Obstacle Avoidance via Hybrid Feedback

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***Abstract*—In this paper we present a hybrid feedback approach to solve the navigation problem in the $n-$dimensional space containing an arbitrary number of ellipsoidal obstacles. The proposed algorithm guarantees both global asymptotic stabilization to a target position and avoidance of the obstacles. The controller, exploiting hysteresis regions, employs a Zeno-free switching between two modes of control: stabilization and avoidance. Simulation results illustrate the performance of the proposed approach for 2-dimensional and 3-dimensional scenarios.**

I. Introduction

For decades, the obstacle avoidance problem has been an active area of research in the robotics and control communities [1]. In a typical robot navigation scenario, the robot is required to reach a given goal (destination) while not colliding with a set of obstacle regions in the workspace. Since the pioneering work by Khatib [2], artificial potential fields have been widely used in the obstacle avoidance problem since they offer the possibility to combine the solution to the global find-path problem with a feedback controller for the robot, thus, allowing the high-level planner to address more abstract tasks. The idea is to generate an artificial potential field that renders the goal attractive and the obstacles repulsive. Then, by considering trajectories that navigate along the negative gradient of the artificial potential field, one can ensure that the robot will reach the desired target while avoiding to collide with the obstacles. However, artificial potential field-based algorithms suffer from 1) the presence of local minima preventing the successful navigation to the target point and 2) arbitrarily large repulsive potential near the obstacles, which is in conflict with the inevitable actuator saturations. The navigation-function approach, initiated by Koditscheck and Rimon [3] for sphere worlds [3, p. 414], solves both problems. It allows obtaining artificial potential fields with the nice property that all but one of the critical points are saddles and the remaining critical point is the desired reference. Since then, the navigation function-based approach has been extended in many different directions; *e.g.,* for multi-agent systems [4]–[6], for unknown sphere words [7], and for focally admissible obstacles [8]. The major drawback of navigation functions is that they are not correct by construction. In fact, navigation functions are theoretically guaranteed to exist, but their explicit computation is not straightforward since they require an unknown tuning of a given parameter to eliminate local minima. Recently, Loizou [9] introduced the navigation transform that diffeomorphically maps the workspace to a trivial domain called the *point world* consisting of a closed ball with a finite number of points removed. Once this transformation is found, the navigation problem is solved from almost all initial conditions without requiring any tuning. In addition, the trajectory duration is explicitly available, which provides a timed-abstraction solution to the motion-planning problem. Similarly, the recent work in [10] uses the so-called prescribed performance control to design a time-varying control law that drives the robot, in finite time, from all initial conditions to some neighborhood of the target while avoiding the obstacles. Another approach to the navigation problem is through barrier functions (see [11] and references therein), which are developed for nonlinear systems with state-space constraints and ensure safety. Model predictive control approaches have been also used for reactive robot navigation, *e.g.*, [12], [13].

However, by using any of the approaches described above, it is not possible to ensure safety from all initial conditions in the obstacle-free state space. As pointed out in [3], the appearance of additional undesired equilibria is unavoidable when considering continuous time-invariant vector fields. Furthermore, this problem is more far-reaching since it is always possible to find arbitrarily small adversarial (noise) signals acting on the vector field, such that a set of initial conditions different from the target, possibly of measure zero, can be rendered stable [14, Thm. 6.5]. To deal with such limitations, the authors in [15] proposed a hybrid state feedback controller, using Lyapunov-based hysteresis switching, to achieve robust global asymptotic regulation in $\mathbb{R}^2$ to a target while avoiding a single obstacle. This approach has been exploited in [16] to steer a planar vehicle to the source of an unknown but measurable signal while avoiding an obstacle. In [17] and [18], a hybrid control law was proposed to globally asymptotically stabilize a class of linear systems while avoiding neighbourhoods of unsafe isolated points in $\mathbb{R}^n$. Although such hybrid approaches are promising, they are still challenged by constructing the suitable hybrid feedback for higher dimensions and with more complex obstacles shapes.

In this work, we propose a hybrid control algorithm for the global asymptotic stabilization of a point mass moving in an arbitrary $n-$dimensional space while safely avoiding obstacles that have generic ellipsoidal shapes, based on the preliminary treatment of this problem for a single spherical obstacle in [19]. The ellipsoids provide a tighter bounding volume than spheres, and in our scheme this volume can be arbitrarily flat and close to the target, which leads to a significant reduction in the level of conservatism compared, *e.g.*, to [20, Thm. 3] as we show in Section VI. Our proposed hybrid algorithm employs a hysteresis-based switching between the avoidance controller and the stabilizing controller to guarantee forward invariance of the obstacle-free region (corresponding to safety) and global asymptotic stability of the target position. We consider trajectories in an $n-$dimensional Euclidean space and we resort to tools from higher-dimensional geometry to provide a construction of the flow and jump sets where the different modes of operation of the hybrid controller are activated. Furthermore, the hybrid control law guarantees a bounded control input, it matches the stabilizing controller in arbitrarily large subsets of the obstacle-free region by a suitable tuning of its parameters (hence qualifying as minimally invasive), it can be readily extended to a non-point mass vehicle and enjoys some level of inherent robustness to perturbations.

Structure. Preliminaries are in Section II. The navigation problem is formulated in Section III. Our proposed hybrid control scheme is discussed in Section IV. Section V presents the main results. Numerical examples are in Section VI. An extended version and all the proofs of the lemmas are in [21].

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II. Preliminaries

$\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_{\geq}$ denote, respectively, the set of nonnegative integers, reals and nonnegative reals. $\mathbb{R}^n$ is the n-dimensional Euclidean space and $\mathbb{S}^n$ is the n-dimensional unit sphere embedded in $\mathbb{R}^{n+1}$. Given the column vectors $v_1 \in \mathbb{R}^{n_1}$ and $v_2 \in \mathbb{R}^{n_2}$, (v_1, v_2) denotes the stacked vector $\begin{bmatrix} v_1^\top & v_2^\top \end{bmatrix}^\top$. The Euclidean norm of $x \in \mathbb{R}^n$ is defined as $\|x\| := \sqrt{x^\top x}$. For an arbitrary matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_i(A)$ denotes the i-th eigenvalue of A. If A is a symmetric matrix, then $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote, respectively, the smallest and largest eigenvalue of A. The closure, interior and boundary of a set $\mathcal{A} \subset \mathbb{R}^n$ are denoted as $\overline{\mathcal{A}}, \mathcal{A}^\circ$ and $\partial\mathcal{A}$, respectively. The relative complement of a set $\mathcal{B} \subset \mathbb{R}^n$ with respect to a set $\mathcal{A}$ is denoted by $\mathcal{A}\backslash\mathcal{B}$ and contains the elements of $\mathcal{A}$ which are not in $\mathcal{B}$. The tangent cone to a set $\mathcal{K} \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted $\mathbf{T}_{\mathcal{K}}(x)$, is defined as in [22, Def. 5.12 and Fig. 5.4]. For $z \in \mathbb{R}^n\backslash\{0\}$, we define the three projection maps

$$\pi^{\|}(z) := \tfrac{zz^\top}{\|z\|^2},\ \pi^{\perp}(z) := I_n - \tfrac{zz^\top}{\|z\|^2},\ \rho(z) := I_n - 2\tfrac{zz^\top}{\|z\|^2} \tag{1}$$

where I_n is the $n \times n$ identity matrix. The map $\pi^{\|}(\cdot)$ is the parallel projection map, $\pi^{\perp}(\cdot)$ is the orthogonal projection map [23], and $\rho(\cdot)$ is the reflector map (also called Householder transformation). For $v \neq 0$, $r \geq 0$, $2\theta \in [0, \pi]$ and E positive definite, we define the next geometric subsets of $\mathbb{R}^n$:

- line $\quad \mathcal{L}(c, v) := \{x \in \mathbb{R}^n : x = c + \lambda v, \lambda \in \mathbb{R}\}, \quad (2)$
- hyperplane $\quad \mathcal{P}(c, v) := \{x \in \mathbb{R}^n : v^\top(x - c) = 0\}, \quad (3)$
- sphere $\quad \mathcal{S}(c, r) := \{x \in \mathbb{R}^n : \|x - c\| = r\}, \quad (4)$
- ellipsoid $\quad \mathcal{E}(c, E) := \{x \in \mathbb{R}^n : \|E(x - c)\|^2 = 1\}, \quad (5)$
- cone $\quad \mathcal{C}(c,v,\theta,E) := \{x \in \mathbb{R}^n : \cos(\theta)\|Ev\|\|E(x - c)\| = v^\top E^2(x - c)\}. \quad (6)$

In (3)-(6), we add subscripts $\leq$ or $\geq$ to refer to the set obtained by substituting the $=$ with $\leq$ or $\geq$. *E.g.*, $\mathcal{P}_{\leq}(c, v)$ and $\mathcal{P}_{\geq}(c, v)$ are the two closed sets into which the hyperplane $\mathcal{P}(c, v)$ divides $\mathbb{R}^n$.

