Perimeter surveillance based on set-invariance

Luis Guerrero-Bonilla, and Dimos V. Dimarogonas

***Abstract*—A solution to the perimeter surveillance problem for one intruder and multiple surveillance robots based on set-invariance is presented. The surveillance robots, constrained to move on the perimeter of a polygonal region, intercept the intruders as they cross the perimeter. The proposed closed-form control laws only depend on the maximum speed of the robots and their distances to the endpoints of the line segments that make the sides of the polygon. The presented results allow for groups of robots with members of different characteristics, such as size and maximum speed, to defend polygonal regions. Simulations are used to show the application and effectiveness of the theoretical results.**

***Index Terms*—Multi-Robot Systems, Surveillance Robotic Systems**

I. INTRODUCTION

THE target guarding problem, first introduced in the seminal work [1], consists of an *evader* or *intruder*, which tries to reach a target location, and a *pursuer* or *defender*, trying to intercept the evader before it reaches the target. In this paper, we study a version of the target guarding problem where multiple defenders are confined to the perimeter of a region that must be protected from undetected intrusions by intercepting the intruder as it crosses the perimeter.

The perimeter surveillance problem belongs to the general category of pursuit-evasion problems, for which there is a vast body of literature [2]. A powerful approach to solve these problems is to formulate the problem in the context of differential games and compute the reachable sets of the pursuers and the evaders through the Hamilton-Jacobi-Isaacs (HJI) equation [3], [4]. However, computing solutions to HJI equations is computationally infeasible for large-scale problems; the work in [5]–[7] attempts to alleviate the dimensionality problem by approximating the solution of the HJI equations in low dimensions, while the work in [8], [9] presents an open-loop formulation that circumvents the need of solving HJI. Using the open-loop formualtion, cooperative evasion of multiple evaders against a single intruder has been studied in [10]. Related to the approach we present, a cooperative pursuit strategy based on geometric arguments is proposed in [11]. Other approaches applied to solve pursuit-evation problems are multi-robot coordination through linear and mixed-integer programming [12], [13], and the use of Voronoi diagrams to partition the environment [14]–[17].

Regarding the target guarding problem [1], [18], [19], optimal strategies have been obtained based on approaches such as differential games, optimal control, and optimization [20]–[22]. For perimeter and area defense, continuous space [23] and graph-based [24] patrolling schemes have been studied, with a non-deterministic patrolling scheme presented in [25], [26]. A geometric analysis of the two-player perimeter defense leading to cooperative defense strategies with multiple intruders and defenders is studied in [27]–[29].

Our contribution in this paper is a solution to the perimeter surveillance problem for one intruder and multiple defenders, based on set-invariance methods. We model a set of positions that guarantee the defenders will always be able to intercept the intruders as they attempt to cross the perimeter of the defended region, and ensure the forward invariance of such set. The interception is guaranteed at the instant the intruder crosses the perimeter. Our solution has a closed form, allows for the intruders to be faster than the defenders, and can be applied to the surveillance of polygonal regions of different sizes through the cooperation of multiple defenders, each of which can have different maximum speed and size. Our solution is based on Zeroing Control Barrier Functions [30], [31], which ensure the forward invariance of the desired set.

The paper is structured as follows. Section II describes the problem formulation and states the control objectives to be satisfied by our proposed control laws. Section III presents our solution to the single defender case, which is then used as a building block for the multi-defender solution in Section IV. Section V focuses on the application of the solutions to polygonal perimeters, and presents simulations that showcase our theoretical results.

II. PROBLEM FORMULATION

Let $\boldsymbol{x}_P \in \mathbb{R}^2$ denote the position of point P on the plane, and let $\widehat{\boldsymbol{\ell}}$ be a unit vector parallel to a line which contains a line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ with endpoints at the constant positions $\boldsymbol{x}_L$ and $\boldsymbol{x}_R$, such that

$$\begin{aligned}\overline{\boldsymbol{x}_L\boldsymbol{x}_R} &= \{\boldsymbol{x}_P \,:\, \boldsymbol{x}_P = \boldsymbol{x}_L + \lambda_1 (\boldsymbol{x}_R - \boldsymbol{x}_L),\ \forall \lambda_1 \in [0,1]\} \\ &= \{\boldsymbol{x}_P \,:\, \boldsymbol{x}_P = \boldsymbol{x}_R - \lambda_2 (\boldsymbol{x}_R - \boldsymbol{x}_L),\ \forall \lambda_2 \in [0,1]\}.\end{aligned} \tag{1}$$

Note that the sets have the same elements, with λ_1 and λ_2 related by the equation $\lambda_1 + \lambda_2 = 1$. The line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ has length $\ell = \|\boldsymbol{x}_R - \boldsymbol{x}_L\|$ where $\|\cdot\|$ is the 2-norm, allowing for $\widehat{\boldsymbol{\ell}}$ to be expressed as $\widehat{\boldsymbol{\ell}} = \frac{(\boldsymbol{x}_R - \boldsymbol{x}_L)}{\|\boldsymbol{x}_R - \boldsymbol{x}_L\|}$. Let $\boldsymbol{x}_D(t)$ be the position of the surveillance and defense robot D, with dynamics given by

$$\dot{\boldsymbol{x}}_D(t) = u(t)\,\widehat{\boldsymbol{\ell}}, \tag{2}$$

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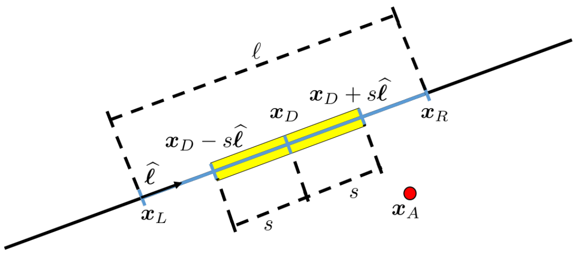

Fig. 1: Robot D and robot A are show in yellow and red respectively. Their positions as well as the quantities related to the blue line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ are also shown.

where $u(t) \in \mathbb{R}$ is the scalar control input. Robot D has an initial condition $\boldsymbol{x}_D(0) \in \overline{\boldsymbol{x}_L\boldsymbol{x}_R}$, so that it is confined to the line containing the line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$. We assume that its maximum speed is $v_D > 0$, so that $\|\dot{\boldsymbol{x}}_D(t)\| = |u(t)| \leq v_D$. Robot D has guards of length $s \in \mathbb{R}$, making it a line-shaped robot with center at $\boldsymbol{x}_D(t)$ spanning the line segment $\{\boldsymbol{x}_{sD}(t) : \boldsymbol{x}_{sD}(t) = \boldsymbol{x}_D(t) + (2\lambda - 1)\, s\widehat{\boldsymbol{\ell}},\ \forall\lambda \in [0,1]\}$. $\boldsymbol{x}_A(t)$ denotes the position of the intruder robot A. It is assumed that $\boldsymbol{x}_A(t)$ is continuously differentiable and can be measured, and that the velocity $\dot{\boldsymbol{x}}_A(t)$ is bounded by $v_A > 0$ such that $\|\dot{\boldsymbol{x}}_A(t)\| \leq v_A$, but it is otherwise unknown. It is also assumed that the maximum speed of robot D is limited to be no greater than that of robot A, such that $v_D \leq v_A$.

The purpose of robot D is to intercept robot A as it crosses the line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ by ensuring that the crossing point is within its span. The control objective is formalized as follows:

Control Objective 1. *Given a line parallel to the unit vector $\widehat{\boldsymbol{\ell}}$, the position $\boldsymbol{x}_A(t)$ of robot A with a continuous velocity and with maximum speed $v_A > 0$, and the position $\boldsymbol{x}_D(t)$ of robot D with dynamics given by* (2)*, maximum speed $0 < v_D \leq v_A$, and guard length s, determine a line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ on the line parallel to $\widehat{\boldsymbol{\ell}}$ as defined in* (1) *and design a control law $u(t)$ for robot D such that $-s \leq (\boldsymbol{x}_A(t) - \boldsymbol{x}_D(t)) \cdot \widehat{\boldsymbol{\ell}} \leq s$ whenever $\boldsymbol{x}_A(t) \in \overline{\boldsymbol{x}_L\boldsymbol{x}_R}$.*

For a single robot D, the line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ can be located anywhere on the line parallel to $\widehat{\boldsymbol{\ell}}$, but the line segment length ℓ, and therefore the selection of $\boldsymbol{x}_L$ and $\boldsymbol{x}_R$ relative to each other, depend on the parameters v_A, v_D and s. If the line segment is required to be longer than what a single robot can handle, a control strategy involving multiple robots can be used. The following control objective is stated to address this scenario:

Control Objective 2. *Given a line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ as defined in* (1)*, the position $\boldsymbol{x}_A(t)$ of robot A with continuous velocity and maximum speed $v_A > 0$, and the positions $\boldsymbol{x}_{Di}(t)$ of a sufficiently large number n of robots each with dynamics as in* (2)*, maximum speed $0 < v_{Di} \leq v_A$, and guard length s_i, design control laws $u_i(t)$ for each robot i such that $-s_i \leq (\boldsymbol{x}_A(t) - \boldsymbol{x}_{D,i}(t)) \cdot \widehat{\boldsymbol{\ell}} \leq s_i$ for some robot i whenever $\boldsymbol{x}_A(t) \in \overline{\boldsymbol{x}_L\boldsymbol{x}_R}$.*

In this case, the length of the line segment is fixed, and the number for robots n is to be determined according to the characteristics of the robots.

