A Feedback Stabilization and Collision Avoidance Scheme for Multiple Independent Non-point Agents ⋆

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Abstract

A navigation functions' based methodology, established in our previous work for centralized multiple robot navigation, is extended to address the problem of decentralized navigation. In contrast to the centralized case, each agent plans its actions without knowing the destinations of the other agents. Asymptotic stability is guaranteed by the existence of a global Lyapunov function for the whole system, which is actually the sum of the separate navigation functions. The collision avoidance and global convergence properties are verified through simulations.

Key words: Decentralized Control; Autonomous Systems; Motion Planning .

1 Introduction

Multi-agent Navigation is a field that has recently gained increasing attention both in the robotics and the control communities, due to the need for autonomous control of multiple robotic agents in the same workspace. While most efforts in the past had focused on centralized planning, specific real-world applications have lead researchers throughout the globe to turn their attention to decentralized concepts. The basic motivation for this work comes from two application domains: (i) decentralized conflict resolution in air traffic management (ATM) and (ii) the field of micro robotics where a team of autonomous microrobots must cooperate to achieve manipulation precision in the sub micron level.

Decentralized approaches are more appealing to centralized ones, due to their reduced computational complexity and increased robustness with respect to agent failures. The main focus in this domain has been cooperative and formation control of multiple agents, where so much effort has been devoted to the design of systems with variable degree of autonomy (e.g. Gupta et al. (2003); Egerstedt and Hu (2002)). There have been many different approaches to the decentralized motion planning problem. Open loop approaches pre-compute the trajectory of each agent off-line based on the initial conditions while closed loop ones are updated on line based on the knowledge of the system state at each time instant. Open loop approaches use game theoretic and optimal control theory to solve the problem taking the constraints of vehicle motion into account; see for example Bicchi and Pallottino (2000) and Inalhan et al. (2002). On the other hand, closed loop approaches use Lyapunov theory to design control laws and achieve the convergence of the distributed system to a desired configuration both in the concept of cooperative (Jadbabaie et al. (2003); Ren et al. (2004); Smith et al. (2005)) and formation control (Olfati-Saber and Murray (2003); Stipanovic et al. (2004); Tanner et al. (2005)).

Closed loop strategies are apparently preferable to open loop ones, mainly because they provide robustness with respect to modelling uncertainties and guaranteed convergence to the desired configurations. However, a common point of most work in this area is devoted to the case of point agents. Although this allows for variable

⋆ This work was partially presented in Dimarogonas et al. (2003); Zavlanos and Kyriakopoulos (2003)and is supported by the EU through contracts HYBRIDGE(IST-2001-32460), MICRON(IST-2001-33567) and I-SWARM (IST-2004-507006). Corresponding author K.J. Kyriakopoulos Tel. ++30-210-772-3595. Fax ++30-210-772-3657.

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degree of decentralization, it is far from realistic in real world applications. For example, in conflict resolution in ATM, two aircraft are not allowed to approach each other closer than a specific "alert" distance. The construction of closed loop methods for distributed non-point multi-agent systems is both evident and appealing.

A closed loop approach for single robot navigation was proposed in the seminal work of Koditschek and Rimon (1990). This navigation functions' framework had all the sought qualities but could only handle single, point-sized, robot navigation. In Loizou and Kyriakopoulos (2002) this method was successfully extended to take into account the volume of each robot while a decentralized version of this work has been presented by the authors in Zavlanos and Kyriakopoulos (2003), Dimarogonas et al. (2003).

In this paper we make the following assumptions:

- Each agent has global knowledge of the position of the others at each time instant.
- Each agent has knowledge only of its own desired destination but not of the others.
- We consider spherical agents.
- The workspace is bounded and spherical.
- The dynamics of each agent are holonomic.

Our assumption regarding the spherical shape of the agents does not constrain the generality of this work since it has been proven that navigation properties are invariant under diffeomorphisms (Koditschek and Rimon (1990)). Arbitrarily shaped agents diffeomorphic to spheres can be taken into account. Methods for constructing such analytic diffeomorphisms are discussed in Tanner et al. (2003) and Rimon and Koditschek (1992).

The second assumption makes the problem decentralized. In the centralized case a central authority has knowledge of everyones goals and positions at each time instant and coordinates the whole team so that the desired specifications (destination convergence and collision avoidance) are fulfilled. In the current situation no such authority exists and we have to deal with the limited knowledge of each agent. This is of course the first step towards a variable degree of decentralization. This paper presents the first to the authors knowledge extension of centralized multi-agent control using navigation functions, to a decentralized scheme. An extension of this work to nonholonomic agents was provided in Loizou et al. (2004).

The rest of the paper is organized as follows: section 2 presents the system definition and problem statement. Section 3 outlines the concept of navigation functions and describes their extension to the decentralized case to obtain the feedback control law. In section 4 simulation results are presented for two non-trivial multi agent navigational tasks. Section 5 includes the conclusions and current research issues. Sketches of proofs of the Propositions of section 3 are provided in the Appendix.

2 System and Problem Definition

Consider a system of N agents operating in the same workspace $W \subset \mathcal{R}^2$. Each agent i occupies a disc: $R = \{q \in \mathcal{R}^2 : \| q - q_i \| \leq r_i\}$ in the workspace where $q_i \in \mathcal{R}^2$ is the center of the disc and r_i is the radius of the agent. The configuration space is spanned by $q = [q_1, \ldots, q_N]^T$. The motion of each agent are described by:

$$\dot{q}_i = u_i, i \in \mathcal{N} = [1, \ldots, N] \qquad (1)$$

The desired destinations of the agents are denoted by the index d: $q_d = [q_{d1}, \ldots, q_{dN}]^T$. Figure 1 shows a three-agent conflict situation. The multi-agent navigation problem treated in this paper can be stated as follows: "*Derive a set of control laws (one for each agent) that drives the team of agents from any initial configuration to a desired goal configuration avoiding, at the same time, collisions. Each agent has global knowledge of the team configuration but is unaware of the other agents desired destinations*".

