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Decentralized abstractions for multi-agent systems under coupled constraints

Dimitris Boskos*, Dimos V. Dimarogonas

Department of Automatic Control, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Malvinas väg 10, Stockholm 10044, Sweden

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ABSTRACT

The goal of this paper is to define abstractions for multi-agent systems with feedback interconnection in their dynamics. In the proposed decentralized framework, we specify a finite or countable transition system for each agent which only takes into account the discrete positions of its neighbors. The dynamics of each agent consist of a feedback component which can guarantee certain system and network requirements and induces the coupled constraints, and additional input terms, which can be exploited for high level planning. In this work, we provide sufficient conditions for space and time discretizations which enable the abstraction of the system's behavior through a discrete transition system. Furthermore, these conditions include design parameters whose tuning provides the possibility for multiple transitions, and hence, the construction of transition systems with motion planning capabilities.

1. Introduction

High level task planning for multi-agent systems constitutes an active area of research which lies in the interface between computer science and modern control theory. A challenge in this new interdisciplinary direction constitutes the problem of defining appropriate abstractions for continuous time multi-agent control systems, which can be used as a tool for the analysis and control of large scale systems and the synthesis of high level plans [25,26]. Robot motion planning and control constitutes a central field where this line of work is applied [11,14]. In particular, the use of a suitable discrete system's model allows the automatic synthesis of discrete plans that guarantee satisfaction of the high level specifications. Then, under appropriate relations between the continuous system and its discrete analogue, these plans can be converted to low level primitives such as sequences of feedback controllers, and hence, enable the continuous system to implement the corresponding tasks. Such tasks in the case of multiple mobile robots in an industrial workspace could include for example the following scenario. Robot 1 should periodically visit regions A, B, while avoiding C and after collecting an item of type X from robot 2 at location D, store it at location E.

In order to synthesize high level plans, it is required to specify an abstraction of the original system, namely a system that preserves some properties of interest of the initial system, while ignoring detail. Results in this direction for the nonlinear single plant case have been obtained in the papers [30] and [42], where the notions of approximate bisimulation and simulation are exploited for certain classes of nonlinear systems, under appropriate stability, and completeness assumptions, respectively. The notion of bisimulation, has its origin in computer science [5], and guarantees that if the initial system and its abstraction are bisimilar, then the task of checking feasibility of high level plans for the original system reduces to the same task for its abstraction and vice versa. Bisimulation relations between transition system models of discrete or continuous time linear control systems with finite affine observation maps were explicitly characterized and constructed in [29], providing also a generalization of the notion of state space equivalence between continuous time systems [13]. Another abstraction tool for a general class of systems is the hybridization approach [4], where the behavior of a nonlinear system is captured by means of a piecewise affine hybrid system on simplices. Motion planing techniques for the latter case have been developed in the recent works [15,16], which are also based on the abstraction and controller synthesis framework provided in [20,21], and further studied in [9]. Other abstraction techniques for nonlinear systems include [33], where discrete time systems are studied in a behavioral framework, the sign based abstraction methodology introduced in [40], which is based on Lie-algebraic type conditions and [1], where box abstractions are studied for polynomial and other classes of systems (for a literature survey on the subject see also the monograph [38]). Furthermore, certain of the aforementioned

* Corresponding author.
E-mail address: boskos@kth.se (D. Boskos).

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approaches have been extended to switched [17,18] and networked control systems [41].

In this work, we focus on single integrator continuous time multi-agent systems and consider dynamics consisting of feedback terms, which induce the coupled constraints and additional bounded input terms, which we call free inputs and provide the ability for high level planning. The feedback interconnection between the agents can represent internal dynamics of the system, or alternatively, a control design guaranteeing certain system properties (e.g., network connectivity or collision avoidance), which appears often in the multi-agent literature. The results of this paper generalize our recent work [6], where sufficient conditions for well posed abstractions of the multi-agent system are derived for the case where the agents' workspace is $\mathbb{R}^n$. A well posed abstraction refers to a partition of the workspace into cells and the selection of a time step, which provide for each agent a discrete model with at least one outgoing transition from each state. The extension in this work is twofold. First, the results on well posed space and time discretizations in [6], are now valid when the agents' workspace is a general domain D of $\mathbb{R}^n$, provided that D is invariant for the dynamics of the system. Also, the corresponding framework is extended for motion planning, and sufficient conditions are provided which guarantee that each agent can perform multiple transitions from each initial discrete state. It is noted that compositional approaches for symbolic models of interconnected systems have been studied recently in [12,28,31,32,34,35,39]. However, Refs. [12,31,32,39] are focused on discrete time systems and require in most cases that the subsystems satisfy certain small-gain criteria. For the continuous time case, the approach in [28] leverages monotonicity of the dynamics and an assume guarantee reasoning for compositional synthesis, whereas in [34,35], the authors are focused on the construction of lower dimensional continuous control systems and thus, of an infinite state space, with explicit sufficient conditions provided in the linear case. Thus, the techniques of the aforementioned works are not in general applicable for the derivation of finite abstract models for the systems considered in this paper, where the assumptions on the interconnection terms are global Lipschitz continuity and boundedness. Finally, we note that the transitions of the discrete models are realized by control laws which navigate each agent from all points in its initial cell to a common point in its successor cell, which in part relates to the notion of In-Block Controllability [22, Definition 3.1], that has been introduced in [10] and explicitly characterized in the recent papers [22,23].

The rest of the paper is organized as follows. Basic notation and preliminaries are introduced in Section 2. Section 3 is devoted to the formulation of the problem and motivates the control design that will be utilized for the derivation of the symbolic models. In Section 4, we define well posed abstractions for single integrator multi-agent systems by means of hybrid feedback controllers and prove that the latter provide solutions consistent with the design requirement on the system's free inputs. Section 5 is devoted to specific properties of the control laws that realize the transitions of the proposed discrete models. In Section 6 we quantify space and time discretizations which provide transition systems with motion planning capabilities. Section 7 describes how the results can be utilized for control synthesis under high level specifications. The framework is illustrated through an example in Section 8 including simulation results. Finally, we conclude and indicate directions of further research in Section 9.

2. Preliminaries and notation

We use the notation $|x|$ for the Euclidean norm of a vector $x \in \mathbb{R}^n$. For a subset A of $\mathbb{R}^n$, we denote by $\mathrm{cl}(A)$, $\mathrm{int}(A)$ and ∂A its closure, interior and boundary, respectively, where $\partial A := \mathrm{cl}(A) \setminus \mathrm{int}(A)$. Given $R > 0$ and $x \in \mathbb{R}^n$, we denote by $B(R)$ the closed ball with center $0 \in \mathbb{R}^n$ and radius R, namely $B(R) := \{x \in \mathbb{R}^n : |x| \leq R\}$ and $B(x; R) := \{y \in \mathbb{R}^n : |x - y| \leq R\}$. For two nonempty sets $A, B \subset \mathbb{R}^n$ their Minkowski sum is given as $A + B := \{x + y : x \in A, y \in B\}$. Also, for a nonempty set $A \subset \mathbb{R}^n$ its diameter is defined as $\mathrm{diam}(A) := \sup\{|x - y| : x, y \in A\}$. Given a measurable subset A of $\mathbb{R}^n$ we denote by $\mathrm{Vol}(A)$ its Lebesgue measure (volume) and by $\beta(n) := \mathrm{Vol}(B(\frac{1}{2}))$ the volume of a ball with diameter 1 in $\mathbb{R}^n$. For a real number $a \in \mathbb{R}_{\geq 0}$, we use the notation $\lceil a \rceil$ for its ceiling, i.e., $\lceil a \rceil := \min\{n \in \mathbb{N} : a \leq n\}$, under the convention that $0 \notin \mathbb{N}$. Finally, the cardinality of a set X is denoted by $\#X$.

Consider a multi-agent system with N agents. For each agent $i \in \mathcal{N} := \{1, \ldots, N\}$ we use the notation $\mathcal{N}_i$ for the set of its neighbors and $N_i := \#\mathcal{N}_i$ for its cardinality. We also consider an ordering of the agent's neighbors which is denoted by $j_1, \ldots, j_{N_i}$, and define the N_i-tuple $j(i) = (j_1, \ldots, j_{N_i})$. Whenever it is clear from the context, the argument i in the latter notation will be omitted. Given an index set $\mathcal{I}$ and an agent $i \in \mathcal{N}$ with neighbors $j_1, \ldots, j_{N_i} \in \mathcal{N}$, define the mapping $\mathrm{pr}_i : \mathcal{I}^N \to \mathcal{I}^{N_i+1}$ which assigns to each N-tuple $(l_1, \ldots, l_N) \in \mathcal{I}^N$ the $N_i + 1$-tuple $(l_i, l_{j_1}, \ldots, l_{j_{N_i}}) \in \mathcal{I}^{N_i+1}$, i.e., the indices of agent i and its neighbors.

We proceed by providing a formal definition for the notion of a transition system (see for instance [5,29,30]).

Definition 2.1. A transition system is a tuple $TS := (Q, Act, \longrightarrow, O, H)$, where:

- Q is a set of states.
- Act is a set of actions.
- $\longrightarrow$ is a transition relation with $\longrightarrow \subset Q \times Act \times Q$.
- O is a set of outputs.
- H: $Q \to O$ is an output map.

The transition system is said to be finite, if Q and Act are finite sets. We also use the (standard) notation $q \xrightarrow{u} q'$ to denote an element $(q, u, q') \in \longrightarrow$. The transition system is called deterministic if for each $q \in Q$ and $u \in Act$, $q \xrightarrow{u} q'$ and $q \xrightarrow{u} q''$ implies that $q' = q''$. When no output set and map are specified, we will refer to a transition system TS as a triple $(Q, Act, \longrightarrow)$.

Definition 2.2. Consider the transition systems $TS_a := (Q_a, Act_a, \longrightarrow_a, O_a, H_a)$ and $TS_b := (Q_b, Act_b, \longrightarrow_b, O_b, H_b)$, with $O_a = O_b$. A relation $\mathcal{R} \subset Q_a \times Q_b$ is a simulation relation from TS_a to TS_b if the following three conditions hold:

(S1) For every $q_a \in Q_a$, there exists $q_b \in Q_b$ with $(q_a, q_b) \in \mathcal{R}$.
(S2) For every $(q_a, q_b) \in \mathcal{R}$ it holds that $H_a(q_a) = H_b(q_b)$.
(S3) For every $(q_a, q_b) \in \mathcal{R}$ and $q_a \xrightarrow{u_a} q_a'$, there exists $q_b \xrightarrow{u_b} q_b'$ such that $(q_a', q_b') \in \mathcal{R}$.

3. Problem formulation

In this section we provide the agents' dynamic models, describe the main requirements of their discrete representations, and provide the control laws which enable the continuous system to implement the discrete transitions.

3.1. Agent dynamics

We focus on multi-agent systems with single integrator dynamics

$$\dot{x}_i = f_i(x_i, \mathbf{x}_j) + v_i, i \in \mathcal{N}. \tag{3.1}$$

The dynamics of each agent are decentralized and consist of a feedback term $f_i(\cdot)$, which depends on i's state x_i and the states of its neighbors, which we compactly denote by $\mathbf{x}_j (= \mathbf{x}_{j(i)}) :=$

$(x_{j_1},\dots,x_{j_{N_i}})$ (see Section 2 for the notation $j(i)$), and an additional input term v_i, which we call free input. We assume that for each $i\in\mathcal{N}$ it holds $x_i\in D$ where D is a domain of $\mathbb{R}^n$ and that each $f_i:D^{N_i+1}\to\mathbb{R}^n$ is locally Lipschitz. We also assume that the feedback terms $f_i(\cdot)$ are globally bounded, namely, there exists a constant $M>0$ such that

$$|f_i(x_i,\mathbf{x}_j)|\le M, \forall (x_i,\mathbf{x}_j)\in D^{N_i+1}. \tag{3.2}$$

Furthermore, we consider piecewise continuous free inputs v_i that satisfy the bound

$$|v_i(t)|\le v_{\max}, \forall t\ge 0, i\in\mathcal{N}. \tag{3.3}$$

The coupling terms $f_i(x_i,\mathbf{x}_j)$ are encountered in a large set of multi-agent protocols [27], including consensus, connectivity maintenance, collision avoidance and formation control. In addition, (3.1) may represent internal dynamics of the system as for instance in the case of smart buildings (see e.g., [2]), where the temperature T_i, $i\in\mathcal{N}$ of each room evolves according to $\dot{T}_i=\sum_{j\in\mathcal{N}_i}a_{ij}(T_j-T_i)+v_i$, with a_{ij} representing the heat conductivity between rooms i and j and v_i the heating/cooling capabilities of the room. In the subsequent analysis, it is assumed that the maximum magnitude of the feedback terms is higher than that of the free inputs, namely, that

$$v_{\max}<M. \tag{3.4}$$

This assumption is motivated by the fact that we are primarily interested in maintaining the property that the feedback is designed for, and secondarily, in exploiting the free inputs in order to accomplish high level tasks. A class of multi-agent systems of the form (3.1) which justifies this assumption has been studied in our companion work [7,8]. In particular, sufficient conditions are provided, which guarantee both connectivity of the network and forward invariance of the system's trajectories inside a given bounded domain, for an appropriate selection of $v_{\max}$ in (3.3) which necessitates $v_{\max}$ to satisfy (3.4). The latter forward invariance property is formally stated in the Invariance Assumption (IA) below, which we assume that the multi-agent system (3.1) satisfies for the rest of the paper.

(IA) For every initial condition $x(0)\in D^N$ and any piecewise continuous input $v=(v_1,\dots,v_n):\mathbb{R}_{\ge 0}\to\mathbb{R}^{Nn}$ satisfying (3.3), the (unique) solution of system (3.1) is defined and remains in D^N for all $t\ge 0$.

It is noted that this assumption is always satisfied for a forward complete system when $D=\mathbb{R}^n$. Recall that system (3.1) is forward complete (see e.g., [3]) if for each initial condition in $\mathbb{R}^{Nn}$ and each measurable locally essentially bounded input $v=(v_1,\dots,v_n):\mathbb{R}_{\ge 0}\to\mathbb{R}^{Nn}$, its solution exists for all positive times. In addition, due to the above bounds on the dynamics and the free input terms, system (3.1) is forward complete when $D=\mathbb{R}^n$. Finally, when D is bounded, as is the case in [8], a finite partition of the workspace will lead to a transition system which captures the behavior of the continuous system through a finite number of states.

3.2. Discretization requirements

In what follows, we consider a cell decomposition of the state space D (which can be regarded as a partition of D) and a time step $\delta t>0$. We will refer to this selection as a space and time discretization. For the definition of a cell decomposition we adopt a modification of the corresponding definition from [19, p 129-called cell covering].

Definition 3.1. Let D be a domain of $\mathbb{R}^n$. A *cell decomposition* $\mathcal{S}=\{S_l\}_{l\in\mathcal{I}}$ of D, where $\mathcal{I}$ is a finite or countable index set, is a family of nonempty connected sets S_l, $l\in\mathcal{I}$, such that $\sup\{\mathrm{diam}(S_l), l\in\mathcal{I}\}<\infty$, $\mathrm{int}(S_l)\cap\mathrm{int}(S_{\hat{l}})=\emptyset$ for all $l\ne\hat{l}$ and $\cup_{l\in\mathcal{I}}S_l=D$.

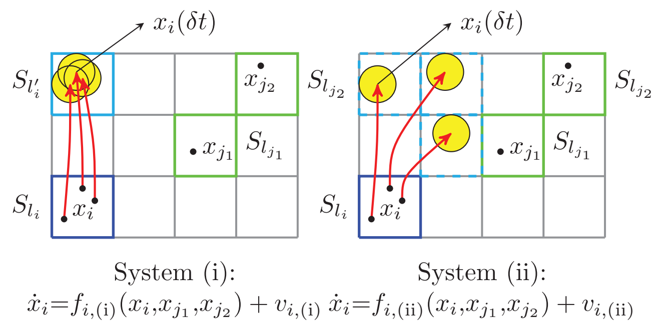

Fig. 1. Illustration of a space-time discretization which is well posed for system (i) but non-well posed for system (ii).

Given a cell decomposition $\mathcal{S}=\{S_l\}_{l\in\mathcal{I}}$ of D, we use the notation $\mathbf{l}_i=(l_i,l_{j_1},\dots,l_{j_{N_i}})\in\mathcal{I}^{N_i+1}$ to denote the cells where agent i and its neighbors belong at a certain time instant and call it the cell configuration of agent i. Similarly, we use the notation $\mathbf{l}=(l_1,\dots,l_N)\in\mathcal{I}^N$ to specify the indices of the cells where all the N agents belong at a given time instant and call it the cell configuration (of all agents). Thus, given a cell configuration $\mathbf{l}$, it is possible to determine the cell configuration of agent i as $\mathbf{l}_i=\mathrm{pr}_i(\mathbf{l})$ (see Section 2 for the definition of $\mathrm{pr}_i(\cdot)$).