***Definition** 1: Two ellipsoids $\mathcal{E}_{\leq}(c_1, E_1)$ and $\mathcal{E}_{\leq}(c_2, E_2)$ are weakly disjoint if $\mathcal{E}_{\leq}(c_1, E_1) \cap \mathcal{E}_{\leq}(c_2, E_2) = \emptyset$, and are strongly disjoint if $(\lambda_{\min}(E_1))^{-1} + (\lambda_{\min}(E_2))^{-1} < \|c_1 - c_2\|$.*

Strong disjointness means that the two smallest spherical balls containing the ellipsoids are disjoint and is more conservative than weak disjointness. We use hybrid dynamical systems [22], *i.e.*,

$$\begin{cases} \dot{X} \in \mathbf{F}(X), & X \in \mathcal{F}, \\ X^+ \in \mathbf{J}(X), & X \in \mathcal{J}, \end{cases} \tag{7}$$

where $X \in \mathbb{R}^n$ is the state, the (set-valued) *flow map* $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and *jump map* $\mathbf{J} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ govern continuous and discrete evolution, which can occur respectively in the *flow set* $\mathcal{F} \subset \mathbb{R}^n$ and the *jump set* $\mathcal{J} \subset \mathbb{R}^n$. The notions of solution ϕ to a hybrid system, its hybrid time domain $\operatorname{dom}\phi$, maximal and complete solution are, respectively, as in [22, Def. 2.6, Def. 2.3, Def. 2.7, p. 30].

III. Problem Formulation

We consider a point mass vehicle moving in the n-dimensional Euclidean space containing $I \in \mathbb{N}$ obstacles denoted by $\mathcal{O}_1, \cdots, \mathcal{O}_I$. For each $i \in \{1, \cdots, I\} =: \mathbb{I}$, the obstacle $\mathcal{O}_i$ has an ellipsoidal shape such that $\mathcal{O}_i := \mathcal{E}_{\leq}(c_i, E_i)$, for some center $c_i \in \mathbb{R}^n$ and some positive definite matrix $E_i \in \mathbb{R}^{n\times n}$ defining the shape of the obstacle. The free workspace is then defined by the closed set

$$\mathcal{W} := \bigcap_{i\in\mathbb{I}} \mathcal{E}_{\geq}(c_i, E_i). \tag{8}$$

The vehicle is moving according to the dynamics

$$\dot{x} = u, \tag{9}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^n$ is the control input. The vehicle is required to stabilize its position to a target position while avoiding the obstacles. Without loss of generality we consider the target position to be the origin $x = 0$.

***Assumption** 1: $n \geq 2$.*

We consider $n \geq 2$ since for $n = 1$ (*i.e.*, the state space is a line), global asymptotic stabilization with obstacle avoidance is infeasible.

***Assumption** 2: For all $i \in \mathbb{I}$, $\|E_i c_i\| > 1$.*

Assumption 2 requires that the target position $x = 0$ is not inside any of the obstacle regions $\mathcal{O}_i$, otherwise the considered navigation problem would be infeasible.

***Assumption** 3: $\{\mathcal{O}_i\}_{i\in\mathbb{I}}$ are weakly pairwise disjoint.*

In Assumption 3 we impose that there is no intersection region between any two obstacles. Otherwise, the union of the two intersecting obstacles forms another region which can have a different shape than an ellipsoid. Our *objectives* in designing a control strategy are:

i) the obstacle-free region $\mathcal{W}$ in (8) is forward invariant,
ii) the target $x = 0$ is globally asymptotically stable.

Objective i) guarantees that all solutions of the closed-loop system are safely avoiding the obstacles by remaining in the free workspace $\mathcal{W}$ while objective ii) corresponds to global stabilization of the target.

IV. Hybrid Control for Obstacle Avoidance

In this section, we propose a hybrid controller that switches suitably between a *stabilizing* and an *avoidance* controller.

A. Control Input

In this section we propose the feedback law for the control input u in (9). We define a discrete variable

$$m \in \{-1, 0, 1\} =: \mathbb{M}.$$

The value $m = 0$ corresponds to the activation of the stabilizing controller and the values $m = -1$, $m = 1$ correspond to the activation of one of the two configurations of the avoidance controller. The proposed control input u depends on the state $x \in \mathbb{R}^n$, the obstacle $i \in \mathbb{I}$ and the control mode $m \in \mathbb{M}$ as

$$\begin{aligned} u &= \kappa(x, i, m) \\ &:= \begin{cases} -k_0 x, & m = 0, \\ -k_m E_i^{-1}\pi^{\perp}(E_i(x - c_i))E_i(x - p_m^i), & m \in \{-1, 1\}, \end{cases} \end{aligned} \tag{10}$$

where k_{-1}, k_0, $k_1 > 0$ are the control gains for each control mode $m \in \mathbb{M}$ and the points $p_m^i \in \mathbb{R}^n$, $m \in \{-1, 1\}$ and $i \in \mathbb{I}$, are design parameters defined below. In the stabilization mode ($m = 0$), the control input in (10) steers x towards the origin through state feedback. In the avoidance mode depicted in Fig. 1, the control input minimizes the distance to the *auxiliary* attractive point p_m^i *while* maintaining a constant distance to the obstacle $\mathcal{O}_i$. Indeed, the time derivative of $\|E_i(x - c_i)\|^2$ along solutions of $\dot{x} = \kappa(x, i, m)$ for $m \in \{-1, 1\}$ and $i \in \mathbb{I}$, is zero. Then, if we activate the avoidance mode sufficiently away from the obstacle, the avoidance feedback $u = \kappa(x, i, m)$ guarantees that the vehicle does not hit the obstacle. Whereas the logic variable i corresponds to obstacle $\mathcal{O}_i$, the logic variable m is selected according to a hybrid mechanism that exploits a suitable construction of flow and jump sets, detailed in Section IV-B.

In order to clear the obstacle while approaching the desired target position at the origin, we select the points p_1^i and p_{-1}^i in the region between the obstacle and the origin, see Fig. 1. More precisely, for

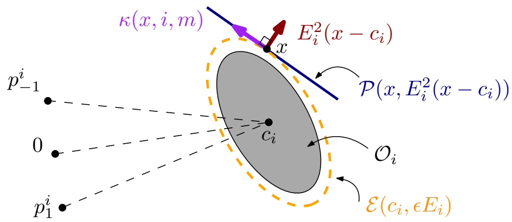

Fig. 1. Illustration of the projection-based avoidance controller. The vehicle is attracted to an auxiliary point p_m^i while sliding on a neighbouring ellipsoid.

$\theta_i > 0$ (which will be further bounded in Lemma 3), the points p_1^i and p_{-1}^i are selected as

$$p_1^i \in \mathcal{C}(c_i, -c_i, \theta_i, E_i) \backslash \{c_i\}, \tag{11a}$$

$$p_{-1}^i := -E_i^{-1}\rho(E_i c_i) E_i p_1^i. \tag{11b}$$

By (11), p_{-1}^i opposes p_1^i diametrically with respect to the cone axis (for $E_i = I_n$, p_{-1}^i is obtained by an orthogonal reflection) and also belongs to $\mathcal{C}(c_i, -c_i, \theta_i, E_i) \backslash \{c_i\}$ as shown in the next lemma.

***Lemma** 1:* $p_{-1}^i \in \mathcal{C}(c_i, -c_i, \theta_i, E_i) \backslash \{c_i\}$.

Note that the results of the paper hold for any selection of the point p_1^i as long as it lies on the surface of the cone as in (11a). An explicit guided choice for the points p_1^i is given in Section VI for the 2D and 3D cases. The motivation for the choice of the avoidance controller mode in (10) is that the avoidance task is analogous (up to a linear transformation) to a stabilization problem on the unit sphere $\mathbb{S}^{n-1}$. Therefore, as pointed out for instance in [24], global asymptotic stabilization cannot be accomplished by only one continuous time-invariant controller, but it can be by a hybrid feedback with at least two configurations. For this reason, we consider two avoidance modes with $m = -1$ and $m = 1$ and, hence, the points p_1^i and p_{-1}^i must be distinct. Finally, further motivation for this construction is detailed in Section IV-B and, in particular, in Lemma 2, which is important for the construction of flow and jump sets.

B. Geometric Construction of the Flow and Jump sets

In this section we construct explicitly the flow and jump sets where the stabilization and avoidance controllers are activated.