III. SINGLE DEFENDER CASE

The control laws are inspired by the time it takes for the robots to reach points on the line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ while moving at their maximum speeds v_A and v_D. Based on this analysis, the length ℓ as well as the control $u(t)$ to satisfy the Control Objective 1 are determined.

The time it takes for robot A to reach some point $\boldsymbol{x}_P$ on the line with line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ is given by

$$T_{A,P}(\boldsymbol{x}_A(t), \boldsymbol{x}_P) = \frac{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\|}{v_A}. \tag{3}$$

Equation (3) is not differentiable at $\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\| = 0$. We will use instead a continuously differentiable approximation, given by

$$\underline{T}_{A,P}(\boldsymbol{x}_A(t), \boldsymbol{x}_P) = \frac{\sqrt{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\|^2 + \epsilon^2} - \epsilon}{v_A} \tag{4}$$

with some constant $\epsilon > 0$. The closer ϵ is to 0, the better the approximation. In the following, we drop the dependencies on the positions in the notation for better readability, and proceed to show that $T_{A,P} \geq \underline{T}_{A,P}$, and thus that $\underline{T}_{A,P}$ provides an equal or smaller time of arrival, which is an acceptable approximation for time-critical surveillance purposes.

Proposition 1. $T_{A,P} \geq \underline{T}_{A,P}$.

Proof. $T_{A,P} - \underline{T}_{A,P}$

$$\begin{aligned}
&= \frac{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\| - \sqrt{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\|^2 + \epsilon^2} + \epsilon}{v_A} \\
&= \frac{1}{v_A} \frac{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\|^2 - \left(\sqrt{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\|^2 + \epsilon^2}\right)^2}{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\| + \sqrt{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\|^2 + \epsilon^2}} + \frac{\epsilon}{v_A} \\
&= \frac{\epsilon}{v_A} - \frac{\epsilon}{v_A} \frac{\epsilon}{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\| + \sqrt{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_P\|^2 + \epsilon^2}} \geq 0
\end{aligned}$$

Then, $T_{A,P} - \underline{T}_{A,P} \geq 0$ and therefore, $T_{A,P} \geq \underline{T}_{A,P}$. $\square$

The following result is used in later proofs.

Proposition 2. *If $\boldsymbol{x}_P \in \overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ for a line segment as in* (1)*, then $\|\boldsymbol{x}_P - \boldsymbol{x}_L\|$ and $\|\boldsymbol{x}_P - \boldsymbol{x}_R\|$ can be calculated as*

$$\|\boldsymbol{x}_P - \boldsymbol{x}_L\| = (\boldsymbol{x}_P - \boldsymbol{x}_L) \cdot \widehat{\boldsymbol{\ell}}, \tag{5}$$

$$\|\boldsymbol{x}_R - \boldsymbol{x}_P\| = (\boldsymbol{x}_R - \boldsymbol{x}_P) \cdot \widehat{\boldsymbol{\ell}}. \tag{6}$$

Proof. From (1), we have

$$\begin{aligned}
\boldsymbol{x}_P - \boldsymbol{x}_L &= \lambda_1 (\boldsymbol{x}_R - \boldsymbol{x}_L) \\
&= \lambda_1 \|\boldsymbol{x}_R - \boldsymbol{x}_L\| \widehat{\boldsymbol{\ell}} = \|\boldsymbol{x}_P - \boldsymbol{x}_L\| \widehat{\boldsymbol{\ell}},
\end{aligned} \tag{7}$$

$$\begin{aligned}
\boldsymbol{x}_R - \boldsymbol{x}_P &= \lambda_2 (\boldsymbol{x}_R - \boldsymbol{x}_L) \\
&= \lambda_2 \|\boldsymbol{x}_R - \boldsymbol{x}_L\| \widehat{\boldsymbol{\ell}} = \|\boldsymbol{x}_R - \boldsymbol{x}_P\| \widehat{\boldsymbol{\ell}}
\end{aligned} \tag{8}$$

for some values λ_1 and λ_2. Taking inner products with $\widehat{\boldsymbol{\ell}}$, we obtain

$$(\boldsymbol{x}_P - \boldsymbol{x}_L) \cdot \widehat{\boldsymbol{\ell}} = \|\boldsymbol{x}_P - \boldsymbol{x}_L\| \widehat{\boldsymbol{\ell}} \cdot \widehat{\boldsymbol{\ell}} = \|\boldsymbol{x}_P - \boldsymbol{x}_L\|, \tag{9}$$

$$(\boldsymbol{x}_R - \boldsymbol{x}_P) \cdot \widehat{\boldsymbol{\ell}} = \|\boldsymbol{x}_R - \boldsymbol{x}_P\| \widehat{\boldsymbol{\ell}} \cdot \widehat{\boldsymbol{\ell}} = \|\boldsymbol{x}_R - \boldsymbol{x}_P\| \tag{10}$$

$\square$

Based on the time it takes for the edges of robot D to reach the positions $\boldsymbol{x}_L$ and $\boldsymbol{x}_R$, we define the functions $T_{D,L}\left(\boldsymbol{x}_D\left(t\right),\boldsymbol{x}_L\right)$ and $T_{D,R}\left(\boldsymbol{x}_D\left(t\right),\boldsymbol{x}_R\right)$ as

$$T_{D,L}\left(\boldsymbol{x}_D\left(t\right),\boldsymbol{x}_L\right)=\frac{\left(\left(\boldsymbol{x}_D\left(t\right)-s\widehat{\boldsymbol{\ell}}\right)-\boldsymbol{x}_L\right)\cdot\widehat{\boldsymbol{\ell}}}{v_D} \tag{11}$$

$$T_{D,R}\left(\boldsymbol{x}_D\left(t\right),\boldsymbol{x}_R\right)=\frac{\left(\boldsymbol{x}_R-\left(\boldsymbol{x}_D\left(t\right)+s\widehat{\boldsymbol{\ell}}\right)\right)\cdot\widehat{\boldsymbol{\ell}}}{v_D} \tag{12}$$

A. Conditions for line segment surveillance

The following result shows that ensuring $\underline{T}_{A,L}-T_{D,L}\geq 0$ and $\underline{T}_{A,R}-T_{D,R}\geq 0$ is sufficient to solve the Control Objective 1.

Proposition 3. *Let $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ be a line segment as defined in* (1)*, and let $\underline{T}_{A,P}$, $T_{D,L}$ and $T_{D,R}$ be defined as in* (4)*,* (11) *and* (12) *respectively. If $\underline{T}_{A,L}-T_{D,L}\geq 0$ and $\underline{T}_{A,R}-T_{D,R}\geq 0$ for all $t\geq 0$, then $-s\leq\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_D\left(t\right)\right)\cdot\widehat{\boldsymbol{\ell}}\leq s$ whenever $\boldsymbol{x}_A\left(t\right)\in\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$.*

Proof. If $\underline{T}_{A,L}-T_{D,L}\geq 0$ and $\underline{T}_{A,R}-T_{D,R}\geq 0$ for all $t\geq 0$, then by Proposition 1 we have $T_{A,L}-T_{D,L}\geq 0$ and $T_{A,R}-T_{D,R}\geq 0$. Multiplying both sides of the inequalities by v_D leads to

$$0\leq\frac{v_D}{v_A}\left\|\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_L\right\|-\left(\left(\boldsymbol{x}_D\left(t\right)-s\widehat{\boldsymbol{\ell}}\right)-\boldsymbol{x}_L\right)\cdot\widehat{\boldsymbol{\ell}}, \tag{13}$$

$$0\leq\frac{v_D}{v_A}\left\|\boldsymbol{x}_R-\boldsymbol{x}_A\left(t\right)\right\|-\left(\boldsymbol{x}_R-\left(\boldsymbol{x}_D\left(t\right)+s\widehat{\boldsymbol{\ell}}\right)\right)\cdot\widehat{\boldsymbol{\ell}}. \tag{14}$$