Through the space and time discretization we aim at capturing reachability properties of the original continuous time system, by means of a discrete state transition system. Informally, we consider for each agent i, its individual transition system with state set the cells of the state partition, actions determined through the agent's control capabilities and its cell configuration, i.e., the ordered N_i+1-tuple corresponding to the cells of i and its neighbors, and transition relation specified as follows. Given the initial cells of agent i and its neighbors, it is possible for i to perform a transition to a final cell, *if for all states in its initial cell there exists a free input satisfying the constraint (3.3), such that its trajectory will reach the final cell at time δt, for all possible initial states of its neighbors in their cells, and irrespectively of their corresponding evolution during the transition interval*. In order to synthesize high level plans, we will require the discretization to be well posed, in the sense that for each agent and any initial cell it is possible to perform a transition to at least one final cell.

We next provide an illustrative example of a well posed space-time discretization. Both well posed discretizations and the associated individual transition systems of the agents will be formally defined in the next section. Consider a cell decomposition as depicted in Fig. 1 and a time step δt. The arrows in the figure represent trajectories of agent i from the depicted initial conditions in cell S_{l_i}. Also, the circles centered at the tips of the arrows contain the agent's reachable states at δt from the corresponding initial states. In both cases in the figure we focus on agent i and consider the same cell configuration for i and its neighbors. However, we consider different dynamics for Cases (i) and (ii). In Case (i), we observe that for the three distinct initial positions in cell S_{l_i}, it is possible to drive agent i to cell $S_{l_i'}$ at time δt. We assume that this is possible for all initial conditions in this cell and irrespectively of the initial conditions of i's neighbors in their cells and the inputs they choose. We also assume that this property holds for all possible cell configurations of i and for all the agents of the system. Thus, we have a well posed discretization for system (i). On the other hand, for the same cell configuration and system (ii), we observe that for three distinct initial conditions of i the corresponding reachable sets at δt lie in different cells. Thus, given this cell configuration of i it is not possible to find a cell in the decomposition which is reachable from every point in the initial

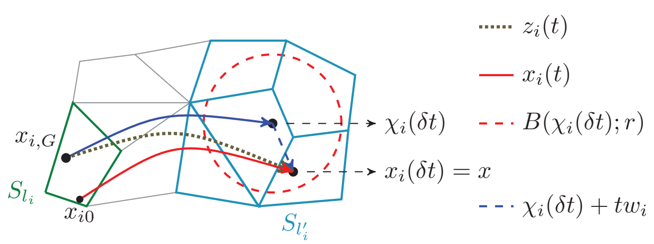

Fig. 2. Consider any point x inside the ball with center $\chi_i(\delta t)$. Then, for each initial condition x_{i0} in the cell S_{l_i}, the feedback law (3.12) ensures that the endpoint of agent's i trajectory $x_i(\cdot)$ coincides with the endpoint of the curve $z_i(\cdot)$, which is precisely x, and lies in $S_{l'_i}$, namely, $x_i(\delta t) = z_i(\delta t) = x = \chi_i(\delta t) + \delta t w_i \in S_{l'_i}$.

cell and we conclude that the discretization is not well posed for system (ii).

3.3. Associated control laws

In order to enable each agent of system (3.1) to perform its desired transitions under the effect of the $f_i(\cdot)$ terms, we design appropriate hybrid control laws in place of the free inputs v_i. We next provide the specific feedback laws that are utilized therefore in the paper. In particular, these control laws guarantee that for well posed discretizations, as the latter are described above, each agent can reach a final cell form a given cell configuration at the end of the transition step. The formal definition of well posed discretizations and their concise relation to the selected controllers will be the subject of the later sections. Consider first a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of D. For each agent $i \in \mathcal{N}$ and cell configuration $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{N_i}})$ of i select an arbitrary $N_i + 1$-tuple of reference points

$$(x_{i,G}, \mathbf{x}_{j,G}) \in S_{l_i} \times (S_{l_{j_1}} \times \cdots \times S_{l_{j_{N_i}}}), \tag{3.5}$$

define

$$F_{i,\mathbf{l}_i}(x_i) := f_i(x_i, \mathbf{x}_{j,G}), x_i \in D, \tag{3.6}$$

and let $\chi_i(\cdot)$ be the solution of the initial value problem

$$\dot{\chi}_i = F_{i,\mathbf{l}_i}(\chi_i), \chi_i(0) = x_{i,G}. \tag{3.7}$$

We also denote as $[0, T_{\max})$, with $T_{\max} \in (0, \infty]$, the maximum right interval on which $\chi_i(\cdot)$ is defined and remains inside D. The reference trajectory of i is obtained by "freezing" agent i's neighbors at their corresponding reference points through the feedback term

$$v_i = k_{i,\mathbf{l}_i,1}(x_i, \mathbf{x}_j) := f_i(x_i, \mathbf{x}_{j,G}) - f_i(x_i, \mathbf{x}_j), \tag{3.8}$$

in place of the agent's free input v_i. It is noted that this controller selection will impose restrictions on the admissible discretizations, since the magnitude of the term evaluated along the solution of the system, needs to respect the bound (3.3) on the available control. Next, consider also a time step $\delta t < T_{\max}$ and select a vector w_i from the set

$$W := B(\lambda v_{\max}), \lambda \in (0, 1). \tag{3.9}$$

By considering the modification $z_i(\cdot)(= z_i(\cdot; w_i))$ of the reference trajectory, defined as

$$z_i(t) := \chi_i(t) + tw_i, t \in [0, T_{\max}), \tag{3.10}$$

and informally assuming that we can move along $z_i(\cdot)$ by an appropriate control law, it is possible to reach the point $x = \chi_i(\delta t) + \delta t w_i$ inside the dashed ball at time δt from the reference point $x_{i,G}$, as depicted in Fig. 2. The parameter λ in (3.9) stands for the part of the free input that is used to increase the degree of freedom in the transition choices. In a similar way, it is possible to reach any point inside the dashed ball by a different selection of w_i. This ball has radius

$$r := \lambda v_{\max} \delta t, \tag{3.11}$$

namely, the distance that the agent can cross in time δt by exploiting the part of the free input that is available for reachability purposes. *For the abstraction, we require the ability to reach each point inside the ball at time δt from every initial state in cell S_{l_i} by a suitable control law. Therefore, it is possible to perform a transition to each cell which has nonempty intersection with $B(\chi_i(\delta t); r)$* (this will be verified for system (3.1) through the establishment of condition (6.17) in Theorem 6.3 for appropriate space-time discretizations). In order to enable these transitions we use an appropriate modification of the control law in (3.8). In particular, consider the family of feedback laws $k_{i,\mathbf{l}_i} : [0, T(x_{i0}, w_i)) \times D^{N_i+1} \to \mathbb{R}^n$ parameterized by $x_{i0} \in S_{l_i}$, $w_i \in W$ and defined as

$$\begin{aligned} v_i = k_{i,\mathbf{l}_i}(t, x_i, \mathbf{x}_j; x_{i0}, w_i) &:= k_{i,\mathbf{l}_i,1}(x_i, \mathbf{x}_j) \\ &+ k_{i,\mathbf{l}_i,2}(x_{i0}) + k_{i,\mathbf{l}_i,3}(t; x_{i0}, w_i), \end{aligned} \tag{3.12}$$

with $k_{i,\mathbf{l}_i,1}(\cdot)$ as given in (3.8) and

$$k_{i,\mathbf{l}_i,2}(x_{i0}) := \frac{1}{\delta t}(x_{i,G} - x_{i0}), \tag{3.13}$$

$$\begin{aligned} k_{i,\mathbf{l}_i,3}(t; x_{i0}, w_i) := F_{i,\mathbf{l}_i}(\chi_i(t)) + w_i - F_{i,\mathbf{l}_i}(\chi_i(t) + tw_i \\ + (1 - \tfrac{t}{\delta t})(x_{i0} - x_{i,G})), \end{aligned} \tag{3.14}$$

$$\begin{aligned} T(x_{i0}, w_i) := \sup\{t \in [0, T_{\max}) : \chi_i(s) + sw_i \\ + \left(1 - \tfrac{s}{\delta t}\right)(x_{i0} - x_{i,G}) \in D, \forall s \in [0, t]\}, \\ t \in [0, T(x_{i0}, w_i)), (x_i, \mathbf{x}_j) \in D^{N_i+1}, x_{i0} \in S_{l_i}, w_i \in W. \end{aligned} \tag{3.15}$$

The time $T(x_{i0}, w_i)$ in (3.15) is the right endpoint of the maximal right interval for which the designated modification of the reference trajectory that depends on x_{i0} and w_i remains inside the domain D (recall that $T_{\max}$ is given after (3.7)). By selecting a parameter w_i from (3.9) and leveraging the control law $k_{i,\mathbf{l}_i}(\cdot)$ in (3.12), agent i can reach the point $x = \chi_i(\delta t) + \delta t w_i$ in Fig. 2 and hence, the cell $S_{l'_i}$ at time δt, from any initial condition $x_{i0} \in S_{l_i}$. This is possible through the extra terms $k_{i,\mathbf{l}_i,2}(\cdot)$ and $k_{i,\mathbf{l}_i,3}(\cdot)$, which enforce the agent to move with the velocity of the reference trajectory plus two constant velocity terms, one analogous to the displacement between the agent's initial state and the reference point, and the other analogous to the difference between x and the endpoint of $\chi_i(\cdot)$. Specifically, these extra velocity terms of each such trajectory $x_i(\cdot)$ are $k_{i,\mathbf{l}_i,2}(\cdot)$, and w_i from $k_{i,\mathbf{l}_i,3}(\cdot)$, respectively, with the remaining terms in $k_{i,\mathbf{l}_i,3}(\cdot)$ guaranteeing that the agent will evolve according to $x_i(\cdot)$ through the suggested control scheme. It is noted that due the term $k_{i,\mathbf{l}_i,1}(\cdot)$ in the feedback law (3.12), the transition is possible irrespectively of i's neighbors' evolution on $[0, \delta t]$, which are initially located in the corresponding cells of the configuration $\mathbf{l}_i$. This evolution can be appropriately quantified through the size of the cells and the transition duration, by using the agents' dynamics' bounds. *Therefore, the derivation of well posed discretizations will be based on the choice of cell decompositions and associated time steps δt which can ensure that the magnitude of the feedback law apart from the term w_i in $k_{i,\mathbf{l}_i,3}(\cdot)$ does not exceed $(1 - \lambda)v_{\max}$ during the transition interval. Thus, due to (3.9), which implies that $|w_i| \le \lambda v_{\max}$, it follows that the total magnitude of the applied control law will be consistent with assumption (3.3) on the free inputs' bound.* Notice also that due to the assumption $v_{\max} < M$ in (3.4), it is in principle not possible to cancel the interconnection terms. Furthermore, the control laws $k_{i,\mathbf{l}_i}(\cdot)$ are decentralized, since they only use information of agent i's neighbors states and they depend on the cell configuration $\mathbf{l}_i$, through the reference points $(x_{i,G}, \mathbf{x}_{j,G})$ which are involved in (3.8), (3.13), and (3.14).

4. Abstractions for multi-agent systems

In this section we formalize the discussion in Section 3, by exploiting a class of hybrid feedback laws containing the ones introduced in (3.12). One reason for employing the subsequent analysis in an abstract framework is that the control laws in (3.8), (3.13), (3.14) are not the only possible choice, which provides the flexibility for alternative control designs. In the sequel, given a time step δt and the bounds M and $v_{\max}$ on the feedback and input terms provided by (3.2) and (3.3), respectively, it is convenient to introduce the following lengthscale

$$R_{\max} := \delta t(M + v_{\max}). \tag{4.1}$$

It follows from (3.1), (3.2), (3.3), and (4.1) that $R_{\max}$ is the maximum distance an agent can travel within time δt.

Before defining the notion of a well posed space-time discretization we define the class of hybrid feedback laws which are assigned to the free inputs v_i in order to obtain the discrete transitions. For each agent, these control laws are parameterized by the agent's initial conditions and a set of auxiliary parameters belonging to a nonempty subset W of $\mathbb{R}^n$. These parameters, as discussed in the previous section, are exploited to increase the transition choices of the abstract model. In particular, for every agent i, each vector $w_i \in W$ is in a one-to-one correspondence with a point inside a reachable ball for i, and the agent can reach this point by selecting the control law corresponding to the specific parameter w_i. The latter provides the possibility for the agent to perform transitions to different cells, namely, all cells which have nonempty intersection with that ball. Furthermore, we note that in accordance to the control laws introduced in (3.12) for each agent i, the feedback laws in the following definition depend on the selection of the cells where i and its neighbors belong. One basic requirement for this class of controllers consists of conditions that guarantee well posed solutions for the system (condition (P1) in Definition 4.1, below). We also impose the consistency requirement (condition (P2) in Definition 4.1) that their magnitude does not exceed the maximum bound on the free inputs (3.3), when the states of the agent and its neighbors lie in an appropriate overapproximation of their reachable states over the time interval $[0, \delta t]$.

Definition 4.1. Given a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of D, a time step δt and a nonempty subset W of $\mathbb{R}^n$, consider an agent $i \in \mathcal{N}$ and a cell configuration $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{N_i}})$ of i. Also, consider a mapping $k_{i,\mathbf{l}_i}(\cdot; x_{i0}, w_i) : [0, T(x_{i0}, w_i)) \times D^{N_i+1} \to \mathbb{R}^n$, parameterized by $x_{i0} \in S_{l_i}$ and $w_i \in W$. We say that $k_{i,\mathbf{l}_i}(\cdot)$ satisfies *Property (P)*, if the following conditions are satisfied.

(P1) For each $x_{i0} \in S_{l_i}$ and $w_i \in W$, the mapping $k_{i,\mathbf{l}_i}(\cdot; x_{i0}, w_i)$ is locally Lipschitz continuous.

(P2) It holds

$$\begin{aligned} &|k_{i,\mathbf{l}_i}(t, x_i, \mathbf{x}_j; x_{i0}, w_i)| \le v_{\max},\\ &\forall t \in [0, \delta t] \cap [0, T(x_{i0}, w_i)), x_\ell \in (S_{l_\ell} + B(R_{\max})) \cap D,\\ &\ell \in \mathcal{N}_i \cup \{i\}, x_{i0} \in S_{l_i}, w_i \in W, \end{aligned} \tag{4.2}$$

with $v_{\max}$ as given in (3.3) and $R_{\max}$ as in (4.1).

(P3) It holds $T(x_{i0}, w_i) > \delta t$, for all $x_{i0} \in S_{l_i}$, $w_i \in W$.

The motivation for considering the time interval $[0, T(x_{i0}, w_i))$ in Definition 4.1 comes from the maximal right interval on which the modification of agent's i reference trajectory in (3.15) remains inside the domain D. We next provide an extra Condition (C) for the feedback laws provided in the above definition, which is needed in order to define well posed discretizations.

Definition 4.2. Consider a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of D, a time step δt and a nonempty subset W of $\mathbb{R}^n$. Given an agent $i \in \mathcal{N}$, a cell configuration $\mathbf{l}_i$ of i, a control law

$$v_i = k_{i,\mathbf{l}_i}(t, x_i, \mathbf{x}_j; x_{i0}, w_i) \tag{4.3}$$

as in Definition 4.1 that satisfies Property (P), a vector $w_i \in W$, and a cell index $l'_i \in \mathcal{I}$, we say that $k_{i,\mathbf{l}_i}(\cdot)$, w_i, l'_i satisfy *Condition (C)*, if the following hold. For each initial cell configuration $\mathbf{l} = (l_1, \ldots, l_N)$ with $\mathrm{pr}_i(\mathbf{l}) = \mathbf{l}_i$ and feedback laws

$$v_\ell = k_{\ell,\mathrm{pr}_\ell(\mathbf{l})}(t, x_\ell, \mathbf{x}_{j(\ell)}; x_{\ell 0}, w_\ell), \ell \in \mathcal{N} \setminus \{i\}, \tag{4.4}$$

that satisfy Property (P), the solution of the closed-loop system (3.1), (4.3)-(4.4) is well defined on $[0, \delta t]$ and satisfies $x_i(\delta t, x(0)) \in S_{l'_i}$, for all initial conditions $x(0) \in D^N$ with $x_i(0) = x_{i0} \in S_{l_i}$, $x_\ell(0) = x_{\ell 0} \in S_{l_\ell}$, $\ell \in \mathcal{N} \setminus \{i\}$ and $w_\ell \in W$, $\ell \in \mathcal{N} \setminus \{i\}$.

Notice that when Condition (C) is satisfied, agent i will have reached cell $S_{l'_i}$ at time δt under the feedback law $k_{i,\mathbf{l}_i}(\cdot)$ corresponding to the given parameter w_i in the definition. In particular, Condition (C) ensures that the latter holds for any choice of feedback laws in place of the other agents' free inputs, as long as these control laws satisfy Property (P). We next provide the definition of a well posed space-time discretization. This definition formalizes our discussion on the possibility to assign a feedback law to each agent, in order to enable a transition from any initial discrete configuration to at least one successor cell.

Definition 4.3. Consider a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of D, a time step δt and a nonempty subset W of $\mathbb{R}^n$. We say that the *space-time discretization $\mathcal{S} - \delta t$ is well posed* (for system (3.1)), if for each agent $i \in \mathcal{N}$ and cell configuration $\mathbf{l}_i$ of i, there exist a control law $k_{i,\mathbf{l}_i}(\cdot)$ satisfying Property (P), a vector $w_i \in W$ and a cell index $l'_i \in \mathcal{I}$ such that $k_{i,\mathbf{l}_i}(\cdot)$, w_i, l'_i satisfy Condition (C) of Definition 4.2.

