mathcal{N}, (i,j),(j,i) \in E$. Please note however that the conflicting inter-agent objectives condition is only sufficient for formation infeasibility. For example, for an undirected formation graph and a four agent team with communication sets given by $N_1 = \{2,3,4\}, N_2 = \{1,4\}, N_3 = \{1\}, N_4 = \{1,2\}$, it is easily seen that if the desired interagent positions are $c_{12} = -c_{21} = c_{14} = -c_{41} = c_{24} - c_{42} = [-1, 0]^T, c_{13} = -c_{31} = [0, -1]^T$ then the desired formation is nowhere realizable, i.e. infeasible.

B. Tools from Algebraic Graph Theory

In this subsection we review some tools from algebraic graph theory that we shall use in the stability analysis of the next sections. The following can be found in any standard textbook on algebraic graph theory(e.g. [9]).

For a graph G with n vertices the *adjacency matrix* $A = A(G) = (a_{ij})$ is the $n \times n$ matrix given by $a_{ij} = 1$, if $(i,j) \in E$ and $a_{ij} = 0$, otherwise. If there is an edge connecting two vertices i,j, i.e. $(i,j) \in E$, then i,j are called *adjacent*. When there is an orientation defined on each edge $(i,j) \in E$, the graph is called *directed* otherwise it is called *undirected*. A *path* of length r from a vertex i to a vertex j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent and that respects the orientation of the edges in the case of a directed graph. If there is a path between any two vertices of the graph G, then G is called *strongly connected* in the case of directed, and simply *connected* in the case of undirected graphs. A directed graph has a *spanning tree* if there exists at least one vertex to which there exists a path from all other vertices respecting the edge orientation. A directed graph is called *balanced* [17] when $\sum_j a_{ij} = \sum_j a_{ji}$, for all i. The *degree* d_i of vertex i is defined as the number of its neighboring vertices, i.e. $d_i = \#j : (i,j) \in E$. Let Δ be the $n \times n$ diagonal matrix of d_i's. The (combinatorial) *Laplacian* of G is the matrix $\mathcal{L} = \Delta - A$. For an undirected graph the Laplacian matrix is symmetric positive semidefinite. When the directed graph has a spanning tree, the Laplacian has a single zero eigenvalue and the corresponding eigenvector is the vector of ones, $\overrightarrow{\mathbf{1}}$. This result was established in [13],[18]. For the case of undirected graphs, a necessary and sufficient condition for zero to be a simple eigenvalue of the Laplacian matrix, is that the undirected graph is connected.

III. The Leaderless Case

We now consider the case where no leaders are present in the multi-agent team. We first review the results for the single integrator case and proceed to provide an analytic expression for the common velocity vector in the case of formation infeasibility. The results are then extended to the cases of directed graphs and nonholonomic agents.

A. Previous Results

In this section, we consider N single integrator agents whose motion obeys the model:

$$\dot{q}_i = u_i, i \in \mathcal{N} = [1, \ldots, N] \tag{1}$$

We first review the results regarding the connection between formation infeasibility and velocity alignment that first appeared in [3],[5]. We note that in these papers we included the collision avoidance objective between the agents as a control design goal. This specification is omitted in this paper as the emphasis is given in the resulting equilibria of the system, but we point out that the results presented here can be extended to include the collision avoidance objective.

The formation objective for agent i is encoded by the cost function $\gamma_i = \frac{1}{2} \sum_{j \in N_i} \|q_i - q_j - c_{ij}\|^2$. The control law for agent i is given by:

$$u_i = -\frac{\partial \gamma_i}{\partial q_i} \tag{2}$$

The analysis of [3],[5] involves the case of undirected formation graphs and the main result is summarized as:

Theorem 1: Let the open loop system (1) be driven by the control law (2). Assume that the undirected formation graph is connected. Then the following hold:

1) If the desired formation is feasible in the sense of Definition 1, then the system reaches the desired formation at steady state.
2) The system reaches a configuration in which all agents have the same velocities and orientations even if the formation feasibility assumption does not hold. □

In the sequel, an analytic expression of the resulting common velocity vector in the formation infeasibility case is derived and the results are extended to the case of directed graphs.

B. The common velocity vector

In both subcases of Theorem 1, the agents converge to a common velocity, which is zero in the formation feasibility case, and not necessarily zero otherwise. We now derive an analytic expression for this common velocity vector.