If $\boldsymbol{x}_A\left(t\right)\in\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$, then by Proposition 2 we have $\left\|\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_L\right\|=\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_L\right)\cdot\widehat{\boldsymbol{\ell}}$ and $\left\|\boldsymbol{x}_R-\boldsymbol{x}_A\left(t\right)\right\|=\left(\boldsymbol{x}_R-\boldsymbol{x}_A\left(t\right)\right)\cdot\widehat{\boldsymbol{\ell}}$. Substituting and using $\frac{v_D}{v_A}\leq 1$ leads to

$$\begin{aligned}0&\leq\frac{v_D}{v_A}\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_L\right)\cdot\widehat{\boldsymbol{\ell}}-\left(\left(\boldsymbol{x}_D\left(t\right)-s\widehat{\boldsymbol{\ell}}\right)-\boldsymbol{x}_L\right)\cdot\widehat{\boldsymbol{\ell}}\\&\leq\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_L\right)\cdot\widehat{\boldsymbol{\ell}}-\left(\left(\boldsymbol{x}_D\left(t\right)-s\widehat{\boldsymbol{\ell}}\right)-\boldsymbol{x}_L\right)\cdot\widehat{\boldsymbol{\ell}}\\&=\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_D\left(t\right)\right)\cdot\widehat{\boldsymbol{\ell}}+s\end{aligned} \tag{15}$$

$$\begin{aligned}0&\leq\frac{v_D}{v_A}\left(\boldsymbol{x}_R-\boldsymbol{x}_A\left(t\right)\right)\cdot\widehat{\boldsymbol{\ell}}-\left(\boldsymbol{x}_R-\left(\boldsymbol{x}_D\left(t\right)+s\widehat{\boldsymbol{\ell}}\right)\right)\cdot\widehat{\boldsymbol{\ell}}\\&\leq\left(\boldsymbol{x}_R-\boldsymbol{x}_A\left(t\right)\right)\cdot\widehat{\boldsymbol{\ell}}-\left(\boldsymbol{x}_R-\left(\boldsymbol{x}_D\left(t\right)+s\widehat{\boldsymbol{\ell}}\right)\right)\cdot\widehat{\boldsymbol{\ell}}\\&=-\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_D\left(t\right)\right)\cdot\widehat{\boldsymbol{\ell}}+s\end{aligned} \tag{16}$$

The inequalities (15) and (16) together imply $-s\leq\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_D\left(t\right)\right)\cdot\widehat{\boldsymbol{\ell}}\leq s$ whenever $\boldsymbol{x}_A\left(t\right)\in\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$. □

B. Ensuring the conditions for surveillance

In this section, the conditions mentioned in Proposition 3 to guarantee the surveillance of the line segment are ensured for all $t\geq 0$. This is done by defining a set that contains the positions of robots A and D which satisfy the conditions, and ensuring its forward invariance. To accomplish this, we use results from the literature on Zeroing Control Barrier Functions (ZCBF). A brief introduction is given next, but the reader is referred to [30]. Consider a system of the form

$$\dot{\boldsymbol{x}}=\boldsymbol{f}\left(\boldsymbol{x}\right)+\boldsymbol{g}\left(\boldsymbol{x}\right)\boldsymbol{u} \tag{17}$$

where $\boldsymbol{x}\in\mathbb{R}^n$, $\boldsymbol{u}\in\mathcal{U}\subset\mathbb{R}^m$, with $\boldsymbol{f}$ and $\boldsymbol{g}$ locally Lipschitz. For any initial condition $\boldsymbol{x}\left(0\right)$, there exists a maximum time interval $I\left(\boldsymbol{x}\left(0\right)\right)=\left[0,\tau_{max}\right)$ such that $\boldsymbol{x}\left(t\right)$ is the unique solution to (17) on $I\left(\boldsymbol{x}\left(0\right)\right)$. In the case when (17) is forward complete, $\tau_{max}=\infty$. A set $\mathcal{S}$ is called forward invariant with respect to (17) if for every $\boldsymbol{x}\left(0\right)\in\mathcal{S}$, $\boldsymbol{x}\left(t\right)\in\mathcal{S}$ for all $t\in I\left(\boldsymbol{x}\left(0\right)\right)$.

Let the set $\mathcal{C}$ be defined as

$$\mathcal{C}=\left\{\boldsymbol{x}\in\mathbb{R}^n:h\left(\boldsymbol{x}\right)\geq 0\right\} \tag{18}$$

where $h:\mathbb{R}^n\to\mathbb{R}$ is continuously differentiable. It is assumed that $\mathcal{C}$ is non-empty and has no isolated points.

Definition 1 (Definition 5, [30]). *Given a set $\mathcal{C}\subset\mathbb{R}^n$ as defined in* (18) *for a continuously differentiable function h, the function h is called a zeroing control barrier function $(ZCBF)$ defined on a set $\mathcal{D}$ with $\mathcal{C}\subseteq\mathcal{D}\subset\mathcal{R}^n$, if there exists an extended class $\mathcal{K}$ function α such that*

$$\sup_{\boldsymbol{u}\in U}\left(L_f h\left(\boldsymbol{x}\right)+L_g h\left(\boldsymbol{x}\right)\boldsymbol{u}+\alpha\left(h\left(\boldsymbol{x}\right)\right)\right)\geq 0,\forall\boldsymbol{x}\in\mathcal{D}. \tag{19}$$

The Lie derivative notation is used, so that $\dot{h}\left(\boldsymbol{x}\right)=\frac{\partial h(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{f}\left(\boldsymbol{x}\right)+\frac{\partial h(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{g}\left(\boldsymbol{x}\right)\boldsymbol{u}=L_{\boldsymbol{f}}h\left(\boldsymbol{x}\right)+L_{\boldsymbol{g}}h\left(\boldsymbol{x}\right)\boldsymbol{u}$. Given a ZCBF h, define the set $K=\{\boldsymbol{u}\in\mathcal{U}:L_{\boldsymbol{f}}h\left(\boldsymbol{x}\right)+L_{\boldsymbol{g}}h\left(\boldsymbol{x}\right)\boldsymbol{u}+\alpha\left(h\left(\boldsymbol{x}\right)\right)\geq 0\}$ for each $\boldsymbol{x}\in\mathbb{R}^n$.

Theorem 1 (Corollary 2, [30]). *Given a set $\mathcal{C}\subset\mathbb{R}^n$ as defined in* (18) *for a continuously differentiable function h, if h is a ZCBF on $\mathcal{D}$, then any Lipschitz continuous controller $\boldsymbol{u}:\mathcal{D}\to\mathcal{U}$ for the dynamics* (17) *such that $\boldsymbol{u}\left(\boldsymbol{x}\right)\in K$ will render the set $\mathcal{C}$ forward invariant.*

These results can be applied to time varying systems, as described in [31]. In the following, the time dependencies of $\boldsymbol{x}$ and u are implied. Let the state $\boldsymbol{x}$ be defined as $\boldsymbol{x}=\begin{bmatrix}\boldsymbol{x}_D\left(t\right) & \boldsymbol{x}_A\left(t\right)\end{bmatrix}^{\mathsf{T}}$, with dynamics

$$\dot{\boldsymbol{x}}=\begin{bmatrix}\boldsymbol{0}\\\dot{\boldsymbol{x}}_A\left(t\right)\end{bmatrix}+\begin{bmatrix}\widehat{\boldsymbol{\ell}}\\\boldsymbol{0}\end{bmatrix}u \tag{20}$$

where $u\in\mathcal{U}=\left[-v_D,v_D\right]\subset\mathbb{R}$. We use the constraints from Proposition 3 to define the ZCBF candidates $h_L\left(\boldsymbol{x}\right)$ and $h_R\left(\boldsymbol{x}\right)$ as follows

$$h_L\left(\boldsymbol{x}\right)=\frac{\sqrt{\left\|\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_L\right\|^2+\epsilon^2}-\epsilon}{v_A}-\frac{\left(\left(\boldsymbol{x}_D\left(t\right)-s\widehat{\boldsymbol{\ell}}\right)-\boldsymbol{x}_L\right)\cdot\widehat{\boldsymbol{\ell}}}{v_D}, \tag{21}$$

$$h_R\left(\boldsymbol{x}\right)=\frac{\sqrt{\left\|\boldsymbol{x}_R-\boldsymbol{x}_A\left(t\right)\right\|^2+\epsilon^2}-\epsilon}{v_A}-\frac{\left(\boldsymbol{x}_R-\left(\boldsymbol{x}_D\left(t\right)+s\widehat{\boldsymbol{\ell}}\right)\right)\cdot\widehat{\boldsymbol{\ell}}}{v_D}. \tag{22}$$

The following property of the sum of $h_L\left(\boldsymbol{x}\right)$ and $h_R\left(\boldsymbol{x}\right)$ will be used in the proofs to follow.