Assume a well posed space-time discretization $\mathcal{S} - \delta t$ is given. Based on Definition 4.3, it is possible to provide a discrete transition system for each agent, which serves as an abstract model for its behavior. In particular, for each $i \in \mathcal{N}$ and cell configuration $\mathbf{l}_i$ of i we pick a control law $k_{i,\mathbf{l}_i}(\cdot)$ *which generates at least one transition*, i.e., such that $k_{i,\mathbf{l}_i}(\cdot)$, w_i, l'_i satisfy Condition (C) for certain $w_i \in W$ and $l'_i \in \mathcal{I}$ (this is always possible since the discretization is well posed) and define for all $l \in \mathcal{I}$

$$[w_i]_{(\mathbf{l}_i, l)} := \{w \in W : k_{i,\mathbf{l}_i}(\cdot), w, l \text{ satisfy Condition (C)}\}. \tag{4.5}$$

Note that $[w_i]_{(\mathbf{l}_i, l)}$ represents the set of all parameters in W under which the control law $k_{i,\mathbf{l}_i}(\cdot)$ can drive agent i to the successor cell l at δt, according to Condition (C). By exploiting (4.5), we next provide the individual transition system of each agent.

Definition 4.4. Consider a well posed space-time discretization and select for each agent i and cell configuration $\mathbf{l}_i$ a control law $k_{i,\mathbf{l}_i}(\cdot)$ which generates at least one transition. Then, the *individual transition system* $TS_i := (Q_i, Act_i, \longrightarrow_i)$ of each agent i is defined as follows

- $Q_i := \mathcal{I}$
- $Act_i := \mathcal{I}^{N_i+1} \times 2^W$
- $l_i \overset{(\mathbf{l}_i, [w_i])}{\longrightarrow_i} l'_i$ iff $[w_i] = [w_i]_{(\mathbf{l}_i, l'_i)}$ and $[w_i]_{(\mathbf{l}_i, l'_i)} \neq \emptyset$, for each $l_i, l'_i \in Q_i$, $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{N_i}}) \in \mathcal{I}^{N_i+1}$ and $[w_i] \in 2^W$, with $[w_i]_{(\mathbf{l}_i, l'_i)}$ as defined in (4.5).

  In addition, let

  $$\mathrm{Post}_i(\mathbf{l}_i) := \{l'_i \in \mathcal{I} : \exists [w_i] \in 2^W \text{ with } l_i \overset{(\mathbf{l}_i, [w_i])}{\longrightarrow_i} l'_i\}, \tag{4.6}$$

  which provides all successor cells from cell configuration $\mathbf{l}_i$ and note that $\mathrm{Post}_i(\mathbf{l}_i) \neq \emptyset$.

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From Definition 4.4 it follows that agent's i transitions from a given cell are affected by the discrete positions of i and its neighbors, and the parameters w_i which tune its corresponding control law, resulting in tuples $(\mathbf{l}_i, [w_i])$ as actions. This is also in accordance with the control design introduced in Section 3, since the cell configuration $\mathbf{l}_i$ of the agent affects the endpoint of its reference trajectory, and the parameters $[w_i]$ enable the transition to a cell which intersects the ball around this endpoint.

Remark 4.5.

(i) Note that due to (4.5), for any transition of an agent, the action term $[w_i] \in 2^W$ is uniquely determined by the agent's successor cell and represents all the parameters under which the transition can be enabled by tuning accordingly the specific control law. In addition, it follows that the cardinality of the $[w_i]$'s that are included in the actions from a cell configuration does not exceed the cardinality of the agents' corresponding successor cells.

(ii) The transition system of each agent i is not necessarily uniquely defined, since the transitions in TS_i are associated to the specific controller selection for the cell configuration. In particular, given an agent i and a cell configuration $\mathbf{l}_i$, it might be possible to perform different transitions by choosing an alternative control law. For example, it is possible for the control laws considered in (3.12) to obtain a different reference trajectory in (3.7) by selecting another set of points $(x_{i,G}, \mathbf{x}_{j,G})$ and hence, reach a ball which intersects different cells (see Fig. 2).

(iii) If we additionally assume that the cell decomposition $\{S_l\}_{l\in\mathcal{I}}$ of D forms also a partition of D, i.e., it holds that $S_l \cap S_{\hat{l}} = \emptyset$ and not necessarily only that $\mathrm{int}(S_l) \cap \mathrm{int}(S_{\hat{l}}) = \emptyset$, for each $l, \hat{l} \in \mathcal{I}$ with $l \neq \hat{l}$, then the transition system TS_i of each agent is also deterministic. In the general case, nondeterministic transitions may "unlikely" occur from a given cell configuration $\mathbf{l}_i$, if the parameters $[w_i]$ enable the agent to reach the common boundary of two distinct cells $S_{l'}$ and $S_{\hat{l}'}$ at δt. However, nondeterminism is interpreted as the property that the same action can drive the agent to both cells simultaneously, precisely due to the fact that they are not disjoint, and not as any uncertainty with respect to the cell that can be reached.

Notice that according to Definition 4.3, a well posed space-time discretization requires the existence of an outgoing transition for each agent from any discrete position. A transition is guaranteed for every agent i individually and is based on the selection of an appropriate feedback controller for i satisfying Property (P), and the requirement that the control laws of the other agents also satisfy (P). From the definition of the agents' individual transition systems and Condition (C), it follows rather directly that for any initial cell configuration of the team and corresponding individual transition selection for every agent, it is possible to choose a feedback law for each, so that the resulting closed-loop system will guarantee all these transitions. This is summarized in the following remark.

Remark 4.6. Consider a cell decomposition $\mathcal{S}$ of D, a time step δt, a nonempty subset W of $\mathbb{R}^n$ and assume that the space-time discretization $\mathcal{S} - \delta t$ is well posed for system (3.1). Also, consider for each agent i and cell configuration $\mathbf{l}_i$ a control law $k_{i,\mathbf{l}_i}(\cdot)$ which generates at least one transition, implying by virtue of Definition 4.4 that $\mathrm{Post}_i(\mathbf{l}_i) \neq \emptyset$. Then, given a cell configuration $\mathbf{l} = (l_1, \ldots, l_N)$ and any $\mathbf{l}' = (l'_1, \ldots, l'_N) \in \mathrm{Post}_1(\mathrm{pr}_1(\mathbf{l})) \times \cdots \times \mathrm{Post}_N(\mathrm{pr}_N(\mathbf{l}))$, there exist $w_1, \ldots, w_N \in W$, such that for each initial condition $x(0) \in D^N$ with $x_i(0) = x_{i0} \in S_{l_i}, i \in \mathcal{N}$, the solution of the closed-loop system (3.1) with

$$v_i = k_{i,\mathrm{pr}_i(\mathbf{l})}(t, x_i, \mathbf{x}_j; x_{i0}, w_i), i \in \mathcal{N}, \tag{4.7}$$

is well defined on $[0, \delta t]$ and satisfies

$$x_i(\delta t, x(0)) \in S_{l'_i}, \forall i \in \mathcal{N}. \tag{4.8}$$

The result of the following proposition guarantees that the selection of the controllers introduced in Definition 4.1 provides well posed solutions for the closed-loop system on the time interval $[0, \delta t]$. Furthermore, it is shown that the magnitude of the hybrid feedback laws does not exceed the maximum allowed magnitude $v_{\max}$ of the free inputs on $[0, \delta t]$, as required by (3.3). In addition, it follows that every solution of the closed-loop system on $[0, \delta t]$ is identical to a solution of the original system (3.1), generated by the same initial condition and an open loop control signal $u(\cdot)$ that satisfies $|u_i(t)| \leq v_{\max}$, for all $t \geq 0$ and $i \in \mathcal{N}$. Finally, in order to verify that the control laws introduced in (3.12) satisfy property (P3) for appropriate selections of the time step δt in the next section, it is convenient to obtain the first result of the proposition for feedback laws that only satisfy Properties (P1) and (P2) of Definition 4.1.

Proposition 4.7. *Consider a cell decomposition $\mathcal{S}$ of D, a time step δt and a nonempty subset W of $\mathbb{R}^n$. Let $\mathbf{l} = (l_1, \ldots, l_N)$ be an initial cell configuration and consider any feedback laws of the form*

$$v_i = k_{i,\mathrm{pr}_i(\mathbf{l})}(t, x_i, \mathbf{x}_j; x_{i0}, w_i), i \in \mathcal{N} \tag{4.9}$$

assigned to the agents that satisfy Properties (P1) and (P2). Then:

(i) For each $w_i \in W$, $i \in \mathcal{N}$ and initial condition $x(0) \in D^N$ with $x_i(0) = x_{i0} \in S_{l_i}$, $i \in \mathcal{N}$, the solution of the closed-loop system (3.1), (4.9) is defined and remains in D^N for all $t \in [0, \tau)$, where

$$\tau := \min\{\delta t, \min\{T(x_{i0}, w_i) : i \in \mathcal{N}\}\} \tag{4.10}$$

and

$$\lim_{t\to\tau^-} x(t) \in D^N. \tag{4.11}$$

Assume additionally that (P3) also holds, namely, that (P) is satisfied. Then:

(iia) The solution $x(t)$ of (3.1), (4.9) above remains in D^N for all $t \in [0, \delta t]$ and satisfies

$$|k_{i,\mathrm{pr}_i(\mathbf{l})}(t, x_i(t), \mathbf{x}_j(t); x_{i0}, w_i)| \leq v_{\max}, \forall t \in [0, \delta t], i \in \mathcal{N}, \tag{4.12}$$

which provides the desired consistency with the design requirement (3.3) on the v_i's.

(iib) There exists an open loop control signal given by a piecewise continuous function $u = (u_1, \ldots, u_N) : [0, \infty) \to \mathbb{R}^{Nn}$ and satisfying $|u_i(t)| \leq v_{\max}$, $\forall t \geq 0$, $i \in \mathcal{N}$, such that the solution $x(\cdot)$ above and the solution $\xi(\cdot)$ of (3.1), with the same initial condition as $x(\cdot)$ and input $u(\cdot)$, coincide on $[0, \delta t]$.

Proof. The proof is given in the Appendix. □

Remark 4.8. Note that the result of part (i) of Proposition 4.7 holds for any selection of feedback laws $v_i = k_{i,\mathrm{pr}_i(\mathbf{l})}(\cdot)$ that satisfy Properties (P1) and (P2). Respectively, the results of parts (iia) and (iib) hold for all selections of feedback laws $v_i = k_{i,\mathrm{pr}_i(\mathbf{l})}(\cdot)$ that satisfy Property (P).

Given a well posed space-time discretization and the agents' individual abstract models, we next define their product transition system, which captures the coupled behavior of the team. To obtain meaningful transitions in the product model, we require that the agents' actions $(\mathbf{l}_i, [w_i])$ are compliant in their first argument, which corresponds to their cell configurations. This is due to the fact that each global discrete state, i.e., cell configuration

$\mathbf{l} = (l_1, \ldots, l_N)$ of all agents, automatically also fixes the cell configuration $\mathbf{l}_i = \mathrm{pr}_i(\mathbf{l})$ of each agent i through the projection operator $\mathrm{pr}_i(\cdot)$. Therefore, for each transition from a global cell configuration, the agents' actions are formed by the N-tuples of their individual control parameters, which enable them to reach synchronously their successor cells at the end of time step.

Definition 4.9.

(i) Consider a well posed space-time discretization $\mathcal{S} - \delta t$, and for each agent $i \in \mathcal{N}$, its individual transition system TS_i as provided by Definition 4.4. The *product transition system* $TS_{\mathcal{P}} := (Q_{\mathcal{P}}, Act_{\mathcal{P}}, \longrightarrow_{\mathcal{P}})$ is defined as follows:
- $Q_{\mathcal{P}} := \mathcal{I}^N$ (all possible cell configurations)
- $Act_{\mathcal{P}} := \{[w_1] \times \cdots \times [w_N] : [w_i] \in 2^W, i \in \mathcal{N}\}$
- $\mathbf{l} \xrightarrow{[w]}_{\mathcal{P}} \mathbf{l}'$, where $\mathbf{l} = (l_1, \ldots, l_N)$, $\mathbf{l}' = (l'_1, \ldots, l'_N)$, and $[w] = [w_1] \times \cdots \times [w_N]$, iff $l_i \overset{(\mathrm{pr}_i(\mathbf{l}), [w_i])}{\longrightarrow_i} l'_i$, $\forall i \in \mathcal{N}$, for each $\mathbf{l}, \mathbf{l}' \in \mathcal{I}^N$, $[w] \in Act_{\mathcal{P}}$.

(ii) Given an initial cell configuration $\mathbf{l}^0 \in \mathcal{I}^N$, a path originating from $\mathbf{l}^0$ in $TS_{\mathcal{P}}$, is an infinite sequence of states $\mathbf{l}^0\mathbf{l}^1\mathbf{l}^2\ldots$ such that $\mathbf{l}^i \xrightarrow{[w]}_{\mathcal{P}} \mathbf{l}^{i+1}$ for all $i \in \mathbb{N} \cup \{0\}$.

Remark 4.10. Given a well posed space-time discretization $\mathcal{S} - \delta t$ and an initial cell configuration $\mathbf{l}^0 \in \mathcal{I}^N$, it follows from Definitions 4.4 and 4.9 that there exists at least one path $\mathbf{l}^0\mathbf{l}^1\mathbf{l}^2\ldots$ in $TS_{\mathcal{P}}$ originating from $\mathbf{l}^0$.

Additionally, consider also an output set O which contains desirable attributes of the coupled system, as for instance labels of regions to be reached, or safe/unsafe parts of the workspace. By assigning an appropriate output map H: $D^N \to O$ we can establish behavioral inclusion of the product abstract model by the coupled continuous system through a suitable simulation relation. In particular, consider a well posed discretization $\mathcal{S} - \delta t$ and assume that the output map H is compliant with $\mathcal{S}$, i.e., for every $\mathbf{l} = (l_1, \ldots, l_N) \in \mathcal{I}^N$ and $x, y \in D^N$ with $x, y \in S_{l_1} \times \cdots \times S_{l_N}$ it holds that $H(x) = H(y)$. Also, consider the transition systems with outputs $TS_{\mathcal{P}}^O$ and $TS_{\delta t}^O$, corresponding to the product discrete model and the sampled continuous system, respectively, which are provided in the following definition.

Definition 4.11.

(i) The *product transition system with outputs* $TS_{\mathcal{P}}^O$ is the 5-tuple $(Q_{\mathcal{P}}, Act_{\mathcal{P}}, \longrightarrow_{\mathcal{P}}, O_{\mathcal{P}}, H_{\mathcal{P}})$, with $Q_{\mathcal{P}}$, $Act_{\mathcal{P}}$, and $\longrightarrow_{\mathcal{P}}$ as given in $TS_{\mathcal{P}}$, $O_{\mathcal{P}} = O$, where O is give above, and $H_{\mathcal{P}} : Q_{\mathcal{P}} \to O$ defined as $H_{\mathcal{P}}(\mathbf{l}) = H(x)$ for some $x \in S_{l_1} \times \cdots \times S_{l_N}$, for any $\mathbf{l} = (l_1, \ldots, l_N) \in Q_{\mathcal{P}}$.

(ii) The *δt-sampled system with outputs* $TS_{\delta t}^O$ is the 5-tuple $(Q_{\delta t}, Act_{\delta t}, \longrightarrow_{\delta t}, O_{\delta t}, H_{\delta t})$ with
- $Q_{\delta t} := D^N$ (all possible initial states of the continuous system)
- $Act_{\delta t} :=$ {piecewise continuous $u = (u_1, \ldots, u_N)$: $[0, \delta t] \to \mathbb{R}^{Nn} : |u_i(t)| \le v_{\max}, \forall i \in \mathcal{N}, t \in [0, \delta t]$}
- $x \xrightarrow{u}_{\delta t} x'$ iff $x' = \xi(\delta t, x; u)$, with $\xi(\delta t, x; u)$ denoting the solution of (3.1) at δt with initial condition x and input $u(\cdot)$.
- $O_{\delta t} := O$, $H_{\delta t} := H$, with O and H as given above.

By exploiting Remark 4.6 and Proposition 4.7, we obtain the following result which establishes that $TS_{\mathcal{P}}^O$ is simulated by $TS_{\delta t}^O$.