The partial derivative of γ_i with respect to q_i is given by $\frac{\partial \gamma_i}{\partial q_i} = \sum_{j \in N_i} (q_i - q_j - c_{ij}) = \sum_{j \in N_i} (q_i - q_j) + c_{ii}$, where

$c_{ii} \triangleq -\sum_{j\in N_i} c_{ij}$. Let $\mathcal{L}$ denote the Laplacian of the formation graph. The dynamics of the closed loop system are given by

$$\dot{q} = -\left[\begin{array}{ccc} \left(\frac{\partial\gamma_1}{\partial q_1}\right)^T & \dots & \left(\frac{\partial\gamma_N}{\partial q_N}\right)^T \end{array}\right]^T - (Lq + c_l) \quad (3)$$

where $L = \mathcal{L} \otimes I_2$, $c_l \triangleq [c_{11}, \dots, c_{NN}]^T$, and $\otimes$ denotes the standard Kronecker product between two matrices. The common velocity vector is given by the following Theorem:

Theorem 2: Let the system (1) be driven by the control law (2). Assume that the undirected formation graph is connected. Then the agents attain a common velocity vector $\dot{q}_i = \dot{q}^*$ for all $i \in \mathcal{N}$ which is given by $\dot{q}^* = -\frac{1}{N}\sum_i c_{ii}$. □

Proof: The fact that the agents reach a common constant velocity is derived from Theorem 1. Denoting this common velocity by $\dot{q}^*$ and using the notation $\tilde{c}_l = [\dot{q}^*, \dots, \dot{q}^*]^T$,, equation (3) yields $\dot{q} = -(Lq + c_l) = \tilde{c}_l$. Hence $Lq + c_l = -\tilde{c}_l \Rightarrow Lq = -c_l - \tilde{c}_l$. The fact that the formation graph is undirected implies that the vectors that belong to the range of the corresponding Laplacian matrix have zero row sums. Thus, $Lq = -c_l - \tilde{c}_l \Rightarrow \sum_i (c_{ii} + \dot{q}^*) = 0 \Rightarrow \dot{q}^* = -\frac{1}{N}\sum_i c_{ii}$, and this concludes the proof. ♢

An immediate corollary of Theorem 2 is the fact that the norm of the common velocity vector is given by $\|\dot{q}^*\| = \frac{1}{N}\left\|\sum_i c_{ii}\right\|$. Hence the orientation and volume of the velocity of the resulting flock are completely determined by the number N of the team members and the term $\sum_i c_{ii}$.

C. The case of a directed formation graph

In this section, we show that a result similar to that of Theorem 1 holds also for a directed formation. In particular, the following extension of the undirected case holds:

Theorem 3: Let the open loop system (1) be driven by the control law (2). Assume that the directed formation graph has a spanning tree. Then the following hold:

1) If the desired formation is feasible in the sense of Definition 1, then the system reaches the desired formation at steady state.
2) The system reaches a configuration in which all agents have the same velocities and orientations even if the formation feasibility assumption does not hold. □

Proof: 1) Since in this case the formation is feasible, there exists at least one configuration in the state space that realizes the desired formation. For all $i \in \mathcal{N}$, let c_i denote the configuration of agent i in the desired formation configuration with respect to the global coordinate frame. It is then obvious that $c_{ij} = c_i - c_j \ \forall (i,j) \in E$ for all possible desired final formations. Using now the notation $\tilde{q}_i = q_i - c_i$ for all $i \in \mathcal{N}$, we have $\tilde{q}_i = q_i - c_i \Rightarrow \dot{\tilde{q}}_i = \dot{q}_i$ and thus $\dot{\tilde{q}}_i = -\sum_{j\in N_i}(q_i - q_j - c_{ij}) = -\sum_{j\in N_i}(q_i - q_j - (c_i - c_j)) = -\sum_{j\in N_i}(\tilde{q}_i - \tilde{q}_j)$. In stack vector form we then have $\dot{\tilde{q}} = -L\tilde{q}$, where $L = \mathcal{L} \otimes I_2$ and $\mathcal{L}$ the Laplacian of the formation graph. Since the directed formation graph has (at least one) spanning tree, the elements of the vector $\tilde{q}$ attain a common value at steady state, i.e. $\tilde{q}_i = \tilde{q}_j$ for all $i, j \in \mathcal{N}$. Hence $\tilde{q}_i = \tilde{q}_j \Rightarrow q_i - q_j = c_{ij}$ for all $(i,j) \in E$ and the agents reach the desired formation.