1) Safety Helmets: Our proposed construction of flow and jump sets is based on regions that have the shape of a *helmet*, whose construction is now motivated. In the stabilization mode $m = 0$, the closed-loop system should *not* flow when: 1) x is close enough to any of the obstacle regions $\mathcal{E}_{\le}(c_i, E_i)$ and 2) the vector field $-k_0 x$ points inside $\mathcal{E}_{\le}(c_i, E_i)$. Otherwise, the vehicle ends up hitting the obstacle i. Indeed, by computing the time derivative of $\|E_i(x-c_i)\|^2$ along solutions of the vector field $-k_0 x$, we obtain

$$\tfrac{1}{2}\tfrac{d}{dt}\|E_i(x-c_i)\|^2 = k_0\|E_i\bar{c}_i\|^2\left(1 - \|\bar{E}_i(x-\bar{c}_i)\|^2\right), \tag{12}$$

where $\bar{c}_i := c_i/2$ and $\bar{E}_i := 2E_i/(\|E_i c_i\|)$. (12) implies that the distance function $\|E_i(x-c_i)\|^2$ decreases for all x in the closed set $\mathcal{E}_{\ge}(\bar{c}_i, \bar{E}_i)$. Consider now Fig. 2 for a sketch of the next sets. For obstacle i, define the *helmet*-shaped set

$$\mathcal{H}_i^* := \mathcal{E}(c_i, E_i) \cap \mathcal{E}_{\ge}(\bar{c}_i, \bar{E}_i). \tag{13}$$

$\mathcal{H}_i^*$ is the set of all points that lie on the boundary of the obstacle $\mathcal{O}_i$ and are associated with a vector field pointing towards the obstacle. Then, for obstacle i, we define the *safety helmet* as:

$$\mathcal{H}_i(\epsilon,\nu) := \mathcal{E}_{\le}(c_i, \epsilon E_i) \cap \mathcal{E}_{\ge}(c_i, E_i) \cap \mathcal{E}_{\ge}(\bar{c}_i, \nu\bar{E}_i) \tag{14}$$

for some parameters ϵ, $\nu > 0$. ϵ and ν determine the thickness of the safety helmet by tuning the dilation/shrinking of the ellipsoids $\mathcal{E}(c_i, E_i)$ and $\mathcal{E}(\bar{c}_i, \bar{E}_i)$, thereby generating a dilated version of $\mathcal{H}_i^*$.

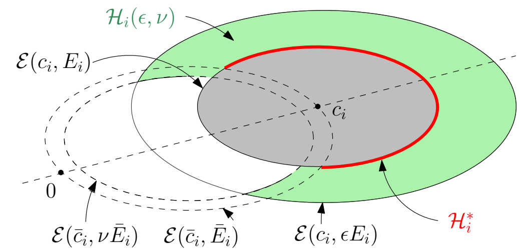

Fig. 2. The helmet $\mathcal{H}_i^*$ in (13) (red) corresponds to all boundary points where the stabilization vector field is pointing inside the obstacle (grey). The safety helmet $\mathcal{H}_i(\epsilon,\nu)$ in (14) (green) corresponds to a dilated version of $\mathcal{H}_i^*$.

The safety helmet $\mathcal{H}_i(\epsilon,\nu)$ constitutes the main ingredient of our following constructions.

2) Stabilization Mode $m = 0$: Consider Fig. 3 from now on for a visualization of the sets we are introducing in our construction. In stabilization mode ($m = 0$), we create around each obstacle $\mathcal{O}_i$ a safety helmet $\mathcal{H}_i(\epsilon_i,\nu_i)$ that adds a safety layer to the given obstacle. The controller mode must be switched to the avoidance mode whenever the vehicle reaches this safety helmet. Specifically, we define for each $i \in \mathbb{I}$, a jump set

$$\mathcal{J}_0^i := \mathcal{H}_i(\epsilon_i,\nu_i) \cap \mathcal{W}, \tag{15}$$

where $\epsilon_i \in (0,1)$ dilates $\mathcal{E}_{\le}(c_i, E_i)$ to $\mathcal{E}_{\le}(c_i, \epsilon_i E_i)$, $\nu_i \in (1,\infty)$ shrinks $\mathcal{E}_{\ge}(\bar{c}_i, \bar{E}_i)$ to $\mathcal{E}_{\ge}(\bar{c}_i, \nu_i\bar{E}_i)$, and $\mathcal{W}$ is the free workspace defined in (8). We emphasize that we consider the intersection with $\mathcal{W}$ in (15) for convenience, but later we tune the parameters such that $\mathcal{H}_i(\epsilon_i,\nu_i) \subset \mathcal{W}$, which implies $\mathcal{J}_0^i$ will equal to $\mathcal{H}_i(\epsilon_i,\nu_i)$. The selection of $\mathcal{J}_0^i$ in (15) leads naturally to the next set (corresponding to the closed complement of $\mathcal{J}_0^i$ in the free workspace)

$$\mathcal{F}_0^i := \Big(\mathcal{E}_{\ge}(c_i, \epsilon_i E_i) \cup \mathcal{E}_{\le}(\bar{c}_i, \nu_i\bar{E}_i)\Big) \cap \mathcal{W}, \tag{16}$$

which we use for the flow set of the stabilization mode. Finally, from (15) and (16), we take all the obstacles into account and define the flow and jump sets for the stabilization mode as

$$\mathcal{F}_0 := \Big(\bigcap_{i\in\mathbb{I}} \mathcal{F}_0^i\Big) \times \mathbb{I}, \qquad \mathcal{J}_0 := \Big(\bigcup_{i\in\mathbb{I}} \mathcal{J}_0^i\Big) \times \mathbb{I}. \tag{17}$$

Indeed, the stabilization mode will be selected when the state x belongs to the intersection of the sets $\mathcal{F}_0^i$, and a jump to the avoidance mode will occur when the state x belongs to the union of the sets $\mathcal{J}_0^i$. In other words, if during the stabilization mode the vehicle reaches any one of the safety helmets, then the controller jumps to one of the avoidance modes with m equal to -1 or 1.

3) Avoidance Mode $m \in \{-1,1\}$: We consider now the construction of flow and jump sets for the avoidance modes $m \in \{-1,1\}$ and the specific obstacle $i \in \mathbb{I}$ with the aid of Fig. 3. To highlight their motivation, we first define such flow sets and state later in (20) the corresponding jump sets. For each $i \in \mathbb{I}$ and $m \in \{-1,1\}$, the avoidance flow set is

$$\mathcal{F}_m^i := \mathcal{H}_i(\delta_i,\mu_i) \cap \mathcal{C}_{\ge}(c_i, c_i - p_m^i, \psi_i, E_i) \cap \mathcal{W} \tag{18}$$

with $\delta_i \in (0,\epsilon_i)$ dilating $\mathcal{E}_{\le}(c_i, \epsilon_i E_i)$ to $\mathcal{E}_{\le}(c_i, \delta_i E_i)$, $\mu_i \in (\nu_i,\infty)$ shrinking $\mathcal{E}_{\ge}(\bar{c}_i, \nu_i\bar{E}_i)$ to $\mathcal{E}_{\ge}(\bar{c}_i, \mu_i\bar{E}_i)$, and $\psi_i \in (0,\pi/2]$. In the two configurations $m \in \{-1,1\}$ of the avoidance of obstacle $i \in \mathbb{I}$, we want the vehicle to slide on the safety helmet $\mathcal{H}_i(\delta_i,\mu_i)$ while maintaining a constant distance to the obstacle. By selecting $\delta_i \in (0,\epsilon_i)$ and $\mu_i \in (\nu_i,\infty)$, one obtains a dilated version of $\mathcal{H}_i(\epsilon_i,\nu_i)$ used in $\mathcal{J}_0^i$ and, thus, creates a hysteresis region useful to prevent infinitely many consecutive jumps (Zeno behavior). However, the avoidance vector field $\kappa(x,i,m)$ in (10) has some undesirable