Proposition 4.12. *Consider the transition systems $TS_{\mathcal{P}}^O$, $TS_{\delta t}^O$, and the relation $\mathcal{R} \subset Q_{\mathcal{P}} \times Q_{\delta t} (= \mathcal{I}^N \times D^N)$ given as $(\mathbf{l}, x) \in \mathcal{R}$, iff $x \in S_{l_1} \times \cdots \times S_{l_N}$, where $\mathbf{l} = (l_1, \ldots, l_N)$. Then, $\mathcal{R}$ is a simulation relation from $TS_{\mathcal{P}}^O$ to $TS_{\delta t}^O$.*

Proof. In order to prove the result, we need to show that properties (S1), (S2), and (S3) of a simulation relation are satisfied. Property (S1) follows directly from the definition of $\mathcal{R}$ and the fact that the cells are subsets of D. For (S2), it suffices to recall that the output map is compliant with $\mathcal{S}$, which implies by the definition of $\mathcal{R}$ that for any $(\mathbf{l}, x) \in \mathcal{R}$ it holds that $H_{\mathcal{P}}(\mathbf{l}) = H_{\delta t}(x)$. Finally, in order to show (S3), let $(\mathbf{l}, x) \in \mathcal{R}$ and assume that $\mathbf{l} \xrightarrow{[w]}_{\mathcal{P}} \mathbf{l}'$. Since $(\mathbf{l}, x) \in \mathcal{R}$, with $\mathbf{l} = (l_1, \ldots, l_N)$ and $x = (x_1, \ldots, x_N)$, we have that $x_i \in S_{l_i}$, for all $i \in \mathcal{N}$. Thus, given that $\mathbf{l} \xrightarrow{[w]}_{\mathcal{P}} \mathbf{l}'$, with $\mathbf{l}' = (l'_1, \ldots, l'_N)$, we deduce from Definition 4.9 and Remark 4.6 that there exist feedback laws $k_{i,\mathrm{pr}_i(\mathbf{l})}(\cdot)$ as in (4.7) which satisfy Property (P), and $w_1, \ldots, w_N \in W$ such that (4.8) holds. Hence, it follows from Proposition 4.7(iib) and strict causality[1] of the solutions of (3.1) that there exists a piecewise continuous input $u : [0, \delta t] \to \mathbb{R}^{Nn}$ satisfying $|u_i(t)| \le v_{\max}$, $\forall t \in [0, \delta t]$, $i \in \mathcal{N}$ and such that $x'_i := \xi_i(\delta t, x; u) \in S_{l'_i}$ for all $i \in \mathcal{N}$. Hence, we derive that $x' = (x'_1, \ldots, x'_N)$ satisfies $x' = \xi(\delta t, x; u)$ which implies that $x \xrightarrow{u}_{\delta t} x'$. Finally, since $x'_i \in S_{l'_i}$ for all $i \in \mathcal{N}$, we get that $(\mathbf{l}', x') \in \mathcal{R}$, which establishes (S3) and the proof is complete. □

It is noted that due to the fact that $TS_{\delta t}^O$ is deterministic, a satisfying plan or discrete controller that is synthesized for $TS_{\mathcal{P}}^O$ can be refined to a corresponding sequence of transitions or controller for $TS_{\delta t}^O$. Further details on the exploitation of the abstractions for synthesis purposes are provided in Section 7.

5. Time domain properties of the control laws

In this section we use the results of Section 4 in order to prove certain useful properties of the reference trajectory $\chi_i(\cdot)$ and the time domain $[0, T_i(x_{i0}, w_i))$ of the control laws (3.12) as specified by (3.15). For the sequel, we additionally assume that the f_i terms in (3.1) are globally Lipschitz functions. Furthermore, in order to achieve more accurate bounds for the dynamics of the feedback controllers in (3.12) that are assigned to the free inputs v_i, we assume Lipschitz constants $L_1, L_2 > 0$ such that

$$|f_i(x_i, \mathbf{x}_j) - f_i(x_i, \mathbf{y}_j)| \le L_1 |(x_i, \mathbf{x}_j) - (x_i, \mathbf{y}_j)|, \tag{5.1}$$

$$|f_i(x_i, \mathbf{x}_j) - f_i(y_i, \mathbf{x}_j)| \le L_2 |(x_i, \mathbf{x}_j) - (y_i, \mathbf{x}_j)|,$$
$$\forall x_i, y_i \in D, \mathbf{x}_j, \mathbf{y}_j \in D^{N_i}, i \in \mathcal{N}. \tag{5.2}$$

In particular, the constant L_1 is exploited to bound the feedback term (3.8) which compensates for the deviation of agent's i dynamics from its corresponding dynamics along the reference trajectory, due to the time evolution of its neighbors' states. On the other hand, it follows from (3.6) that the constant L_2 is utilized to bound the feedback term (3.14) which compensates for the deviation of the agent's desired trajectory with respect to its reference trajectory.

Based on the global Lipschitz assumption, we establish uniqueness of the reference trajectory $\chi_i(\cdot)$ and provide a lower bound for the right endpoint $T_{\max}$ of its maximal interval of existence, which is independent of the selection of $(x_{i,G}, \mathbf{x}_{j,G})$ in (3.5).

Lemma 5.1. *For each tuple of reference points $(x_{i,G}, \mathbf{x}_{j,G})$ as in (3.5), the initial value problem (3.7) has a unique solution which is defined and remains in D on the right maximal interval $[0, T_{\max})$. Furthermore, it holds*

$$T_{\max} > \frac{v_{\max}}{2ML_1 \max\{\sqrt{N_i} : i \in \mathcal{N}\}}. \tag{5.3}$$

Proof. The proof is given in the Appendix. □

[1] Strict causality refers to the property that the value of the solution $\xi(t, x; u)$ depends only on the values of the input $u(\cdot)$ on $[0, t)$ (see e.g., [24, Chapter 1] or [36, Chapter 2])

By exploiting Lemma 5.1, it will be shown in the next section that $T_{\max}$ is always greater than the maximum possible selection of the time step δt for a well posed discretization. The latter in conjunction with the result of Lemma 5.2 below enables us to prove that in this case the control law $k_{i,\mathbf{l}_i,3}(\cdot)$ and hence also $k_{i,\mathbf{l}_i}(\cdot)$ are well defined on $[0, \delta t]$.

Lemma 5.2. *Consider a cell decomposition $\mathcal{S}$ of D, a time step δt and select an agent $i\in\mathcal{N}$ and a cell configuration $\mathbf{l}_i=(l_i,l_{j_1},\ldots,l_{j_{N_i}})$ of i. Also, consider a tuple of reference points $(x_{i,G},\mathbf{x}_{j,G})$ as in (3.5) and the control law $k_{i,\mathbf{l}_i}(\cdot)$ in (3.12). We assume that $k_{i,\mathbf{l}_i}(\cdot)$ satisfies Properties (P1) and (P2) of Definition 4.1, and that the right endpoint $T_{\max}$ of the interval where the reference trajectory (3.7) is defined, satisfies $T_{\max}>\delta t$. Then, for all $x_{i0}\in S_{l_i}$ and $w_i\in W$, the time $T(x_{i0},w_i)$ satisfies $T(x_{i0},w_i)>\delta t$, which implies that $k_{i,\mathbf{l}_i}(\cdot)$ also satisfies Property (P3) of Definition 4.1.*

Proof. Indeed, let $x_{i0}\in S_{l_i}$ and $w_i\in W$. By defining

$$X_i(t):=z_i(t)+\left(1-\frac{t}{\delta t}\right)(x_{i0}-x_{i,G}),\, t\in[0,T_{\max}), \tag{5.4}$$

with $z_i(t)=\chi_i(t)+tw_i$ as given in (3.10), and taking into account the definition of $T(x_{i0},w_i)$ in (3.15), we want to show that $X_i(\cdot)$ remains in D for more than time δt. By virtue of our assumption that $T_{\max}>\delta t$, the latter is meaningful to verify and implies that $T(x_{i0},w_i)>\delta t$. We next show that $X_i(\cdot)$ coincides on a suitable time interval with the ith component of the solution of (3.1) by choosing appropriate initial conditions and feedback laws that satisfy (P1) and (P2).

Let $x_{i0}\in S_{l_i}$, $w_i\in W$, consider an arbitrary initial cell configuration $\mathbf{l}$ with $\mathrm{pr}_i(\mathbf{l})=\mathbf{l}_i$, $\mathbf{l}=(l_1,\ldots,l_N)$, and assign the feedback law $k_{i,\mathrm{pr}_i(\mathbf{l})}=k_{i,\mathbf{l}_i}$ (as the latter is given by (3.12)) to i and the feedback laws $k_{\ell,\mathrm{pr}_\ell(\mathbf{l})}:=0$ to the rest of the agents $\ell\in\mathcal{N}\setminus\{i\}$. It also follows from the assumptions of the lemma for i, and trivially for the other agents, that the feedback laws satisfy Properties (P1) and (P2). Thus, we can use the result of Proposition 4.7(i). By selecting an initial condition $x(0)\in D^N$ with $x_i(0)=x_{i0}$ and $x_\ell(0)\in S_{l_\ell}, \ell\in\mathcal{N}\setminus\{i\}$, and recalling that $w_i\in W$, we get from Proposition 4.7(i) that the ith component of the solution satisfies

$$x_i(t)\in D, \forall t\in[0,\tau), \tau:=\min\{\delta t, T(x_{i0},w_i)\}, \tag{5.5}$$

$$\lim_{t\to\tau^-} x_i(t)\in D. \tag{5.6}$$

We proceed by showing that $x_i(t)=X_i(t)$, for all $t\in[0,\tau)$, with τ as given in (5.5), or equivalently, that

$$x_i(t)=\chi_i(t)+tw_i+\left(1-\frac{t}{\delta t}\right)(x_{i0}-x_{i,G}), \forall t\in[0,\tau) \tag{5.7}$$

Indeed, from (3.10), (3.7), (3.1), (3.12), (3.8), and (3.6) we have that $\dot z_i(t)=F_{i,\mathbf{l}_i}(\chi_i(t))+w_i$, $\dot x_i(t)=F_{i,\mathbf{l}_i}(x_i(t))+k_{i,\mathbf{l}_i,2}(x_{i0})+k_{i,\mathbf{l}_i,3}(t;x_{i0},w_i)$. By recalling that $z_i(0)=x_{i,G}$, $x_i(0)=x_{i0}$, and that due to (3.15) and (5.5) it holds $\tau\le T(x_{i0},w_i)\le T_{\max}$, and thus $\chi_i(\cdot)$, $x_i(\cdot)$ and $k_{i,\mathbf{l}_i,3}(\cdot)$ are well defined on $[0,\tau)$, it follows from (3.13), (3.14), and (3.10) that $x_i(t)-z_i(t)=x_{i0}-x_{i,G}+\int_0^t[F_{i,\mathbf{l}_i}(x_i(s))-F_{i,\mathbf{l}_i}(\chi_i(s))+k_{i,\mathbf{l}_i,2}(x_{i0})+k_{i,\mathbf{l}_i,3}(s;x_{i0},w_i)-w_i]ds=(1-\frac{t}{\delta t})(x_{i0}-x_{i,G})+\int_0^t[F_{i,\mathbf{l}_i}(x_i(s))-F_{i,\mathbf{l}_i}(z_i(s)+(1-\frac{s}{\delta t})(x_{i0}-x_{i,G}))]ds, \forall t\in[0,\tau)$. Hence, we get from (A.14) that for all $t\in[0,\tau)$ it holds $|x_i(t)-z_i(t)-\left(1-\frac{t}{\delta t}\right)(x_{i0}-x_{i,G})|\le\int_0^t L_2\left|x_i(s)-z_i(s)-\left(1-\frac{s}{\delta t}\right)(x_{i0}-x_{i,G})\right|ds$. Application of the Gronwall Lemma to the continuous function $t\mapsto|x_i(t)-z_i(t)-\left(1-\frac{t}{\delta t}\right)(x_{i0}-x_{i,G})|$, $t\in[0,\tau)$ implies that $x_i(t)-z_i(t)-\left(1-\frac{t}{\delta t}\right)(x_{i0}-x_{i,G})=0$ on $[0,\tau)$. Hence, from (3.10) and the fact that $\tau\le T_{\max}$, i.e., that $z_i(t)=\chi_i(t)+tw_i$ for all $t\in[0,\tau)$ we derive that $x_i(t)-\chi_i(t)-tw_i-\left(1-\frac{t}{\delta t}\right)(x_{i0}-x_{i,G})=0$ on $[0,\tau)$. Thus, (5.7) holds.

We are now in position to prove that $T(x_{i0},w_i)>\delta t$. Indeed, suppose on the contrary that $T(x_{i0},w_i)\le\delta t$, which by virtue of the assumption that $T_{\max}>\delta t$ and (5.5), implies that $T(x_{i0},w_i)<T_{\max}$ and $\tau=T(x_{i0},w_i)$. From the latter, together with (5.4), (5.5), (5.6), and continuity of $X_i(\cdot)$, we get that $X_i(T(x_{i0},w_i))=\lim_{t\to T(x_{i0},w_i)^-}X_i(t)=\lim_{t\to T(x_{i0},w_i)^-}x_i(t)\in D$. Hence, from the deduction that $T(x_{i0},w_i)<T_{\max}$ and continuity of $X_i(\cdot)$, it follows from (5.4) that there exists $\varepsilon\in(0,T_{\max}-T(x_{i0},w_i)]$ such that $X_i(t)\in D$ for $t\in[T(x_{i0},w_i),T(x_{i0},w_i)+\varepsilon)$. Consequently, we get from (5.4) and (3.10) that $\chi_i(t)+tw_i+\left(1-\frac{t}{\delta t}\right)(x_{i0}-x_{i,G})\in D$ for $t\in[T(x_{i0},w_i),T(x_{i0},w_i)+\varepsilon)$ which contradicts (3.15). Thus, we conclude that $T(x_{i0},w_i)>\delta t$, which establishes (P3). □

6. Well posed space-time discretizations

In this section, we exploit the controllers introduced in (3.12) to provide sufficient conditions for well posed space-time discretizations. Consider again system (3.1), a cell decomposition $\mathcal{S}=\{S_l\}_{l\in\mathcal{I}}$ of D, a time step δt and let

$$d_{\max}:=\sup\{\mathrm{diam}(S_l), l\in\mathcal{I}\}, \tag{6.1}$$

which due to Definition 3.1 is well defined. We will call $d_{\max}$ the *diameter* of the cell decomposition. *Our goal is to determine sufficient conditions relating the Lipschitz constants L_1, L_2, the bounds M, $v_{\max}$ for the system's dynamics, as well as the space and time scales $d_{\max}$ and δt, which guarantee that the discretization $\mathcal{S}-\delta t$ is well posed.* According to Definition 4.3, establishment of a well posed discretization is based on the selection of appropriate feedback laws which guarantee outgoing transitions for all agents and their possible cell configurations. For each agent $i\in\mathcal{N}$ and cell configuration $\mathbf{l}_i=(l_i,l_{j_1},\ldots,l_{j_{N_i}})$ of i let $(x_{i,G},\mathbf{x}_{j,G})$ be a tuple of reference points as in (3.5). We consider the family of feedback laws given in (3.8), (3.13), (3.14), and parameterized by $x_{i0}\in S_{l_i}$ and $w_i\in W$. The function $F_{i,\mathbf{l}_i}(\cdot)$ is given in (3.6), and $\chi_i(\cdot)$ is the reference solution of the initial value problem (3.7), defined on $[0, T_{\max})$. Recall that the parameter λ in (3.9) provides the part of the free input that is exploited in order to increase the states that the agent can reach at the end of the transition interval. We also introduce an additional parameter μ which imposes a design requirement on the minimum number of transitions from each agent's cell configuration. Specifically, in Corollary 6.5 we show that the increasing integer valued function $\theta:\mathbb{R}_{\ge0}\to\mathbb{N}$, given as $\theta(\mu)=\lceil\mu^n\rceil$, is a lower bound for the number of possible successor cells in terms of μ. Thus, the parameters λ and μ provide a quantifiable tuning of the control design in terms of the input magnitude that is chosen for reachability purposes and the number of successor cells that are required from each configuration, respectively. Before proceeding to the desired sufficient conditions for well posed discretizations and their reachability properties, we prove the auxiliary Propositions 6.1 and 6.2. Proposition 6.1 below provides bounds on the hybrid control laws $k_{i,\mathbf{l}_i}(\cdot)$ in (3.12).

