Consensus with Quantized Relative State Measurements ⋆

Meng Guo [a,1], Dimos V. Dimarogonas [a,b]

[a] *ACCESS Linnaeus Center, School of Electrical Engineering, KTH Royal Institute of Technology, SE-100 44,Stockholm, Sweden*

[b] *Center of Autonomous Systems, KTH Royal Institute of Technology, SE-100 44,Stockholm, Sweden*

Abstract

In this paper, cooperative control of multi-agent systems under limited communication between neighboring agents is investigated. In particular, quantized values of the relative states are used as the control parameters. By taking advantage of tools from nonsmooth analysis, explicit convergence results are derived for both uniform and logarithmic quantizers under static and time-varying communication topologies. Compared with our previous work, less conservative conditions that ensure global convergence are provided. Moreover, second order dynamical systems under similar constraints are taken into account. Computer simulations are provided to demonstrate the validity of the derived results.

Key words: Multi-agent systems; Quantization; Consensus; Nonsmooth Analysis.

1 Introduction

In the vast recent literature concerning the consensus problem of multi-agent systems, many results on stability and convergence rate have been obtained by utilizing the spectral properties of the graph Laplacian matrix and under the assumption of perfect communication, as in Dimarogonas & Kyriakopoulos (2007); Olfati-Saber & Murray (2004); Ren & Atkins (2007). Specifically regarding static communication topologies, global convergence to a consensus point is guaranteed if and only if the corresponding Laplacian matrix has nonnegative eigenvalues with exactly one of them being zero (Dimarogonas & Johansson, 2010). But imperfect information exchange and communication constraints may have a considerable impact on the performance of a multi-agent system and also the implementation of the control algorithms. Relevant topics have received attention for different system dynamics and different constrained models, as in Carli, Fagnani, & Zampieri (2006); Fagnani, Johansson, Speranzon, & Zampieri (2004); Kashyap, Basar & Srikant (2007); Nedic, Olshevsky, Ozdaglar & Tsitsiklis (2008) for discrete-time dynamics and Ceragioli, Persis, & Frasca (2010); Dimarogonas & Johansson (2010) for continuous-time models. In particular, communication delays [Seuret, Dimarogonas, & Johansson (2008); Xiao & Wang (2006)] and quantized information [Carli, Persis & Frasca (2010); Dimarogonas & Johansson (2010); Fagnani, Johansson, Speranzon, & Zampieri (2004); Shevitz & Paden (1994)] are two of the most common constraints considered not only in theoretical research but also in practice. Here we will put emphasis on multi-agent systems involving quantized information.

Compared to our previous work in Dimarogonas & Johansson (2010), this paper considers the Filippov solution of both first-order and second-order closed-loop systems under quantization by using the tools from nonsmooth analysis (Shevitz & Paden, 1994), since the classical or Carathedory solutions may not exist from a set of initial conditions of measure zero (Carli, Persis & Frasca, 2010). More importantly, the convergence results of first-order systems established in this paper are less conservative than in Dimarogonas & Johansson (2010), since they hold for both tree graphs and general undirected graphs. Finally, the second order consensus problem involving quantized relative states is not treated in Dimarogonas & Johansson (2010), whereas in this paper we derived explicit results on the convergence set and the stability constraints.

⋆ This work was supported by the Swedish Research Council (VR), the EU STREP Feednetback project, and HYCON2 NOE. Preliminary versions of this paper appear in Guo & Dimarogonas (2011) and Guo (2011). The second author is supported by the VR contract 2009-3948.

Email addresses: mengg@kth.se (Meng Guo), dimos@kth.se (Dimos V. Dimarogonas).
[1] Tel.: +46 07 07252925; fax: +46 8 7907329.

Reference Carli, Persis & Frasca (2010) addresses the case where relative quantized states $(q(x_i) - q(x_j))$, instead of quantized relative states $(q(x_i - x_j))$ as in this paper, are used as control parameters for the first-order system. This difference alters the techniques available for deriving the closed-loop dynamics: in Carli, Persis & Frasca (2010), it has the form $\dot{x} = -L\,q(x)$ and consequently the well-known spectral properties of the Laplacian matrix L are used for showing convergence; however, in our case it is not possible to include L in the closed-loop dynamics, which is also the motivation behind introducing the edge dynamics. The results derived in Theorem 6 of Liu, Cao, & Persis (2011) are similar to Theorem 4 in this paper, where convergence properties of second-order systems with the presence of uniform or logarithmic quantizers are derived. We tackle the problem here using different theoretical tools from Liu, Cao, & Persis (2011). In particular, instead of introducing an integral term (over the quantized relative positions) in the Lyapunov function candidate, we design a quadratic Lyapunov function that utilizes the structure of the underlying communication topology and the spectral properties of the corresponding edge Laplacian matrix. Thus the derivations of the current paper are also novel with respect to the analytical contribution.

The rest of the paper is organized as follows: Section II presents some background on algebraic graph theory and Filippov solutions. In Section III, we treat the first order system under static tree topology, then extend the results to switching trees and general undirected graphs. Section IV is devoted to quantized second order systems under general undirected topologies. The paper concludes with computer simulations in Section V and a summary of the results in Section VI.