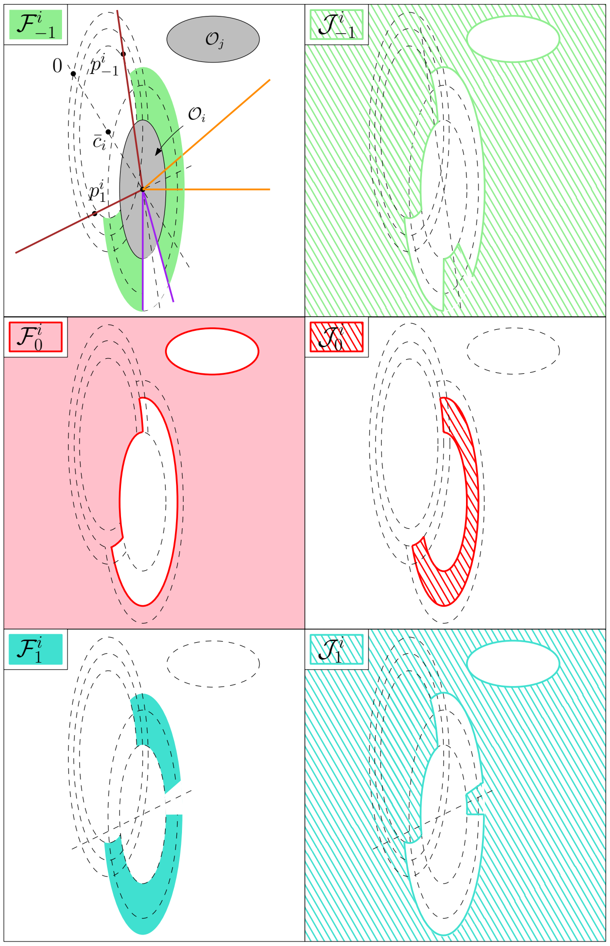

Fig. 3. 2D illustration of flow and jump sets considered in Sections IV-V corresponding to obstacle $\mathcal{O}_i$ (in the presence of a second obstacle $\mathcal{O}_j$). The stabilization-mode jump set $\mathcal{J}_0^i$ (hatched red) is constructed by using the helmet $\mathcal{H}_i(\epsilon_i, \nu_i)$, while the corresponding flow set $\mathcal{F}_0^i$ is the complement of $\mathcal{J}_0^i$ in the free workspace. For the avoidance mode we select p_1^i and p_{-1}^i to lie on the cone $\mathcal{C}(c_i, -c_i, \theta_i, E_i)$ (solid brown line). The avoidance flow set $\mathcal{F}_m^i$, with $m \in \{-1, 1\}$, corresponds to the helmet $\mathcal{H}_i(\delta_i, \mu_i)$ deprived of the interior of the the cone region defined by $\mathcal{C}(c_i, c_i - p_m^i, \psi_i, E_i)$ (solid purple line for $m = -1$ and solid orange line for $m = 1$). The corresponding jump set $\mathcal{J}_m^i$ is the complement of $\mathcal{F}_m^i$ in the free workspace.

equilibria, which we need to rule out from the flow sets $\mathcal{F}_1^i$ and $\mathcal{F}_{-1}^i$ and we characterize in the next lemma.

***Lemma** 2: Let $c \in \mathbb{R}^n$, $p \in \mathbb{R}^n \backslash \{c\}$ and $E \in \mathbb{R}^{n\times n}$ positive definite. For each $x \in \mathbb{R}^n \backslash \{c\}$, $\pi^\perp(E(x-c))E(x-p) = 0$ if and only if $x \in \mathcal{L}(c, p-c)$.*

For each $m \in \{-1, 1\}$, $i \in \mathbb{I}$, we want solutions to eventually leave the set $\mathcal{F}_m^i$ of the avoidance mode, so it is necessary to select point p_m^i and flow set $\mathcal{F}_m^i$ such that $\mathcal{L}(c_i, p_m^i - c_i) \cap \mathcal{F}_m^i = \emptyset$ based on Lemma 2, otherwise solutions could stay in avoidance mode indefinitely. This motivates the intersection with the cone in (18), and the next lemma.

***Lemma** 3: For each $i \in \mathbb{I}$, define the quantities*

$$\underline{\delta}_i := \|E_i c_i\|^{-\frac{1}{2}} \tag{19a}$$

$$\bar{\mu}_i(\delta_i) := \left(1 - 4\underline{\delta}_i^2(1 - \underline{\delta}_i^2/\delta_i^2)\right)^{-\frac{1}{2}} \tag{19b}$$

$$\bar{\theta}_i(\delta_i, \mu_i) := \arccos\left(\frac{\underline{\delta}_i^2}{\delta_i^2} + \frac{1}{4\underline{\delta}_i^2}\left(1 - \frac{1}{\mu_i^2}\right)\right) \tag{19c}$$

and select the parameters $\delta_i, \mu_i, \theta_i, \psi_i$ as in Table I so that $\bar{\mu}_i(\delta_i)$ and $\bar{\theta}_i(\delta_i, \mu_i)$ are well-defined. Then, for each $m \in \{-1, 1\}$, $\mathcal{L}(c_i, p_m^i - c_i) \cap \mathcal{F}_m^i = \emptyset$.

TABLE I
SELECTION OF THE DESIGN PARAMETERS OF (22), WITH $i \in \mathbb{I}$.

Parameter	Selection	Parameter	Selection
δ_i	$(\underline{\delta}_i, 1)$	ϵ_i	$(\delta_i, 1)$
μ_i	$(1, \bar{\mu}_i(\delta_i))$	ν_i	$(1, \mu_i)$
θ_i	$(0, \bar{\theta}_i(\delta_i, \mu_i))$	$\bar{\psi}_i$	$(0, \theta_i)$
$k_0,\ k_1,\ k_{-1}$	$(0, +\infty)$	ψ_i	$(0, \bar{\psi}_i)$
		$p_1^i,\ p_{-1}^i$	as in (11)

From the flow set in (18), we suitably define the jump set for the avoidance mode, of an obstacle $i \in \mathbb{I}$ with configuration $m \in \{-1, 1\}$, to be the closed complement of $\mathcal{F}_m^i$ in the free workspace. For $i \in \mathbb{I}$ and $m \in \{-1, 1\}$,

$$\mathcal{J}_m^i := \Big(\mathcal{E}_{\geq}(c_i, \delta_i E_i) \cup \mathcal{E}_{\leq}(\bar{c}_i, \mu_i \bar{E}_i) \cup \mathcal{C}_{\leq}(c_i, c_i - p_m^i, \psi_i, E_i)\Big) \cap \mathcal{W}. \tag{20}$$

Finally, the avoidance mode has overall flow and jump sets

$$\mathcal{F}_1 := \bigcup_{i\in\mathbb{I}} \left(\mathcal{F}_1^i \times \{i\}\right), \qquad \mathcal{J}_1 := \bigcup_{i\in\mathbb{I}} \left(\mathcal{J}_1^i \times \{i\}\right), \tag{21a}$$

$$\mathcal{F}_{-1} := \bigcup_{i\in\mathbb{I}} \left(\mathcal{F}_{-1}^i \times \{i\}\right), \qquad \mathcal{J}_{-1} := \bigcup_{i\in\mathbb{I}} \left(\mathcal{J}_{-1}^i \times \{i\}\right), \tag{21b}$$

where $\mathcal{F}_m^i$ and $\mathcal{J}_m^i$ ($m \in \{-1, 1\}$) are defined in (18) and (20). Indeed, *each* obstacle i gives rise, for the avoidance mode, to a specific flow (jump) set with two configurations $\mathcal{F}_1^i$ and $\mathcal{F}_{-1}^i$ ($\mathcal{J}_1^i$ and $\mathcal{J}_{-1}^i$), as we motivated in this section.

C. Hybrid Mode Selection

In this section we define the hybrid switching strategy that permits a Zeno-free transition between the different control modes. The hybrid selection of the logical variables $i \in \mathbb{I}$ and $m \in \mathbb{M}$ is implemented in the hybrid system

$$\begin{cases} \dot{x} = \kappa(x, i, m) \\ \dot{\widehat{i}} = 0 \\ \dot{m} = 0 \end{cases} \qquad (x, i, m) \in \mathcal{F} \tag{22a}$$

$$\begin{cases} x^+ = x \\ \begin{bmatrix} i^+ \\ m^+ \end{bmatrix} \in \mathbf{L}(x, i, m) \end{cases} \qquad (x, i, m) \in \mathcal{J} \tag{22b}$$

where $\kappa(x, i, m)$ is the control input defined in (10) and the flow and jump sets are given by

$$\mathcal{F} := \bigcup_{m\in\mathbb{M}} \left(\mathcal{F}_m \times \{m\}\right), \qquad \mathcal{J} := \bigcup_{m\in\mathbb{M}} \left(\mathcal{J}_m \times \{m\}\right) \tag{22c}$$

with $\mathcal{F}_m$ and $\mathcal{J}_m$ being defined in (17) for $m = 0$ and in (21a)-(21b) for $m \in \{-1, 1\}$. We define now the (set-valued) jump map $\mathbf{L}$ in (22b). To this end, for $i \in \mathbb{I}$ and $m \in \{-1, 1\}$, define the sets $\mathcal{C}_m^i$ as

$$\mathcal{C}_m^i := \mathcal{C}_{\geq}(c_i, c_i - p_m^i, \bar{\psi}_i, E_i), \tag{22d}$$

which corresponds to the region outside the cone with vertex at c_i, axis $c_i - p_m^i$ and aperture $2\bar{\psi}_i$, where $\bar{\psi}_i$ is a design parameter selected below. The jump map $\mathbf{L}$ for $m \in \{-1, 1\}$ is then defined as

$$\mathbf{L}(x, i, -1) := \mathbf{L}(x, i, 1) := \{\left[\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right]\}, \tag{22e}$$

i.e., when jumping to stabilization mode, the obstacle index i is not used in the control law κ in (10) and consequently is not updated. The jump map $\mathbf{L}$ for $m = 0$ is

$$\mathbf{L}(x, i, 0) := \left\{\begin{bmatrix} i' \\ m' \end{bmatrix} : x \in \mathcal{J}_0^{i'}, m' \in \mathbf{M}(x, i')\right\} \tag{22f}$$

where $\mathbf{M}$ is defined, based on (22d), as

$$\mathbf{M}(x,i) := \begin{cases} \{-1\} & x \in \mathcal{C}_{-1}^i \backslash \mathcal{C}_1^i \\ \{1\} & x \in \mathcal{C}_1^i \backslash \mathcal{C}_{-1}^i \\ \{-1,1\} & x \in \mathcal{C}_{-1}^i \cap \mathcal{C}_1^i. \end{cases} \tag{22g}$$

$\mathbf{L}(\cdot,\cdot,0)$ captures that when jumping from the stabilization mode $m = 0$, the suitable avoidance mode of obstacle $i' \in \mathbb{I}$ with configuration $m' \in \{-1,1\}$ is selected based on the position x of the vehicle (m', in particular, is selected based on whether x is within the cone region $\mathcal{C}_{-1}^{i'}$ or $\mathcal{C}_1^{i'}$). A necessary condition to implement our hybrid controller is that the jump map is nonempty, for which we have the next lemma.