Proposition 6.1. *Consider a cell decomposition $\mathcal{S}$ of D with diameter $d_{\max}$ and a time step δt. Also, for each agent $i\in\mathcal{N}$ and cell configuration $\mathbf{l}_i=(l_i,l_{j_1},\ldots,l_{j_{N_i}})$ of i let $(x_{i,G},\mathbf{x}_{j,G})$ be a tuple of reference points as in (3.5) and consider the feedback law $k_{i,\mathbf{l}_i}(\cdot)$ in (3.12). Then, its components $k_{i,\mathbf{l}_i,1}(\cdot)$, $k_{i,\mathbf{l}_i,2}(\cdot)$, and $k_{i,\mathbf{l}_i,3}(\cdot)$ as given in (3.8), (3.13), and (3.14), respectively, satisfy the bounds*

$$|k_{i,\mathbf{l}_i,1}(x_i,\mathbf{x}_j)|\le L_1\sqrt{N_i}(R_{\max}+d_{\max}), \forall x_i\in D, \\ x_{j_m}\in(S_{l_{j_m}}+B(R_{\max}))\cap D, m=1,\ldots,N_i, \tag{6.2}$$

$$|k_{i,\mathbf{l}_i,2}(x_{i0})|\le\frac{1}{\delta t}d_{\max}, \forall x_{i0}\in S_{l_i}, \tag{6.3}$$

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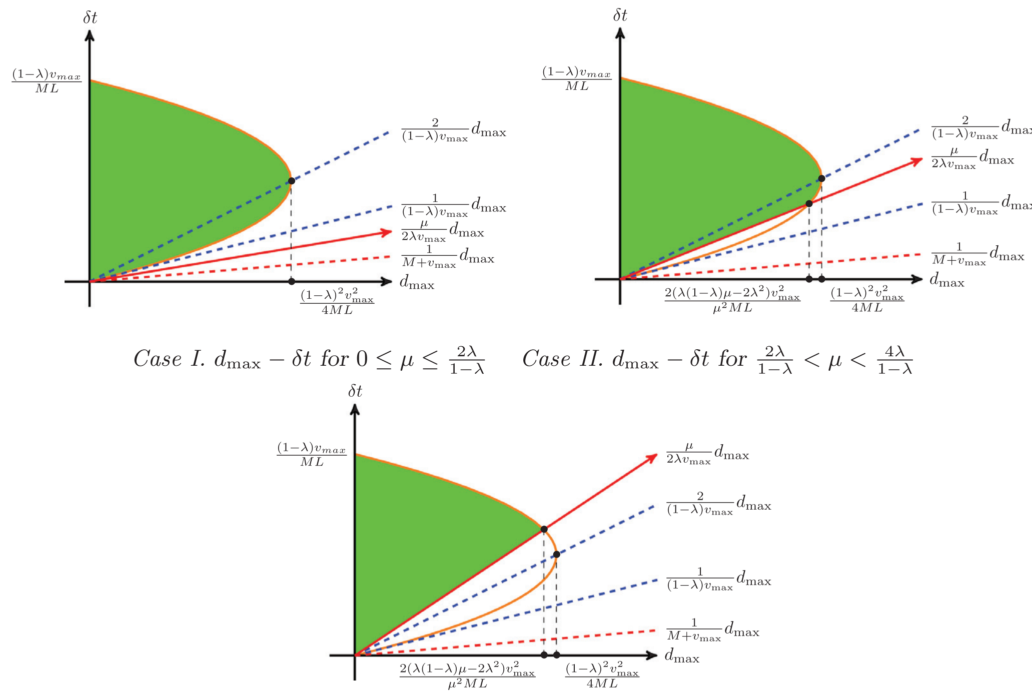

Case I. $d_{\max} - \delta t$ *for* $0 \leq \mu \leq \frac{2\lambda}{1-\lambda}$ *Case II.* $d_{\max} - \delta t$ *for* $\frac{2\lambda}{1-\lambda} < \mu < \frac{4\lambda}{1-\lambda}$

Case III. $d_{\max} - \delta t$ *for* $\mu \geq \frac{4\lambda}{1-\lambda}$

Fig. 3. Feasible $d_{\max} - \delta t$ regions.

$$|k_{i,\mathbf{l}_i,3}(t;x_{i0},w_i)| \leq L_2(\delta t\lambda v_{\max} + d_{\max}) + \lambda v_{\max},$$
$$\forall t \in [0,\delta t] \cap [0, T(x_{i0},w_i)), x_{i0} \in S_{l_i}, w_i \in W. \tag{6.4}$$

with $R_{\max}$ *as given in (4.1).*

Proof. Indeed, in order to show (6.2) let $\mathbf{x}_j \in D^{N_i}$ satisfying $x_{j_m} \in (S_{l_{j_m}} + B(R_{\max})) \cap D, m = 1, \ldots, N_i$. Then, for each $m = 1, \ldots, N_i$ there exists $y_{j_m} \in S_{l_{j_m}}$ with $|y_{j_m} - x_{j_m}| \leq R_{\max}$. Hence, from the latter together with (3.8) and (5.1), we get that $|k_{i,\mathbf{l}_i,1}(x_i,\mathbf{x}_j)| \leq L_1|(x_{j_1} - x_{j_1,G}, \ldots, x_{j_{N_i}} - x_{j_{N_i},G})| \leq L_1(\sum_{m=1}^{N_i}(|x_{j_m} - y_{j_m}| + |y_{j_m} - x_{j_m,G}|)^2)^{\frac{1}{2}} \leq L_1(\sum_{m=1}^{N_i}(R_{\max} + d_{\max})^2)^{\frac{1}{2}} = L_1\sqrt{N_i}(R_{\max} + d_{\max})$, which establishes (6.2). Furthermore, by recalling that $x_{i,G} \in S_{l_i}$, it follows directly from (3.13) that $|k_{i,\mathbf{l}_i,2}(x_{i0})| = \frac{1}{\delta t}|x_{i0} - x_{i,G}|$ and hence, that (6.3) is satisfied. Finally, for $k_{i,\mathbf{l}_i,3}(\cdot)$ we get from (3.14) and (A.14) that $|k_{i,\mathbf{l}_i,3}(t;x_{i0},w_i)| \leq L_2|(\chi_i(t) + tw_i + (1 - \frac{t}{\delta t})(x_{i0} - x_{i,G})) - \chi_i(t)| + |w_i|$, which due to (3.9) implies (6.4). □

Based on the result of Proposition 6.1 we next provide conditions on $d_{\max}$ and δt which guarantee that the feedback laws $k_{i,\mathbf{l}_i}(\cdot)$ satisfy Property (P). Additionally, it is shown that the radius r introduced in (3.11) satisfies a design requirement which is related later in Corollary 6.5 to a lower bound on the number of possible transitions through the parameter μ.

Proposition 6.2. *Consider a cell decomposition* $\mathcal{S}$ *of* D *with diameter* $d_{\max}$, *a time step* δt, *the parameters* $\lambda \in (0,1)$, $\mu \geq 0$ *and define*

$$L := \max\{3L_2 + 4L_1\sqrt{N_i}, i \in \mathcal{N}\}, \tag{6.5}$$

with L_1 *and* L_2 *as given in (5.1) and (5.2). We assume that* λ, μ, $d_{\max}$ *and* δt *satisfy the following restrictions, as provided by the three cases below (see also Fig. 3):*

Case I. $0 \leq \mu \leq \frac{2\lambda}{1-\lambda}$.

$$d_{\max} \in \left(0, \frac{(1-\lambda)^2 v_{\max}^2}{4ML}\right], \tag{6.6}$$

$$\delta t \in \left[\frac{(1-\lambda)v_{\max} - \sqrt{(1-\lambda)^2 v_{\max}^2 - 4MLd_{\max}}}{2ML}, \frac{(1-\lambda)v_{\max} + \sqrt{(1-\lambda)^2 v_{\max}^2 - 4MLd_{\max}}}{2ML}\right]. \tag{6.7}$$

Case II. $\frac{2\lambda}{1-\lambda} < \mu < \frac{4\lambda}{1-\lambda}$.

$$d_{\max} \in \left(0, \frac{2(\lambda(1-\lambda)\mu - 2\lambda^2)v_{\max}^2}{\mu^2 ML}\right], \tag{6.8}$$

$$\delta t \in \left[\frac{\mu}{2\lambda v_{\max}}d_{\max}, \frac{(1-\lambda)v_{\max} + \sqrt{(1-\lambda)^2 v_{\max}^2 - 4MLd_{\max}}}{2ML}\right], \tag{6.9}$$

or

$$d_{\max} \in \left(\frac{2(\lambda(1-\lambda)\mu - 2\lambda^2)v_{\max}^2}{\mu^2 ML}, \frac{(1-\lambda)^2 v_{\max}^2}{4ML}\right] \tag{6.10}$$

and δt *satisfies (6.7).*

Case III. $\mu \geq \frac{4\lambda}{1-\lambda}$. $d_{\max}$ *and* δt *satisfy (6.8) and (6.9), respectively.*

Then, the intervals in Cases I, II, III are well defined, and for each agent $i \in \mathcal{N}$, *cell configuration* $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{N_i}})$ *of* i, *and tuple of reference points* $(x_{i,G}, \mathbf{x}_{j,G})$ *as in (3.5) the solution* $\chi_i(t)$ *of (3.7) is defined and remains in* D *for all* $t \in [0,\delta t]$. *In addition, the feedback law* $k_{i,\mathbf{l}_i}(\cdot)$ *in (3.12) satisfies property (P) and the distance* r *as defined in (3.11) satisfies the design requirement*

$$r \geq \frac{\mu}{2}d_{\max}. \tag{6.11}$$

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Proof. The proof of the fact that the intervals in Cases I, II, III are well defined follows from elementary calculations. In addition, we deduce from Lemma 5.1 that for any tuple of reference points $(x_{i,G}, \mathbf{x}_{j,G})$ as in (3.5) the solution $\chi_i(t)$ of (3.7) is defined and remains in D for all $t\in[0, T_{\max})$, and by virtue of (5.3) and the assumed bounds on δt in Cases I, II, III, that

$$T_{\max} > \delta t, \tag{6.12}$$

establishing thus that $\chi_i(t)\in D$ for all $t\in[0, \delta t]$. We break the subsequent proof in the following steps.

STEP 1: Verification of Properties (P1) and (P2) for the feedback law (3.12) for $d_{\max} - \delta t$ as given by Cases I, II, III, in conjunction with the design requirement (6.11). In this step we prove that the proposed feedback law (3.12) satisfies Properties (P1) and (P2). Verification of (P1) is straightforward. Thus, we proceed to show that (4.2) holds, which implies (P2), and simultaneously, that (6.11) is fulfilled. By taking into account (3.12) and the result of Proposition 6.1, namely, (6.2), (6.3), and (6.4), we need to prove that

$$L_1\sqrt{N_i}(R_{\max}+d_{\max}) + \frac{1}{\delta t}d_{\max} + L_2(\delta t\lambda v_{\max} + \ d_{\max}) \\ +\lambda v_{\max} \le v_{\max}. \tag{6.13}$$

By recalling (4.1), (3.4) and the fact that $\lambda\in(0,\ 1)$ we get that $\delta t\lambda v_{\max} \le \frac{R_{\max}}{2}$. Also, from the fact that $d_{\max}$ and δt are selected according to the Cases I, II, III, it is not hard to deduce that $d_{\max} \le R_{\max}$. Hence, it suffices instead of (6.13) to show that $(2L_1\sqrt{N_i} + \frac{3}{2}L_2)R_{\max} + \frac{1}{\delta t}d_{\max} \le (1-\lambda)v_{\max}$, which by virtue of (4.1) is equivalent to

$$(M+v_{\max})\left(2L_1\sqrt{N_i} + \frac{3}{2}L_2\right)\delta t^2 - (1-\lambda)v_{\max}\delta t + d_{\max} \le 0. \tag{6.14}$$

By taking into account (3.4), it suffices instead of (6.14) to show that $M(3L_2 + 4L_1\sqrt{N_i})\delta t^2 - (1-\lambda)v_{\max}\delta t + d_{\max} \le 0$ which by virtue of (6.5) follows from

$$ML\delta t^2 - (1-\lambda)v_{\max}\delta t + d_{\max} \le 0. \tag{6.15}$$

It then follows from elementary calculations that for all Cases I, II, and III as in the statement of the proposition the requirements (6.11) and (6.15) are satisfied and hence, that (P2) holds.

STEP 2: Verification of Property (P3). In order to show (P3), it suffices to prove that for the given selection of $\lambda\in(0,\ 1)$, $\mu\ge 0$, $d_{\max}$ and δt as provided by Cases I, II, III, the agent i, and the cell configuration $\mathbf{l}_i$ it holds $T(x_{i0}, w_i) > \delta t$, for all $x_{i0}\in S_{l_i}$ and $w_i\in W$. The latter follows directly from Lemma 5.2, by taking into account (6.12) and that due to Step 1, the feedback law $k_{i,\mathbf{l}_i}(\cdot)$ in (3.12) satisfies Properties (P1) and (P2). □

We are now in position to state our main result on sufficient conditions for well posed abstractions.

Theorem 6.3. *Consider a cell decomposition $\mathcal{S}$ of D with diameter $d_{\max}$, a time step δt, the parameters $\lambda\in(0,\ 1)$, $\mu\ge 0$ and assume that λ, μ, $d_{\max}$ and δt satisfy the restrictions of Proposition 6.2. Then, the space-time discretization is well posed for the multi-agent system (3.1). In particular, by selecting for each agent $i\in\mathcal{N}$ and cell configuration $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{N_i}})$ of i a tuple of reference points $(x_{i,G}, \mathbf{x}_{j,G})$ as in (3.5) and the corresponding control law $k_{i,\mathbf{l}_i}(\cdot)$ in (3.12), it holds $\mathrm{Post}_i(\mathbf{l}_i)\ne\emptyset$. In addition, each corresponding reference trajectory $\chi_i(\cdot)$ of i as given by (3.7) satisfies*

$$B(\chi_i(\delta t); r)\subset D \tag{6.16}$$

and it holds

$$\mathrm{Post}_i(\mathbf{l}_i) = \{l\in\mathcal{I} : S_l\cap B(\chi_i(\delta t); r)\ne\emptyset\}, \tag{6.17}$$

where r is defined in (3.11). Furthermore, for each i, $\mathbf{l}_i$ and $l\in\mathcal{I}$ the set of parameters $[w_i]_{(\mathbf{l}_i,l)}$ in (4.5) for the specification of the transitions in TS_i is given as

$$[w_i]_{(\mathbf{l}_i,l)} = \left\{\frac{x-\chi_i(\delta t)}{\delta t} : x\in S_l\cap B(\chi_i(\delta t); r)\right\}. \tag{6.18}$$

Proof. For the proof, pick $i\in\mathcal{N}$, $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{N_i}})$, $(x_{i,G}, \mathbf{x}_{j,G})$ as in (3.5) and notice that by virtue of Proposition 6.2, the reference trajectory $\chi_i(\cdot)$ is well defined on $[0,\ \delta t]$. In addition, consider the control law $k_{i,\mathbf{l}_i}(\cdot)$ in (3.12). Then, it follows again from Proposition 6.2 that the latter satisfies Property (P). Next, notice that by virtue of (3.9) and (3.11) it holds $w_i\in W \iff \chi_i(\delta t) + \delta t w_i\in B(\chi_i(\delta t); r)$, which implies that

$$B(\chi_i(\delta t); r)\ni x\mapsto w_i := \frac{x-\chi_i(\delta t)}{\delta t}\in W \text{ is a bijection.} \tag{6.19}$$

In order to prove the theorem, we need to verify that (6.16), (6.17), and (6.18) are fulfilled.

Proof of (6.16). In order to show (6.16), pick $x\in B(\chi_i(\delta t);\ r)$, w_i as in (6.19) and recall that the control law $k_{i,\mathbf{l}_i}(\cdot)$ satisfies Property (P). Then we have from (6.19) that $w_i\in W$ and thus, we get from Property (P3) applied with $x_{i0} = x_{i,G}$ and the selected parameter w_i that $T(x_{i,G}, w_i) > \delta t$. From the latter and (3.15) we obtain that $\chi_i(\delta t) + \delta t w_i\in D$, which by virtue of (6.19) implies that $x\in D$ and establishes validity of (6.16).

Proof of (6.17). Note that due to Definition 4.4, it holds that $\mathrm{Post}(\mathbf{l}_i) = \{l\in\mathcal{I} : \exists w_i\in W$ such that $k_{i,\mathbf{l}_i}(\cdot), w_i, l$ satisfy Condition (C)$\}$. Hence, in order to show (6.17), it suffices to prove that

$$\exists\ w_i\in W \text{ s.t. } k_{i,\mathbf{l}_i}(\cdot), w_i, l \text{ satisfy Condition (C)} \\ \iff S_l\cap B(\chi_i(\delta t); r)\ne\emptyset. \tag{6.20}$$

In order to prove (6.20) we first establish the following claim.

Claim II. Consider the control law $k_{i,\mathbf{l}_i}(\cdot)$ above and pick any $w_i\in W$. Then, for any initial cell configuration $\mathbf{l}$ with $\mathrm{pr}_i(\mathbf{l}) = \mathbf{l}_i$ and selection of feedback laws in (4.4) which satisfy (P) the following hold. The solution of the closed-loop system (3.1), (3.12), (4.3), (4.4), with the selected parameter w_i for $k_{i,\mathbf{l}_i}(\cdot)$, is well defined on $[0,\ \delta t]$ and satisfies

$$x_i(\delta t)(:= x_i(\delta t, x(0))) = \chi_i(\delta t) + \delta t w_i = z_i(\delta t), \tag{6.21}$$

for all $x(0)\in D^N$ with $x_{m0}\in S_{l_m}$, $m\in\mathcal{N}$ and $w_m\in W$, $m\in\mathcal{N}\setminus\{i\}$, with $z_i(\cdot)$ as given by (3.10) (see also Fig. 2 in Section 3).

Proof of Claim II. Let $k_{i,\mathbf{l}_i}(\cdot)$ and w_i as in the statement of Claim II. We first note that due to Proposition 4.7(iia), the solution of the closed-loop system is defined and remains in D^N on the whole interval $[0,\ \delta t]$. In addition, the fact that $x_i(\delta t) = z_i(\delta t)$ follows directly if we show that $x_i(t) = z_i(t) + \left(1-\frac{t}{\delta t}\right)(x_{i0} - x_{i,G})$, $\forall t\in[0, \delta t]$. The proof of the latter is based precisely on the arguments used for the proof of (5.7) in Lemma 5.2 and is therefore omitted. Hence, we conclude that $x_i(\delta t) = z_i(\delta t)$ and the proof of Claim II is complete.