2 System Model and Background

2.1 Graph Theory and Consensus Preliminaries

For an undirected graph $G = (\mathcal{V}, E)$, denote by $\mathcal{V} = 1, \cdots, N$ the set of vertices and by $E = \{(i, j) \in V \times V \,|\, i \in \mathcal{N}_j\}$ the set of edges, where $\mathcal{N}_j$ denotes agent j's *communication set* that includes the agents with which it can communicate. G is *undirected* if $i \in \mathcal{N}_j \Leftrightarrow j \in \mathcal{N}_i, \forall (i,\, j) \in E$. Its *Laplacian matrix* (Godsil & Royle, 2001) $L = \{l_{ij}\}$, where $l_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j, \forall i,\, j = 1, \cdots, N$. Note here we do not restrict a_{ij} to be uniformly one or equal as in Guo & Dimarogonas (2011). A *path* of length r from i to j is a sequence of $r + 1$ vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices, then G is called *connected*. A connected graph is called a *tree* if it contains no cycles. For a connected graph, L has nonnegative eigenvalues and a single zero eigenvalue (Ren & Atkins, 2007) with the corresponding eigenvector $\mathbf{1} = (1, \cdots, 1)^T$. Denote by $\lambda_k(M)$ the k_{th} eigenvalue of M in ascending order. An *orientation* on G is an assignment of a direction to each edge. The *incidence matrix* $B = B(G) = \{b_{ij}\}$ is the $\{0, \pm 1\}$ matrix, where $b_{ij} = 1$ if the vertex i is the head of the edge j, $b_{ij} = -1$ if vertex i is the tail of the edge j, and $b_{ij} = 0$ otherwise. Denote by Γ the $m \times m$ diagonal matrix of w_k, $k = 1, \cdots, m$. w_k stands for the weight of the k_{th} undirected edge of G (Godsil & Royle, 2001). Then $L = B\Gamma B^T$, where Γ is uniquely defined by the sequence of edges in B. Denote by $\bar{x} = B^T x$ the stack vector of relative states (*head–tail*) between neighboring agents. If G is connected, $Lx = 0$ if and only if x has all its elements equal (Fax & Murray, 2002), implying that $\bar{x} = B^T x = 0$.

2.2 System Model

We first consider N single-integrator agents:

$$\dot{x}_i = u_i, \; i = 1, \cdots, N, \tag{1}$$

where $x_i \in \mathbb{R}$ denotes the position and $u_i \in \mathbb{R}$ the control input of agent i. The goal is to construct distributed feedback controllers that lead the system to an agreement point. On the other hand, since a broad class of vehicles requires a second-order dynamics, a double-integrator model is also considered

$$\dot{x}_i = v_i, \; \dot{v}_i = u_i, \; i = 1, \cdots, N, \tag{2}$$

where $x_i, v_i \in \mathbb{R}$ denotes the position and velocity, and $u_i \in \mathbb{R}$ the acceleration input. The desired configuration is that all agents moving with the common speed as one point. In this paper, we treat only the system behavior in the x-coordinate but the analysis that follows holds in higher dimensions.

The consensus protocol in Fax & Murray (2002), Olfati-Saber & Murray (2004) for system (1) is given by $u_i = -\sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j)$ and the closed-loop nominal system (without quantization) is $\dot{x}_i = -\sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j,)$, $i = 1, \cdots, N$, or equivalently $\dot{x} = -Lx$. On the other hand, the protocol defined in Ren & Atkins (2007) for system (2) is $u_i = -\sum_{j \in \mathcal{N}_i} a_{ij}[(x_i - x_j) + \gamma(v_i - v_j)]$, where $\gamma > 0$ is the control gain. Similarly we have $\dot{x} = v$, $\dot{v} = -Lx - \gamma L v$. As in Dimarogonas & Johansson (2010), we assume that each agent i has only quantized measurements of the relative position $q(x_i - x_j), \forall j \in \mathcal{N}_i$, where $q(\cdot) : \mathbb{R} \to \mathbb{R}$ is the *quantization function*. In this paper, we mainly consider two quantization models: uniform and logarithmic. The *uniform* quantizer, q_u, $\mathbb{R} \to \mathbb{R}$ is defined as: $q_u(x) = \delta_u\, [\frac{x}{\delta_u}]$, where $[\cdot]$ denotes the nearest integer operation and $[\frac{1}{2}] = 1$. The following relations hold: (i) $x\, q_u(x) \geq 0$, (ii) $|q_u(x) - x| \leq \frac{\delta_u}{2}$, (iii) $q_u(-x) = -q_u(x)$, (iv) $q_u(0) = 0$. The *logarithmic* quantizer q_l, $\mathbb{R} \to \mathbb{R}$ Speranzon (2006) is defined as: $q_l(x) = \text{sign}(x) \cdot \exp(q_u(\ln(|x|)))$ when $x \neq 0$, where $q_u(\cdot)$ is the uniform quantizer with gain δ_u and $q_l(0) = 0$.

Preprint submitted to Automatica
Received April 30, 2013 01:19:40 PST

Similarly, we have (i) $x\, q_l(x) \geq 0$, (ii) $q_l(-x) = -q_l(x)$, (iii) $|q_l(x) - x| \leq \delta_l\, |x|$, where $\delta_l = e^{\frac{\delta_u}{2}} - 1$.

2.3 Filippov Solution

Given the system $\dot{x} = f(x(t))$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is measurable and essentially locally bounded, a Filippov set-valued map $\mathcal{F}[f](x(t)) : \mathbb{R}^m \to \mathcal{B}(\mathbb{R}^d)$ is defined in Shevitz & Paden (1994) and Filippov & Arscott (1988) by $\mathcal{F}[f](x(t)) = \cap_{\delta>0} \cap_{\mu(S)=0} \overline{\text{co}}\{f(B(x(t),\, \delta) \setminus S)\},\ x(t) \in \mathbb{R}^m$, where $B(x(t),\, \delta)$ is an open ball centered at x with radius δ, $\overline{\text{co}}$ denotes convex closure, μ denotes Lebesgue measure, and $\mathcal{B}(\mathbb{R}^d)$ denotes the collection of subsets of $\mathbb{R}^d$. Moreover, a Filippov solution defined on $[t_0,\, t_1) \subset \mathbb{R}$ is an absolutely continuous map $x : [t_0,\, t_1) \to \mathbb{R}^m$ that satisfies the differential inclusion $\dot{x} \in \mathcal{F}[f](x(t))$ for almost all $t \in [t_0,\, t_1)$. For simplicity, we use the notation $\mathcal{S} \geq 0$ for a set $\mathcal{S} \subset \mathbb{R}$ if $v \geq 0,\ \forall v \in \mathcal{S}$.