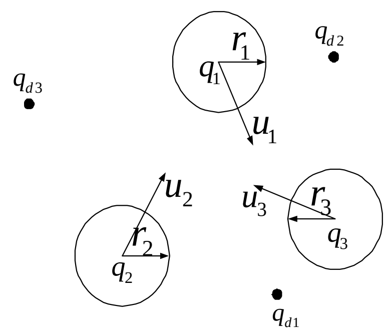

Fig. 1. A conflict scenario with three agents.

3 Decentralized Navigation Functions(DNF's)

3.1 Preliminaries

In this section we review the navigation function method introduced in the seminal paper by Koditschek and Rimon (1990) for single point robot navigation.

Navigation functions (NF's) are real valued maps realized through cost functions $\varphi(q)$, whose negated gradient field is attractive towards the goal configuration and repulsive wrt obstacles. It has been shown by Koditschek and Rimon that strict global navigation (i.e. the system $\dot{q} = u$ under a control law of the form $u = -\nabla\varphi$ admits a globally attracting equilibrium state) is not possible, and a smooth vector field on any sphere world with a unique attractor, must have at least as many saddles as obstacles (Koditschek and Rimon (1990)). Our assumption about spherical agents and obstacles does not constrain the generality of this work since it has been proven that navigation properties are invariant under diffeomorphisms. A navigation function can be defined as follows:

Definition 1 Koditschek and Rimon (1990):*Let $F \subset \mathcal{R}^{2N}$ be a compact connected analytic manifold with boundary. A map $\varphi : F \rightarrow [0,1]$ is a navigation function if:(1) it is analytic on F, (2) it has only one minimum at $q_d \in int(F)$, (3) its Hessian at all critical points (zero gradient vector field) is full rank, and (4) $\lim_{q \rightarrow \vartheta F} \varphi(q) = 1$.*

Strictly speaking, the continuity requirements for the navigation functions are to be C^2. The property 1 of Definition 1 follows the intuition provided in Koditschek and Rimon (1990), that is preferable to use closed form mathematical expressions to encode actuator commands instead of "patching together" closed form expressions on different portions of space, so as to avoid branching and looping in the control algorithm. In this paper, we further relax this requirement by using merely C^1 navigation function. The discontinuity however, takes place outside of the region where critical points occur, so it does not affect the navigation properties of the proposed function.

A function φ with a unique minimum q_d on F is called *polar*. Using a polar function on a compact connected manifold with boundary, all initial conditions are either brought to a saddle point or to the unique minimum q_d.

A scalar valued function φ whose Hessian at all critical points is full rank is called *Morse*. The corresponding critical points are called *non-degenerate*. This property establishes that the initial conditions that bring the system to saddle points are sets of measure zero (Koditschek and Rimon (1990)). In view of this property, all initial conditions away from sets of measure zero are brought to the unique minimum.

The last property of Definition 1 guarantees that the resulting vector field is transverse to the boundary of F. This fact is established by Lemma 3 in section 3.7. This establishes that the system will be safely brought to qd, avoiding collisions.

3.2 DNF's vs MRNF's

In Loizou and Kyriakopoulos (2002), the navigation functions method was extended to the case of multiple mobile robots with the use of Multi-Robot navigation functions (MRNF's). The decentralized extension of this work was initially presented in Zavlanos and Kyriakopoulos (2003).

In the form of a centralized setup of Loizou and Kyriakopoulos (2002), where a central authority has knowledge of the current positions and desired destinations of all agents, the sought control law is of the form: $u = -K\nabla\varphi(q)$ where K is a gain. In the decentralized case addressed in this work, each agent has knowledge of only the current positions of the others, and not of their desired destinations. Hence each agent has a different navigation law. Following the procedure of Koditschek and Rimon (1990),Loizou and Kyriakopoulos (2002), we consider the following class of decentralized navigation functions(*DNF's*):

$$\varphi_i \triangleq \sigma_d \circ \sigma \circ \hat{\varphi}_i = \left(\frac{\gamma_i}{\gamma_i + G_i}\right)^{1/k} \qquad (2)$$

which is a composition of $\sigma_d \triangleq x^{1/k}$, $\sigma \triangleq \frac{x}{1+x}$ and the cost function $\hat{\varphi}_i \triangleq \frac{\gamma_i}{G_i}$,where k is a positive scalar parameter, $\gamma_i^{-1}(0)$ denotes the desirable set(i.e. the goal configuration) and $G_i^{-1}(0)$ the set that we want to avoid(i.e. collisions with other agents). A suitable choice is:

$$\gamma_i = (\gamma_{di} + f_i)^k \qquad (3)$$

where $\gamma_{di} = \| q_i - q_{di} \|^2$, is the squared metric of the current agent's configuration q_i from its destination q_{di}. The definition of the function f_i will be given later. Function G_i has as arguments the coordinates of all agents, i.e. $G_i = G_i(q)$, in order to express all possible collisions of agent i with the others. Hence the proposed DNF is

$$\varphi_i(q) = \frac{\gamma_{di} + f_i}{\left((\gamma_{di} + f_i)^k + G_i\right)^{1/k}} \qquad (4)$$

Figure 2 shows a contour plot of a DNF of an agent in an environment of 3 (other) agents denoted by A-i. The DNF is maximized on the boundary of the free space and minimized at the goal configuration. Using

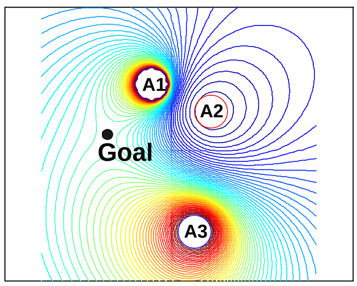

Fig. 2. Contour graph of a DNF as an agent moves amongst other agents

the notation $\tilde{q}_i \triangleq [q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N]^T$, the DNF can be rewritten as

$$\varphi_i = \varphi_i(q_i, \tilde{q}_i(t))$$

that is, the potential function in hand contains a *time-varying* element which corresponds to the movement in time of all the other agents apart from i. The time derivative of φ_i is given by $\dot{\varphi}_i = \frac{\partial \varphi_i}{\partial q_i}\dot{q}_i + \sum_{j \neq i} \frac{\partial \varphi_i}{\partial q_j}\dot{q}_j = \frac{\partial \varphi_i}{\partial q_i}\dot{q}_i + \frac{\partial \varphi_i}{\partial \tilde{q}_i}\dot{\tilde{q}}_i$. Since we consider moving agents we have $\dot{\tilde{q}}_i \neq 0$. This term is neglected whenever $\tilde{q}_i$ represents static obstacles as in Koditschek and Rimon (1990) because in this case obviously $\dot{\tilde{q}}_i = 0$. In this case it can not be neglected in the stability analysis of the system.