From Claim II we deduce for any cell configuration $\mathbf{l}_i$ with corresponding control law $k_{i,\mathbf{l}_i}(\cdot)$, vector $w_i\in W$ and cell index $l\in\mathcal{I}$ that

$$k_{i,\mathbf{l}_i}(\cdot), w_i, l \text{ satisfy Condition (C)} \iff \chi_i(\delta t) + \delta t w_i\in S_l. \tag{6.22}$$

In order to verify (6.22) assume that $k_{i,\mathbf{l}_i}(\cdot)$, w_i, l satisfy Condition (C), which implies that for each cell configuration $\mathbf{l}$ with $\mathrm{pr}_i(\mathbf{l}) = \mathbf{l}_i$ and selection of feedback laws in (4.4) which satisfy (P) the solution of (3.1), (3.12), (4.3), (4.4) satisfies $x_i(\delta t)\in S_l$. In addition, from Claim II, (6.21) holds, and hence, $\chi_i(\delta t) + \delta t w_i\in S_l$. Conversely, assume that for $\mathbf{l}_i$, w_i, l it holds $\chi_i(\delta t) + \delta t w_i\in S_l$. Then, given the corresponding control law $k_{i,\mathbf{l}_i}(\cdot)$ and the vector w_i, it follows from Claim II that for each cell configuration $\mathbf{l}$ with $\mathrm{pr}_i(\mathbf{l}) = \mathbf{l}_i$ and selection of feedback laws in (4.4) which satisfy (P) the solution of (3.1), (3.12), (4.3), (4.4) satisfies $x_i(\delta t) = \chi_i(\delta t) + \delta t w_i$. Hence, we obtain that $x_i(\delta t) \in S_l$ which completes the proof of (6.22).

In order to show (6.20), assume that there exists $w_i \in W$ such that $k_{i,\mathbf{l}_i}(\cdot)$, w_i, l satisfy Condition (C). Then, it follows from (6.22) that $\chi_i(\delta t) + \delta t w_i \in S_l$. In addition, since $w_i \in W$, we get from (6.19) that also $\chi_i(\delta t) + \delta t w_i \in B(\chi_i(\delta t); r)$ and hence, that $S_l \cap B(\chi_i(\delta t); r) \neq \emptyset$. Conversely, assume that $S_l \cap B(\chi_i(\delta t); r) \neq \emptyset$ and pick $x \in S_l \cap B(\chi_i(\delta t); r) \neq \emptyset$. Let w_i as given by (6.19) and note that $\chi_i(\delta t) + \delta t w_i = x \in S_l$. Thus, we obtain from (6.22) that $k_{i,\mathbf{l}_i}(\cdot)$, w_i, l satisfy Condition (C), which also verifies the inverse implication of (6.20).

Proof of (6.18). In order to show (6.18), note that due to Definition 4.4 it holds that $\mathrm{Post}_i(\mathbf{l}_i) = \{l \in \mathcal{I} : [w_i]_{(\mathbf{l}_i,l)} \neq \emptyset\}$. Thus, form (6.17), we get that

$$[w_i]_{(\mathbf{l}_i,l)} \neq \emptyset \iff S_l \cap B(\chi_i(\delta t); r) \neq \emptyset, \tag{6.23}$$

which directly implies (6.18) for the case where $S_l \cap B(\chi_i(\delta t); r) = \emptyset$. Next, assume that $S_l \cap B(\chi_i(\delta t); r) \neq \emptyset$ and let $w_i \in [w_i]_{(\mathbf{l}_i,l)}$. Then, it follows from (4.5) that $k_{i,\mathbf{l}_i}(\cdot)$, w_i, l satisfy Condition (C) and thus, from (6.22), that $x := \chi_i(\delta t) + \delta t w_i \in S_l$. Also, due to (6.19) it holds $x \in B(\chi_i(\delta t); r)$. Thus we get that $w_i \in \{\frac{x - \chi_i(\delta t)}{\delta t} : x \in S_l \cap B(\chi_i(\delta t); r)\}$. Conversely, assume that $w_i = \frac{x - \chi_i(\delta t)}{\delta t}$ for certain $x \in S_l \cap B(\chi_i(\delta t); r)$, implying that $\chi_i(\delta t) + \delta t w_i = x \in S_l$. Then, it follows from (6.22) that $k_{i,\mathbf{l}_i}(\cdot)$, w_i, l satisfy Condition (C), which by virtue of (4.5) implies that $w_i \in [w_i]_{(\mathbf{l}_i,l)}$. The proof is now complete. □

Remark 6.4. Given a well posed discretization and a transition $l_i \overset{(\mathbf{l}_i,[w_i])}{\longrightarrow}_i l_i'$ of agent i, the set $[w_i] = [w_i]_{(\mathbf{l}_i,l_i')}$ of parameters which enable this transition through the control law $k_{i,\mathbf{l}_i}(\cdot)$ can be geometrically visualized with the aid of Fig. 2. Specifically, $[w_i]$ is the subset of $W = B(\lambda v_{\max})$ given by the depicted intersection of the ball $B(\chi_i(\delta t); r)$ and the successor cell $S_{l_i'}$ in Fig. 2, being translated by $-\chi_i(\delta t)$ and then dilated by $\frac{1}{\delta t}$.

The following corollary provides a lower bound for the minimum number of cells each agent can reach in time δt, depending on the selection of the design parameter μ for the space-time discretization.

Corollary 6.5. *Consider a cell decomposition $\mathcal{S}$ of D with diameter $d_{\max}$, a time step δt, and parameters $\lambda \in (0, 1)$, $\mu \geq 0$ such that the hypotheses of Theorem 6.3 are fulfilled. Then, for each agent $i \in \mathcal{N}$ and each cell configuration of i, there exist at least $\theta(\mu) := \lceil \mu^n \rceil$ possible discrete transitions.*

Proof. The proof is given in the Appendix. □

7. Exploitation of the abstractions for control synthesis

In this section, we clarify how the results of the paper can be leveraged for controller synthesis under high level specifications assigned to the agents of system (3.1). We also discuss how the individual discrete models of the agents can reduce the computational burden of centralized solutions in specific cases.

For the general case, assume that certain tasks have been assigned to the agents. Then, the proposed abstractions can be utilized for the derivation of satisfying plans and their execution via sequences of feedback controllers through the following procedure:

Step 1. Given the Lipschitz constants L_1, L_2 and the bounds M, $v_{\max}$ on the agents' dynamics, pick design parameters λ, μ and select a well posed space-time discretization $\mathcal{S} - \delta t$ for the multi-agent system based on Theorem 6.3.

Step 2. Fix a reference point for each cell S_l of the decomposition $\{S_l\}_{l\in\mathcal{I}}$. Then, derive the transition system TS_i of each agent i as follows. For each cell configuration $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{N_i}})$ compute the endpoint $\chi_i(\delta t)$ of the reference trajectory (3.7) at time δt, corresponding to the reference points $x_{i,G}, x_{j_1,G}, \ldots, x_{j_{N_i},G}$ of the cells $S_{l_i}, S_{l_{j_1}}, \ldots, S_{l_{j_{N_i}}}$, as selected at the beginning of Step 2. Then, specify the cells which have nonempty intersection with $B(\chi_i(\delta t); r)$, in order to obtain all the transitions $l_i \overset{(\mathbf{l}_i,[w_i])}{\longrightarrow}_i l_i'$ to the cells in (6.17), where $[w_i] = [w_i]_{(\mathbf{l}_i,l_i')}$ is given by (6.18). Also, recall that according to Remark 4.5(i) each action term $[w_i]$ of a transition is uniquely determined through the agent's successor cell and note that it does not need to be explicitly specified, as will be clarified in Step 4.

Step 3. Find a path $\mathbf{l}^0\mathbf{l}^1\mathbf{l}^2\cdots$ in the product transition system $TS_{\mathcal{P}}$ of Definition 4.9 which satisfies the plan and project it for each agent i to a sequence of transitions $l_i^0 \overset{(\mathbf{l}_i^0,[w_i]^0)}{\longrightarrow}_i l_i^1 \overset{(\mathbf{l}_i^1,[w_i]^1)}{\longrightarrow}_i l_i^2 \ldots$.

Step 4. Select the control laws to implement the individual transitions by the continuous system as follows. For each transition $l_i^m \overset{(\mathbf{l}_i^m,[w_i]^m)}{\longrightarrow}_i l_i^{m+1}$ pick any $w_i \in [w_i]^m = [w_i]_{(\mathbf{l}_i^m, l_i^{m+1})}$, with the latter as given by (6.18), and apply the control law (3.12) with the selected parameter w_i. Note that according to (6.18), a desired w_i can be obtained by any convenient selection of a point x in the intersection of $B(\chi_i(\delta t); r)$ and the successor cell $S_{l_i^{m+1}}$.

We next provide certain cases where due to the network structure or the nature of the tasks, it is either required to compose a strict subset of subsystems, or to exploit the individual discrete models in a sequential manner. Thus, it is possible to avoid their global composition and reduce the memory and time resources required for the derivation of satisfying plans.

As a first case, assume that the agents' network forms an acyclic graph and without any loss of generality that it is a directed tree, with agent i as the root, which has no couplings in its dynamics. Then, we can first select the set of discrete paths of i which satisfy its specification and as a next step, use all these paths as actions for the transition systems of i's children in order to determine the paths which satisfy their plans. Proceeding analogously, we use for each descendant of i, the selected paths of its ancestor in order to derive all the satisfying paths of its specification. This approach can reduce significantly the memory storage that is required for the transitions compared to the centralized case and lead to a linear complexity of task verification with respect to the number of agents.

Assume next the case where a task assigned to agent i needs to be verified within a few number of time steps, as for instance a short term reachability goal. For a large network, this implies that for most agents, their distance in the graph from i will be larger than the number of time steps required for the verification of the task, which are e.g., $\kappa \in \mathbb{N}$. Notice next, that by using the transition systems of the agent and its neighbors, we can find all the one step successor cells of i and their corresponding one step actions, i.e., the possible successor cells of its neighbors. Then, by taking all combinations of the one step successor cells and their corresponding actions, it is possible to evaluate the agent's reachable cells in two time steps. This approach can be recursively applied up to κ time steps ahead and requires the discrete positions of the agents with distance up to κ in the network from i. Hence, it provides a partially decentralized framework, suitable to address motion planning problems from a bottom up perspective, i.e., at the agent level by leveraging local network information.

The above cases are only two examples of when the proposed distributed abstractions' scheme can be used for high level multi-agent task planning and yield guaranteed reduced computational complexity with respect to the centralized approach that utilizes the full product composition. Investigating the full spectrum of high level specification classes that can be treated with guaranteed reduced complexity under the proposed distributed framework is a topic of ongoing work.

8. Example and simulation results

As an illustrative example we consider a system of four agents whose states x_1, x_2, x_3, x_4 lie inside the circular domain $\mathrm{int}(B(R))(=\{x\in\mathbb{R}^2:|x|<R\})$ with center zero and radius $R>0$. Their dynamics are given as:

$$\begin{aligned}\dot{x}_1&=\mathrm{sat}_\rho(x_2-x_1)+g(x_1)+\upsilon_1,\\ \dot{x}_2&=g(x_2)+\upsilon_2,\\ \dot{x}_3&=\mathrm{sat}_\rho(x_2-x_3)+g(x_3)+\upsilon_3,\\ \dot{x}_4&=\mathrm{sat}_\rho(x_3-x_4)+g(x_4)+\upsilon_4,\end{aligned}\tag{8.1}$$

where the function $\mathrm{sat}_\rho:\mathbb{R}^2\to\mathbb{R}^2$ is defined as $\mathrm{sat}_\rho(x):=x$ if $|x|\le\rho$; $\mathrm{sat}_\rho(x):=\frac{\rho}{|x|}x$, if $|x|>\rho$. The agents' neighbors' sets in this example are $\mathcal{N}_1=\{2\}$, $\mathcal{N}_2=\emptyset$, $\mathcal{N}_3=\{2\}$, $\mathcal{N}_4=\{3\}$ and specify the corresponding network topology. The constant $\rho>0$ in (8.1) satisfies $\rho\le R$ and represents a bound on the distance between agents 1, 2, and agents 2, 3, that we will require the system to satisfy during its evolution. The function $g:\mathrm{int}(B(R))\to\mathbb{R}^2$ is defined as

$$g(x):=\begin{cases}0, & \text{if } |x|<R-\frac{\rho}{2}-\varepsilon,\\ ((R-\frac{\rho}{2}-\varepsilon)-|x|)\frac{x}{|x|}, & \text{if } R-\frac{\rho}{2}-\varepsilon\le|x|<R-\varepsilon,\\ -\frac{\rho}{2}\frac{x}{|x|}, & \text{if } R-\varepsilon\le|x|<R\end{cases}\tag{8.2}$$

for certain $\varepsilon<\rho$ and determines for each agent a repulsive vector field from the boundary of $\mathrm{int}(B(R))$. Then, by selecting $\upsilon_{\max}=\frac{\rho}{2}$, it can be deduced (along the lines of the corresponding result in [7]) that the circular domain remains invariant for the dynamics of the system.

We next show that if the initial distances between agents 1 and 2 (and similarly for agents 2 and 3) is less than ρ, it will also remain less than ρ for all positive times, for an appropriate bound on the magnitude of the free input terms υ_i. By selecting the energy function $V(x_1,x_2):=\frac{1}{2}|x_1-x_2|^2$ and evaluating its derivative along the right hand side of (8.1), we obtain that

$$\begin{aligned}&\dot{V}\le-(\rho-2\upsilon_{\max})|x_1-x_2|+\langle x_1-x_2,g(x_1)-g(x_2)\rangle,\\ &\text{if } |x_1-x_2|\ge\rho\end{aligned}\tag{8.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^2$. By recalling that $\upsilon_{\max}=\frac{\rho}{2}$, we get from (8.3) that

$$\dot{V}\le\langle x_1-x_2,g(x_1)-g(x_2)\rangle,\ \text{if } |x_1-x_2|\ge\rho\tag{8.4}$$

In addition, from the definition of $g(\cdot)$, we have for any x, $y\in\mathrm{int}(B(R))$ with $|x|\ge|y|$, that $g(x)=-ax$, $g(y)=-by$ for some $a\ge b\ge0$. Thus, we obtain that $\langle x-y,g(x)-g(y)\rangle=\langle x-y,-ax+by\rangle=\langle x-y,-(a-b)x-b(x-y)\rangle=-b|x-y|^2-(a-b)(|x|^2-\langle x,y\rangle)\le-(a-b)(|x|^2-|x||y|)\le0$. Consequently, it follows from (8.4) that $\dot{V}\le0$ when $|x_1-x_2|\ge\rho$, which implies that $|x_1(t)-x_2(t)|\le\rho$ for all positive times, given that $|x_1(0)-x_2(0)|\le\rho$. Analogously, by considering the function $V(x_2,x_3):=\frac{1}{2}|x_2-x_3|^2$ it follows that the same holds for $|x_2(t)-x_3(t)|$. Finally, we obtain from (8.1) and (8.2) the following dynamics bounds and Lipschitz constants in (3.2), (5.1) and (5.2), respectively: $M=\frac{3}{2}\rho$, $L_1=1$, $L_2=2$. Thus, it follows that system (8.1) satisfies all requirements for the derivation of well posed discretizations.

In this example, it is also assumed that the reference point of each cell of the square partition is the center of the square. This enables us to obtain the following improved bounds on the feedback laws in (3.12), for their corresponding values of t, x_i, $\mathbf{x}_j$ and w_i: $|k_{i,\mathbf{l}_i,1}(x_i,\mathbf{x}_j)|\le L_1((M+\upsilon_{\max})\delta t+\frac{d_{\max}}{2})$, $|k_{i,\mathbf{l}_i,2}(x_{i0})|\le\frac{d_{\max}}{2\delta t}$, $|k_{i,\mathbf{l}_i,3}(t;x_{i0},w)|\le L_2(\lambda\upsilon_{\max}\delta t+\frac{d_{\max}}{2})+\lambda\upsilon_{\max}$. Thus, in order to verify Property (P2), we need to select $d_{\max}$ and δt satisfying $L_1((M+\upsilon_{\max})\delta t+\frac{d_{\max}}{2})+\frac{d_{\max}}{2\delta t}+L_2(\lambda\upsilon_{\max}\delta t+\frac{d_{\max}}{2})+\lambda\upsilon_{\max}\le\upsilon_{\max}$. Equivalently, by virtue of the selected bound on $\upsilon_{\max}$, the bound on the system's feedback terms and the corresponding Lipschitz constants, it is required that $d_{\max}\le\rho\frac{(1-\lambda)\delta t-4\delta t^2}{3\delta t+1}$. Hence, by elementary calculations we obtain the time $\bar{\delta t}=\frac{-2+\sqrt{4+3(1-\lambda)}}{6}$, corresponding to the maximum possible diameter $\bar{d}_{\max}=\rho\frac{(1-\lambda)\bar{\delta t}-4\bar{\delta t}^2}{3\bar{\delta t}+1}$.