3 First-order Quantized Agreement under Time-varying Topology

The control law for system (1) with quantized relative states is given by $u_i = -\sum_{j \in \mathcal{N}_i} a_{ij}\, q\,(x_i - x_j)$, $i = 1, \cdots, N$. Thus the closed-loop system becomes

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} a_{ij}\, q\,(x_i - x_j), \qquad i = 1, \cdots, N. \tag{3}$$

It was observed in Proposition 1 in Carli, Persis & Frasca (2010) that quantized consensus controllers may have a set of initial conditions of measure zero from which no classic or Carathédory solutions exist. Thus we consider here more general solutions of (3) in the Filippov sense. The local existence of Filippov solution is guaranteed as the right hand side of (3) is measurable and locally bounded (Shevitz & Paden, 1994). Since $q\,(-a) = -q\,(a)\,, \forall a \in \mathbb{R}$ holds for both quantizers, the Filippov solution of (3) is given by

$$\begin{aligned}\dot{x} &\in \mathcal{F}[-B\Gamma\, q](B^T x)\\ &= -B\Gamma\mathcal{F}[q](B^T x) = -B\Gamma\mathcal{F}[q](\bar{x}),\end{aligned} \tag{4}$$

where $q(\bar{x})$ is the stack vector of all pairs $q\,(x_i - x_j)\,, \forall (i,\, j) \in E$. The matrix Γ is defined in Section 2.1. Note that $\mathcal{F}(-B\Gamma\, q(B^T x)) = -B\Gamma\mathcal{F}[q](B^T x)$ follows from statement 5 of Theorem 1 in Shevitz & Paden (1994).

3.1 Static Tree Topology

We start from the case that the underlying communication topology is a static tree. By multiplying B^T at both sides of equation (4), we get $\dot{\bar{x}} = B^T \dot{x} \in -B^T B\, \Gamma\mathcal{F}[q](\bar{x}) = -M\, \Gamma\mathcal{F}[q](\bar{x})$, where $M = B^T B$. By Lemma 1 in Dimarogonas & Johansson (2010), M is always positive definite with a tree graph. Let $V = \bar{x}^T M^{-1} \bar{x}$ be a candidate Lyapunov function. Since M is positive definite, M^{-1} exists and is also positive definite (Horn & Johnson, 1990). Since V is smooth and regular, the generalized time derivative (Paden & Sastry, 1987) of V satisfies

$$\begin{aligned}\dot{\tilde{V}} &\subset (2\, M^{-1}\bar{x})^T(-M\Gamma\mathcal{F}[q](\bar{x})) = -2\, \bar{x}^T\, \Gamma\mathcal{F}[q](\bar{x})\\ &= -2\sum_{i=1}^{m} w_i\, \bar{x}_i\, \mathcal{F}[q](\bar{x}_i).\end{aligned} \tag{5}$$

In the case of $q(x)$ being uniform, the Filippov set-valued map for $q_u(x)$ is given by: $\mathcal{F}[q_u](x) = q_u(x)$ when $x \neq (k - \frac{1}{2})\delta_u,\ k \in \mathbb{Z}$; $\mathcal{F}[q_u](x) = [(k-1)\delta_u,\ k\delta_u]$, otherwise. Note that $a\, \mathcal{F}[q_u](a) \geq 0, \forall a \in \mathbb{R}$ and the equality holds when $|a| \leq \frac{\delta_u}{2}$. Thus $\bar{x}^T\, \Gamma\mathcal{F}[q_u](\bar{x}) \geq 0$ and $\dot{\tilde{V}} \leq 0$ where the equality holds only when $|x_i - x_j| \leq \frac{\delta_u}{2},\ \forall (i,\, j) \in E$. Since the level sets of V are compact, we can apply the nonsmooth version of the LaSalle's invariance principle (Shevitz & Paden, 1994). System (3) converges to the consensus set $\mathcal{I} = \{x |\, |x_i - x_j| \leq \frac{\delta_u}{2}, (i, j) \in E\}$, which implies $\{x |\, |\bar{x}| \leq \frac{\delta_u}{2}\sqrt{m}\}$, a ball centered in the desired equilibrium point $|\bar{x}| = 0$, with radius $\frac{\delta_u}{2}\sqrt{m}$ (Liberzon, 2003). This point coincides with the average of the initial states by virtue of Lemma 5 which will be stated in the sequel. When $x \in \mathcal{I}$, we have $u = 0$, thus all agents stay in the set $\mathcal{I}$.

In the case of $q(x)$ being logarithmic, the Filippov set-valued map for $q_l(x)$ is given as: $\mathcal{F}[q_l](x) = q_l(x)$ when $x \geq 0$ and $x \neq e^{(k-\frac{1}{2})\delta_u},\ k \in \mathbb{Z}$; $\mathcal{F}[q_l](x) = [e^{(k-1)\delta_u}, e^{k\delta_u}]$ when $x = e^{(k-\frac{1}{2})\delta_u},\ k \in \mathbb{Z}$. Moreover, $\mathcal{F}[q_l](-x) = -\mathcal{F}[q_l](x)$. Since $a\, \mathcal{F}[q_l](a) > 0,\ \forall a \neq 0$, $\bar{x}\, \Gamma\mathcal{F}[q_l](\bar{x}) \leq 0$ and $\dot{\tilde{V}} \leq 0$, where the equality holds when $x_i = x_j,\ \forall (i,\, j) \in E$. For a connected tree graph, this corresponds to a consensus point. The nonsmooth version of LaSalle's invariance principle guarantees that (3) converges to the consensus point asymptotically for any $\delta_l = e^{\frac{\delta_u}{2}} - 1 > 0$. The previous analysis is summarized as:

Theorem 1 *Assume that G is a weighted static tree. Let $x(t)$ be a Filippov solution of system (3).*

(1) In the case of uniform quantizers, $x(t)$ converges to the consensus set $\{x |\, |x_i - x_j| \leq \frac{\delta_u}{2}, (i, j) \in E\}$.
(2) In the case of logarithmic quantizers, $x(t)$ asymptotically converges to the average consensus for all $\delta_l > 0$.

Remark: Compared to Dimarogonas & Johansson (2010), a smaller convergence set is obtained for uniform quantizers and the bound on the logarithmic gain

Preprint submitted to Automatica
Received April 30, 2013 01:19:40 PST

is less conservative. It will be shown in Theorem 6 that the convergence in case 1 occurs in finite time.

3.2 Time-varying Communication Topology

In this section we treat the case when the communication topology is time-varying or in particular switching among different tree topologies. Note that the stack vector $\bar{x}$ changes discontinuously whenever edges are added or deleted. In the following, we will show that the same function $V = \bar{x}^T(B^TB)^{-1}\bar{x}$ can serve as a common Lyapunov function under time varying topologies.