Proposition 4. *For $h_L(\boldsymbol{x})$ and $h_R(\boldsymbol{x})$ as defined in* (21) *and* (22)*, the minimum value of their sum is given by*

$$\min_{\boldsymbol{x}} \left(h_L(\boldsymbol{x}) + h_R(\boldsymbol{x})\right) = 2\left(\frac{\sqrt{\left(\frac{\ell}{2}\right)^2+\epsilon^2}-\epsilon}{v_A} - \frac{\left(\frac{\ell}{2}\right)-s}{v_D}\right), \tag{23}$$

and it is strictly positive if $s > \epsilon\frac{v_D}{v_A}$ and: for $v_A > v_D$, ℓ satisfies

$$2s \leq \ell < \ell^*(v_D, s), \tag{24}$$

where $\ell^*(v_D,s)$ is given by

$$\ell^*(v_D,s) = 2\left(s-\epsilon\frac{v_D}{v_A}\right)\left(\frac{v_A^2}{v_A^2-v_D^2}\right) + \sqrt{\left(2\left(s-\epsilon\frac{v_D}{v_A}\right)\left(\frac{v_A v_D}{v_A^2-v_D^2}\right)\right)^2 + (2\epsilon)^2\left(\frac{v_D^2}{v_A^2-v_D^2}\right)}; \tag{25}$$

for $v_A = v_D$, ℓ satisfies

$$\ell \geq 2s. \tag{26}$$

Proof. It is straightforward to show that $\min_{\boldsymbol{x}}(h_L(\boldsymbol{x}) + h_R(\boldsymbol{x}))$ can be found at $\boldsymbol{x}_A = \frac{\boldsymbol{x}_L+\boldsymbol{x}_R}{2}$, and is given by (23). We proceed to calculate the range of ℓ that ensures $\min_{\boldsymbol{x}}(h_L(\boldsymbol{x}) + h_R(\boldsymbol{x})) > 0$. We consider $\ell \geq 2s$, since $2s$ is the span of robot D. Multiplying and dividing (23) by $\frac{\sqrt{\left(\frac{\ell}{2}\right)^2+\epsilon^2}-\epsilon}{v_A} + \frac{\left(\frac{\ell}{2}\right)-s}{v_D}$ and rearranging terms, we obtain

$$\min_{\boldsymbol{x}}(h_L(\boldsymbol{x}) + h_R(\boldsymbol{x})) = 2\frac{\frac{\left(\frac{\ell}{2}\right)^2+\epsilon^2}{v_A^2} - \frac{\left(\left(\frac{\ell}{2}\right)-\left(s-\epsilon\frac{v_D}{v_A}\right)\right)^2}{v_D^2}}{\frac{\sqrt{\left(\frac{\ell}{2}\right)^2+\epsilon^2}}{v_A} + \frac{\left(\frac{\ell}{2}\right)-\left(s-\epsilon\frac{v_D}{v_A}\right)}{v_D}}. \tag{27}$$

For $\ell \geq 2s$ and $s > \epsilon\frac{v_D}{v_A}$, (27) is positive as long as

$$\frac{\left(\frac{\ell}{2}\right)^2+\epsilon^2}{v_A^2} - \frac{\left(\left(\frac{\ell}{2}\right)-\left(s-\epsilon\frac{v_D}{v_A}\right)\right)^2}{v_D^2} > 0. \tag{28}$$

If $v_A > v_D$, expanding and rearranging the terms in (28) leads to

$$\left(\frac{\ell}{2}\right)^2 - 2\left(s-\epsilon\frac{v_D}{v_A}\right)\left(\frac{v_A^2}{v_A^2-v_D^2}\right)\left(\frac{\ell}{2}\right) + \left(s-\epsilon\frac{v_D}{v_A}\right)^2\left(\frac{v_A^2}{v_A^2-v_D^2}\right) - \epsilon^2\left(\frac{v_D^2}{v_A^2-v_D^2}\right) < 0. \tag{29}$$

Equating (29) to zero and solving for ℓ, we obtain its upper bound as a function of the parameters v_D and s of robot D,

$$\ell^*(v_D,s) = 2\left(s-\epsilon\frac{v_D}{v_A}\right)\left(\frac{v_A^2}{v_A^2-v_D^2}\right) + \sqrt{\left(2\left(s-\epsilon\frac{v_D}{v_A}\right)\left(\frac{v_A v_D}{v_A^2-v_D^2}\right)\right)^2 + (2\epsilon)^2\left(\frac{v_D^2}{v_A^2-v_D^2}\right)}, \tag{30}$$

so that the values for ℓ are bounded by

$$2s \leq \ell < \ell^*(v_D, s). \tag{31}$$

If $v_A = v_D$, then (28) simplifies to $2(s-\epsilon)\frac{\ell}{2} - (s-\epsilon)^2 + \epsilon^2 > 0$, which for $s > \epsilon$ leads to

$$\ell \geq 2s > \frac{s(s-2\epsilon)}{s-\epsilon}. \tag{32}$$

□

Since satisfying $h_L(\boldsymbol{x}) \geq 0$ and $h_R(\boldsymbol{x}) \geq 0$ guarantees the surveillance of the line, we define the set $\mathcal{C}$ as

$$\mathcal{C} = \{\boldsymbol{x} : h_L(\boldsymbol{x}) \geq 0\} \bigcap \{\boldsymbol{x} : h_R(\boldsymbol{x}) \geq 0\} \tag{33}$$

The derivatives $\dot{h}_L(\boldsymbol{x})$ and $\dot{h}_R(\boldsymbol{x})$ are given by

$$\dot{h}_L(\boldsymbol{x}) = \frac{(\boldsymbol{x}_A(t) - \boldsymbol{x}_L)\cdot\dot{\boldsymbol{x}}_A(t)}{v_A\sqrt{\|\boldsymbol{x}_A(t)-\boldsymbol{x}_L\|^2+\epsilon^2}} - \frac{u}{v_D}, \tag{34}$$

$$\dot{h}_R(\boldsymbol{x}) = \frac{(\boldsymbol{x}_A(t) - \boldsymbol{x}_R)\cdot\dot{\boldsymbol{x}}_A(t)}{v_A\sqrt{\|\boldsymbol{x}_R-\boldsymbol{x}_A(t)\|^2+\epsilon^2}} + \frac{u}{v_D}, \tag{35}$$

Since $(\boldsymbol{x}_A(t) - \boldsymbol{x}_P)\cdot\dot{\boldsymbol{x}}_A(t) \geq -\|\boldsymbol{x}_A(t)-\boldsymbol{x}_P\|v_A$, $\dot{h}_L(\boldsymbol{x})$ and $\dot{h}_R(\boldsymbol{x})$ can be bounded by

$$\dot{h}_L(\boldsymbol{x}) \geq -\frac{\|\boldsymbol{x}_A(t)-\boldsymbol{x}_L\|}{\sqrt{\|\boldsymbol{x}_A(t)-\boldsymbol{x}_L\|^2+\epsilon^2}} - \frac{u}{v_D} \geq -1 - \frac{u}{v_D}, \tag{36}$$

$$\dot{h}_R(\boldsymbol{x}) \geq -\frac{\|\boldsymbol{x}_R-\boldsymbol{x}_A(t)\|}{\sqrt{\|\boldsymbol{x}_R-\boldsymbol{x}_A(t)\|^2+\epsilon^2}} + \frac{u}{v_D} \geq -1 + \frac{u}{v_D}. \tag{37}$$

To verify that $h_R(\boldsymbol{x})$ and $h_L(\boldsymbol{x})$ are ZCBFs, we proceed to verify that the inequalities obtained from the lower bounds of (36) and (37),

$$-1 - \frac{u}{v_D} + \gamma h_L(\boldsymbol{x}) \geq 0 \Rightarrow u \leq -v_D(1-\gamma h_L(\boldsymbol{x})), \tag{38}$$

$$-1 + \frac{u}{v_D} + \gamma h_R(\boldsymbol{x}) \geq 0 \Rightarrow u \geq v_D(1-\gamma h_R(\boldsymbol{x})), \tag{39}$$

can be satisfied simultaneously for all $\boldsymbol{x}$ satisfying $h_L(\boldsymbol{x}) \geq 0$ and $h_R(\boldsymbol{x}) \geq 0$. For both of them to satisfy equation (19) simultaneously, defined as *control-sharing property* and studied in [31], the control input u is required to satisfy

$$v_D(1-\gamma h_R(\boldsymbol{x})) \leq u \leq -v_D(1-\gamma h_L(\boldsymbol{x})). \tag{40}$$

Proposition 5. *If $h_L(\boldsymbol{x}) \geq 0$ and $h_R(\boldsymbol{x}) \geq 0$, ℓ and s satisfy the conditions of Proposition 4, and γ is given by*

$$\gamma \geq \frac{2}{\min_{\boldsymbol{x}}(h_L(\boldsymbol{x}) + h_R(\boldsymbol{x}))}, \tag{41}$$

then the inequality (40) *can be satisfied by some control input $u \in [-v_D, v_D]$.*