For the simulation results, we select the distance $\rho=10$, $\varepsilon=0.01$ and the radius of the circular domain $R=10$. We also assume that the agents 1, 2, 3 and 4, are initially located at $x_{10}=(5,-3)$, $x_{20}=(5,3)$, $x_{30}=(0,6)$ and $x_{40}=(-4,6)$, respectively. Thus, it follows that agents 1, 2, and 2, 3, satisfy the requirement on their initial relative distance. In the sequel we will focus on the behavior of the system for times $t\in[0,2]$. Given this time interval and a selection of the planning parameter $\lambda\in(0,1)$, we choose the time step δt as the largest possible time step not exceeding $\bar{\delta t}$ above, in such a way that the number of time steps $NT:=\frac{2}{\delta t}$ is a positive integer. We also choose the largest possible cell diameter $d_{\max}$ corresponding to δt and consider a square grid in $\mathbb{R}^2$. Each square has side length d, where d is the largest number not exceeding $\frac{\sqrt{2}}{2}d_{\max}$, such that the quotient $\frac{2R}{d}$ is an integer. Thus, we can form a cell decomposition of the circular domain D by defining as a cell each square in the grid which has nonempty intersection with D. In Figs. 4 and 5 we have plotted (half of) the grid lines, in order to illustrate how the grid is affected by the choice of λ. We next consider two cases for the motion of agent 2, which is unaffected by the coupled constraints.

Case I: It holds $\upsilon_2(t)=\upsilon_{2c}$, $\forall t\in[0,2]$, with $\upsilon_{2c}=(-3,-3)$.

Case II: It holds $\upsilon_2(t)=\upsilon_{2c}+\upsilon_{2d}(t)$, $\forall t\in[0,2]$, with υ_{2c} as above and $\upsilon_{2d}\in\mathcal{U}_d$, where $\mathcal{U}_d$ is the set of all piecewise continuous functions $\tilde{\upsilon}:[0,2]\to\mathbb{R}^2$ that satisfy $\tilde{\upsilon}(t)=\gamma(t)(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})$, with $-1\le\gamma(t)\le1$ for all $t\in[0,0.9]$ and $\gamma(t)=0$, for all $t\in(0.9,2]$.

Notice that in Case I we consider a pre-specified path for agent 2, by selecting a constant control, whereas in Case II we allow for the possibility to modify this path and superpose a motion perpendicular to it (up to certain bound) over the time interval [0,0.9]. Furthermore, in both cases the magnitude of $\upsilon_2(\cdot)$ is bounded by $\upsilon_{\max}(=5)$.

For Case I, we assign reachability goals to agents 1, 3 and 4 which should be fulfilled at the end of the time interval [0,2], given the selected path for agent 2. Specifically, we want agents 1, 3 and 4 to reach the corresponding boxes in the workspace that are depicted in Fig. 4. First, we sample the trajectory of 2 at the time instants $\kappa\delta t$, $\kappa=0,1\ldots,NT$ and specify the sequence $l_2^0l_2^1\ldots l_2^{NT}$ corresponding to the cells $S_{l_2^\kappa}$ with $x_2(\kappa\delta t,x_{20})\in S_{l_2^\kappa}$. Then, we exploit the individual transition systems of agents 1 and 3, in order to determine their reachable cells for the given sampled trajectory of agent 2. In particular, by denoting as l_1^0 the index of the cell where the initial state x_{10} of agent 1 belongs, we can evaluate the indices of its reachable cells at time $\kappa\delta t$ as $Q_1^\kappa=\mathrm{Post}_1(Q_1^{\kappa-1},l_2^{\kappa-1})$, $\kappa=0,1,\ldots,NT$, where $Q_1^0:=\{l_1^0\}$ and we have used the notational convention $\mathrm{Post}_1(Q_1,l_2):=\cup_{l_1\in Q_1}\mathrm{Post}_1(l_1,l_2)$ (recall that (l_1, l_2) stands for a cell configuration of agent 1). The approach fol-

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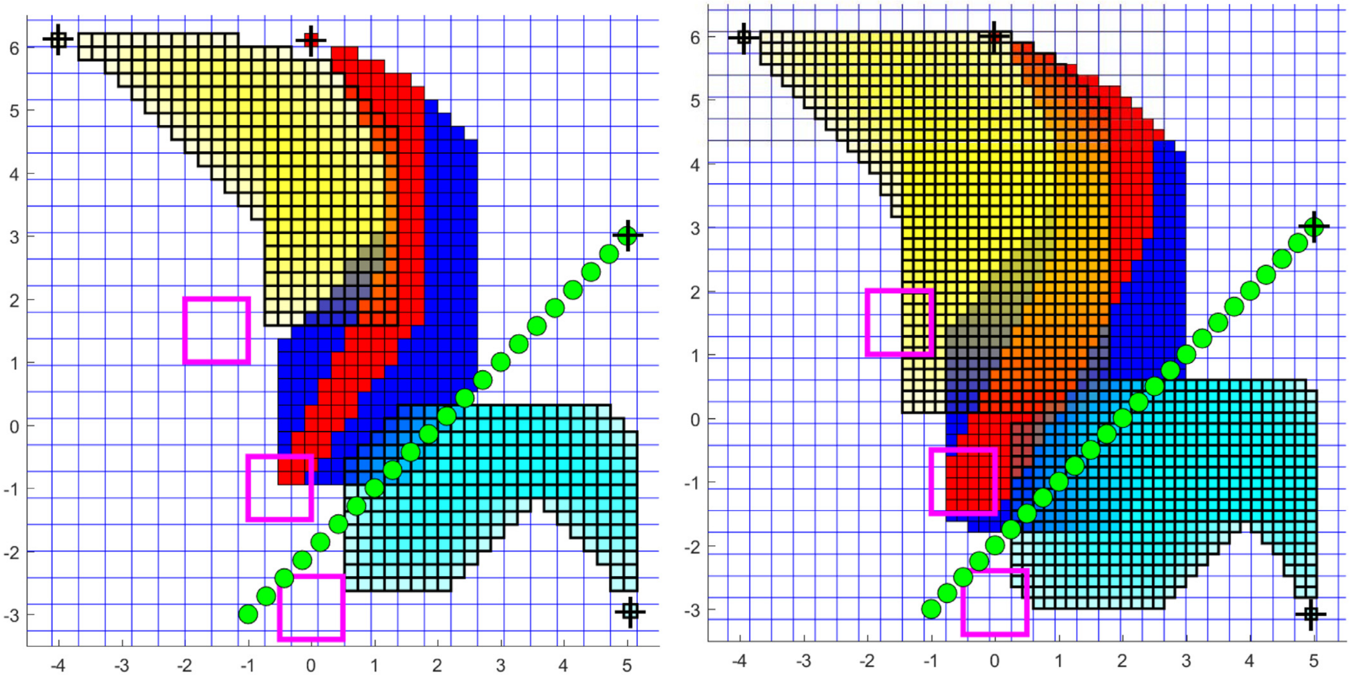
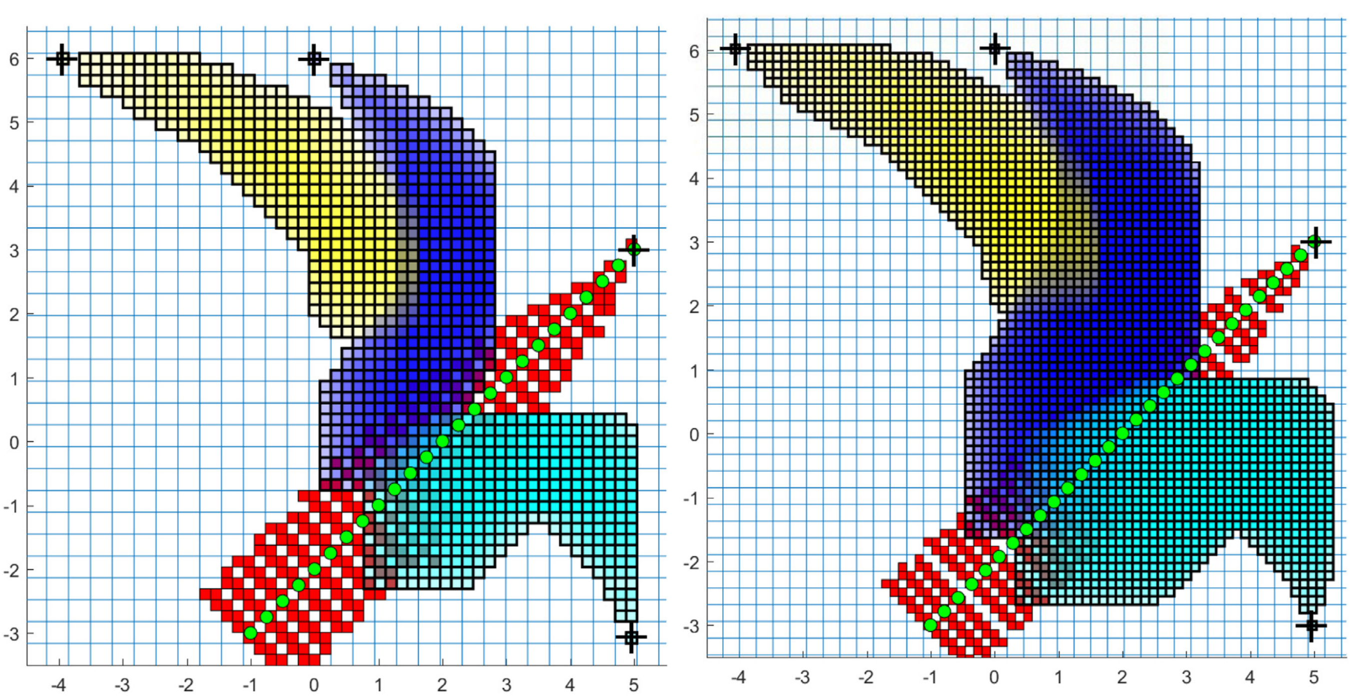


(i) (ii)

Fig. 4. Reachable cells of the agents for (i) $\lambda = 0.2$ and (ii) $\lambda = 0.3$. Agents 1, 3 and 4 are initially located at the bottom right, top center, and top left of the illustrated workspace, and their reachable cells are depicted with cyan, blue, and yellow, respectively. The circles denote the sampled trajectory of agent 2 as determined by Case I and the boxes the corresponding target sets of agents 1, 3 and 4. The union of all discrete paths of agent 3 which end in its target box are highlighted within the union of its reachable cells. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(i) (ii)

Fig. 5. Reachable cells of the agents for (i) $\lambda = 0.3$ and (ii) $\lambda = 0.4$. Agents 1, 3 and 4 are initially located at the bottom right, top center and top left of the illustrated workspace, and their reachable cells are depicted with cyan, blue, and yellow, respectively. The circles denote the nominal sampled trajectory of agent 2 and their nearby red cells represent the cells where agent 2 can lie at the sampling times, for all possible inputs of Case II. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

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lowed in this case is possible because agent 2 is decoupled from the other agents and the individual transition system of agent 1 depends only on the cell indices of agent 2. Similarly, we can evaluate the reachable cells of agent 1 and check whether it fulfills its reachability task. Next, by computing the reachable cells of agent 3 which lie in its target box at the final time step NT, we calculate the backward reachable cells of the agent in order to encode the discrete trajectories which fulfill its reachability goal, which are depicted with the red cells in Fig. 4. Then, we exploit the individual transition system of agent 4 in order to determine its reachable cells for all the possible trajectories of agent 3 that satisfy its reachability task. The corresponding simulation results are depicted in Fig. 4 for $\lambda = 0.2$ (left) and $\lambda = 0.3$ (right). The figure also illustrates the effect of the parameter λ in the accomplishment of the reachability goals, since for $\lambda = 0.2$ only agent 3 reaches its target box, whereas for $\lambda = 0.3$ all agents achieve their corresponding task.

Remark 8.1. It is noted that for Case I, the reachability goals of agents 1 and 3 can also be computed by exploiting existing reachability tools. However, one main advantage of exploiting the derived distributed symbolic models comes from the evaluation of the corresponding cells for agent 4. For this agent, the reachable cells are specified for all possible paths of agents 3 which lead to its target box. These paths are determined by exploiting agent's 3 individual transition system and their union is depicted through the red cells in Fig. 4. Thus, they provide enhanced reachability capabilities to agent 4 compared to the case where only one continuous trajectory of 3 which satisfies its task would have been used. In addition, for every selection of a discrete path, the controllers that realize the corresponding continuous trajectory can be designed in a straightforward way based on the four step procedure of Section 6. Finally, the same approach can be followed if we assume that there are more agents in the network and that the latter has still a tree structure (as in this example where agent 2 is the root). In that case, the complexity of determining the corresponding analogous reachability goals will be linear and not exponential to the number of agents in the network.

For Case II, we exploit the individual transition system of agents 1, 3 and 4 in order to obtain (an underapproximation of) the cells these agents can reach, irrespectively of the choice of v_{2d} for the free input of agent 2. In particular, we define the finite cell sequence $\{Q_2^\kappa\}_{\kappa\in\{0,1,\ldots,NT\}}$ as $Q_2^\kappa = \{l \in \mathcal{I} : \exists v_{2,d} \in \mathcal{U}_d$ with $x_2(\kappa\delta t, x_{20}; v_{2c} + v_{2d}(\cdot)) \in S_l\}$, which is depicted with the red cells in Fig. 5. Also, we inductively define for $\kappa = 0, 1, \ldots, NT$ the sets $Q_1^\kappa = \cup_{l_1\in Q_1^{\kappa-1}} \cap_{l_2\in Q_2^{\kappa-1}} \mathrm{Post}_1(l_1, l_2)$, $Q_3^\kappa = \cup_{l_3\in Q_3^{\kappa-1}} \cap_{l_2\in Q_2^{\kappa-1}} \mathrm{Post}_3(l_3, l_2)$ and $Q_4^\kappa = \cup_{l_4\in Q_4^{\kappa-1}} \cap_{l_3\in Q_3^{\kappa-1}} \mathrm{Post}_4(l_4, l_3)$, with $Q_1^0 = \{l_1^0\}$, $Q_3^0 = \{l_3^0\}$ and $Q_4^0 = \{l_4^0\}$ (we use the same notational convention as above for the operators $\mathrm{Post}_i(\cdot)$, and the notation l_i^0 for the initial cells of the agents $i = 1, 3, 4$). Next, consider any selection of sequences $l_1^0 l_1^1 \cdots l_1^{NT}$, $l_3^0 l_3^1 \cdots l_3^{NT}$ and $l_4^0 l_4^1 \cdots l_4^{NT}$, of agents 1, 3 and 4, that satisfy $l_1^\kappa \in Q_1^\kappa$, $l_3^\kappa \in Q_3^\kappa$ and $l_4^\kappa \in Q_4^\kappa$, respectively. Then, by taking into account the definition of the sets Q_i^κ, $i = 1, 3, 4$, the definition of the individual transition systems of agents 1, 3, 4, and the particular coupling between the agents in this example, we arrive at the following conclusion. For each agent 1, 3 and 4, it is possible to assign a sequence of control laws, such that each corresponding agent will reach the cells with indices l_1^κ, l_3^κ and l_4^κ at time $\kappa\delta t$, respectively, for any selection of the input v_{2d} of agent 2. In Fig. 5 we illustrate the union of the reachable cells of agents 1, 3 and 4 for $\lambda = 0.3$ and $\lambda = 0.4$, respectively. Notice that the underapproximation of agents' 1 and 3 reachable cells increases with the selection of the larger parameter λ, namely, with the exploitation of a larger part of the free input for planning. However, the same observation does not necessarily hold for the reachable cells of agent 4. The reason why the area covered by the reachable cells of agent 4 remains approximately the same, is that the corresponding area increases for agent 3 for larger values of λ. Thus, although the reachability properties of agent 4 are improved, this is compensated by the fact that each illustrated transition of agent 4 to a certain cell needs to be possible for an increasing number of different positions of agent 3.

The code for the simulation results has been implemented in MATLAB and the worst case running time for the illustrated results is of the order of 45 min, on a PC with an Intel(R) Core(TM) i7-4600U CPU @ 2.10 GHz processor.

9. Conclusions

We have provided a decentralized abstraction methodology for multi-agent systems and quantified space and time discretizations in order to obtain for each agent an individual transition system with multiple transition possibilities. The abstraction framework is based on the design of hybrid feedback control laws that take into account the agents' coupled constraints and guarantee the implementation of the discrete transitions by the continuous system.

Ongoing work includes the improvement of the acceptable choices of $d_{\max}$ and δt in order to obtain coarser abstractions and reduce the size of each agent's transition system. Another possible direction for complexity reduction is the modification of the current framework by considering event based online abstractions with updated choices of $d_{\max}$ and δt. Finally, it should be noted that while this paper provides informal indicators of how the results can be used for planning, we are currently formalizing a distributed planning methodology from high level specifications that builds on the derived abstractions.

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Appendix

In the Appendix we provide the proofs of Proposition 4.7, Lemma 5.1 and Corollary 6.5.