There are always $N-1$ edges for undirected tree graphs with N vertices, i.e., $m = N-1$. Thus the incidence matrix B of any tree topology has dimension $N \times (N-1)$. We assume its singular value decomposition to be $B = U\Sigma W^T$, where U and V are orthonormal and $\Sigma_{(N\times(N-1))}$ has zero entries but b_i for $i = (N-1), \cdots, 1$ on the upper diagonal, which are the singular values of B in descending order.

Lemma 2 *BB^T has N nonnegative eigenvalues; one of them is zero, corresponding to the eigenvector $\frac{1}{\sqrt{N}}\mathbf{1}$ and the others are the same as the eigenvalues of B^TB.*

PROOF. The proof follows directly from the fact that BB^T of a tree graph is positive semidefinite and has only one zero eigenvalue with eigenvector $\frac{1}{\sqrt{N}}\mathbf{1}$. Moreover, BB^T has the same nonzero eigenvalues as B^TB (Horn & Johnson, 1990).

Lemma 3 *$H = B(B^TB)^{-1}B^T$ is identical for all incidence matrices B corresponding to undirected trees.*

PROOF. Inserting $B = U\Sigma W^T$ into H we have

$$\begin{aligned} H &= B(B^TB)^{-1}B^T = U\Sigma W^T(W\Sigma^T\Sigma W^T)^{-1}W\Sigma^TU^T \\ &= U\Sigma W^T(WTW^T)^{-1}W\Sigma^TU^T = U\Sigma\, T^{-1}\Sigma^TU^T. \end{aligned}$$

Then we denote $G = \Sigma\, T^{-1}\Sigma^T$, which has the first $N-1$ diagonal elements as 1. Consider the matrix $U_{N\times N} = [u_{N-1}\ u_{N-2}\cdots u_0]$, where u_k are the normalized column eigenvectors of BB^T, $k = 0, \cdots, N-1$. Denote by $u_k(i)$ the i_{th} element of u_k. Since U is orthonormal , we get $\sum_{k=0}^{N-1} u_k(i)\ u_k(i) = 1$ and $\sum_{k=0}^{N-1} u_k(i)\ u_k(j) = 0$ for $i \neq j$, for all $i,\ j = 1, \cdots, N$. In Lemma 2 we have shown that the last eigenvector corresponding to the eigenvalue zero is $u_0 = \frac{1}{\sqrt{N}}\mathbf{1}$. Computing each entry of $H = UGU^T$ element-wise, we have $H(i,j) = \sum_{k=1}^{N-1} u_k(i)\, u_k(j) = \sum_{k=0}^{N-1} u_k(i)\, u_k(j) - u_0(i)\, u_0(j)$

$$= \begin{cases} 1 - \frac{1}{\sqrt{N}}\frac{1}{\sqrt{N}} = \frac{N-1}{N} & \text{for } i = j \\ 0 - \frac{1}{\sqrt{N}}\frac{1}{\sqrt{N}} = -\frac{1}{N} & \text{for } i \neq j\,, \end{cases}$$

by which we can conclude that $H = B(B^TB)^{-1}B^T$ is identical for any B corresponding to a tree graph. This completes the proof.

Let $\mathcal{T} = \{t_1, t_2, \cdots, t_K\}$ denote the set of switching instants of $G(t)$, where $K \to \infty$ as we assume that $G(t)$ remains a tree for an infinite switching sequence. To avoid infinitely frequent switching we define a strictly positive dwell time (Bacciotti & Mazzi, 2005) $\tau > 0$ such that $t_{k+1} - t_k > \tau$, $k = 1, \cdots$. Proposition S2 in Cortes (2008) guarantees that there exists a Caratheodory solution of system (4). We consider the Lyapunov function candidate $V = x^THx$, where $H = B(B^TB)^{-1}B^T$ from Lemma 3. We will show that V serves as a common Lyapunov function (Bacciotti & Mazzi, 2005) for system (4).

Denote the tree topology during time interval $t \in [t_k,\, t_{k+1})$ as $\mathcal{T}_k$ and the corresponding incidence matrix as B_k, the edge set as E_k, the edge vector as $\bar{x}_k = B_kx$, $k \in \mathbb{Z}^+$. Since H is invariant by Lemma 3, V is continuously differentiable with respect to state x. Furthermore, V is positive semidefinite and its time derivative can be computed as in (5), $\dot{\tilde{V}} \subset -2\,\bar{x}_k^T\,\Gamma\,\mathcal{F}[q](\bar{x}_k) \leq 0$. Namely, $v \leq 0$, $\forall v \in \dot{\tilde{V}}$. In particular, $\dot{\tilde{V}} = 0$ only when $\mathcal{F}[q](\bar{x}_k) = 0$, $k \in \mathbb{Z}^+$. Thus V serves as a common Lyapunov function for system (4). Based on the invariance principle for nonlinear switched system, namely Theorem 1 in Bacciotti & Mazzi (2005), the solutions of system (4) is attracted by the union of all weakly invariant sets contained in $\{x \in \mathbb{R}^n |\, \mathcal{F}[q](\bar{x}_k) = 0\}$, $\forall k \in \mathbb{Z}^+$. Even though the sequence of tree graphs is infinite, the set of all possible tree graphs with N vertices is finite. For uniform quantizers, $\mathcal{F}[q_u](\bar{x}_k) = 0$ implies $|x_i - x_j| \leq \frac{\delta_u}{2}, \forall (i,\, j) \in E_k$. Given any undirected tree with N vertices, there always exists a path with maximal length $N-1$ connecting any two nodes (Godsil & Royle, 2001). Thus the union of the invariant sets $\{x \in \mathbb{R}^N |\, |x_i - x_j| \leq \frac{\delta_u}{2}, \forall (i,\, j) \in E_k\}$, $k \in \mathbb{Z}^+$, is given by $\{x \in \mathbb{R}^N |\, |x_i - x_j| \leq (N-1)\frac{\delta_u}{2}, \forall (i,\, j) \in V \times V\}$. For logarithmic quantizers, as $q_l(a) = 0$ if and only if $a = 0$, $\mathcal{F}[q_l](\bar{x}_k) = 0$ implies $x_i = x_j, \forall (i,\, j) \in E_k$. Since a tree graph is always connected, $x_i = x_j, \forall (i,\, j) \in E_k$ implies $x_i = x_j, \forall (i,\, j) \in V \times V$. The union of the invariant sets $\{x \in \mathbb{R}^n |\, x_i = x_j, \forall (i,\, j) \in V \times V\}$, $k \in \mathbb{Z}^+$ is uniquely the invariant set $\{x \in \mathbb{R}^n |\, x_1 = x_2 \cdots = x_N\}$. Hence asymptotic convergence of system (3) to consensus is achieved instead when logarithmic quantizers are used. The above results are summarized in the following theorem:

Theorem 4 *Assume that $G(t)$ remains a tree with infinite switching sequences and positive dwell time. Let $x(t)$ be a Filippov solution of system (3).*

(1) In the case of uniform quantizers, $x(t)$ converges to the invariant set $\{x |\, |x_i - x_j| \leq \frac{\delta_u}{2}(N-1), \forall (i,\, j) \in V \times V\}$.