3.3 Control Strategy

The proposed feedback control strategy for agent i is

$$u_i = -K_i \frac{\partial \varphi_i}{\partial q_i} \tag{5}$$

where $K_i > 0$ a positive gain.

3.4 Construction of the G function

In the proposed decentralized control law, each agent has a different G_i which represents its relative position with all the other agents. As mentioned in section 3.2, the function G_i is minimized on the boundary of the free space of agent i. Mathematically, $G_i = 0$ whenever $\|q_i - q_j\|^2 - (r_i + r_j)^2 = 0$ for at least one $j \neq i$. The construction of the G_i function is held in such a way to ensure that the resulting φ_i satisfies the properties of a navigation function. This among others, guarantees that the boundary of the free space of each agent is repulsive with respect to the control law (5). Hence the collision avoidance specification $\|q_i - q_j\| > r_i + r_j, \forall j \neq i$ is satisfied for all i. In the sequel we review the construction of G_i for each agent i, which was introduced in Zavlanos and Kyriakopoulos (2003),Dimarogonas et al. (2003).

In contrast to the centralized case, in which a central authority has global knowledge of the positions and desired destinations of the whole team and plans a global G function accordingly, in the decentralized case, each member i of the team has its own G_i function, which encodes the different proximity situations with the rest. The main difference of the DNF's and the MRNF's in Loizou and Kyriakopoulos (2002)from the NF's introduced in Koditschek and Rimon (1990) lies in the structure of the function G. The collision scheme in Koditschek and Rimon (1990) involved a single moving point agent in an environment with static obstacles. A collision with more than one obstacle was impossible and G was simply the product of the distances of the agent from each obstacle. In our case however, this is inappropriate, as can be seen in the next figure. The control law of agent A should distinguish when agent A is in conflict with B, C, or B and C simultaneously. Each of these situations is a different possible *proximity situation* with respect to agent A.

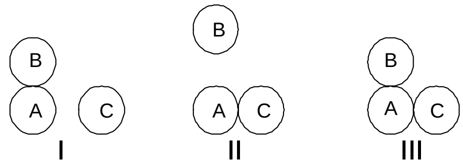

Fig. 3. Different Proximity Situations with respect to agent A

To encode all possible inter-agent proximity situations, the multi-agent team is associated with an (undirected) graph whose vertices are indexed by the team members. The following notions were firstly introduced in Loizou and Kyriakopoulos (2002):

Definition 2: *A* binary relation *with respect to an agent R is an edge between agent R and another agent.*

Definition 3: *A* relation *with respect to agent R is defined as a set of binary relations with respect to agent R.*

Definition 4: *The* relation level *is the number of binary relations in a relation with respect to agent R.*

Consider now a multi-agent scenario where we have five agents. We proceed to define the function G_R for a specific agent R. We denote by O_1, O_2, O_3, O_4 the remaining four agents in this scenario and by $(R_j)_l$ the jth relation of level-l with respect to agent R. We use the notation

$$(R_j)_l = \{\{R, A\}, \{R, B\}, \{R, C\}, \ldots\}$$

to denote the set of binary relations in a relation with respect to agent R, where $\{A, B, C, ...\}$ the set of agents that participate in the specific relation. With this terminology in hand, the level-3 relations with respect to agent R in the five-agent scenario are

$$(R_1)_3 = \{\{R, O_1\}, \{R, O_2\}, \{R, O_3\}\}$$

$$(R_2)_3 = \{\{R, O_1\}, \{R, O_2\}, \{R, O_4\}\}$$

$$(R_3)_3 = \{\{R, O_1\}, \{R, O_3\}, \{R, O_4\}\}$$

$$(R_4)_3 = \{\{R, O_2\}, \{R, O_3\}, \{R, O_4\}\}$$

where we have indexed relations $1, ..., 4$ in lexicographic order with respect to the participating agents. Please note that the indexing of the relations of the same level can be done arbitrarily and should not lead to confusion.

The complementary set $(R_j^C)_l$ of relation j is the set that contains all the relations of the same level apart from the specific relation j. In our example, the set $\left(R_1^C\right)_3$ is $\left(R_1^C\right)_3 = \{(R_2)_3, (R_3)_3, (R_4)_3\}$.

The "Proximity Function"between agents i and j is

$$\beta_{\{i,j\}} = \|q_i - q_j\|^2 - (r_i + r_j)^2 \tag{6}$$

We also use the notation $\beta_{\{i,j\}} \equiv \beta_{ij}$ for simplicity.

The purpose of the G function is to act as an indicator of a specific relation. This is achieved by associating a specific metric to each relation. This metric, which is called *Relation Verification Function*(RVF), is defined in the sequel. The key property of RVF's is that the RVF of one and only one relation can tend to zero at each time instant, namely, the RVF of the relation that holds at the highest level. A relation holds when the Proximity Functions of all its binary relations tend to zero.

A "Relation Proximity Function" (RPF) provides a measure of the distance between agent i and the other agents involved in the relation. Each relation has its own RPF. Let R_k denote the k^{th} relation of level l. The RPF of this relation is given by:

$$(b_{R_k})_l = \sum_{j\in(R_k)_l} \beta_{\{R,j\}} \tag{7}$$

where $j \in (R_k)_l$ denotes the agents that participate in the specific relation of agent R. In the proofs, we also use the simplified notation $b_r = \sum_{j\in P_r} \beta_{ij}$ for simplicity, where r denotes a relation and P_r denotes the set of agents participating in the specific relation wrt agent i. Continuing with our previous example, we have

$$(b_{R_1})_3 = \sum_{m\in(R_1)_3} \beta_{\{R,m\}} = \beta_{\{R,O_1\}} + \beta_{\{R,O_2\}} + \beta_{\{R,O_3\}}$$

A "Relation Verification Function" (RVF) is defined by:

$$(g_{R_k})_l = (b_{R_k})_l + \frac{\lambda(b_{R_k})_l}{(b_{R_k})_l + (B_{R_k^C})_l^{1/h}} \tag{8}$$

where $\lambda, h > 0$ and $(B_{R_k^C})_l = \prod_{m\in(R_k^C)_l}(b_m)_l$ where as previously defined, $(R_k^C)_l$ is the complementary set of relations of level-l, i.e. all the other relations with respect to agent i that have the same number of binary relations with the relation R_k. Continuing with the previous example we could compute, for instance, $\left(B_{R_1^C}\right)_3 = (b_{R_2})_3 \cdot (b_{R_3})_3 \cdot (b_{R_4})_3$, which refers to level-3 relations of agent R.