Proof of Proposition 4.7. *Proof of (i).* Let $w_i \in W$, $i \in \mathcal{N}$ and $x(0) \in D^N$ with $x_i(0) \in S_{l_i}$, $i \in \mathcal{N}$ be the initial condition of the closed-loop system. Then, it follows from the local Lipschitz property on the functions $f_i(\cdot)$ and the corresponding property on the mappings $k_{i,\mathrm{pr}_i(\mathbf{l})}(\cdot; x_{i0}, w_i)$ provided by (P1), that the dynamics of the closed-loop system are given by a locally Lipschitz function on D^N. Hence, there exists a unique solution $x(\cdot) = x(\cdot, x(0))$ to the initial value problem, which is defined and remains in D^N for all times in its right maximal interval of existence $[0, T_{\max})$. We proceed by proving that each component $x_i(\cdot)$, $i \in \mathcal{N}$ of the solution satisfies

$$x_i(t) \in (S_{l_i} + B(R_{\max})) \cap D, \forall t \in [0, \min\{T_{\max}, \tau\}). \tag{A.1}$$

Indeed, suppose on the contrary that (A.1) is violated, and hence, by taking into account that $x_i(t) \in D$ for all $t \in [0, T_{\max})$, that there exists $\iota \in \mathcal{N}$ and a time T with

$$T \in (0, \min\{T_{\max}, \tau\}) \text{ and } x_\iota(T) \notin S_{l_\iota} + B(R_{\max}). \tag{A.2}$$

By recalling that $x_i(0) \in S_{l_i}$, $i \in \mathcal{N}$, we may define

$$\tau_0 := \max\{t \in [0, T] : x_i(s) \in \mathrm{cl}(S_{l_i} + B(R_{\max})), \forall s \in [0, t], i \in \mathcal{N}\}. \tag{A.3}$$

Then, it follows from (4.10), (A.2), and (A.3) that there exists $\ell \in \mathcal{N}$ such that

$$x_\ell(\tau_0) \in \partial(S_{l_\ell} + B(R_{\max})) \tag{A.4}$$

and that

$$\tau_0 \le T < \tau \le \delta t. \tag{A.5}$$

It also follows from (A.3), (4.10), (A.5), and Property (P2) that for all $t \in [0, \tau_0]$ it holds

$$|k_{\ell,\mathrm{pr}_\ell(\mathbf{l})}(t, x_\ell(t), \mathbf{x}_{j(\ell)}(t); x_{\ell 0}, w_\ell)| \le v_{\max} \tag{A.6}$$

Hence, we get from (3.1), (4.1), (4.9), (A.6), and (A.5), which implies that $\tau_0 < \delta t$, that

$$\begin{aligned} |x_\ell(\tau_0) - x_{\ell 0}| &\le \int_0^{\tau_0} (|f_\ell(x_\ell(s), \mathbf{x}_{j(\ell)}(s))| \\ &\quad + |k_\ell, \mathrm{pr}_\ell(\mathbf{l})(s, x_\ell(s), \mathbf{x}_{j(\ell)}(s); x_{\ell 0}, w_\ell)|)ds \\ &\le \int_0^{\tau_0} (M + v_{\max}) ds < \delta t (M + v_{\max}) = R_{\max}. \end{aligned} \tag{A.7}$$

In order to finish the proof of (A.1) we exploit the following elementary fact.

Fact I. Consider a nonempty set $S \subset \mathbb{R}^n$ and a constant $R > 0$. Then for every $x \in \partial(S + B(R))$ it holds $|x - y| \ge R$, $\forall y \in S$.

Proof of Fact I. Indeed, suppose on the contrary that there exists $\bar{y} \in S$ with $|x - \bar{y}| \le R - \varepsilon$ for certain $\varepsilon > 0$. Then for all $\bar{x} \in \mathrm{int}(B(x; \varepsilon))$ we have

$$|\bar{x} - \bar{y}| \le |\bar{x} - x| + |x - \bar{y}| < \varepsilon + R - \varepsilon = R,$$

and hence, $\bar{x} \in S + B(R)$ for all $\bar{x} \in \mathrm{int}(B(x; \varepsilon))$, which implies that $x \notin \partial(S + B(R))$ and contradicts our statement.

By exploiting Fact I with $S = S_{l_\ell}$, $R = R_{\max}$, $y = x_{\ell 0}$ and $x = x_\ell(\tau_0)$ we deduce from (A.7) that $x_\ell(\tau_0) \notin \partial(S_{l_\ell} + B(R_{\max}))$ which contradicts (A.4), and provides validity of (A.1).

We now prove the following claim:

Claim I. It holds $T_{\max} \ge \tau$.

Proof of Claim I. Indeed, suppose on the contrary that

$$T_{\max} < \tau. \tag{A.8}$$

For each $i \in \mathcal{N}$ let $u_i : [0, \infty) \to \mathbb{R}^n$ be a piecewise continuous function satisfying

$$u_i(t) = k_{i,\mathrm{pr}_i(\mathbf{l})}(t, x_i(t), \mathbf{x}_j(t); x_{i0}, w_i), \forall t \in [0, T_{\max}). \tag{A.9}$$

Notice that due to (4.10) and (A.8) we have that $T_{\max} < \min\{\delta t, \min\{T(x_{i0}, w_i) : i \in \mathcal{N}\}\}$, and thus, we get from (A.1) and (P2) that $|u_i(t)| \le v_{\max}$, $\forall t \in [0, T_{\max})$. Hence, we may select $u_i(\cdot)$ to satisfy $|u_i(t)| \le v_{\max}$, $\forall t \ge 0$ (select for instance $u_i(t) = 0$ for $t \ge T_{\max}$). Thus, if we denote by $\xi(\cdot)$ the solution of (3.1) with free inputs $u_i(\cdot)$, $i \in \mathcal{N}$ and the same initial condition with $x(\cdot)$, it follows from the Invariance Assumption (IA) that $\xi(t)$ is defined and remains in D^N for all $t \ge 0$. Furthermore, it follows from standard arguments from the theory of ODEs that $\xi(t) = x(t), \forall t \in [0, T_{\max})$. Hence, since $\xi(t)$ belongs to a compact subset of D^N for all $t \in [0, T_{\max}]$, the same holds for $x(t)$ on $[0, T_{\max})$. The latter contradicts maximality of $[0, T_{\max})$ since by (A.8) and (4.10) it holds $T_{\max} < \tau \le \min\{T(x_{i0}, w_i) : i \in \mathcal{N}\}$ and the mappings $k_{i,\mathrm{pr}_i(\mathbf{l})}(\cdot)$ are defined for $t \in [0, \min\{T(x_{i0}, w_i) : i \in \mathcal{N}\})$. Hence, we have shown Claim I.

From Claim I, it follows that $x(t)$ is defined and remains in D^N for all $t \in [0, \tau)$ and that (A.1) holds for all $t \in [0, \tau)$. Thus, by applying the same arguments with those in the proof of Claim I, we can determine a continuous function $\xi(\cdot)$ with $\xi(t) = x(t)$ for all $t \in [0, \tau)$ and $\xi(\tau) \in D^N$, which establishes (4.11).

Proof of (iia). In the case where (P3) also holds, and hence by (4.10) we have that $\tau = \delta t$, it follows from part (i) of the proposition and standard arguments, that the solution $x(\cdot)$ is defined on $[0, T_{\max})$, with $T_{\max} > \delta t$. From the latter, we conclude that $x(t) \in D^N$ for all $t \in [0, \delta t]$. Moreover, since $T_{\max} > \delta t = \tau$, it follows that (A.1) is satisfied for all $t \in [0, \delta t)$. The latter, by virtue of (P2), (P3) and continuity of $x(\cdot)$ implies (4.12).

Proof of (iib). By exploiting the result of part (iia) of the proposition and defining $u_i(t) = k_{i,\mathrm{pr}_i(\mathbf{l})}(t, x_i(t), \mathbf{x}_j(t); x_{i0}, w_i)$, $\forall t \in [0, \delta t)$ we can extend $u_i(\cdot)$ to a piecewise continuous function on $[0, \infty)$ which satisfies (3.3). Hence, by applying the same arguments with those in the proof of Claim I, we conclude that the solutions $x(\cdot)$ of (3.1), (4.9) and $\xi(\cdot)$ of (3.1) (with input $u(\cdot)$) coincide on $[0, \delta t]$. $\square$

Proof of Lemma 5.1. For the proof of the lemma we exploit the result of Proposition 4.7. In particular, we show that the solution $\chi_i(\cdot)$ of (3.7) coincides on a suitable time interval with the ith component of the solution of the multi-agent system (3.1) under an appropriate selection of the initial conditions and feedback controllers for the v_i's. Hence, by implicitly exploiting the Invariance Assumption (IA) that leads to the result of Proposition 4.7(iia), which is valid for any choice of feedback laws that satisfy Property (P), we will verify that (5.3) is fulfilled.

In order to proceed with the proof, let $(x_{i,G}, \mathbf{x}_{j,G})$ be a tuple of reference points as in (3.5), corresponding to a cell decomposition $\{S_l\}_{l \in \mathcal{I}}$ of D and a cell configuration $\mathbf{l}_i$ of agent i, and consider another cell decomposition $\{\bar{S}_{\bar{l}}\}_{\bar{l} \in \bar{\mathcal{I}}}$ of D and an initial cell configuration $\bar{\mathbf{l}} = (\bar{l}_1, \ldots, \bar{l}_N) \in \bar{\mathcal{I}}^N$ with $\mathrm{pr}_i(\bar{\mathbf{l}}) = (\bar{l}_i, \bar{l}_{j_1}, \ldots, \bar{l}_{j_{N_i}})$, such that

$$x_{i,G} \in \bar{S}_{\bar{l}_i} \text{ and } \bar{S}_{\bar{l}_{j_\kappa}} = \{x_{j_\kappa,G}\}, \kappa = 1, \ldots, N_i. \tag{A.10}$$

We have selected the auxiliary cell decomposition $\{\bar{S}_{\bar{l}}\}_{\bar{l} \in \bar{\mathcal{I}}}$ with the sets $\bar{S}_{\bar{l}_{j_\kappa}}$ consisting of a single element, because this slightly simplifies the subsequent analysis and also allows obtaining a greater uniform lower bound for the time $T_{\max}$. Next, define the time step

$$\bar{\delta t} := \frac{v_{\max}}{2ML_1 \max\{\sqrt{N_i} : i \in \mathcal{N}\}} \tag{A.11}$$

and consider the feedback laws $k_{i,\mathrm{pr}_i(\bar{\mathbf{l}})} : D^{N_i+1} \to \mathbb{R}^n$ given by

$$k_{i,\mathrm{pr}_i(\bar{\mathbf{l}})}(x_i, \mathbf{x}_j) := f_i(x_i, \mathbf{x}_{j,G}) - f_i(x_i, \mathbf{x}_j) = F_{i,\mathbf{l}_i}(x_i) - f_i(x_i, \mathbf{x}_j), \tag{A.12}$$

with $F_{i,\mathbf{l}_i}(\cdot)$ as in (3.6) and $k_{\ell,\mathrm{pr}_\ell(\bar{\mathbf{l}})} : D^{N_\ell+1} \to \mathbb{R}^n$ for $\ell \in \mathcal{N} \setminus \{i\}$ given by

$$k_{\ell,\mathrm{pr}_\ell(\bar{\mathbf{l}})}(x_\ell, \mathbf{x}_{j(\ell)}) := 0. \tag{A.13}$$

Note that the feedback laws $k_{\ell,\mathrm{pr}_\ell(\bar{\mathbf{l}})}(\cdot)$ for $\ell \in \mathcal{N} \setminus \{i\}$ satisfy Property (P) by default, with the auxiliary cell decomposition $\{\bar{S}_{\bar{l}}\}_{\bar{l} \in \bar{\mathcal{I}}}$ and the selected time step $\bar{\delta t}$ in (A.11) (if viewed as mappings $\mathbb{R}_{\ge 0} \times D^{N_\ell+1} \ni (t, x_\ell, \mathbf{x}_{j(\ell)}) \mapsto k_{\ell,\mathrm{pr}_\ell(\bar{\mathbf{l}})}(t, x_\ell, \mathbf{x}_{j(\ell)}; x_{\ell 0}, w_\ell) \in \mathbb{R}^n$ parameterized by $x_{\ell 0} \in \bar{S}_{\bar{\ell}}$, $w_\ell \in W$, with any $\emptyset \ne W \in \mathbb{R}^n$ and being independent of t, $x_{\ell 0}$ and w_ℓ). Hence, in order to invoke Proposition 4.7(iia), we show that $k_{i,\mathrm{pr}_i(\bar{\mathbf{l}})}(\cdot)$ also satisfies (P). Property (P3) is obvious, since $k_{i,\mathrm{pr}_i(\bar{\mathbf{l}})}(\cdot)$ is independent of t. Property (P1) follows from the corresponding Lipschitz property for $f_i(\cdot)$ and $F_{i,\mathbf{l}_i}(\cdot)$, since by virtue of (5.2) and (3.6), the latter satisfies the Lipschitz condition

$$|F_{i,\mathbf{l}_i}(x) - F_{i,\mathbf{l}_i}(y)| \le L_2|x - y|, \forall x, y \in D. \tag{A.14}$$

In order to show (P2), notice that due to (A.11) we get

$$v_{\max} \ge 2M\bar{\delta t}L_1\sqrt{N_i}, \text{ for all } i \in \mathcal{N}. \tag{A.15}$$

Hence, we get from (4.1), (5.1), (3.4), (A.10) and (A.15) that for every $x_i \in (\bar{S}_{\bar{l}_i} + B(R_{\max})) \cap D$ and $x_{j_\kappa} \in B(x_{j_\kappa,G}, R_{\max}) \cap D$, $\kappa = 1, \ldots, N_i$, it holds $|k_{i,\mathrm{pr}_i(\bar{\mathbf{l}})}(x_i, \mathbf{x}_j)| \le L_1|\mathbf{x}_j - \mathbf{x}_{j,G}| = L_1(\sum_{\kappa=1}^{N_i}(x_{j_\kappa} -$

$x_{j_\kappa,G})^2)^{\frac{1}{2}} \leq L_1\sqrt{N_i}\bar{\delta}t(M+v_{\max}) < 2M\bar{\delta}tL_1\sqrt{N_i} \leq v_{\max}$, with $R_{\max}$ corresponding to the selected time step $\bar{\delta}t$. Thus, (P2) holds as well, since $k_{i,\mathrm{pr}_i(\bar{\mathbf{l}})}(\cdot)$ is independent of t, x_{i0} and w_i. Then, it follows from Proposition 4.7(iia) that the solution $x(t)$ of the closed-loop system (3.1), (A.12)-(A.13) with initial condition $x(0)\in D^N$ satisfying $x_i(0)=x_{i,G}$, $x_{j_1}(0)=x_{j_1,G},\ldots,x_{j_{N_i}}(0)=x_{j_{N_i},G}$ (and the initial state of each other agent ℓ belonging to $\bar{S}_{\bar{l}_\ell}$) is defined and remains in D^N for all $t\in[0,\bar{\delta}t]$. Hence, the ith component of the solution $x(\cdot)$ satisfies

$$x_i(t)\in D, \forall t\in[0,\bar{\delta}t], \tag{A.16}$$

and by virtue of (3.1) and (A.12), it holds

$$\dot{x}_i(t)=F_{i,\mathbf{l}_i}(x_i(t)), \forall t\in[0,\bar{\delta}t], x_i(0)=x_{i,G}. \tag{A.17}$$

Hence, it follows from (A.17) that $x_i(\cdot)$ coincides with the unique solution $\chi_i(\cdot)$ of (3.7) on $[0,\bar{\delta}t]\cap[0,T_{\max})$, which in conjunction with (A.16) implies that $\chi_i(t)$ remains in a compact subset of D for $t\in[0,\bar{\delta}t]\cap[0,T_{\max})$. From the latter, we deduce that $T_{\max}>\bar{\delta}t$. Indeed, otherwise $\chi_i(t)$ would remain in a compact subset of D for $t\in[0, T_{\max})$, contradicting maximality of $[0, T_{\max})$. Thus, we conclude that (5.3) is satisfied. □

Proof of Corollary 6.5. In order to prove the result, we need by virtue of (6.17) to show that

$$\#\{l\in\mathcal{I}: S_l\cap B(\chi_i(\delta t);r)\neq\emptyset\}\geq\lceil\mu^n\rceil \tag{A.18}$$

(recall that that # denotes the cardinality of a set). In addition, it follows from (6.1) and the iso-diametric inequality (see e.g., [37]) that for each $l\in\mathcal{I}$ it holds

$$\mathrm{Vol}(S_l)\leq\mathrm{Vol}\left(B\left(\frac{d_{\max}}{2}\right)\right)=d_{\max}^n\beta(n)=:S_{\max}, \tag{A.19}$$

(recall that $\mathrm{Vol}(\cdot)$ denotes volume and that $\beta(n):=\mathrm{Vol}(B(\frac{1}{2})))$. It then follows from (6.16), namely, that $B(\chi_i(\delta t);r)\subset D$, (A.19) and the fact that due to Definition 3.1 it holds $\cup_{l\in\mathcal{I}}S_l=D$, that

$$\#\{l\in\mathcal{I}: S_l\cap B(\chi_i(\delta t);r)\neq\emptyset\}\geq\left\lceil\frac{\mathrm{Vol}(B(\chi_i(\delta t);r))}{S_{\max}}\right\rceil. \tag{A.20}$$

By taking into account (6.11) and (A.19), we get that $\frac{\mathrm{Vol}(B(\chi_i(\delta t);r))}{S_{\max}}\geq\frac{(\mu d_{\max})^n\beta(n)}{d_{\max}^n\beta(n)}=\mu^n$. Thus, (A.18) is a direct consequence of the latter and (A.20). The proof is now complete. □

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejcon.2018.10.002.

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