***Lemma** 4: Select the parameters $\bar{\psi}_i$ and ψ_i as in Table I. Then, the set $\mathbf{L}(x,i,m)$ is nonempty for all $(x,i,m) \in \mathcal{J}$.*

For compact notation, we write flow and jump maps as

$$(x,i,m) \mapsto \mathbf{F}(x,i,m) := (\kappa(x,i,m),0,0) \tag{22h}$$

$$(x,i,m) \rightrightarrows \mathbf{J}(x,i,m) := (x,\mathbf{L}(x,i,m)), \tag{22i}$$

and the overall state of the hybrid system as

$$\xi := (x,i,m) \in \mathbb{R}^n \times \mathbb{I} \times \mathbb{M}. \tag{22j}$$

This completes the description of the hybrid controller in (22). The selections we made in this section for the parameters of (22) are summarized in Table I.

V. Main Results

In this section, we show that the hybrid controller achieves forward invariance and global asymptotic stability, as well as some complementary properties. The mild regularity conditions satisfied by the hybrid system (22), as in the next lemma, allows us to invoke useful results on hybrid systems for proving our results.

***Lemma** 5: The hybrid system with data $(\mathcal{F},\mathbf{F},\mathcal{J},\mathbf{J})$ satisfies the hybrid basic conditions in [22, Assumption 6.5].*

A. Forward invariance

Since the state x must evolve always within the free workspace $\mathcal{W}$ in (8) regardless of the logic variables i and m, we seek forward invariance of the set $\mathcal{K}$ defined as:

$$\mathcal{K} := \bigcap_{i\in\mathbb{I}} \mathcal{E}_{\geq}(c_i,E_i) \times \mathbb{I} \times \mathbb{M} = \mathcal{W} \times \mathbb{I} \times \mathbb{M}. \tag{23}$$

The next lemma shows that the union of flow and jump sets covers exactly the obstacle-free state space $\mathcal{K}$ and that solutions cannot leave $\mathcal{K}$ through jumps.

***Lemma** 6: $\mathcal{F} \cup \mathcal{J} = \mathcal{K}$ and $\mathbf{J}(\mathcal{J}) \subset \mathcal{K}$.*

Forward invariance of $\mathcal{K}$ holds by the next theorem, proven in the Appendix.

***Theorem** 1: Under Assumptions 1-3, consider the hybrid system (22) with parameters selected as in Table I. Assume also that the controller parameters δ_i are tuned so that the ellipsoids $\{\mathcal{E}_{\leq}(c_i,\delta_i E_i)\}_{i\in\mathbb{I}}$ are weakly pairwise disjoint. Then, the obstacle-free set $\mathcal{K}$ in (23) is forward invariant.*

The existence of tuning parameters $\delta_1,\ldots,\delta_I$ satisfying the weak pairwise disjointness of the sets $\{\mathcal{E}_{\leq}(c_i,\delta_i E_i)\}_{i\in\mathbb{I}}$ is guaranteed by Assumption 3, which implies that weak pairwise disjointness holds when $\delta_i = 1$ for all $i \in \mathbb{I}$. Hence, by a continuity argument, we can always tune each δ_i sufficiently close to 1 in order to guarantee the weak pairwise disjointness of the dilated obstacles $\{\mathcal{E}_{\leq}(c_i,\delta_i E_i)\}_{i\in\mathbb{I}}$. Algebraic tests of weak pairwise disjointness (in [25, Thm. 6] for $n = 2$ and in [26, Thm. 8] for $n = 3$) can be used for this tuning.

B. Global asymptotic stability

We show that from all initial conditions in the free workspace, all solutions converge asymptotically to the origin. To this end, we define the notion of *sufficient disjointness* of a set of ellipsoids, which is slightly stronger than weak disjointness but less conservative than strong disjointness, and guarantees that each obstacle is avoided at most one time. The motivation behind the assumption of sufficient disjointness is that the ellipsoids considered here can be arbitrarily large and flat, which might lead to long avoidance-mode detours that take the vehicle far away from the origin. In this case, specific configurations of the obstacles exist such that from a set of initial conditions, the vehicle does not converge to the origin although it remains safe. Similarly, in the Bug 0 planning algorithm [27], termination (*i.e.*, convergence to the target) is not *always* guaranteed since the algorithm is designed to "walk toward the target whenever you can" [27]. Our hybrid feedback shares a similar philosophy since the vehicle jumps from avoidance to stabilization mode whenever the stabilization controller generates a vector field not pointing towards the obstacle, see (12). To proceed, the next lemma characterizes the intersection of two ellipsoids of interest.

***Lemma** 7: Consider an arbitrary $i \in \mathbb{I}$. For $\underline{\delta}_i$, $\delta \mapsto \bar{\mu}_i(\delta)$ and $(\delta,\mu) \mapsto \bar{\theta}_i(\delta,\mu)$ defined in (19), let $\delta \in [\underline{\delta}_i,1]$, $\mu \in [1,\bar{\mu}_i(\delta)]$ and $\vartheta_i(\delta,\mu)$ be such that*

$$\cos(\vartheta_i(\delta,\mu)) := \frac{1-\cos(\bar{\theta}_i(\delta,\mu))\underline{\delta}_i^2}{\sqrt{(1+\mu^{-2})/2-\delta^{-2}\underline{\delta}_i^4}}. \tag{24}$$

The expression in (24) is well-defined and positive, and

$$\mathcal{E}(c_i,\delta E_i) \cap \mathcal{E}(\bar{c}_i,\mu\bar{E}_i) \subset \mathcal{C}(0,c_i,\vartheta_i(\delta,\mu),E_i). \tag{25}$$

Let us consider for each obstacle $i \in \mathbb{I}$ the sphere $\mathcal{S}(0,\bar{r}_i)$ with center at the origin and radius $\bar{r}_i$ defined by the next quadratic optimization problem

$$\bar{r}_i^2 := \min\{\|x\|^2 \colon x \in \mathcal{H}_i^*\} \tag{26}$$

where $\mathcal{H}_i^*$ is the helmet defined in (13). The radius $\bar{r}_i$ defines the minimum distance from the helmet $\mathcal{H}_i^*$ to the origin. Let x be a point belonging to the intersection of the two ellipsoids $\mathcal{E}(c_i,E_i)$ and $\mathcal{E}(\bar{c}_i,\bar{E}_i)$. Taking δ and μ equal to 1 in Lemma 7, one obtains $x \in \mathcal{C}(0,c_i,\bar{\vartheta}_i,E_i)$ with

$$\cos(\bar{\vartheta}_i) := \cos(\vartheta_i(1,1)) = \sqrt{1-\|E_i c_i\|^{-2}}, \tag{27}$$

from (24), (19c) and (19a). Now, let us define the set

$$\mathcal{R}_i^* := \mathcal{C}(0,c_i,\bar{\vartheta}_i,E_i) \cap \mathcal{S}_{\geq}(0,\bar{r}_i) \cap \mathcal{E}_{\geq}(c_i,E_i) \cap \mathcal{E}_{\leq}(\bar{c}_i,\bar{E}_i), \tag{28}$$

whose geometry is sketched in Fig. 4. In particular, it is contained in the set of points on the cone $\mathcal{C}(0,c_i,\bar{\vartheta}_i,E_i)$ that have a distance to the origin greater than the distance $\bar{r}_i$ of the helmet $\mathcal{H}_i^*$ to the origin. The idea is that the vehicle should not start avoiding another obstacle while it is still in $\mathcal{R}_i^*$, otherwise there is no guarantee that the number of times the vehicle avoids obstacles is bounded and that global attractivity holds. This leads to the next definition.