For simplicity we also use the notation $(B_{R_k^C})_l \equiv \tilde{b}_i = \prod_{m\in(R_k^C)_l} b_m$. The RVF can be written as $g_i = b_i + \frac{\lambda b_i}{b_i+\tilde{b}_i^{1/h}}$ It is obvious that for the highest level $l = n-1$ only one relation is possible so that $(R_k^C)_{n-1} = \emptyset$ and $(g_{R_k})_l = (b_{R_k})_l$ for $l = n-1$. The basic property that we demand from RVF is that it assumes the value of zero if a relation holds, while no other relations of the same or other higher levels hold. In other words it should indicate which of all possible relations holds. We have he following limits of RVF (using the simplified notation): (a) $\lim\limits_{b_i\to 0}\lim\limits_{\tilde{b}_i\to 0} g_i\left(b_i,\tilde{b}_i\right) = \lambda$ (b) $\lim\limits_{\substack{b_i\to 0\\ \tilde{b}_i\neq 0}} g_i\left(b_i,\tilde{b}_i\right) = 0$. These limits guarantee that RVF will behave in the way we want it to, as an indicator of a specific collision.

The function G_i is now defined as

$$G_i = \prod_{l=1}^{n_L^i}\prod_{j=1}^{n_{R_l}^i}(g_{R_j})_l \tag{9}$$

where n_L^i the number of levels and $n_{R_l}^i$ the number of relations in level-l with respect to agent i. Hence G_i is the product of the RVF's of all relations wrt i.

3.5 An example

As an example, we will present steps to construct the function G with respect to a specific agent in a team of 4 agents indexed 1 through 4. We construct the function G_1 wrt agent 1. We begin by defining the RPF's (eq.(7)) in every level in the following table::

Relation	Level 1	Level 2	Level 3
1	$(b_1)_1 = \beta_{12}$	$(b_1)_2 = \beta_{12} + \beta_{13}$	$(b_1)_3 = \beta_{12}+$ $+\beta_{13} + \beta_{14}$
2	$(b_2)_1 = \beta_{13}$	$(b_2)_2 = \beta_{12} + \beta_{14}$	-
3	$(b_3)_1 = \beta_{14}$	$(b_3)_2 = \beta_{13} + \beta_{14}$	-

It is now easy to calculate the RVF of each relation based on equation (8). For example, for the second relation of level 2, the complement (term $(B_{R_k^C})_l$ in eq.(8)) is given by $(B_{2^C})_2 = (b_1)_2 \cdot (b_3)_2$ and substituting in (8), we have

$$(g_2)_2 = (b_2)_2 + \frac{\lambda\,(b_2)_2}{(b_2)_2 + ((b_1)_2\cdot(b_3)_2)^{1/h}}$$

The function G_1 is then calculated as the product of the Relation Verification Functions of all relations.

3.6 The f function

The key difference of the decentralized method with respect to the centralized is that the control law of each agent ignores the destinations of the others. Using $\varphi_i = \frac{\gamma_{di}}{\left((\gamma_{di})^k+G_i\right)^{1/k}}$ as a navigation function for agent i, there is no potential for i to cooperate in a possible collision scheme when its initial condition coincides with its final destination. In order to overcome this limitation,we add a function f_i to γ_i so that the cost function φ_i attains positive values in proximity situations even when i has already reached its destination. A preliminary definition for this function was given in Dimarogonas et al.

(2003), Zavlanos and Kyriakopoulos (2003).This is modified here to ensure that the destination point is a non-degenerate local minimum of φ_i. We define f_i by:

$$f_i(G_i) = \begin{cases} a_0 + \sum_{j=1}^{3} a_j G_i^j, \ G_i \le X \\ 0, \ G_i > X \end{cases} \tag{10}$$

where $X, Y = f_i(0) > 0$ are positive parameters the role of which will be made clear in the following. The parameters a_j are evaluated so that f_i is maximized when $G_i \to 0$ and minimized when $G_i = X$. We also require that f_i is continuously differentiable at X. Thus we have: $a_0 = Y, a_1 = 0, a_2 = \frac{-3Y}{X^2}, a_3 = \frac{2Y}{X^3}$. We require that $Y \le \frac{\Theta_1}{k}$ where Θ_1 is an arbitrarily large positive gain. This will help in obtaining a lower bound of k analytically in the stability analysis that follows. The parameter X serves as a sensing parameter that activates the f_i function whenever possible collisions are bound to occur. The only requirement we have for X is that it must be small enough to guarantee that f_i vanishes whenever the system has reached its equilibrium, i.e. when everyone has reached its destination. In mathematical terms:

$$X < G_i \left(q_{d1}, \ldots, q_{dN}\right) \forall i \tag{11}$$

That's the minimum requirement we have regarding knowledge of the destinations of the team. The resulting navigation function is no longer analytic but merely C^1 at $G_i = X$. However, by choosing X large enough, the resulting function is analytic in a neighborhood of the boundary of the free space so that the characterization of its critical points can be made by the evaluation of its Hessian. Hence, the parameter X must be chosen small enough in order to satisfy (11) but large enough to include the region described above.

Equation (11) is a feasibility requirement for the system equilibrium. The function G_i of each agent is strictly positive whenever all agents have reached their destinations, since by definition, the agents do not collide when they are at their destinations. Clearly, this makes the solution of the problem feasible. Hence, the parameter X can be chosen small enough in order to satisfy (11), but each agent does not have to know the destinations of the others. In practical situations, X can be interpreted as a fixed parameter of the controller, and can be used in situations where the final destinations of agents satisfy (11). This equation also guarantees that the function f_i vanishes whenever each agent has reached its destination. This guarantees that q_{di} is a non-degenerate local minimum of φ_i, as can be seen from the statement and proof of Lemma 2 in section 3.7.