Proof. Since $h_L(\boldsymbol{x}) \geq 0$, multiplying (41) by $-h_L(\boldsymbol{x})$ and adding 1 on both sides leads to

$$1 - \gamma h_L(\boldsymbol{x}) \leq 1 - \frac{2h_L(\boldsymbol{x})}{\min_{\boldsymbol{x}}(h_L(\boldsymbol{x}) + h_R(\boldsymbol{x}))} \tag{42}$$

Similarly, repeating the process with $h_R(\boldsymbol{x})$ leads to

$$1 - \gamma h_R(\boldsymbol{x}) \leq 1 - \frac{2h_R(\boldsymbol{x})}{\min_{\boldsymbol{x}}(h_L(\boldsymbol{x}) + h_R(\boldsymbol{x}))} \tag{43}$$

Adding (42) and (43), we obtain

$$\begin{aligned}(1-\gamma h_L(\boldsymbol{x}))+(1-\gamma h_R(\boldsymbol{x}))&\\ \leq 2\left(1-\frac{(h_L(\boldsymbol{x})+h_R(\boldsymbol{x}))}{\min_{\boldsymbol{x}}(h_L(\boldsymbol{x})+h_R(\boldsymbol{x}))}\right)&\end{aligned} \tag{44}$$

Since $\min_{\boldsymbol{x}}(h_L(\boldsymbol{x})+h_R(\boldsymbol{x}))\leq(h_L(\boldsymbol{x})+h_R(\boldsymbol{x}))$, we have $(1-\gamma h_L(\boldsymbol{x}))+(1-\gamma h_R(\boldsymbol{x}))\leq 0$. Rearranging terms and multiplying both sides by v_D leads to

$$v_D(1-\gamma h_R(\boldsymbol{x}))\leq -v_D(1-\gamma h_L(\boldsymbol{x})) \tag{45}$$

Using a constant γ satisfying (41) ensures that $v_D(1-\gamma h_R(\boldsymbol{x}))\leq -v_D(1-\gamma h_L(\boldsymbol{x}))$, and therefore there exists a value u satisfying (40). The existence of such a u that, in addition, belongs to $\mathcal{U}=[-v_D,v_D]$, is proved next. Since $h_L(\boldsymbol{x})\geq 0$, $h_R(\boldsymbol{x})\geq 0$, and $\gamma>0$, we have

$$\begin{aligned}\gamma h_L(\boldsymbol{x})\geq 0\Rightarrow 1-\gamma h_L(\boldsymbol{x})\leq 1\Rightarrow&\\ -v_D(1-\gamma h_L(\boldsymbol{x}))\geq -v_D,&\end{aligned} \tag{46}$$

$$\begin{aligned}\gamma h_R(\boldsymbol{x})\geq 0\Rightarrow 1-\gamma h_R(\boldsymbol{x})\leq 1\Rightarrow&\\ v_D(1-\gamma h_R(\boldsymbol{x}))\leq v_D,&\end{aligned} \tag{47}$$

so that $\min(u)=-v_D$ and $\max(u)=v_D$, and therefore, there is a $u\in\mathcal{U}=[-v_D,v_D]$ satisfying (40). □

Finally, we compute the control action $u(\boldsymbol{x})$ to solve the Control Objective 1 through ensuring forward invariance of the set $\mathcal{C}$.

Proposition 6. *If the initial state $\boldsymbol{x}(0)\in\mathcal{C}$ with $\mathcal{C}$ as defined in* (33) *and the conditions of Proposition 4 and Proposition 5 hold, then the control law*

$$\begin{aligned}u(\boldsymbol{x})=\max\{v_D(1-\gamma h_R(\boldsymbol{x})),0\}\ +&\\ \min\{0,-v_D(1-\gamma h_L(\boldsymbol{x}))\}&\end{aligned} \tag{48}$$

solves the Control Objective 1.

Proof. Let $m(\boldsymbol{x})=v_D(1-\gamma h_R(\boldsymbol{x}))$ and $M(\boldsymbol{x})=-v_D(1-\gamma h_L(\boldsymbol{x}))$. Equation (40) becomes $m(\boldsymbol{x})\leq u(\boldsymbol{x})\leq M(\boldsymbol{x})$. Since $\boldsymbol{x}(0)\in\mathcal{C}$, then $h_L(\boldsymbol{x}(0))\geq 0$ and $h_R(\boldsymbol{x}(0))\geq 0$. Then, given a γ following Proposition 5, we select $u(\boldsymbol{x})$ as follows:

$$u(\boldsymbol{x})=\begin{cases} m(\boldsymbol{x}) & \textit{for}\ \ m(\boldsymbol{x})\geq 0\\ 0 & \textit{for}\ \ m(\boldsymbol{x})<0<M(\boldsymbol{x})\\ M(\boldsymbol{x}) & \textit{for}\ \ M(\boldsymbol{x})\leq 0,\end{cases} \tag{49}$$

By Proposition 5 it is ensured that $m(\boldsymbol{x})\leq M(\boldsymbol{x})$, and equation (48) satisfies all the cases in (49). Since

$$\|\frac{\partial h_L(\boldsymbol{x})}{\partial \boldsymbol{x}}\|=\|\frac{\partial h_R(\boldsymbol{x})}{\partial \boldsymbol{x}}\|\leq\frac{\sqrt{v_A^2+v_D^2}}{v_Av_D}, \tag{50}$$

by Lemma 3.3 in [32], $h_L(\boldsymbol{x})$ and $h_R(\boldsymbol{x})$ are both globally Lipschitz continuous. Since the max and min functions are also globally Lipschitz continuous, it can be verified that the control law (48) satisfies

$$\|u(\boldsymbol{x}_1)-u(\boldsymbol{x}_2)\|\leq\frac{2\gamma}{v_A}\sqrt{v_A^2+v_D^2}\|\boldsymbol{x}_1-\boldsymbol{x}_2\|, \tag{51}$$

and therefore is globally Lipschitz continuous. Furthermore, the right side of the state equation (20) is globally Lipschitz continuous since

$$\|\begin{bmatrix}u(\boldsymbol{x}_1)\widehat{\boldsymbol{\ell}}\\ \dot{\boldsymbol{x}}_A(t)\end{bmatrix}-\begin{bmatrix}u(\boldsymbol{x}_2)\widehat{\boldsymbol{\ell}}\\ \dot{\boldsymbol{x}}_A(t)\end{bmatrix}\|=\|u(\boldsymbol{x}_1)-u(\boldsymbol{x}_2)\|, \tag{52}$$

and therefore by Theorem 3.2 in [32], there is a unique solution for all $t\geq 0$. By Theorem 1, the set $\mathcal{C}$ is forward invariant for all $t\geq 0$. Since $h_L(\boldsymbol{x})\geq 0$ and $h_R(\boldsymbol{x})\geq 0$ for all $t\geq 0$, by Proposition 3 the Control Objective 1 is solved. □

C. Initial position of $\boldsymbol{x}_D$ on the line, and the intruder detection distance

Given an initial position $\boldsymbol{x}_D(0)\in\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$, it is possible to determine the distances $\|\boldsymbol{x}_A(0)-\boldsymbol{x}_L\|$ and $\|\boldsymbol{x}_R-\boldsymbol{x}_A(0)\|$ between the initial position of robot A and the line segment endpoints that ensure $\boldsymbol{x}(0)\in\mathcal{C}$, as required by Proposition 6. Solving from $h_L(\boldsymbol{x})\geq 0$ and $h_R(\boldsymbol{x})\geq 0$ using (21) and (22), leads to

$$d_L(\boldsymbol{x}_D(0))\geq\sqrt{\left(\frac{v_A}{v_D}(\|\boldsymbol{x}_D(0)-\boldsymbol{x}_L\|-s)+\epsilon\right)^2-\epsilon^2}, \tag{53}$$

$$d_R(\boldsymbol{x}_D(0))\geq\sqrt{\left(\frac{v_A}{v_D}(\|\boldsymbol{x}_R-\boldsymbol{x}_D(0)\|-s)+\epsilon\right)^2-\epsilon^2}. \tag{54}$$

The equalities in (53) and (54) represent the minimum *detection distances* from $\boldsymbol{x}_L$ and $\boldsymbol{x}_R$ at which the position of a robot A must be acquired to ensure its interception, as a function of $\boldsymbol{x}_D(0)$. The initial position of robot D can be anywhere within the line segment, as long as the initial position of robot A is detected sufficiently far away following (53) and (54). To ensure interception for any initial position $\boldsymbol{x}_D(0)\in\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$, we consider the maximum value $\|\boldsymbol{x}_D(0)-\boldsymbol{x}_L\|=\|\boldsymbol{x}_R-\boldsymbol{x}_D(0)\|=\ell$, leading to the minimum detection distance d_{LR} from both endpoints:

$$d_{LR}=\sqrt{\left(\frac{v_A}{v_D}(\ell-s)+\epsilon\right)^2-\epsilon^2} \tag{55}$$