***Definition** 2: The ellipsoids $\{\mathcal{E}(c_i,E_i)\}_{i\in\mathbb{I}}$ are sufficiently pairwise disjoint if they are weakly pairwise disjoint and*

$$\forall i,i' \in \mathbb{I} \text{ with } i \neq i', \quad \mathcal{R}_i^* \cap \mathcal{E}_{\leq}(c_{i'},E_{i'}) = \emptyset. \tag{29}$$

Now, let us introduce the ingredients for a dilated version of $\mathcal{R}_i^*$ as in (31) below and refer to Fig. 5. First, consider the *escape annulus cone* where solutions escape from the avoidance mode by applying the stabilization vector field. This region lies between the two cones $\mathcal{C}(0,c_i,\vartheta_i(1,\mu_i),E_i)$ and $\mathcal{C}(0,c_i,\vartheta_i(\delta_i,\mu_i),E_i)$ which are related, according to Lemma 7, to the intersections $\mathcal{E}(c_i,E_i) \cap \mathcal{E}(\bar{c}_i,\mu_i\bar{E}_i)$ and $\mathcal{E}(c_i,\delta_i E_i) \cap \mathcal{E}(\bar{c}_i,\mu_i\bar{E}_i)$, respectively. Second, consider for each

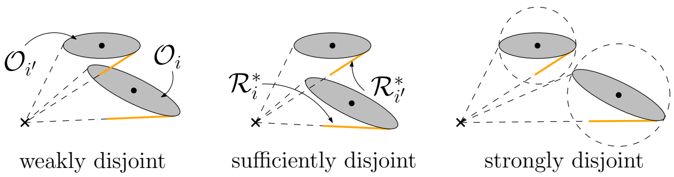

Fig. 4. Different types of disjointness introduced in the paper with set $\mathcal{R}_i^*$ (orange) in (28). For global attractivity, sufficient disjointness is asked.

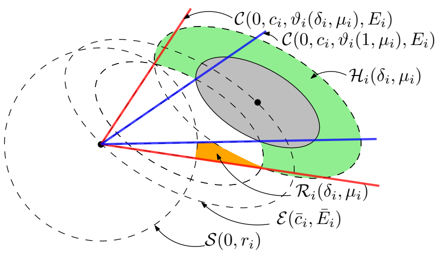

Fig. 5. Safety helmet $\mathcal{H}_i(\delta_i, \mu_i)$ (green) and the corresponding escape region $\mathcal{R}_i(\delta_i, \mu_i)$ (orange). The region $\mathcal{R}_i(\delta_i, \mu_i)$ must not intersect with any other jump set $\mathcal{J}_0^{i'}$, $i' \neq i$, to avoid starting another avoidance while the distance to the target has not decreased yet.

obstacle $i \in \mathbb{I}$ the ball $\mathcal{S}_{\geq}(0, r_i)$ where the radius r_i is defined by the quadratic optimization problem

$$r_i^2 := \min\{\|x\|^2 : x \in \mathcal{H}_i(\delta_i, \mu_i)\}. \tag{30}$$

Note the following on (30). 1) The safety helmet $\mathcal{H}_i(\delta_i, \mu_i)$ is nonempty and compact; hence, a solution to (30) exists. 2) For each $i \in \mathbb{I}$, $r_i > 0$. Indeed, for each $i \in \mathbb{I}$, $\|\delta_i E_i c_i\| = \delta_i \underline{\delta}_i^{-2} > \delta_i \underline{\delta}_i^{-1} > 1$ by Assumption 2 and the selection of δ_i in Table I, so that $0 \notin \mathcal{E}_{\leq}(c_i, \delta_i E_i)$ and in turn $0 \notin \mathcal{H}_i(\delta_i, \mu_i)$ since $\mathcal{H}_i(\delta_i, \mu_i) \subset \mathcal{E}_{\leq}(c_i, \delta_i E_i)$. Finally, we can define the considered "dilated" version of $\mathcal{R}_i^*$ as

$$\begin{aligned}\mathcal{R}_i(\delta_i, \mu_i) := \mathcal{S}_{\geq}(0, r_i) \cap \mathcal{E}_{\geq}(c_i, \delta_i E_i) \cap \mathcal{E}_{\leq}(\bar{c}_i, \bar{E}_i) \\ \cap\, \mathcal{C}_{\geq}(0, c_i, \vartheta_i(1, \mu_i), E_i) \cap \mathcal{C}_{\leq}(0, c_i, \vartheta_i(\delta_i, \mu_i), E_i).\end{aligned} \tag{31}$$

***Lemma** 8: Assume that the obstacles $\{\mathcal{O}_i\}_{i\in\mathbb{I}}$ are sufficiently pairwise disjoint. Then, for each $i \in \mathbb{I}$, there exist δ_i^*, μ_i^* such that for all $\delta_i \in (\delta_i^*, 1)$ and $\mu_i \in (1, \mu_i^*)$, we have*

$$\forall i', i'' \in \mathbb{I}, i' \neq i'', \ \mathcal{R}_{i'}(\delta_{i'}, \mu_{i'}) \cap \mathcal{E}_{\leq}(c_{i''}, \delta_{i''} E_{i''}) = \emptyset. \tag{32}$$

Property (32) of Lemma 8 is used to show global attractivity. Intuitively, we require that after avoiding an obstacle, the distance $\|x\|$ to the target decreases before the vehicle reaches the proximity of another obstacle. Although the bounds δ_i^* and μ_i^* are not defined explicitly for generic ellipsoids, the parameters δ_i and μ_i can be tuned offline. Next is our main result for this section, proven in the Appendix.

***Theorem** 2: Consider the hybrid system* (22) *under the same assumptions as Theorem 1. Assume also that the obstacles $\{\mathcal{O}_i\}_{i\in\mathbb{I}}$ are sufficiently pairwise disjoint, and the δ_i's and μ_i's are tuned so that* (32) *holds. Then, the set $\mathcal{A} := \{0\} \times \mathbb{I} \times \mathbb{M}$ is globally asymptotically stable for* (22) *and the number of jumps is bounded.*

For spherical obstacles, we show next that the extra tuning of the parameters to satisfy (32) is not needed. The proof is in [21].

***Theorem** 3: (Spherical obstacles) Let $E_i = \lambda_i I_n$ for all $i \in \mathbb{I}$. Under the same assumptions as Theorem 1, the set $\mathcal{A} := \{0\} \times \mathbb{I} \times \mathbb{M}$ is globally asymptotically stable for* (22) *and the number of jumps is bounded.*

C. Complementary properties

1) Bounded Control: First, solutions initialized within a certain compact ball always remain there. Indeed, let $\mathcal{S}_{\leq}(0, r_b)$, with $r_b > 0$, be the smallest ball containing all the dilated ellipsoids $\mathcal{E}(c_i, \delta_i E_i)$ (which must exist since these ellipsoids are compact). During stabilization mode the distance $\|x\|$ is decreasing and during avoidance mode the vehicle stays within the dilated ellipsoids $\mathcal{E}(c_i, \delta_i E_i)$. Then, it is guaranteed that from all $x(0,0) \in \mathcal{S}_{\leq}(0, r_b)$, all solutions satisfy $x(t,j) \in \mathcal{S}_{\leq}(0, r_b)$ for all $(t,j) \in \operatorname{dom} x$. Moreover, since the projection matrix $\pi^{\perp}(E_i(x - c_i))$ has eigenvalues in 0 and 1, it follows that we can upper bound the control input in (10) by $\|u\| \leq k\alpha(r_b + p)$ where $k = \max\{k_1, k_0, k_{-1}\}$, $\alpha = \max_{i\in\mathbb{I}}(\lambda_{\max}(E_i)/\lambda_{\min}(E_i))$ and $p = \max_{i\in\mathbb{I}} \|p_1^i\|$. The control gains can then be tuned to satisfy the inherent practical saturation of the actuators.

2) Semiglobal Preservation: This property [17, §II] is desirable when the original controller parameters are optimally tuned and the controller modifications imposed by the presence of the obstacles should be as minimal as possible. Such a property is also accounted for in the quadratic programming formulation of [28, III.A]. In our case we have the next proposition, proven in the Appendix.

***Proposition** 1: Let $\epsilon \in (0,1)$ and $\mathcal{W}_\epsilon := \bigcap_{i\in\mathbb{I}} \mathcal{E}_{\geq}(c_i, \epsilon E_i)$. There exist controller parameters such that the control law matches, in $\mathcal{W}_\epsilon$, the stabilization feedback $u = -k_0 x$ ($k_0 > 0$) used in the absence of obstacles.*

3) Non-point Mass Vehicles: There is no loss of generality in considering a point-mass vehicle in this work. In fact, let us consider a vehicle with some volume, *e.g.,* bounded by $\mathcal{S}_{\leq}(x, r_v)$. Then, in a feasible navigation scenario, the radius r_v of the vehicle needs to be smaller than the smallest distance between the obstacles, *i.e.*, for all $i, i' \in \mathbb{I}$ with $i \neq i'$, $r_v < \mathbf{dist}(\mathcal{E}_{\leq}(c_i, E_i), \mathcal{E}_{\leq}(c_{i'}, E_{i'})) := \inf\{|x - x'| : x \in \mathcal{E}_{\leq}(c_i, E_i), x' \in \mathcal{E}_{\leq}(c_{i'}, E_{i'})\}$. For safety of the whole volume of the vehicle, the selection $\epsilon_i < (1 + \lambda_{\max}(E_i) r_v)^{-1}$ is sufficient (in addition to Table I) to guarantee that the vehicle in stabilization mode starts the avoidance mode away from the obstacle. Indeed, under this condition, it is easy to show that for all $x \in \mathcal{E}_{\geq}(c_i, \epsilon_i E_i)$ (*i.e.*, the vehicle center is outside the dilated ellipsoid $\mathcal{E}(c_i, \epsilon_i E_i)$) and for all $x' \in \mathcal{S}_{\leq}(x, r_v)$, one has $x' \in \mathcal{E}_{\geq}(c_i, E_i)$, which guarantees safety of the whole volume of the vehicle.