3.7 Proof of Correctness

Let $\varepsilon > 0$. Define $B_{j,l}^i(\varepsilon) \equiv \{q : 0 < (g_{R_j}^i)_l < \varepsilon\}$. Following Koditschek and Rimon (1990),Loizou and Kyriakopoulos (2002) we discriminate the following topologies for the function φ_i:

(1) The destination point: q_{di}
(2) The free space boundary: $\partial F(q) = G_i^{-1}(\delta), \delta \to 0$
(3) The set near collisions: $F_0(\varepsilon) = \bigcup_{l=1}^{n_L^i} \bigcup_{j=1}^{n_{R,l}^i} B_{j,l}^i(\varepsilon) - \{q_{di}\}$
(4) The set away from collisions: $F_1(\varepsilon) = F - (\{q_{di}\} \cup \partial F \cup F_0(\varepsilon))$

Theorem 1 allows us to derive results for the function φ_i by examining the simpler function $\hat{\varphi}_i(q) = \frac{\gamma_i}{G_i}$:

Theorem 1 Koditschek and Rimon (1990):*Let I_1, I_2 be intervals, $\hat{\varphi} : F \to I_1$ and $\sigma : I_1 \to I_2$ be analytic. Define the composition $\varphi : F \to I_2$ to be $\varphi = \sigma \circ \hat{\varphi}$. If σ is monotonically increasing on I_1, then the set of critical points of φ and $\hat{\varphi}$ coincide and the (Morse) index of each critical point is identical.*

A key point in the discrimination between centralized and decentralized navigation functions is that the latter contain a time-varying part which depends on the movement of the other agents. Using the same procedure as in Loizou and Kyriakopoulos (2002),Koditschek and Rimon (1990) we first prove that the construction of each φ_i guarantees collision avoidance:

Proposition 1: *For each fixed $\tilde{q}_i$, the function $\varphi_i(q_i, \cdot)$ is a navigation function if the parameters h, k assume values bigger than a finite lower bound.*

Proof Sketch: For the complete proof see Dimarogonas et al. (2004). The set of critical points of φ_i is defined as $C_{\varphi_i} = \{q : \partial\varphi_i/\partial q_i = 0\}$. A critical point is non-degenerate if $\partial^2\varphi_i/\partial^2 q_i$ has full rank at that point. The statement of the proposition is guaranteed by the following Lemmas:

Lemma 2: *If the workspace is valid, the destination point q_{di} is a non-degenerate local minimum of φ_i.*

Lemma 3: *All critical points of φ_i are in the interior of the free space.*

Lemma 4: *For every $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that if $k > N(\varepsilon)$ then there are no critical points of $\hat{\varphi}_i$ in $F_1(\varepsilon)$.*

Lemma 5: *There exists an $\varepsilon_0 > 0$ such that $\hat{\varphi}_i$ has no local minimum in $F_0(\varepsilon)$, as long as $\varepsilon < \varepsilon_0$.*

Lemma 6: *There exist $\varepsilon_1 > 0$ and $h_1 > 0$, such that the critical points of $\hat{\varphi}_i$ are non-degenerate as long as $\varepsilon < \varepsilon_1$ and $h > h_1$.*

The complete proofs of the Lemmas can be found in Dimarogonas et al. (2004). A sketch of the proofs is provided in the Appendix. Lemmas 2-5 guarantee the polarity of the proposed DNF, whilst Lemma 6 guarantees the non-degeneracy of the critical points. By choosing k, h that satisfy the above Lemmas, the statement of Proposition 1 is proved.

This however does not guarantee global convergence of the system state to the destination configuration. This is achieved by using a Lyapunov function for the *whole* system which is *time invariant* that is a function that depends on the positions of all the agents. The candidate Lyapunov function that we use is simply the sum of the DNF's of all agents. Specifically we prove the following:

Proposition 2: *The time-derivative of $\varphi = \sum_{i=1}^{N} \varphi_i$ is negative definite across the trajectories of the system up to a set of initial conditions of measure zero if the parameters h, k assume values bigger than a finite lower bound.* A sketch of the proof is provided in the appendix while the complete proof can be found in Dimarogonas et al. (2004).

4 Simulations

To demonstrate the navigation properties of our decentralized approach, we present two simulations of four holonomic agents that have to navigate from an initial to a final configuration, avoiding collision with each other. Each agent has no knowledge of the desired destinations of the other agents. In the next two figures A-i,T-i denote the initial position and desired destination of agent i respectively. The chosen configurations constitute non-trivial setups since the straight-line paths connecting initial and final positions of each agent are obstructed by other agents. Screenshots I-VI in each simulation show the evolution in time of the four member team. Each agent is navigating under the control law (5) which is updated continuously based on the knowledge of the other agents' positions. The parameters in both simulations are $k = 80, r_1 = r_2 = r_3 = r_4 = 0.05$ cm, $X = 0.001, Y = 0.1$, $\lambda = 1, h = 5$.

In the first simulation agent 4 is forced to participate in the conflict resolution procedure. Although its initial position coincides with its desired destination the inclusion of the f function in the navigation function forces the agent to cooperate with the rest in the collision avoidance maneuver. The collision avoidance specification is also visible, since the discs corresponding to the different agents never overlap. The convergence specification is satisfied since all agent converge to their destinations in the last screenshot.

The same comments hold for the second simulation. The initial positions of agents 2,3,4 coincide with their desired destinations. The f function forces these 3 robots to cooperate in order to let robot 1 reach its target.