2) We have $\dot{q} = -(Lq + c_l) \Rightarrow \ddot{q} = -L\dot{q}$ at steady state. Using the arguments of [13], we deduce that since the directed formation graph has (at least one) spanning tree, the elements of the vector $\dot{q}$ have a common value at steady state, i.e. $\dot{q}_i = \dot{q}^*$ for all $i \in \mathcal{N}$. Now since $\ddot{q} = -L\dot{q} = 0$, the common velocity $\dot{q}^*$ is also constant. ♢

The common velocity vector can be obtained in the directed graph case as well with the additional assumption that the formation graph is balanced:

Theorem 4: Let the open loop system (1) be driven by the control law (2). Assume that the directed formation graph has a spanning tree and is balanced. Then the agents attain a common velocity vector $\dot{q}_i = \dot{q}^*$ for all $i \in \mathcal{N}$ which is given by $\dot{q}^* = -\frac{1}{N}\sum_i c_{ii}$. □

Proof: From Theorem 3, the agents attain a common velocity. Denote by $\tilde{c}_l = [\dot{q}^*, \dots, \dot{q}^*]^T$ the stack vector of all common velocities at steady state, and by $(Lq)_i$ the elements of the $2N \times 1$ vector Lq corresponding to agent i. Then $\sum_i (Lq)_i = \sum_i \sum_{j\in N_i}(q_i - q_j) = \sum_i\sum_j a_{ij}(q_i - q_j) = 0$, since $\sum_j a_{ij} = \sum_j a_{ji}$ for all $i \in \mathcal{N}$ in a balanced graph. Now, since at steady state $\dot{q} = -(Lq + c_l) = \tilde{c}_l$, we get $Lq = -c_l - \tilde{c}_l \Rightarrow \sum_i (c_{ii} + \dot{q}^*) = 0 \Rightarrow \dot{q}^* = -\frac{1}{N}\sum_i c_{ii}$. ♢

D. Nonholonomic Agents

In this section we consider the case of N nonholonomic unicycle type agents. In particular, agent motion is now described by the following nonholonomic kinematics:

$$\begin{array}{l} \dot{x}_i = u_i \cos\theta_i \\ \dot{y}_i = u_i \sin\theta_i \\ \dot{\theta}_i = \omega_i \end{array}, i \in \mathcal{N} = [1, \dots, N] \quad (4)$$

where u_i, ω_i denote the translational and rotational velocity of agent i, respectively.

We first review the results obtained in [4]. The proposed control law for each nonholonomic agent i has the form:

$$\begin{array}{l} u_i = -\mathrm{sgn}\left\{\frac{\partial\gamma_i}{\partial x_i}\cos\theta_i + \frac{\partial\gamma_i}{\partial y_i}\sin\theta_i\right\} \cdot \left(\left(\frac{\partial\gamma_i}{\partial x_i}\right)^2 + \left(\frac{\partial\gamma_i}{\partial y_i}\right)^2\right)^{1/2} \\ \omega_i = \dot{\theta}_{nh_i} - (\theta_i - \theta_{nh_i}) \end{array} \quad (5)$$

where $\theta_{nh_i} = \arctan 2\left(\frac{\partial\gamma_i}{\partial y_i}, \frac{\partial\gamma_i}{\partial x_i}\right)$. The function $\arctan 2(x, y)$ that is used is the same as the arc tangent of the two variables x and y with the distinction that the signs of both arguments are used to determine the quadrant of the result. We have $\arctan 2(0,0) = 0$ by definition. It should also be pointed out that the time derivative of θ_{nh_i} is not defined at $\frac{\partial\gamma_i}{\partial x_i} = \frac{\partial\gamma_i}{\partial y_i} = 0$. In implementation, one can use the modification of the nonholonomic angle used in [7]:

$$\hat{\theta}_{nh_i} = \left\{\begin{array}{l} \theta_{nh_i}, \mathrm{if}\theta_{nh_i} > \varepsilon \\ \frac{\theta_{nh_i}\left(-2\rho_i^3 + 3\varepsilon\rho_i^2\right) + \theta_r\left(-2(\varepsilon-\rho)^3 + 3\varepsilon(\varepsilon-\rho)^2\right)}{\varepsilon^3}, \mathrm{if}\theta_{nh_i} \le \varepsilon \end{array}\right. \quad (6)$$

where $\rho_i = \left(\left(\frac{\partial \gamma_i}{\partial x_i} \right)^2 + \left(\frac{\partial \gamma_i}{\partial y_i} \right)^2 \right)^{1/2}$ and ε is chosen arbitrarily small. The main result of [4] is summarized as:

Theorem 5: Let the multi-agent nonholonomic system (4) be driven by the control law (5). Assume that the undirected formation graph is connected. Then the following hold:

1) If the desired formation is feasible in the sense of Definition 1, then the system reaches the desired formation with zero orientation at steady state.
2) The system reaches a configuration in which all agents have the same velocities and orientations even if the formation feasibility assumption does not hold. □

Similarly to the single integrator case, the agents converge to a common velocity, which is zero if the formation is feasible, and not necessarily zero otherwise. An analytic expression for this common velocity, which is shown to be the same as in the single integrator case, is now derived.

The angular velocity control law implies that θ_i is aligned with θ_{nh_i} as $t \to \infty$. The closed loop kinematics for the x, y-cofficients then become $\dot{x}_i = u_i \cos\theta_{nh_i} = -\mathrm{sgn}\left\{\gamma_{xi}\cos\theta_{nh_i} + \gamma_{yi}\sin\theta_{nh_i}\right\}\gamma_{xi}$ and $\dot{y}_i = u_i \sin\theta_{nh_i} = -\mathrm{sgn}\left\{\gamma_{xi}\cos\theta_{nh_i} + \gamma_{yi}\sin\theta_{nh_i}\right\}\gamma_{yi}$, where for simplicity we used the notation $\gamma_{xi} = \frac{\partial \gamma_i}{\partial x_i}, \gamma_{yi} = \frac{\partial \gamma_i}{\partial y_i}$. But since by definition of θ_{nh_i} we have $\gamma_{xi}\cos\theta_{nh_i} + \gamma_{yi}\sin\theta_{nh_i} \geq 0$, at steady state the previous equations yield:

$$\dot{x}_i = -\gamma_{xi}, \dot{y}_i = -\gamma_{yi} \tag{7}$$

for $i \in \mathcal{N} = \{1, ..., N\}$. Note now that since $\begin{bmatrix} \frac{\partial \gamma_i}{\partial x_i} & \frac{\partial \gamma_i}{\partial y_i} \end{bmatrix}^T = \frac{\partial \gamma_i}{\partial q_i} = \sum\limits_{j \in N_i} (q_i - q_j) + c_{ii}$, equation (7), written in stack vector form is equivalent to

$$\dot{q} = \begin{bmatrix} -\frac{\partial \gamma_1}{\partial q_1} & \ldots & -\frac{\partial \gamma_N}{\partial q_N} \end{bmatrix}^T = -\left(Lq + c_l\right) \tag{8}$$

Hence the nonholonomic system behaves as in the single integrator case in the velocity space. The previous discussion is summarized in the following Theorem:

Theorem 6: Let the multi-agent system (4) be driven by (5). Assume that the undirected formation graph is connected. Then the agents attain a common velocity vector $\dot{q}_i = \dot{q}^*$ for all $i \in \mathcal{N}$ which is given by $\dot{q}^* = -\frac{1}{N}\sum\limits_i c_{ii}$. □

In essence, the same comments at the end of Section III-B hold for the nonholonomic case as well.

IV. Multiple Leaders

In the previous sections we treated the leaderless case in which all agents were assigned specific desired inter-agent position vectors in order to implement the proposed control laws. Hence despite the fact that each agent planned its actions based on distributed knowledge, all agents were aware of the global formation objective. In this section we allow some of the agents to be unaware of the global formation objective. This is achieved by equipping the team with a leader-follower containment control law formulation. This was introduced in [8] for the single integrator case and extended to nonholonomic unicycle agents in [2]. Specifically, the leaders of the team have two performance objectives. The first is convergence to a desired formation configuration encoded by the final desired relative inter-leader positions. The second objective is containment of the followers in the convex hull of the leaders' final positions. The leaders evolve under a formation control law, while the followers under an agreement control law which allows them not to be aware of the leaders' objective. We will show in the sequel that if the leader formation is infeasible, then both leaders and followers converge to a common velocity vector and an analytic expression of this vector is also provided.