Preprint submitted to Automatica
Received April 30, 2013 01:19:40 PST

(2) *In the case of logarithmic quantizers, $x(t)$ asymptotically converges to the average consensus point for all logarithmic gains $\delta_l > 0$.*

3.3 General Undirected Graphs

The above results are useful whenever the communication graph retains the tree structure. A more practical situation however occurs if we allow for the tree assumption to be relaxed.

Lemma 5 *Let $x(t)$ be a Filippov solution of system (3). The average of all agent states $\frac{1}{N}\sum_{i=1}^{N} x_i$ is invariant in the case of undirected topologies.*

PROOF. By definition, the Filippov solution $x(t)$ satisfies $\dot{x} \in -B\Gamma\mathcal{F}[q](\bar{x})$ from (4). The time derivative of $\frac{1}{N}\sum_{i=1}^{N} x_i$ is given by $\frac{1}{N}\sum_{i=1}^{N} \dot{x}_i$. Equivalently we have $\frac{1}{N}\mathbf{1}^T\dot{x} \subset -\frac{1}{N}\mathbf{1}^T B\Gamma\,\mathcal{F}[q](\bar{x}) = \{0\}$, where the final equality is due to the fact that $\mathbf{1}^T B = \mathbf{0}$. Hence the centroid is preserved during the evolution.

We denote the invariant centroid by the constant $C \in \mathbb{R}$ and propose a new Lyapunov function candidate for system (3): $V_g = \sum_{i=1}^{N}(x_i - \frac{1}{N}\sum_{i=1}^{N} x_i)^2 = \sum_{i=1}^{N}(x_i - C)^2$, which is known as the quadratic disagreement function (Olfati-Saber & Murray, 2004) to the invariant centroid. V_g is continuously differentiable and $V_g = 0$ only when all states equal to the initial average. The level sets of V_g define compact sets with respect to the agents' states. Its generalized time derivative is given by $\dot{\tilde{V}}_g \subset (\nabla V_g)^T\dot{x} = -2(x^T - C\mathbf{1}^T)B\Gamma\,\mathcal{F}[q](\bar{x}) = -2\,\bar{x}^T\Gamma\,\mathcal{F}[q](\bar{x})$, where $\bar{x}^T\,\Gamma\mathcal{F}[q](\bar{x})$ has been proved to be positive semidefinite in Theorem 1.

Theorem 6 *Assume that G is undirected and static. Let $x(t)$ be a Filippov solution of system (3). Then $x(t)$ converges to the invariant sets*

(1) $\{x|\,|x_i - x_j| \le \frac{\delta_u}{2}, \forall (i,j) \in E\}$ *with uniform quantizers, in finite time.*
(2) $\{x|\,x_i = x_j, \forall (i,j) \in E\}$ *asymptotically with logarithmic quantizers satisfying $\delta_l > 0$.*

PROOF. As stated above, we have shown that $v \le 0$ for all $v \in \dot{\tilde{V}}_g$. The nonsmooth version of LaSalle's Invariance Principle (Shevitz & Paden, 1994) ensures the convergence of the system to the largest invariant subset $\mathcal{I}_n = \{x|\,\mathcal{F}[q](x_i - x_j) = 0, \forall (i,j) \in E\}$, which depends on the edge set E of the static topology. For uniform quantizers, if at least one pair $(i,\,j) \in E$ satisfies $|x_i - x_j| > \frac{\delta_u}{2}$, it holds that $\mathcal{F}[q_u](x_i - x_j) \ge \delta_u$. Then we have $\dot{\tilde{V}}_g \subset -2\sum_{i=1}^{m} w_i\,\bar{x}_i\,\mathcal{F}[q_u](\bar{x}_i) \le -a_{ij}\,\delta_u^2$, which is strictly negative. Since V_g is bounded from below, there exists a settling time $T \in [0,\infty)$ such that $x(t) \in \{x|\,|x_i - x_j| < \frac{\delta_u}{2}, \forall (i,j) \in E\}$ for $t \ge T$ (Carli, Fagnani, & Zampieri, 2006). For the case of logarithmic quantizers, due to the fact that $q_l(x) \to 0$ when $x \to 0$, asymptotic convergence can be achieved instead.

Remark: The finite-level quantizer model proposed in Li, Fu, Xie & Zhang (2011) is given by $q_f(\cdot) : \mathbb{R} \to \Gamma$, where $\Gamma = \{0, \pm i,\, i = 1, 2, \cdots, K\}$ and $K \in \mathbb{Z}^+$. We refer the interested readers to the above paper for detailed definition. When $q_f(\cdot)$ is used, the closed-loop system becomes $\dot{x}_i = -\sum_{j\in\mathcal{N}_i} a_{ij}\,q_f(x_i - x_j),\ i = 1, \cdots, N$. Similar to the case of infinite-level quantizers, as $q_f(x) = q_f(-x),\ \forall x \in \mathbb{R}$, the Filippov solution satisfies $\dot{x} \in -B\Gamma\mathcal{F}[q_f](\bar{x})$. Consider the same Lyapunov function candidate as in Theorem 3 that $V_g = \sum_{i=1}^{N}(x_i - C)^2$ and $\dot{\tilde{V}}_g \subset -2\bar{x}^T\Gamma\mathcal{F}[q_f](\bar{x})$, where $\bar{x}^T\Gamma\mathcal{F}[q_f](\bar{x}) \ge 0$ as $x\,q_f(x) \ge 0,\ \forall x \in \mathbb{R}$. Thus $v \le 0,\ \forall v \in \dot{\tilde{V}}_g$. The largest invariant subset is given by $\{x|\,\mathcal{F}[q_f](x_i - x_j) = 0, \forall (i,\,j) \in E\} = \{x|\,|x_i - x_j| \le \frac{1}{2}, \forall (i,\,j) \in E\}$. Thus the conclusions obtained for uniform quantizers in Theorem 1, 2 and 3 still hold if the finite-level quantizer is used.