5 Conclusions

In this paper, a methodology for multi-agent navigation is presented. The methodology extends the centralized agent navigation established in Loizou and Kyriakopoulos (2002) to a decentralized approach to the problem. As in Loizou and Kyriakopoulos (2002), the agent potentials are formed by appropriately constructed agent proximity potentials, which capture all the possible multi agent proximity situations. The great advantages of the

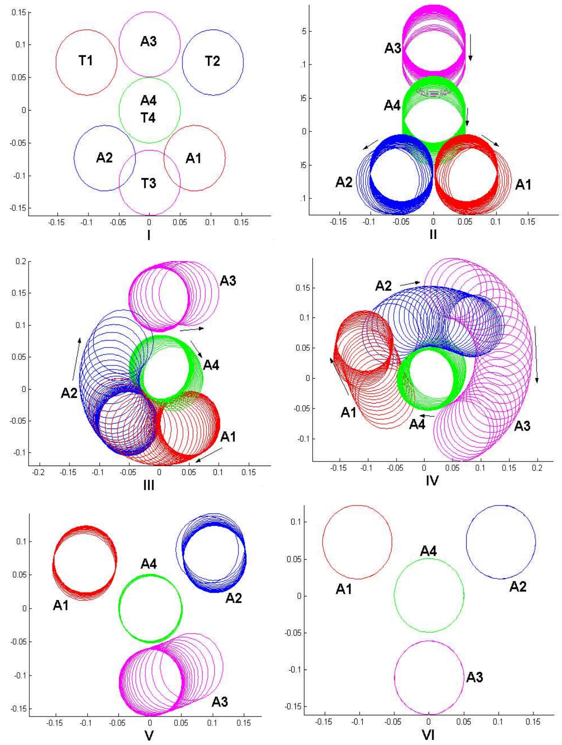

Fig. 4. Simulation A

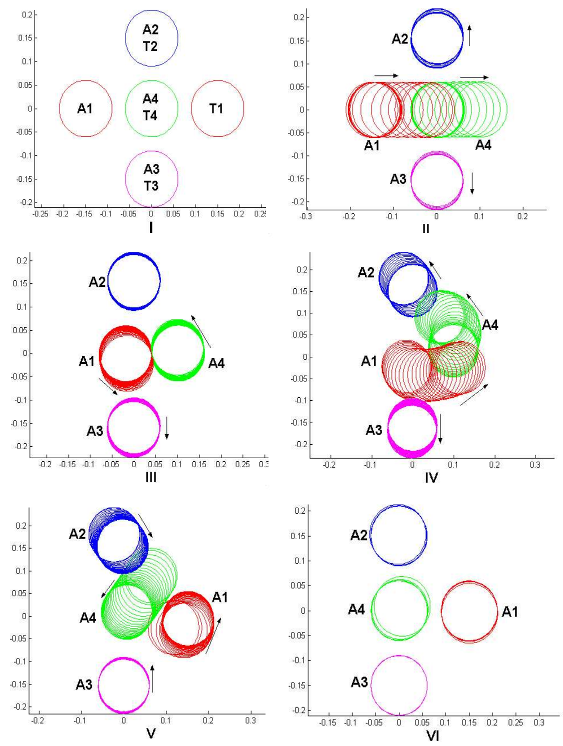

Fig. 5. Simulation B

method are (i) its relatively low complexity with respect to the number of agents, compared to centralized approaches to the problem and (ii) its application to non-point agents. The effectiveness of the methodology is verified through non-trivial computer simulations.

Current research directions are towards applying the methodology to the cases where each agent has limited knowledge of the positions of the others and where there is some form of uncertainty in the agent movement. Extensions of the methodology to more general topologies are also pursued by searching for the appropriate diffeomorphic mappings from the sphere world to such topologies. Moreover, although the complexity of the proposed scheme is lower than the centralized case of Loizou and Kyriakopoulos (2002), it grows exponentially in the number of agents, since all possible relations are taken into account in the computation of the control law of each agent at each time instant. Methods to further reduce the complexity of the decentralized scheme are currently are under investigation. Finally, the convergence of the multiagent system is guaranteed provided that each agent is controlled by the proposed scheme. Hence no static obstacles are considered. This can be seen by the inclusion of the f_i function in the navigation function that forces each agent to cooperate with the rest in the conflict avoidance maneuver even it has already reached its destination. A static obstacle could be considered if an agents initial condition coincided with its final destination and we set $f_i = 0$ for this agent. In this case however, the stability analysis of Proposition 2 would no longer be valid and deadlocks could occur due to the lack of cooperation on behalf of this specific agent. Mathematically, in this case the eigenvalues of matrix M in the proof of Proposition 2 would no longer be proven to be positive definite, but positive semi-definite. Hence only stability, and not asymptotic stability of the overall system would be guaranteed. In many situations of course, the system converges even in the presence of static obstacles. This can be exploited in practice.

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A Proof Sketches

Before proceeding, we introduce some simplifications concerning notation. In the proof sketches of Lemmas 2-6 we denote by q instead of q_i the current agent configuration, by q_d instead of q_{di} its goal configuration, by G instead of G_i its "G" function, by φ instead of φ_i its DNF and by q_j the configurations of the other agents. In the proof sketches of Lemmas 2-6 we also use the notation $\frac{\partial}{\partial q_i}(\cdot) \triangleq \nabla(\cdot)$ and $\frac{\partial^2}{\partial q_i^2}(\cdot) \triangleq \nabla^2(\cdot)$.

A.1 Proof of Lemma 2

At steady state, the function f vanishes due to the constraint $X < G_i(q_{d1}, \ldots, q_{dN})\ \forall i$. Taking the gradient of the definition of φ we have:

$$\nabla\varphi(q_d) = \frac{\left(\gamma_d^k + G\right)^{1/k} \nabla\gamma_d - \gamma_d \nabla\left(\gamma_d^k + G\right)^{1/k}}{\left(\gamma_d^k + G\right)^{2/k}} = 0$$

since both γ_d and $\nabla(\gamma_d)$ vanish by definition at q_d. The Hessian at q_d is

$$\nabla^2\varphi(q_d) = \frac{\left(\gamma_d^k+G\right)^{1/k}\nabla^2\gamma_d - \gamma_d\nabla^2\left(\gamma_d^k+G\right)^{1/k}}{\left(\gamma_d^k+G\right)^{2/k}} = \\ = G^{-1/k}\cdot\nabla^2(\gamma_d) = 2G^{-1/k} I$$