We assume that the agents belong to either the subset of leaders N_l or to the subset of followers N_f, i.e. $N_l \bigcap N_f = \emptyset$ and $N_l \bigcup N_f = \mathcal{N}$. The leaders are aware of an inter-leader formation objective while the followers obey an agreement control law which does not require knowledge of the global formation objective. Each leader is assigned to a specific subset $N_i^l \subseteq N_l$ of the rest of the leaders, called leader i's *leader communication set* with which it can communicate in order to achieve the desired formation. Hence each leader i aims to be stabilized in a desired relative position c_{ij} with respect to each member j of N_i^l. An undirected *Leader-formation graph* $G^l = \{V^l, E^l\}$ is defined based on these communication sets. In particular, $V^l = N_l$, i.e. the vertices of G^l are indexed by the leaders of the multi-agent team, and the set of edges is given by $E^l = \{(i,j) \in V^l \times V^l | i \in N_j^l\}$. The leaders obey a formation control law of the form (2):

$$u_i = -\sum_{j \in N_i^l} (q_i - q_j - c_{ij}), \forall i \in N_l \tag{9}$$

On the other hand, the followers are assigned to a subset $N_i \subseteq \mathcal{N}$ of the rest of the team called agent i's *leader-follower communication set* with which it can communicate in order to achieve the desired objective (containment of the followers in the convex hull of the desired leader formation). An undirected *Leader-follower communication graph* $G = \{V, E\}$ consists of a set of vertices $V = \{1, ..., N\}$ indexed by the team members and a set of edges, $E = \{(i,j) \in V \times V | i \in N_j\}$ containing pairs of nodes that represent inter-agent communication specifications. The followers obey an agreement control law of the form:

$$u_i = -\sum_{j \in N_i} (q_i - q_j), \forall i \in N_f \tag{10}$$

The following result appeared in [8], while the nonholonomic counterpart appeared in [2]:

Theorem 7: [8] Let the multi-agent system (1) be driven by the control laws (9),(10). Assume that both the Leader-formation graph G^l and the Leader-follower communication graph G are connected and that the desired leader formation is feasible in the sense of Definition 1. Then the leaders converge to the desired leader formation, while the followers converge to the convex hull of the leader formation. □

The main result of the current section involves the case when the desired leader formation is infeasible:

Theorem 8: Let the multi-agent system (1) be driven by the control laws (9),(10). Assume that both the Leader-formation graph G^l and the Leader-follower communication graph G are connected. Then all agents (both leaders and followers) attain a common velocity vector $\dot{q}_i = \dot{q}^*$ for all $i \in \mathcal{N}$ which is given by $\dot{q}^* = -\frac{1}{|N_l|} \sum_{i \in N_l} c_{ii}$, where $c_{ii} \triangleq - \sum_{j \in N_i^l} c_{ij}$ and $|N_l|$ is the cardinality of N_l. $\square$

Proof: The fact that all leaders reach a common velocity vector which is given by $\dot{q}^* = -\frac{1}{|N_l|} \sum_{i \in N_l} c_{ii}$ is guaranteed by virtue of Theorem 2. We proceed to show that the followers also reach the same velocity vector. The followers kinematics in the x, y coordinates are rewritten as

$$\dot{x}_i = - \sum_{j \in N_i} (x_i - x_j), \dot{y}_i = - \sum_{j \in N_i} (y_i - y_j), i \in N_f \quad (11)$$

Denoting by $\mathcal{L}$ the Laplacian of the Leader-follower communication graph G we shall use $W = \frac{1}{2} \left(\dot{x}^T \mathcal{L} \dot{x} + \dot{y}^T \mathcal{L} \dot{y} \right)$ as a candidate Lyapunov function for the system (11). Differentiating with respect to time we have $\dot{W} = \dot{x}^T \mathcal{L} \ddot{x} + \dot{y}^T \mathcal{L} \ddot{y} = - \sum_{i \in N_f} \left(\left(\dot{x}^T \mathcal{L} \right)_i \ddot{x}_i + \left(\dot{y}^T \mathcal{L} \right)_i \ddot{y}_i \right)$, since $\ddot{x}_i = \ddot{y}_i = 0, \forall i \in N_l$. We also have $\ddot{x}_i = (\mathcal{L} \dot{x})_i, \ddot{y}_i = (\mathcal{L} \dot{y})_i$, for all $i \in N_f$, so that $\dot{W} = - \sum_{i \in N_f} \left((\mathcal{L} \dot{x})_i^2 + (\mathcal{L} \dot{y})_i^2 \right) \leq 0$. It is easily shown that the level sets of W are compact and invariant with respect to the agents' relative velocity components. Using Lasalle's invariance principle, we deduce that the agents converge to a configuration that is an equilibrium of the partial difference equation