4 Second-order Quantized Agreement under Static topology

The control law for second-order system (2) with quantized relative states is given by $u_i = -\sum_{j\in\mathcal{N}_i} a_{ij}[\,q(x_i - x_j) + \gamma\,q(v_i - v_j)]$, for $i = 1, \cdots, N$. Thus we have the closed-loop system

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= -\sum_{j\in\mathcal{N}_i} a_{ij}[\,q(x_i - x_j) + \gamma\,q(v_i - v_j)]. \end{aligned} \tag{6}$$

Due to similar reasons as in the first order system, the Carathédory solution of system (6) may not exist. Thus we consider its solution in the Filippov sense. Given that $\bar{x} = B^T x$ and $\bar{v} = B^T v$, (6) is equivalent to $\dot{\bar{x}} = \bar{v}$ and $\dot{\bar{v}} = -B^T B\Gamma\,q(\bar{x}) - \gamma B^T B\Gamma\,q(\bar{v})$.

Denote the stack vector $y = [\bar{x}^T\ \ \bar{v}^T]^T$ and $M = B^T B$. By performing a coordinates transformation, we have

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0_{m\times m} & I_m \\ 0_{m\times m} & 0_{m\times m} \end{bmatrix} z + \begin{bmatrix} 0_{m\times m} \\ I_m \end{bmatrix} \tilde{u} \\ \tilde{u} &= \begin{bmatrix} -J & -\gamma J \end{bmatrix} P\,q(y)\,, \end{aligned} \tag{7}$$

Preprint submitted to Automatica
Received April 30, 2013 01:19:40 PST

where $\tilde{u} = Pu$ and

$$P = \begin{bmatrix} \sqrt{\Gamma} & 0 \\ 0 & \sqrt{\Gamma} \end{bmatrix}, \quad z = P\, y = \begin{bmatrix} \sqrt{\Gamma}\bar{x} \\ \sqrt{\Gamma}\bar{v} \end{bmatrix} = \begin{bmatrix} \sqrt{\Gamma}B^T x \\ \sqrt{\Gamma}B^T v \end{bmatrix}.$$

$\sqrt{\Gamma}$ is the element-wise square root of the weight matrix Γ and $J_{m\times m} = \sqrt{\Gamma}M\sqrt{\Gamma} = \sqrt{\Gamma}B^T B\sqrt{\Gamma}$ is symmetric. Denote by $D_{N\times m} = B\sqrt{\Gamma}$, then $J = D^T D$. If the undirected graph G is connected, then $J = \sqrt{\Gamma}B^T B\sqrt{\Gamma}$ and the Laplacian matrix $L = B\Gamma B^T$ both have non-negative eigenvalues and moreover the same positive ones. Let

$$V(z) = z^T \begin{bmatrix} \gamma J & \frac{1}{2}I_m \\ \frac{1}{2}I_m & \frac{\gamma}{2}I_m \end{bmatrix} z$$

be the Lyapunov function candidate for system (7). Note that $V(z)$ is continuously differentiable for a static graph G. The generalized time derivative of $V(z)$ along the Filippov solution of system (6) is given by

$$\dot{\tilde{V}}(z) \subset -z^T Qz + z^T WP\,(\mathcal{F}[q](y) - y), \qquad (8)$$

where

$$Q = \begin{bmatrix} J & 0_{m\times m} \\ 0_{m\times m} & \gamma^2 J - I_m \end{bmatrix}, \quad W = \begin{bmatrix} -J & -\gamma J \\ -\gamma J & -\gamma^2 J \end{bmatrix}.$$

Lemma 7 *$s^T Js \geq \lambda_2(L)|s|^2$ for a connected G, where $s_{m\times 1} = D^T x$, $x \in \mathbb{R}^{n\times 1}$. Moreover, $z^T Qz \geq \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\}|z|^2$ if $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$.*

PROOF. (Sketch) We have $J = \sqrt{\Gamma}B^T B\sqrt{\Gamma} = D^T D$ and $L = DD^T$. $Ds = 0$ only if $s = 0$ because $Ds = 0 \Rightarrow DD^T x = 0 \Rightarrow Lx = 0 \Rightarrow x \in span\{\mathbf{1}\} \Rightarrow s = \sqrt{\Gamma}B^T x = 0$. Denote by c_i the eigenvectors of $D^T D$ associated with the eigenvalue zero, $i = 1, \cdots, m+1-N$. By definition, $Dc_i = 0$. Moreover, since $c_i^T s = c_i^T D^T x = (Dc_i)^T x = 0$, we have $c_i \perp s, \forall i = 1, \cdots, m+1-N$. By the Courant-Fischer Theorem (Horn & Johnson, 1990), $\min_{c_i\perp s,\, s\neq 0 i=1,\cdots,m+1-N} \frac{s^T D^T Ds}{s^T s} = \lambda_{m+2-N}(D^T D)$, where $\lambda_{m+2-N}(D^T D) = d_1^2 = \lambda_2(DD^T) = \lambda_2(L)$. Thus $s^T Js \geq \lambda_2(L)\,|s|^2$. Then since $z = [x^T D\ v^T D]^T$, it is easy to verify that $Q \geq 0$ if $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$. The first part of $\dot{\tilde{V}}(z)$ in (8) can be lower bounded by $z^T Qz \geq \lambda_{\min}(Q)|z|^2$, where $\lambda_{\min}(Q) = \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\}$.