which is non-degenerate.◇

A.2 Proof of Lemma 3

Let q_0 be a point in ϑF and suppose that $(g_{R_a})_b(q_0) = 0$ for some relation a of level b. If the workspace is valid:$\left(g_{R_j}\right)_l(q_0) > 0$ for any level-l and $j \neq a$ since only one RVF can hold at a time. Using the terminology previously defined, and setting $g_i \equiv (g_{R_a})_b(q_0) = 0$, it follows that $\bar{g}_i > 0$ where $\bar{g}_i = G/g_i$. Taking the gradient of φ at q_0 , we obtain:

$$\nabla\varphi(q_0) = \left.\frac{\left((\gamma_d+f)^k+G\right)^{1/k}\nabla(\gamma_d+f) - (\gamma_d+f)\nabla\left((\gamma_d+f)^k+G\right)^{1/k}}{\left((\gamma_d+f)^k+G\right)^{2/k}}\right|_{q_0} \\ \overset{G(q_0)=0}{=} \frac{(\gamma_d+f)\nabla(\gamma_d+f) - (\gamma_d+f)\nabla(\gamma_d+f) - \frac{1}{k}(\gamma_d+f)^{2-k}\nabla G}{(\gamma_d+f)^2} = \\ = -\frac{1}{k}(\gamma_d+f)^{-k}\nabla G = -\frac{1}{k}(\gamma_d+f)^{-k}\bar{g}_i\nabla g_i \neq 0$$

Since ϑF is the set where $G = 0$, then $\nabla\varphi|_{G=0} = \nabla\varphi(q_0) \sim -\nabla G$ is normal to the surface and $-\nabla\varphi$ points towards the interior of the free space.

A.3 Proof of Lemma 4

At a critical point $q \in C_{\hat{\varphi}} \bigcap F_1(\varepsilon)$ we have:

$$\hat{\varphi} = \frac{\gamma}{G} \Rightarrow \nabla\hat{\varphi} = \frac{1}{G^2}(G\nabla\gamma - \gamma\nabla G) \\ \overset{\nabla\hat{\varphi}=0}{\Rightarrow} G\nabla\gamma = \gamma\nabla G \Rightarrow G\nabla(\gamma_d+f)^k = (\gamma_d+f)^k\nabla G \\ \Rightarrow kG\nabla(\gamma_d+f) = (\gamma_d+f)\nabla G$$

Thus, $kG\|\nabla(\gamma_d+f)\| = (\gamma_d+f)\|\nabla G\|$. A sufficient condition for the above equality not to hold is given by:

$$\frac{(\gamma_d+f)\|\nabla G\|}{G\|\nabla(\gamma_d+f)\|} < k, \forall q \in F_1(\varepsilon)$$

Since $\left(g_{R_j}\right)_l \geq \varepsilon$, an upper bound for the left side is:

$$\frac{(\gamma_d+f)\|\nabla G\|}{G\|\nabla(\gamma_d+f)\|} < \frac{(\gamma_d+f)}{\|\nabla(\gamma_d+f)\|}\cdot\sum_{l=1}^{n_L}\sum_{j=1}^{n_{R,l}}\frac{\overline{G}_{j,l}}{G}\left\|\nabla\left(g_{R_j}\right)_l\right\| < \\ < \frac{1}{\varepsilon}\cdot\frac{\left(\max\limits_W\{\gamma_d\}+\max\limits_W\{f\}\right)\cdot\sum\limits_{l=1}^{n_L}\sum\limits_{j=1}^{n_{R,l}}\max\limits_W\left\|\nabla\left(g_{R_j}\right)_l\right\|}{\min\limits_W\|\nabla(\gamma_d+f)\|} = \\ = \frac{1}{\varepsilon}\cdot\frac{\left(\max\limits_W\{\gamma_d\}+Y\right)\cdot\sum\limits_{l=1}^{n_L}\sum\limits_{j=1}^{n_{R,l}}\max\limits_W\left\|\nabla\left(g_{R_j}\right)_l\right\|}{\min\limits_W\|\nabla(\gamma_d+f)\|} \quad \diamond$$

A.4 Proof of Lemma 5

If $q \in F_0(\varepsilon) \cap C_{\hat{\varphi}}$, where $C_{\hat{\varphi}}$ is the set of critical points, then $q \in B_i^L(\varepsilon)$ for at least one set $\{L, i\}, i \in \{1...n_{R,L}\}, L \in \{1...n_L\}$, with n_L the number of levels and $n_{R,L}$ the number of relations in level L. We will use a unit vector as a test direction to demonstrate that $\left(\nabla^2\hat{\varphi}\right)(q)$ has at least one negative eigenvalue. At a critical point,$(\nabla\hat{\varphi})(q) = \frac{kG(\gamma_d+f)^{k-1}\nabla(\gamma_d+f)-(\gamma_d+f)^k\nabla G}{G^2} = 0$. Hence, $k\cdot G\cdot\nabla(\gamma_d+f) = (\gamma_d+f)\cdot\nabla G \Rightarrow$

$$(kG)^2\nabla(\gamma_d+f)\nabla(\gamma_d+f)^T = (\gamma_d+f)^2\nabla G\nabla G^T \quad (A.1)$$

The Hessian at a critical point is:

$$\left(\nabla^2\hat{\varphi}\right)(q) = \frac{1}{G^2}\left(G\cdot\nabla^2(\gamma_d+f)^k - (\gamma_d+f)^k\cdot\nabla^2 G\right) = \\ \frac{(\gamma_d+f)^{k-2}}{G^2}\cdot\left\{kG\begin{bmatrix}(\gamma_d+f)\nabla^2(\gamma_d+f)+ \\ (k-1)\nabla(\gamma_d+f)\nabla(\gamma_d+f)^T\end{bmatrix} - \right. \\ \left. -(\gamma_d+f)^2\nabla^2 G\right\}$$

Substituting (A.1) in the previous equation:

$$\left(\nabla^2\hat{\varphi}\right)(q)=\frac{(\gamma_d+f)^{k-1}}{G^2}\left\{\begin{array}{l}kG\nabla^2(\gamma_d+f)+\\+\left(1-\frac{1}{k}\right)\frac{(\gamma_d+f)}{G}\nabla G\nabla G^T-\\-(\gamma_d+f)\nabla^2 G\end{array}\right\}$$