4) Robustness: The constructed hybrid controller guarantees some level of robustness to perturbations (*e.g.*, in the form of measurement noise). Hysteresis switching is one of the typical ways to ensure robustness to measurement noise, and hysteresis switching is indeed behind the designed hybrid feedback, in particular the hysteresis regions of flow and jump sets in Section IV-B and the logical selections of the jump sets in Section IV-C. More generally, fundamental results in [22, Chap. 7] guarantee structurally that global asymptotic stability of $\mathcal{A}$ in Theorem 2 is also uniform (by [22, Thm. 7.12]) and robust (by [22, Thm. 7.21]) with respect to perturbations since $\mathcal{A}$ is a compact set and the hybrid basic conditions are satisfied as per Lemma 5.

VI. Simulations

We illustrate the effectiveness of the proposed hybrid control strategy through two simulation scenarios. The first scenario considers 9 obstacles in 2D as in Fig. 6 while the second one considers 5 obstacles in 3D as in Fig. 7. For both cases, Table I provides a suitable order to choose the parameters for each $i \in \mathbb{I}$, as follows.

1) For $\underline{\delta}_i$ in (19a), select δ_i and ϵ_i so that $\underline{\delta}_i < \delta_i < \epsilon_i < 1$;
2) For δ_i and $\bar{\mu}_i(\delta_i)$ in (19b), select ν_i and μ_i so that $1 < \nu_i < \mu_i < \bar{\mu}_i(\delta_i)$, possibly iterating steps 1) and 2) so that δ_i and μ_i satisfy (32);

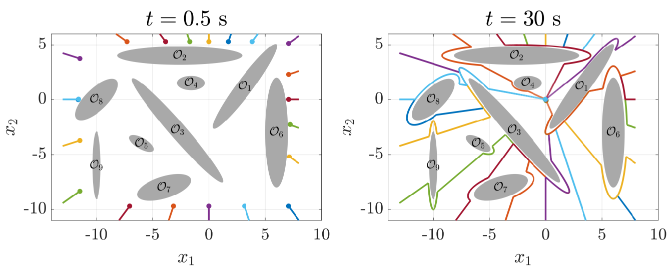

Fig. 6. Plot (at times $t = 0.5$ and $t = 30$ seconds) of the 2-dimensional trajectory of the vehicle starting at different initial conditions.

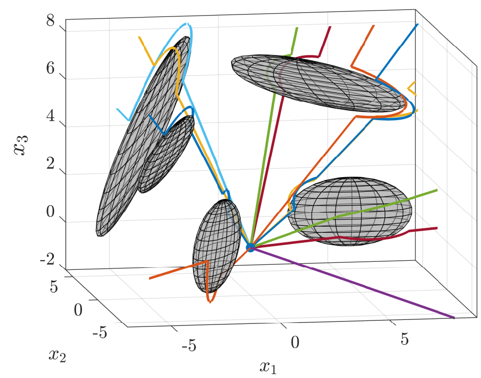

Fig. 7. Plot (at time $t = 30$ seconds) of the 3-dimensional trajectory of the vehicle starting at different initial conditions.

3) For δ_i, μ_i and $\bar{\theta}_i(\delta_i, \mu_i)$ in (19c), select ψ_i, $\bar{\psi}_i$ and θ_i so that $0 < \psi_i < \bar{\psi}_i < \theta_i < \bar{\theta}_i(\delta_i, \mu_i)$.

Any parameter selection according to this guideline guarantees our results, and can be carried out keeping in mind the physical interpretation illustrated in Section IV-B for these parameters. The gains are $k_0 = k_1 = k_{-1} = 1/4$ and determine the speed of convergence of the scheme. By (11a), the point p_1^i can be selected arbitrarily as long as it is on $\mathcal{C}(c_i, -c_i, \theta_i, E_i)\backslash\{c_i\}$. A suitable choice is given by

$$p_1^i = \pi^{\perp}(E_i^{-1}\mathbf{R}(\theta_i)E_i c_i)c_i \tag{33}$$

where $\mathbf{R}(\theta_i)$ is the standard 2×2 rotation matrix with angle θ_i or the standard 3×3 axis-angle rotation matrix with angle θ_i and an arbitrary vector of $\mathbb{S}^2$ as axis. The idea behind (33) is to project c_i on the plane orthogonal to a rotated version of c_i, in order to obtain the point lying on the cone and closest to the origin. Having all points p_m^i close enough to the origin is an effective way so that k_0, k_1, k_{-1} can take the same values and yield comparable speeds for avoidance and stabilization, independently of the obstacles.

Figs. 6 and 7 show that the solutions generated by the closed-loop hybrid system avoid the 2D and 3D obstacles and converge to the origin. The respective complete simulation videos can be found at https://youtu.be/CnXJlhzlzd8, https://youtu.be/4mzTXPR6D9Y.

Finally, we note that for the very obstacle configuration of the 2D scenario, the state-of-the-art approach of navigation functions [3], [20] cannot be applied since the condition [20, Thm. 3, Eq. (23)] is violated for all obstacles except obstacle $\mathcal{O}_5$, where [20, Eq. (23)] intuitively corresponds to the fact that obstacles are not too flat and not too close to the target position. [20, Eq. (23)] is violated for all obstacles of the 3D scenario. Moreover, navigation function approaches require tuning a parameter sufficiently large, namely k in [20, Eq. (17) and Remark 5] and this may conflict with actuator limitations. Instead, our approach provides a clear tuning guideline for all parameters (given in this section) and actuator limitations can be taken into account (see Section V-C1).

VII. Conclusions

We proposed a novel hybrid feedback on $\mathbb{R}^n$ to solve the obstacle avoidance problem for generic ellipsoidal obstacles, in particular flat and close to the target. Our control strategy ensures global asymptotic stabilization to the target and safety (thus, successful navigation from all initial conditions) while guaranteeing a Zeno-free switching between the avoidance and stabilization modes. Moreover, the control input remains bounded (also in arbitrary proximity to obstacles) and matches semi-globally in the free-state space the nominal feedback used in the absence of obstacles. Future work will be devoted to considering more complex vehicle dynamics (*e.g.*, under-actuated and second-order dynamics) and more generic obstacle shapes (*e.g.*, convex obstacles). Further, although our scheme considers static obstacles to obtain formal guarantees for global asymptotic stability and safety, extending this approach to unknown environments is an interesting research direction we aim at pursuing in the future.

Appendix

1) Proof of Theorem 1: Define $\mathbf{S}_{\mathscr{H}}(\mathcal{K})$ as the set of all maximal solutions ϕ to $\mathscr{H} = (\mathcal{F}, \mathbf{F}, \mathcal{J}, \mathbf{J})$ with $\phi(0,0) \in \mathcal{K}$. Each $\phi \in \mathbf{S}_{\mathscr{H}}(\mathcal{K})$ has range $\operatorname{rge}\phi \subset \mathcal{K} = \mathcal{F} \cup \mathcal{J}$ by Lemma 6 and the definition of hybrid solution [22, p. 124], so $\mathcal{K}$ is forward pre-invariant [29, Def. 3.3]. The set $\mathcal{K}$ is in fact forward invariant [29, Def. 3.3] if for each $\xi \in \mathcal{K}$ there exists one solution and each $\phi \in \mathbf{S}_{\mathscr{H}}(\mathcal{K})$ is complete, which we show in the rest of the proof through [22, Prop. 6.10]. In the rest of the proof, let $\mathcal{F}_0^* := \bigcap_{i\in\mathbb{I}} \mathcal{F}_0^i$, $\mathcal{J}_0^* := \bigcup_{i\in\mathbb{I}} \mathcal{J}_0^i$.