IV. MULTIPLE DEFENDERS

The length of the line segment that a robot D can defend is limited by its maximum speed v_D and guard size s through the conditions of Proposition 4, namely equations (24)-(26). The number of robots required for a given line segment length depends on the characteristics of the individual robots. Given the maximum speed v_A of an intruder robot A as described in Section II, and the maximum speed $v_{Di}\leq v_A$ and guard size s_i of the ith robot D_i in a group of surveillance robots, the maximum length $\ell^*(v_{Di},s_i)$ for each robot can be calculated using (25), and the surveillance of a line subsegment of length $\ell_i<\ell^*(v_{Di},s_i)$ can be assign to each robot. Then, a sufficient number n of surveillance robots for a line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ of length ℓ should satisfy

$$\ell=\sum_{i=1}^{n}\ell_i<\sum_{i=1}^{n}\ell^*(v_{Di},s_i), \tag{56}$$

with each robot assigned to the surveillance of a line subsegment $\overline{\boldsymbol{x}_{\ell,i-1}\boldsymbol{x}_{\ell,i}}$ of length ℓ_i. The adjacent subsegments should cover the entire segment such that

$$\overline{\boldsymbol{x}_L\boldsymbol{x}_R} = \bigcup_{i=1}^{n} \overline{\boldsymbol{x}_{\ell,i-1}\boldsymbol{x}_{\ell,i}}. \tag{57}$$

Equations (24)-(26), which determine the maximum length that a single robot can defend, together with (56) and (57) that ensure coverage of the desired line segment, can be used as design equations for the multi-robot surveillance system, either to select the appropriate robots from a given available set, or to build robots with the sufficient velocities and guard sizes to surveil a desired line lenght. In the case of $v_{Di} = v_A$, a single robot can defend a line segment, following equation (26).

For a robot A as described in Section II, and for each surveillance robot D_i assigned to the subsegment $\overline{\boldsymbol{x}_{\ell,i-1}\boldsymbol{x}_{\ell,i}}$, with position $\boldsymbol{x}_{Di}(t)$, initial position $\boldsymbol{x}_{Di}(0) \in \overline{\boldsymbol{x}_{\ell,i-1}\boldsymbol{x}_{\ell,i}}$, dynamics $\dot{\boldsymbol{x}}_{Di}(t) = u_i(t)\,\widehat{\boldsymbol{\ell}}$ confined to the line containing the line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ as defined in (1), maximum speed $0 < v_{Di} \leq v_A$ and guard s_i, we define the state $\boldsymbol{x}_i = \begin{bmatrix}\boldsymbol{x}_{Di}(t) & \boldsymbol{x}_A(t)\end{bmatrix}^{\mathsf{T}}$ with dynamics

$$\dot{\boldsymbol{x}}_i = \begin{bmatrix}\boldsymbol{0}\\ \dot{\boldsymbol{x}}_A(t)\end{bmatrix} + \begin{bmatrix}\widehat{\boldsymbol{\ell}}\\ \boldsymbol{0}\end{bmatrix} u_i \tag{58}$$

and the ZCBFs $h_{Li}(\boldsymbol{x}_i)$ and $h_{Ri}(\boldsymbol{x}_i)$ as follows

$$h_{Li}(\boldsymbol{x}_i) = \frac{\sqrt{\|\boldsymbol{x}_A(t) - \boldsymbol{x}_{\ell,i-1}\|^2 + \epsilon^2} - \epsilon}{v_A} - \frac{\left(\left(\boldsymbol{x}_{Di}(t) - s_i\widehat{\boldsymbol{\ell}}\right) - \boldsymbol{x}_{\ell,i-1}\right)\cdot\widehat{\boldsymbol{\ell}}}{v_{Di}}, \tag{59}$$

$$h_{Ri}(\boldsymbol{x}_i) = \frac{\sqrt{\|\boldsymbol{x}_{\ell,i} - \boldsymbol{x}_A(t)\|^2 + \epsilon^2} - \epsilon}{v_A} - \frac{\left(\boldsymbol{x}_{\ell,i} - \left(\boldsymbol{x}_{Di}(t) + s_i\widehat{\boldsymbol{\ell}}\right)\right)\cdot\widehat{\boldsymbol{\ell}}}{v_{Di}}. \tag{60}$$

Proposition 7. *Given a line segment $\overline{\boldsymbol{x}_L\boldsymbol{x}_R}$ as defined in* (1) *and divided into n adjacent subsegments $\overline{\boldsymbol{x}_{\ell,i-1}\boldsymbol{x}_{\ell,i}}$ satisfying* (57)*, an intruder robot A and surveillance robots D_i for which* (58)*,* (59) *and* (60) *are defined, if $h_{Li}(\boldsymbol{x}_i) \geq 0$ and $h_{Ri}(\boldsymbol{x}_i) \geq 0$ for all $i \in \{1,\ldots,n\}$ and all $t \geq 0$, then $-s_i \leq (\boldsymbol{x}_A(t) - \boldsymbol{x}_{D,i}(t))\cdot\widehat{\boldsymbol{\ell}} \leq s_i$ for some i whenever $\boldsymbol{x}_A(t) \in \overline{\boldsymbol{x}_L\boldsymbol{x}_R}$.*

Proof. By Proposition 3, if $h_{Li}(\boldsymbol{x}_i) \geq 0$ and $h_{Ri}(\boldsymbol{x}_i) \geq 0$, then $-s_i \leq (\boldsymbol{x}_A(t) - \boldsymbol{x}_{Di}(t))\cdot\widehat{\boldsymbol{\ell}} \leq s_i$ for some i whenever $\boldsymbol{x}_A(t) \in \overline{\boldsymbol{x}_{\ell,i-1}\boldsymbol{x}_{\ell,i}}$. Due to (57), the line subsegments cover the entire line segment, and therefore, $-s_i \leq (\boldsymbol{x}_A(t) - \boldsymbol{x}_{Di}(t))\cdot\widehat{\boldsymbol{\ell}} \leq s_i$ for some i whenever $\boldsymbol{x}_A(t) \in \overline{\boldsymbol{x}_L\boldsymbol{x}_R}$. ☐

This strategy requires that each robot D_i defends its corresponding line subsegment $\overline{\boldsymbol{x}_{\ell,i-1}\boldsymbol{x}_{\ell,i}}$. Robots with different properties can be used, adjusting the length of the line subsegments that each one defends according to its velocity and span. Let the set $\mathcal{C}_i$ be defined as

$$\mathcal{C}_i = \{\boldsymbol{x}_i : h_{Li}(\boldsymbol{x}_i) \geq 0\} \bigcap \{\boldsymbol{x}_i : h_{Ri}(\boldsymbol{x}_i) \geq 0\} \tag{61}$$

Selecting $\alpha(h) = \gamma h$, taking derivatives and using lower bound approximations as in the procedure leading to inequality (40), we obtain the inequality that the control input u_i must satisfy to ensure the surveillance of the corresponding line subsegment:

$$v_{Di}(1 - \gamma_i h_{Ri}(\boldsymbol{x}_i)) \leq u_i \leq -v_{Di}(1 - \gamma_i h_{Li}(\boldsymbol{x}_i)). \tag{62}$$

The existence of a control input u_i that satisfies (62) is guaranteed under the conditions of Proposition 4 and Proposition 5. The results are summarized next:

Proposition 8. *The inequality* (62) *can be satisfied by some $u_i \in [-v_{Di}, v_{Di}]$, if*

$$\gamma_i \geq \frac{1}{\frac{\sqrt{\left(\frac{\ell_i}{2}\right)^2 + \epsilon^2} - \epsilon}{v_A} - \frac{\left(\frac{\ell_i}{2}\right) - s_i}{v_{Di}}}, \tag{63}$$

with $s_i > \epsilon\frac{v_{Di}}{v_A}$, and ℓ_i satisfying

$$2s_i \leq \ell_i < \ell^*(v_{Di}, s_i), \tag{64}$$

where $\ell^*(v_{Di}, s_i)$ is given by

$$\ell^*(v_{Di}, s_i) = 2\left(s_i - \epsilon\frac{v_{Di}}{v_A}\right)\left(\frac{v_A^2}{v_A^2 - v_{Di}^2}\right) + \sqrt{\left(2\left(s - \epsilon\frac{v_{Di}}{v_A}\right)\left(\frac{v_A v_{Di}}{v_A^2 - v_{Di}^2}\right)\right)^2 + (2\epsilon)^2\left(\frac{v_{Di}^2}{v_A^2 - v_{Di}^2}\right)}, \tag{65}$$

for $v_A > v_{Di}$, or ℓ_i satisfying

$$\ell_i \geq 2s_i \tag{66}$$

for $v_A = v_{Di}$.