$$\begin{array}{l} (\mathcal{L} \dot{x})_i = (\mathcal{L} \dot{y})_i = 0, \forall i \in N_f \\ \dot{q}_i = \dot{q}^*, \forall i \in N_l \end{array} \quad (12)$$

The solutions of (12) have been studied in [8]. In particular, Theorem 2 in [8] states that for a connected Leader-follower communication graph and a nonempty set of leaders, the velocity of each follower, as given by the solution of (12), lies in the convex hull of the leaders' velocities. Hence the proof is complete. Since the convex hull of the leaders' velocities reduces to a singleton (namely, all leaders share a common velocity q^*), we conclude that the followers' velocities converge to the single point of this singleton, i.e. $\dot{q}_i = \dot{q}^*, \forall i \in N_f$ as well at steady state.$\diamondsuit$

V. SIMULATIONS

To support the results of the previous sections we provide a series of computer simulations.

The first simulation involves seven single integrator agents of the form (1) that evolve under the control law (2). The directed formation graph contains a spanning tree and the communication sets are chosen so that the graph is balanced. The interagent desired relative positions satisfy $\dot{q}^* = -\frac{1}{7} \sum_{i=1}^{N} c_{ii} = \left[\begin{array}{cc} -0.0177 & 0.01 \end{array} \right]^T$. Graphs I-IV of Figure 1 show the evolution in time of the multi-agent team. As can be seen in graph IV, the interagent velocities vectors are stabilized at steady state to a common value. This is also depicted in the velocity diagrams (Figure 2) in both x and y directions, which show that the agents reach the expected velocity volume imposed by Theorem 4 in both directions (-0.0177 in the x and 0.01 in the y-direction.)

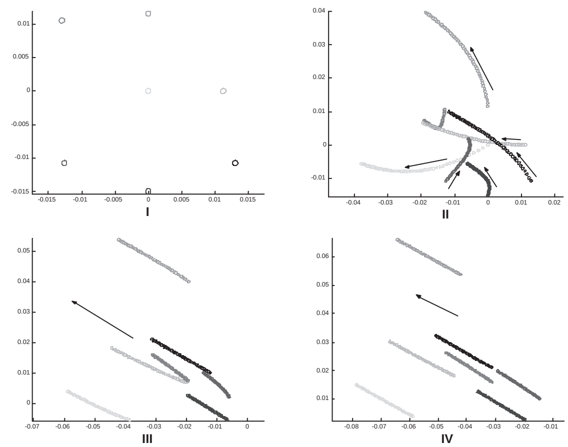

Fig. 1. Formation infeasibility results in velocity alignment.

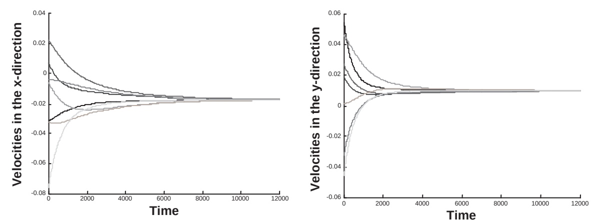

Fig. 2. Velocity diagrams for the first simulation.

The next simulation involves four nonholonomic agents and a connected formation graph. The interagent desired relative positions satisfy $\dot{q}^* = -\frac{1}{7} \sum_{i=1}^{N} c_{ii} = \left[\begin{array}{cc} 0 & 0.02 \end{array} \right]^T$, so that the resulting velocity vector drives the agents to the "north" direction. As can be seen in graphs I-IV of Figure 3, the nonholonomic agents are eventually stabilized to a common velocity. This velocity is equal to q^*, as depicted in the velocity diagram in Figure 4, where the velocity of the agents converges to the zero value in the x-direction and to the expected value 0.02 in the y-direction.