***Lemma** 9: Under the assumptions of Theorem 1, we have for each $i \in \mathbb{I}$ and $m \in \{-1, 1\}$*

$$\mathcal{J}_0^i = \mathcal{H}_i(\epsilon_i, \nu_i), \tag{34a}$$

$$\mathcal{F}_m^i = \mathcal{H}_i(\delta_i, \mu_i) \cap \mathcal{C}_{\geq}(c_i, c_i - p_m^i, \psi_i, E_i), \tag{34b}$$

$$\partial\mathcal{F}_0^* \backslash \mathcal{J}_0^* \subset \bigcup_{i\in\mathbb{I}} \big(\mathcal{E}(c_i, E_i)\backslash\mathcal{E}_{\geq}(\bar{c}_i, \bar{E}_i)\big), \tag{34c}$$

$$\begin{aligned}\partial\mathcal{F}_m^i \backslash \mathcal{J}_m^i \subset &\mathcal{E}(c_i, E_i)\backslash\big(\mathcal{E}_{\leq}(\bar{c}_i, \mu_i\bar{E}_i)\\ &\cup \mathcal{C}_{\leq}(c_i, c_i - p_m^i, \psi_i, E_i)\big).\end{aligned} \tag{34d}$$

First, let us show that the viability condition

$$\mathbf{F}(x, i, m) \cap \mathbf{T}_{\mathcal{F}}(x, i, m) \neq \emptyset \tag{35}$$

holds for all $(x, i, m) \in \mathcal{F}\backslash\mathcal{J}$. Let $(x, i, m) \in \mathcal{F}\backslash\mathcal{J}$, which implies by (22c) that $(x, i) \in \mathcal{F}_m\backslash\mathcal{J}_m$ for some $m \in \mathbb{M}$, and divide into the cases $m = 0$ and $m \in \{-1, 1\}$. When $m = 0$, from (17) there exists $i \in \mathbb{I}$ such that $x \in \mathcal{F}_0^*\backslash\mathcal{J}_0^*$. If $x \in (\mathcal{F}_0^*)^\circ\backslash\mathcal{J}_0^*$ (hence, x is in the interior of $\mathcal{F}_0^*$), then $\mathbf{T}_{\mathcal{F}_0^*}(x) = \mathbb{R}^n$, so that $\mathbf{T}_{\mathcal{F}}(\xi) = \mathbb{R}^n \times \{0\} \times \{0\}$ and (35) holds. If $x \in \partial\mathcal{F}_0^*\backslash\mathcal{J}_0^*$, which satisfies the set inclusion (34c), the weak pairwise disjointness of $\{\mathcal{E}(c_i, E_i)\}_{i\in\mathbb{I}}$ yields:

$$\begin{aligned}&x \in \mathcal{E}(c_i, E_i),\ i \in \mathbb{I}\\ &\mathbf{T}_{\mathcal{F}}(x, i, 0) = \mathcal{P}_{\geq}(0, E_i^2(x - c_i)) \times \{0\} \times \{0\}.\end{aligned} \tag{36}$$

By (12) and $x \notin \mathcal{E}_{\geq}(\bar{c}_i, \bar{E}_i)$ by (34c), we obtain

$$-k_0 x^\top E_i^2(x - c_i) = k_0\|E_i\bar{c}_i\|^2\big(1 - \|\bar{E}_i(x - \bar{c}_i)\|^2\big) > 0, \tag{37}$$

hence, $\kappa(x, i, 0) \in \mathcal{P}_{\geq}(0, E_i^2(x - c_i))$ in (36), and (35) holds for $m = 0$. When $m \in \{-1, 1\}$, we have $i \in \mathbb{I}$ and $x \in \partial\mathcal{F}_m^i\backslash\mathcal{J}_m^i$, which satisfies the set inclusion (34d), and so

$$\mathbf{T}_{\mathcal{F}}(x, i, m) = \mathcal{P}_{\geq}(0, E_i^2(x - c_i)) \times \{0\} \times \{0\}. \tag{38}$$

$\kappa(x,i,m) \in \mathcal{P}_{\geq}(0, E_i^2(x-c_i))$ in (38) because

$$-k_m(x-p_m^i)^\top E_i \pi^\perp(E_i(x-c_i))E_i(x-c_i) = 0, \qquad (39)$$

so the viability condition (35) holds for $m \in \{-1,1\}$ as well.

Second, we apply [22, Prop. 6.10]. By it and (35), there exists a nontrivial solution to $\mathscr{H}$ from each initial condition in $\mathcal{K}$. Finite escape times can only occur through flow. They can neither occur for x in the set $\mathcal{F}_{-1}^i \cup \mathcal{F}_1^i$ because $\mathcal{F}_{-1}^i$ and $\mathcal{F}_1^i$ are bounded by their definitions in (18), nor for x in the set $\mathcal{F}_0^*$ because they would make $x^\top x$ grow unbounded, and this would contradict that $\frac{d}{dt}(x^\top x) \leq 0$ by the definition of $\kappa(x,i,0)$ and by (22a). So, all maximal solutions do not have finite escape times. By Lemma 6, $\mathbf{J}(\mathcal{J}) \subset \mathcal{K} = \mathcal{F} \cup \mathcal{J}$. Hence, by [22, Prop. 6.10], all maximal solutions are complete.

2) Proof of Theorem 2: We prove global asymptotic stability of $\mathcal{A}$ by [22, Def. 7.1]. For each $i \in \mathbb{I}$, $\|\delta_i E_i c_i\| = \delta_i \underline{\delta}_i^{-2} > \delta_i \underline{\delta}_i^{-1} > 1$ by Assumption 2 and the selection of δ_i in Table I, so $0 \notin \mathcal{E}_{\leq}(c_i, \delta_i E_i)$. As a consequence, there exists $\varepsilon^* > 0$ such that the ball $\mathcal{S}_{\leq}(0,\varepsilon^*)$ does not intersect with any of the dilated obstacles $\mathcal{E}_{\leq}(c_i, \delta_i E_i)$. It can be shown easily that for each $\varepsilon \in [0, \varepsilon^*]$, the set $\mathcal{S} := \mathcal{S}_{\leq}(0,\varepsilon) \times \mathbb{I} \times \mathbb{M}$ is forward invariant because $\mathcal{S}_{\leq}(0,\varepsilon)$ is disjoint from $\mathcal{J}_0^*$ and the component x of solutions evolves, after at most one jump, with the stabilization mode $\dot{x} = -k_0 x$. Thanks to forward invariance of $\mathcal{S}$, stability of $\mathcal{A}$ for (22) is immediate from [22, Def. 7.1]. Let us prove global attractivity of $\mathcal{A}$. Before that, we need the next result.

***Lemma** 10: There exists $\sigma > 0$ such that for all solutions $\xi = (x,i,m)$ with $\xi(t,j) \in \mathcal{F}_l \times \{l\}$ for some $l \in \{-1,1\}$ and $(t,j) \in \operatorname{dom}\xi$, there exists $(s,\ell) \in \operatorname{dom}\xi$ such that $(s,\ell) \succeq (t,j)$ and*

$$\|x(s,\ell)\| \leq \|x(t,j)\| - \sigma. \qquad (40)$$

Now, for each solution ξ to (22), there exists a finite time $(T,J) \succeq (0,0)$ after which the solution does not evolve with the avoidance controller any longer, *i.e.*, $m(t,j) = 0$ for all $(t,j) \succeq (T,J)$. Otherwise, there would exist a sequence of hybrid times $\{(t_k,j_k)\}_{k=0}^\infty$ such that $\xi(t_k,j_k) \in \mathcal{F}_{l_k} \times \{l_k\}$ with $l_k \in \{-1,1\}$ and this would imply by Lemma 10 that $\|x(t_{k+1},j_{k+1})\| \leq \|x(t_k,j_k)\| - \sigma$ for all $k \in \mathbb{N}$. This is indeed a contradiction as it would lead to $\|x(\cdot,\cdot)\|$ becoming negative. Then, the solution ξ enters the stabilizing mode $m=0$ after (T,J) and its flow map $\dot{x} = -k_0 x$ guarantees in turn global attractivity. Moreover, J is the maximum number of jumps of the hybrid system since any extra jump will cause m to take values in $\{-1,1\}$, which is not possible after (T,J).

3) Proof of Proposition 1: Note preliminarily that thanks to $\epsilon < 1$, $\mathcal{W}_\epsilon \subset \mathcal{W}$ in (8). It is sufficient to show that the closed loop system under the proposed hybrid feedback cannot flow except with stabilization mode $m=0$ when $x \in \mathcal{W}_\epsilon$. Indeed, if in Table I we further constrain δ_i as $\delta_i \in (\max(\underline{\delta}_i, \epsilon), 1)$ for all $i \in \mathbb{I}$, then we have $\mathcal{F}_m^i \subset \mathcal{H}(\delta_i, \mu_i) \subset \mathcal{E}_{\leq}(c_i, \delta_i E_i) \subset \mathcal{E}_{\leq}(c_i, \epsilon E_i)$, $\mathcal{E}_{\leq}(c_i, \delta_i E_i) \neq \mathcal{E}_{\leq}(c_i, \epsilon E_i)$, and thus $\mathcal{F}_m^i \cap \mathcal{W}_\epsilon = \emptyset$ for all $i \in \mathbb{I}$ and $m \in \{-1,1\}$. This implies that solutions cannot flow in avoidance mode when x belongs to $\mathcal{W}_\epsilon$ and must flow in stabilization mode.

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