Proof. The proof is obtained from the application of Proposition 4 and Proposition 5. ☐

Proposition 9. *If the positions of the n defending robots at $t = 0$ are such that the initial states $\boldsymbol{x}_i(0) \in \mathcal{C}_i$ with $\mathcal{C}_i$ as defined in* (61) *for all $i \in \{1,\ldots,n\}$, and the conditions of Proposition 8 hold, then the control law*

$$u_i(\boldsymbol{x}_i) = \max\{v_{Di}(1 - \gamma_i h_{Ri}(\boldsymbol{x}_i)), 0\} + \min\{0, -v_{Di}(1 - \gamma_i h_{Li}(\boldsymbol{x}_i))\} \tag{67}$$

solves the Control Objective 2.

Proof. The same analysis as in the proof of Proposition 6 shows that (67) satisfies (62) and is globally Lipschitz continuous, and the system (58) is forward complete. By Theorem 1, the set $\mathcal{C}_i$ is forward invariant for all $i \in \{1,\ldots,n\}$ and for all $t \geq 0$. Since $h_{Li}(\boldsymbol{x}_i) \geq 0$ and $h_{Ri}(\boldsymbol{x}_i) \geq 0$ for all $i \in \{1,\ldots,n\}$ and for all $t \geq 0$, by Proposition 7, the Control Objective 2 is satisfied. ☐

V. APPLICATION TO A POLYGONAL PERIMETER

In this section, the presented formulation is applied to polygonal perimeters obtained by joining lines at their endpoints into a closed loop. Suppose we have a polygonal perimeter of N sides, each side $\overline{\boldsymbol{x}_{Lj}\boldsymbol{x}_{Rj}}$ of lenght L_j for $j \in \{1,\ldots,N\}$, and the intruder has a maximum speed $v_A > 0$. Each available surveillance robot has a maximum speed $0 < v_{Di} \le v_A$, guard of size s_i, and can defend a line subsegment $\overline{\boldsymbol{x}_{L,i}\boldsymbol{x}_{R,i}}$ of length $\ell_i \le \ell^*(v_{Di}, s_i)$ according to (65) if $v_{Di} < v_A$. For each jth side, we select a subset $\mathcal{N}_j$ of n_j robots such that

$$L_j = \sum_{i \in \mathcal{N}_j} \ell_i < \sum_{i \in \mathcal{N}_j} \ell^*(v_{Di}, s_i). \tag{68}$$

Then, we divide the jth side into adjacent line subsegments such that

$$\overline{\boldsymbol{x}_{Lj}\boldsymbol{x}_{Rj}} = \bigcup_{i \in \mathcal{N}_j} \overline{\boldsymbol{x}_{L,i}\boldsymbol{x}_{R,i}}. \tag{69}$$

and place the corresponding robot D_i within its line subsegment, making sure that its initial position is in accordance with the detection distances (53)-(54). If a robot has a speed such that $v_{Di} = v_A$, an entire side can be assigned to such robot alone. As an example, consider the case of the non-convex polygon with $N = 5$ shown in Figure 2, with sides of length $L_1 = L_2 = L_3 = 6$, $L_4 = L_5 = 6.245$. We simulate an intruder moving according to $\boldsymbol{x}_A(t) = \begin{bmatrix} 5\cos(5\theta(t)) & 5\sin(6\theta(t)) + 2.7713 \end{bmatrix}^{\mathsf{T}}$, $\dot{\theta}(t) = \frac{10}{\sqrt{(25\sin(5\theta(t)))^2 + (30\sin(6\theta(t)))^2}}$, with $\theta(0) = 0.23$, ensuring a maximum speed of $v_A = 10$. A sufficient number of robots will be placed on each side. The available robots as well as their characteristics are described in Table I. Substituting in

TABLE I: Robots for polygonal perimeter

Robot	$v_{D,i}$	s_i	$\ell^*(v_{D,i}, s_i)$
1	8.5	0.5	6.6553
2,3	7	0.5	3.3287
4,5,6,9	5.5	0.5	2.2198
7	5.5	1.5	6.6642
8	5.5	1.0	4.4420

equation (68), we can verify the following: robot 1 alone is sufficient for side 1 with $\ell_1 = 6$; for side 2, robots 2 and 3 can be assigned each covering an adjacent line subsegment within side 2 of lenght $\ell_2 = \ell_3 = 3$; regarding side 3, robots 4, 5 and 6 can be assigned, each covering a lenght $\ell_4 = \ell_5 = \ell_6 = 2$. Note how all these robots have the same guard size but different speeds. Our approach allows for the customization of the selection of robots. In this example, the fastest robot is assigned alone, while two slower robots are assigned together, and three of the slowliest robots are assigned to the same side. Consider now robot 7 with $\ell_7 = 6.245$, which according to equation (68) is able to surveil and defend side 4 alone in spite of being slow, due to having a large guard. Finally, robot 8 with $\ell_8 = 4.2$ and robot 9 with $\ell_9 = 2.045$ can be assigned for the surveillance of side 5.

Figure 2 shows the 5-sided polygon with robots 1 to 9 assigned to the corresponding sides. The dashed circles denote the minimum detection distance, measured from the endpoints of the line subsegments of each robot. The red dotted path corresponds to the motion of the intruder, shown as a red circle. Figure 3 shows the values of the ZCBFs for each defending robot satisfy $h_{Li}(\boldsymbol{x}_i) \ge 0$ and $h_{Ri}(\boldsymbol{x}_i) \ge 0$, and therefore, by Proposition 7, interception of the intruder is ensured. Figure 4 shows that the control action for each robot satisfies $-v_{Di} \le u_i(\boldsymbol{x}(t)) \le v_{Di}$.

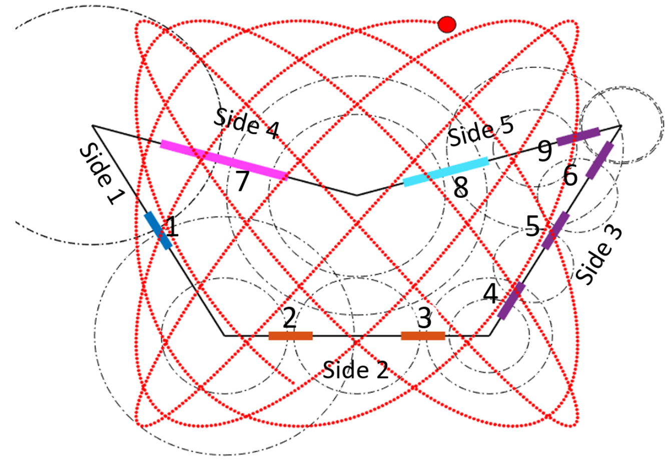

Fig. 2: A polygon with vertices at $(0, 4\sin(\pi/3))$, $(-3, 0)$, $(3, 0)$, $(-3 + 6\cos(\pi/3), 6\sin(\pi/3))$, and $(3 + 6\cos(\pi/3), 6\sin(\pi/3))$, defended by robots numbered from 1 to 9 with initial positions at $(-4.5, 2.5981)$, $(-1.5, 0)$, $(1.5, 0)$, $(3.5, 0.866)$, $(4.5, 2.5981)$, $(5.5, 4.3301)$, $(-3, 4.3301)$, $(2.0176, 4.0465)$, and $(5.0176, 4.9126)$, respectively, and velocity and guard parameters as shown in Table I. The trajectory of the intruder robot is shown in red. The detection distances (53) and (54) for each line subsegment corresponding to each robot are depicted with dashed circles. Note that the initial position of robot A is at a distance larger than the required detection distances.

The selection of robots for each side in this example is not unique, as different assignments and combinations are possible. If more robots with different velocities and guard sizes were available, they could be analyzed similarly and considered for the assignment. Furthermore, given a polygonal perimeter, appropriate speeds and guard sizes could be calculated using equations (64)-(66), (68) and (69) to build a desired number of robots to defend the perimeter. This example showcases the great flexibility of the described solution to the perimeter defense problem to incorporate multiple robots with different characteristics. The assignment of robots on polygons with different number of sides and side lenghts can be done following the same analysis.

VI. CONCLUSION

We present control laws that ensure the surveillance of a polygonal perimeter. The closed-form control laws, based on set-invariance principles, allow for the use of multiple robots with different sizes and maximum speeds for the surveillance of the perimeter with the desired length and polygonal shape. Future work will consider the case of multiple intruders, area surveillance, and the use of robots with higher order dynamics.

REFERENCES

[1] R. Isaacs, *Differential games: a mathematical theory with applications to warfare and pursuit, control and optimization*. Courier Corporation, 1999.

[2] T. H. Chung, G. A. Hollinger, and V. Isler, "Search and pursuit-evasion in mobile robotics," *Autonomous robots*, vol. 31, no. 4, p. 299, 2011.