Lemma 8 $\|W\|_2 = (1+\gamma^2)\lambda_{\max}(L)$

PROOF. (Sketch) Let θ_i be the eigenvalues of J, $i = 1, \cdots, m$. It can be verified that $W^T W$ has an eigenvalue at 0 with multiplicity m and another m non-zero eigenvalues $\theta_i^2(1+\gamma^2)^2$ corresponding to each eigenvalue θ_i of J. The maximal one is $\lambda_{\max}(W^T W) = \lambda_{\max}^2(J)(1+\gamma^2)^2$, yielding $\|W\|_2 = (1+\gamma^2)\lambda_{\max}(J) = (1+\gamma^2)\lambda_{\max}(L)$ by Lemma 7.

When $q = q_u$, since $|\tilde{a} - a| \leq \frac{\delta_u}{2}$, $\forall \tilde{a} \in \mathcal{F}[q_u](a)$, $a \in \mathbb{R}$, $|\tilde{y} - y| \leq \frac{\delta_u}{2}\sqrt{2m}$, $\forall \tilde{y} \in \mathcal{F}[q_u](y)$, $y \in \mathbb{R}^{2m}$. Then by combining Lemmas 7 and 8, we can bound $\dot{\tilde{V}}(z)$ by $\dot{\tilde{V}}(z) \leq -\lambda_{\min}(Q)|z|(|z| - \frac{(1+\gamma^2)\lambda_{\max}(L)}{\lambda_{\min}(Q)}\frac{\delta_u\sqrt{w_{\max}}}{2}\sqrt{2m})$, where $\lambda_{\min}(Q) = \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\}$ and $w_{\max} = \max_{(i,j)\in E}\{a_{ij}\}$. Based on the nonsmooth version of LaSalle's Invariance principle (Shevitz & Paden, 1994), all solutions of system (6) enter the ball

$$\{z|\,|z| \leq \frac{(1+\gamma^2)\lambda_{\max}(L)\sqrt{w_{\max}}}{2\lambda_{\min}(Q)}\sqrt{2m}\,\delta_u\}, \qquad (9)$$

which is centered at the consensus point $\bar{x} = \bar{v} = 0$.

When $q = q_l$, since $|\breve{a} - a| \leq \delta_l\,|a|$, $\forall \breve{a} \in \mathcal{F}[q_l](a)$, $a \in \mathbb{R}$, then $|\breve{y} - y| \leq \delta_l\,|y|$, $\forall \breve{y} \in \mathcal{F}[q_l](y)$, $y \in \mathbb{R}^{2m}$. Moreover, given that P is a positive diagonal matrix, it holds that $|P\,(\breve{y} - y)| \leq \delta_l\,|P\,y| = \delta_l|z|$, $\forall \breve{y} \in \mathcal{F}[q_l](y)$, $y \in \mathbb{R}^{2m}$. Thus $\dot{\tilde{V}}(z) \leq -|z|^2(\lambda_{\min}(Q) - (1+\gamma^2)\lambda_{\max}(L)\,\delta_l)$. If the logarithmic gain δ_l satisfies

$$0 < \delta_l < \frac{\lambda_{\min}(Q)}{(1+\gamma^2)\lambda_{\max}(L)}, \qquad (10)$$

we have $\dot{\tilde{V}}(z) \leq 0$ and equality holds only when $|z| = 0$, i.e., $\bar{x} = \bar{v} = 0$. This implies that the logarithmic gain should be smaller than an upper bound to guarantee asymptotic convergence.

Theorem 9 *Assume that the undirected graph G is static and connected. If $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$, system (6) has the following convergence properties:*

(1) With uniform quantizers, the agents converge to the consensus set (9).
(2) With logarithmic quantizers, the agents asymptotically converge to the a consensus point and move with the same velocity, for all δ_l satisfying (10).

5 Simulations

We now provide computer simulations to support the presented results. Four different communication graphs are used: $G_1 = \{\mathcal{N}_1 = \{2\}, \mathcal{N}_2 = \{1,3\}, \mathcal{N}_3 = \{2,4\}, \mathcal{N}_4 = \{3\}\}$, $G_2 = \{\mathcal{N}_1 = \{4\}, \mathcal{N}_2 = \{3\}, \mathcal{N}_3 = \{2,4\}, \mathcal{N}_4 = \{1,3\}\}$, $G_3 = \{\mathcal{N}_1 = \{3\}, \mathcal{N}_2 = \{4\}, \mathcal{N}_3 = \{1\}, \mathcal{N}_4 = \{2\}\}$, $G_4 = \{\mathcal{N}_1 = \{2\}, \mathcal{N}_2 = \{1\}, \mathcal{N}_3 =$

Preprint submitted to Automatica
Received April 30, 2013 01:19:40 PST

$\{4\}, \mathcal{N}_4 = \{3\}\}$. G_1, G_2 are tree graphs and G_3, G_4 are disconnected. $\delta_l = 10$ for the logarithmic gain and $\delta_u = 0.01$ for the uniform case. The first simulation involves a switching topology from G_1 to G_2, then back to G_1 for both uniform and logarithmic quantizers. The trajectories for uniform quantizers are depicted in Fig. 1 while the case of logarithmic quantizers is shown in Fig. 2. The red circles denote the instants when the communication topology switches. As expected by Theorem 4, in the case of uniform quantizers all agents reach the invariant set while with logarithmic quantizers the average consensus is achieved asymptotically. The same system under more general graphs is tested

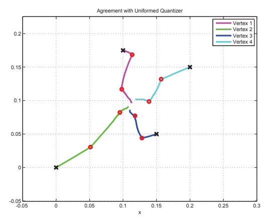

Fig. 1. Switching tree topologies and uniform quantizers.

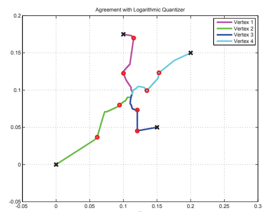

Fig. 2. Switching tree topologies and logarithmic quantizers.

in the second part where the graph G is switching from G_3 to G_1 and finally to G_4. The simulation results in Fig. 3 and 4 illustrate different convergence results with uniform and logarithmic quantizers.

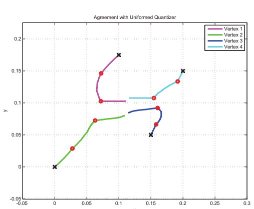

Fig. 3. General topology and uniform quantizers.

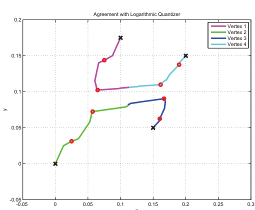

Fig. 4. General topology and logarithmic quantizers.