In the last simulation we apply a leader follower architecture in the seven agent team of the first simulation. In graph I of Figure 5, the red agents denoted by L are the leaders while the black agents denoted by F are the followers. The leaders evolve under the control law (9) while the followers under (10). Both graphs G and G^l (ref. Section IV) are connected. The leader desired formation and according to Theorem 8 both leaders and followers reach a common velocity which in this example is given by $\dot{q}^* = -\frac{1}{|N_l|} \sum_{i \in N_l} c_{ii} = \left[\begin{array}{cc} 0.03 & 0.03 \end{array} \right]^T$. Graphs I-IV show the evolution of the system in time. In graph IV the agents reach the expected common velocity $\dot{q}^* = \left[\begin{array}{cc} 0.03 & 0.03 \end{array} \right]^T$, a fact also depicted in the velocity diagrams of Figure 6.

VI. CONCLUSIONS

We presented new results regarding the connection between formation infeasibility and velocity alignment in

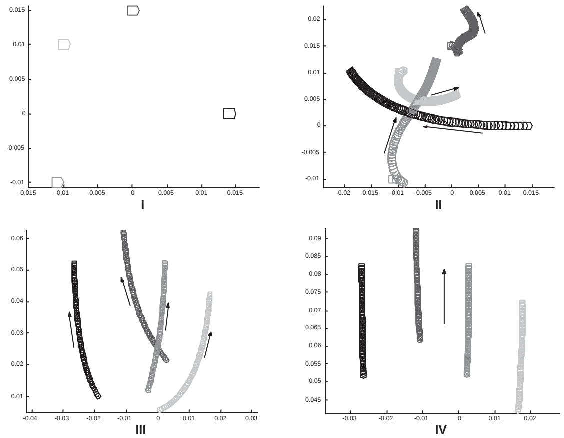

Fig. 3. Formation infeasibility results in velocity alignment for four nonholonomic agents.

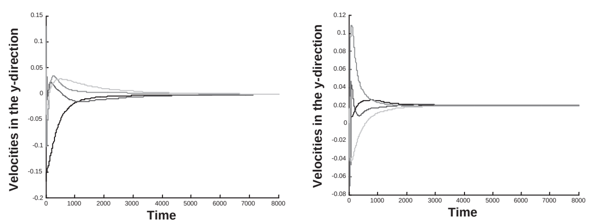

Fig. 4. Velocity diagrams for the second simulation.

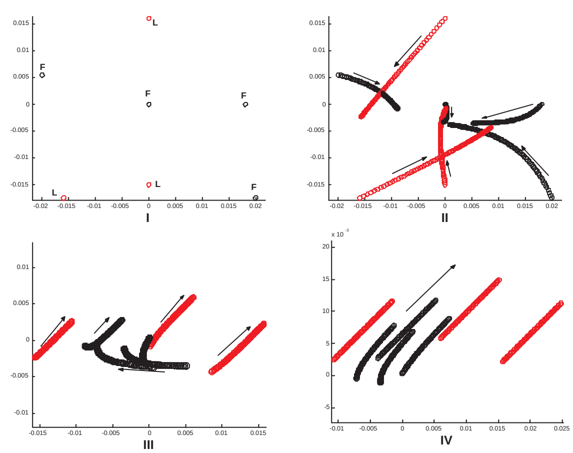

Fig. 5. Formation infeasibility of the leaders results in velocity alignment for the whole group.

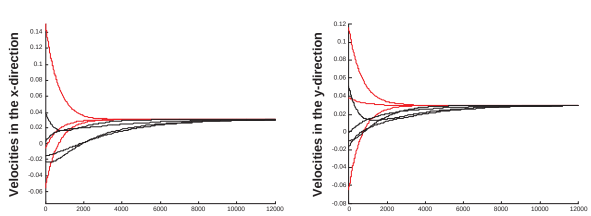

Fig. 6. Velocity diagrams for the last simulation.

kinematic multi-agent systems. Specifically, we obtained an analytic expression for the resulting common velocity vector in the case of formation infeasibility and extended the results to unidirectional communication topology. The results were then extended to the case of a leader-follower architecture in which the followers are not aware of a global objective. Computer simulations supported the derived results.

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