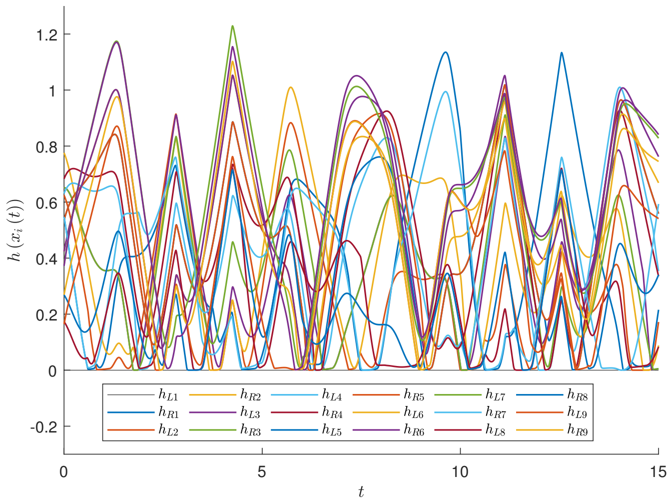

Fig. 3: The ZCBFs $h_{Li}(\boldsymbol{x}_i)$ and $h_{Ri}(\boldsymbol{x}_i)$ corresponding to each line subsegment have a value always greater or equal than zero, which according to Proposition 7, ensures the surveillance and defense of the polygon in the scenario from Figure 2.

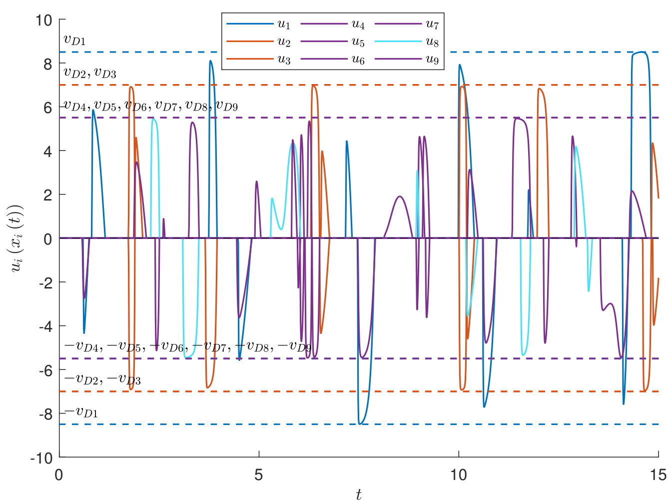

Fig. 4: The control action of each robot in the scenario from Figure 2 is bounded between its corresponding limits, ensuring $-v_{Di} \leq u_i(\boldsymbol{x}(t)) \leq v_{Di}$.

[3] I. M. Mitchell, A. M. Bayen, and C. J. Tomlin, "A time-dependent hamilton-jacobi formulation of reachable sets for continuous dynamic games," *IEEE Transactions on Automatic Control*, vol. 50, no. 7, pp. 947–957, 2005.

[4] H. Huang, J. Ding, W. Zhang, and C. J. Tomlin, "A differential game approach to planning in adversarial scenarios: A case study on capture-the-flag," in *2011 IEEE International Conference on Robotics and Automation*, 2011, pp. 1451–1456.

[5] M. Chen, Z. Zhou, and C. J. Tomlin, "A path defense approach to the multiplayer reach-avoid game," in *53rd IEEE Conference on Decision and Control*, 2014, pp. 2420–2426.

[6] ——, "Multiplayer reach-avoid games via low dimensional solutions and maximum matching," in *2014 American Control Conference*, 2014, pp. 1444–1449.

[7] ——, "Multiplayer reach-avoid games via pairwise outcomes," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1451–1457, 2017.

[8] Z. Zhou, R. Takei, H. Huang, and C. J. Tomlin, "A general, open-loop formulation for reach-avoid games," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, 2012, pp. 6501–6506.

[9] Z. Zhou, J. Ding, H. Huang, R. Takei, and C. Tomlin, "Efficient path planning algorithms in reach-avoid problems," *Automatica*, vol. 89, pp. 28–36, 2018.

[10] S. Liu, Z. Zhou, C. Tomlin, and K. Hedrick, "Evasion as a team against a faster pursuer," in *2013 American Control Conference*, 2013, pp. 5368–5373.

[11] Z. Zhou, J. R. Shewchuk, D. Stipanovic, H. Huang, and C. J. Tomlin, "Smarter lions: Efficient cooperative pursuit in general bounded arenas," *SIAM Journal on Control and Optimization*, vol. 58, no. 2, pp. 1229–1256, 2020.

[12] G. C. Chasparis and J. S. Shamma, "Linear-programming-based multi-vehicle path planning with adversaries," in *Proceedings of the 2005, American Control Conference, 2005.*, 2005, pp. 1072–1077 vol. 2.

[13] M. G. Earl and R. D'Andrea, "Modeling and control of a multi-agent system using mixed integer linear programming," in *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, vol. 1, 2002, pp. 107–111 vol.1.

[14] E. Bakolas and P. Tsiotras, "Optimal pursuit of moving targets using dynamic voronoi diagrams," in *49th IEEE Conference on Decision and Control (CDC)*, 2010, pp. 7431–7436.

[15] Z. Zhou, W. Zhang, J. Ding, H. Huang, D. M. Stipanović, and C. J. Tomlin, "Cooperative pursuit with voronoi partitions," *Automatica*, vol. 72, pp. 64–72, 2016.

[16] A. Pierson, Z. Wang, and M. Schwager, "Intercepting rogue robots: An algorithm for capturing multiple evaders with multiple pursuers," *IEEE Robotics and Automation Letters*, vol. 2, no. 2, pp. 530–537, 2017.

[17] R. Yan, Z. Shi, and Y. Zhong, "Reach-avoid games with two defenders and one attacker: An analytical approach," *IEEE Transactions on Cybernetics*, vol. 49, no. 3, pp. 1035–1046, 2019.

[18] R. H. Venkatesan and N. K. Sinha, "The target guarding problem revisited: Some interesting revelations," *IFAC Proceedings Volumes*, vol. 47, no. 3, pp. 1556–1561, 2014.

[19] M. W. Harris, "Abnormal and singular solutions in the target guarding problem with dynamics," *Journal of Optimization Theory and Applications*, vol. 184, no. 2, pp. 627–643, 2020.

[20] M. Pachter, E. Garcia, and D. W. Casbeer, "Differential game of guarding a target," *Journal of Guidance, Control, and Dynamics*, vol. 40, no. 11, pp. 2991–2998, 2017.

[21] E. Garcia, D. W. Casbeer, and M. Pachter, "Design and analysis of state-feedback optimal strategies for the differential game of active defense," *IEEE Transactions on Automatic Control*, vol. 64, no. 2, pp. 553–568, 2019.

[22] D. Li and J. B. Cruz, "Defending an asset: A linear quadratic game approach," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 47, no. 2, pp. 1026–1044, 2011.

[23] T. G. McGee and J. K. Hedrick, "Guaranteed strategies to search for mobile evaders in the plane," in *2006 American Control Conference*, 2006, pp. 6 pp.–.

[24] F. Pasqualetti, A. Franchi, and F. Bullo, "On cooperative patrolling: Optimal trajectories, complexity analysis, and approximation algorithms," *IEEE Transactions on Robotics*, vol. 28, no. 3, pp. 592–606, 2012.

[25] N. Agmon, S. Kraus, and G. A. Kaminka, "Multi-robot perimeter patrol in adversarial settings," in *2008 IEEE International Conference on Robotics and Automation*, 2008, pp. 2339–2345.

[26] N. Agmon, G. A. Kaminka, and S. Kraus, "Multi-robot adversarial patrolling: facing a full-knowledge opponent," *Journal of Artificial Intelligence Research*, vol. 42, pp. 887–916, 2011.

[27] D. Shishika and V. Kumar, "Local-game decomposition for multiplayer perimeter-defense problem," in *2018 IEEE Conference on Decision and Control (CDC)*, 2018, pp. 2093–2100.

[28] D. Shishika, J. Paulos, and V. Kumar, "Cooperative team strategies for multi-player perimeter-defense games," *IEEE Robotics and Automation Letters*, vol. 5, no. 2, pp. 2738–2745, 2020.

[29] D. Shishika, J. Paulos, M. R. Dorothy, M. Ani Hsieh, and V. Kumar, "Team composition for perimeter defense with patrollers and defenders," in *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 7325–7332.

[30] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2017.

[31] X. Xu, "Constrained control of input–output linearizable systems using control sharing barrier functions," *Automatica*, vol. 87, pp. 195–201, 2018.

[32] H. K. Khalil and J. W. Grizzle, *Nonlinear systems*. Prentice hall Upper Saddle River, NJ, 2002, vol. 3.