In the last part, we simulate a group of second-order agents moving only along x-coordinate to visualize the trajectories of both velocity and position. The communication graph G_1 is used as in the first order case. Figures to the right are zoomed details of the final configuration. Logarithmic quantizers with $\delta_l = 0.05 < \frac{\lambda_{\min}(Q)}{(1+\gamma^2)\lambda_{\max}(L)} = 0.056$ are used and asymptotic convergence in both velocity and position is shown in Fig. 6.

6 Conclusions

In this paper, we analyzed the consensus problem of multi-agent systems under distributed control laws, composed of quantized values of relative states between neighboring agents. In particular, we distinguished between uniform and logarithmic quantizers as well as

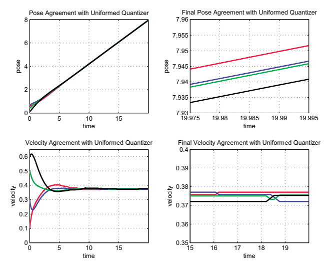

Fig. 5. Second order system with uniform quantizers.

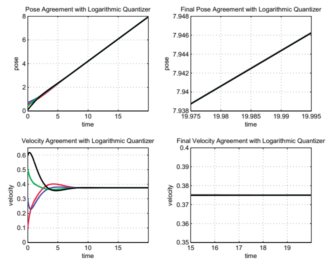

Fig. 6. Second order system with logarithmic quantizers.

between static and time-varying communication topologies. The derived results are less conservative than our previous work with the same quantization constraints. It was established that a tree structure provides convergence guarantees in these cases. Similar conclusions were also shown to hold in the case of general undirected topologies. Second order dynamics were then taken into account and explicit convergence properties were obtained.

Future work includes extensions to the same multi-agent system with directed topologies or agents with non-homogeneous quantizers. Moreover, possible combinations with event-based control techniques may be of great interest in order to reduce the communication load further.

References

Bacciotti, A., & Mazzi, L. (2005). An invariance principle for nonlinear switched systems, *System and Control Letters*, 54(11): 1109-1119.

Carli, R., Fagnani, F., & Zampieri, S. (2006). On the state agreement with quantized information. *17th Intern. Symp. Networks and Systems*, 1500-1508.

Ceragioli, F., Persis, C. D., & Frasca, P. (2010). Discontinuities and hysteresis in quantized average consensus. *Automatica*, 47(9): 1916-1928.

Cortes, J. (2008). Discontinuous dynamical systems. *Control Systems IEEE*, 28: 36-73.

Ceragioli, F., Persis, C. D., & Frasca, P. (2010). Quantized average consensus: discontinuities and hysteresis. *8th IFAC Symposium on Nonlinear Control Systems*.

Dimarogonas, D. V., & Johansson, K. H. (2010). Stability analysis for multi-agent systems using the incidence matrix: quantized communication and formation control *Automatica*, 46(4): 695-700.

Dimarogonas, D. V., & Kyriakopoulos, K. J. (2007). On the rendezvous problem for multiple nonholonomic agents. *IEEE Transaction on Automatic Control*, 52(5): 916-922.

Fax, J. A., & Murray, R. M. (2002). Graph laplacian and stabilization of vehicle formations. *15th IFAC World Congress*, 283-288.

Preprint submitted to Automatica
Received April 30, 2013 01:19:40 PST

Fagnani, F., Johansson, K. H., Speranzon, A., & Zampieri, S.(2004). On multi-vehicle rendezvous under quantized communication. *16th Intern. Symp. Networks and Systems*, Leuven, Belgium.

Liu, H., Cao, M., & Persis, C. D. (2011). Quantization effects on synchronization of mobile agents with second-order dynamics. *18th IFAC World Congress*, 2376-2381, Milano, Italy.

Filippov, A. F., & Arscott, F. M. (1988). *Differential equations with discontinuous righthand sides*. Springer.

Guo, M., & Dimarogonas, D. V. (2011). Quantized cooperative control using relative state measurements, *50th IEEE Conference on Decision and Control and European Control Conference*.

Guo, M. (2011). Quantized cooperative control, *Master Thesis, School of Electrical Engineering (EES), KTH*. Available `http://urn.kb.se/resolve?urn=urn:nbn:se:kth:diva-55852`

Godsil, C., & Royle, G. (2001). Algebraic Graph Theory. *Springer Graduate Texts in Mathematics*, 207.

Horn, R. A., Johnson, C. R. (1990). *Matrix Analysis*. Cambridge University Press.

Kashyap, A., Basar, T., & Srikant, R. (2007). Quantized consensus. *Automatica*, 43(7): 1192-1203.

Liberzon, D. (2003). Hybrid feedback stabilization of systems with quantized signals, *Automatica*, 39(9): 1543-1554.

Li, T., Fu, M., Xie, L., & Zhang, J. F. (2011). Distributed consensus with limited communication data rate. *IEEE Transactions on Automatic Control*, 56(2): 279-292.

Nedic, A., Olshevsky, A., Ozdaglar, A., & Tsitsiklis, J. N. (2008). Distributed subgradient methods and quantization effects. *47th IEEE Conference on Decision and Control*, 4177-4184.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9): 1520-1533.

Paden, B. E., & Sastry, S. S. (1987). A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators. *IEEE Transactions on Circuits and Systems*, 34(1): 73-82.

Ren, W., & Atkins, E. (2007). Distributed multi-vehicle coordinated control via local information exchange, *Int. J. Robust Nonlinear Control*, 17: 1002-1033.

Shevitz, D., & Paden, B. (1994). Lyapunov stability theory of nonsmooth systems. *IEEE Transactions on Automatic Control*, 39(9): 1910-1914.

Speranzon, A. (2006). *Consensus and communication in multi-Robot control systems*. PhD Thesis, KTH.

Seuret, A., Dimarogonas, D. V., & Johansson, K. H. (2008). Consensus under communication delays. *47th IEEE Conference on Decision and Control*, 4922-4927.

Xiao, F., & Wang, L. (2006). State consensus for multi-agent systems with switching topologies and time-varying delays. *International Journal of Control*, 79(1): 1277-1284.

Preprint submitted to Automatica
Received April 30, 2013 01:19:40 PST