We choose the test vector (unit magnitude) to be: $\hat{u}=\frac{\nabla b_i(q_c)^{\perp}}{\left\|\nabla b_i(q_c)^{\perp}\right\|}$. By its definition $\hat{u}$ is orthogonal to ∇b_i at a critical point q_c, and so the following properties hold: $\hat{u}^T\cdot\nabla b_i=0$ and $\nabla b_i^T\cdot\hat{u}=0$. With $\nabla^2(\gamma_d+f)=2\cdot\mathrm{I}+\nabla^2 f$, we form the quadratic form:

$$\frac{G^2}{(\gamma_d+f)^{k-1}}\hat{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\hat{u}=2kG+kG\hat{u}^T\nabla^2 f\hat{u}+\left(1-\frac{1}{k}\right)\frac{(\gamma_d+f)}{G}\hat{u}^T\nabla G\nabla G^T\hat{u}-(\gamma_d+f)\hat{u}^T\nabla^2 G\hat{u}$$

After many nontrivial calculation we get

$$\begin{array}{l}\frac{G^2}{(\gamma_d+f)^{k-1}}\hat{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\hat{u}=\\\bar{g}_i c_i\left(1+\frac{a_0}{\gamma_d}\right)\left(\frac{1}{2}\nabla b_i^T\nabla\gamma_d-\upsilon_i\gamma_d\right)\\+g_i\left\{\begin{array}{l}k\bar{g}_i\hat{u}^T\nabla^2 f\hat{u}+(\gamma_d+f)\eta_i-(\gamma_d+f)\psi_i+\frac{z_2}{2\gamma_d}\\-\upsilon_i\bar{g}_i c_i\left(\sum\limits_{j=1}^{3}a_j g_i^{j-1}\bar{g}_i^j\right)-\zeta_i\end{array}\right\}\end{array}\qquad\text{(A.2)}$$

where $c_i=1+\frac{\lambda}{b_i+\tilde{b}_i^{1/h}}$, $\upsilon_i=2\cdot l$, l the relation level,

$$\eta_i=\left(1-\frac{1}{k}\right)\left[\begin{array}{l}\frac{\hat{u}^T\nabla\bar{g}_i\nabla\bar{g}_i^T\hat{u}}{\bar{g}_i}-2\lambda\frac{\hat{u}^T\nabla\bar{g}_i\left(\nabla\tilde{b}_i^{1/h}\right)^T\hat{u}}{c_i\left(b_i+\tilde{b}_i^{1/h}\right)^2}+\\+\lambda^2\bar{g}_i\frac{\hat{u}^T\nabla\tilde{b}_i^{1/h}\left(\nabla\tilde{b}_i^{1/h}\right)^T\hat{u}}{c_i^2\left(b_i+\tilde{b}_i^{1/h}\right)^4}\end{array}\right]$$

$$\psi_i=\hat{u}^T\cdot\nabla^2\bar{g}_i\cdot\hat{u}+\frac{\bar{g}_i}{c_i}\cdot\hat{u}^T\cdot B_i\cdot\hat{u}-\\-2\frac{\lambda}{c_i\left(b_i+\tilde{b}_i^{1/h}\right)^2}\cdot\hat{u}^T\cdot\nabla\tilde{b}_i^{1/h}\cdot\nabla\bar{g}_i\cdot\hat{u}$$

$$B_i=\lambda\left[\begin{array}{l}2\frac{\left(\nabla b_i+\nabla\tilde{b}_i^{1/h}\right)\left(\nabla b_i+\nabla\tilde{b}_i^{1/h}\right)^T}{\left(b_i+\tilde{b}_i^{1/h}\right)^3}-\\-\frac{\left(\nabla^2 b_i+\nabla^2\tilde{b}_i^{1/h}\right)}{\left(b_i+\tilde{b}_i^{1/h}\right)^2}\end{array}\right]$$

$$z_2(g_i,\bar{g}_i,\nabla g_i,\nabla\bar{g}_i)=\gamma_d\nabla\bar{g}_i^T\nabla\gamma_d+f\nabla\bar{g}_i^T\nabla\gamma_d+\ldots\\-k\bar{g}_i\left(2\nabla\gamma_d^T\cdot\nabla f-\nabla f^T\cdot\nabla f\right)$$

and $\zeta_i=\frac{\lambda\bar{g}_i}{2c_i\left(b+\tilde{b}^{1/h}\right)^2}\left(\nabla b+\nabla\tilde{b}^{1/h}\right)^T\cdot\nabla\gamma_d$. Setting $\tilde{\mu}_i=\left(1+\frac{a_0}{\gamma_d}\right)\cdot\mu_i$ where $\mu_i=\frac{1}{2}\nabla b_i^T\nabla\gamma_d-\upsilon_i\cdot\gamma_d$ equation (A.2) becomes:

$$\frac{G^2}{(\gamma_d+f)^{k-1}}\hat{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\hat{u}=\bar{g}_i\cdot c_i\cdot\tilde{\mu}_i+\\g_i\left\{\begin{array}{l}k\cdot\bar{g}_i\cdot\hat{u}^T\cdot\nabla^2 f\cdot\hat{u}+(\gamma_d+f)\cdot\eta_i-(\gamma_d+f)\cdot\psi_i\\+\frac{z_2}{2\gamma_d}-\upsilon_i\bar{g}_i c_i\left(\sum\limits_{j=1}^{3}a_j g_i^{j-1}\bar{g}_i^j\right)-\zeta_i\end{array}\right\}$$

The second term is proportional to g_i and can be made arbitrarily small by a suitable choice of ε but can still be positive, so the first term should be strictly negative. From Lemma 7 in Dimarogonas et al. (2004), we have:

$$\max_{q\in F_0}\{\mu_i\}=\\=\frac{2}{l}\left(\begin{array}{l}\frac{1}{l}\sqrt{\left\|\sum q_j\right\|^2-l\sum\left\|q_j\right\|^2+l\left(\sum(r+r_j)^2+\varepsilon\right)}\\-\left\|lq_d-\sum q_j\right\|\end{array}\right)\cdot\\\cdot\left\|lq_d-\sum q_j\right\|$$

For ε small enough, $\max\limits_{q\in F_0}\{\mu_i\}$ is negative. Moreover, the term $\left(1+\frac{a_0}{\gamma_d}\right)$ is always greater than one, since we have assumed that $a_0>0$, and $\gamma_d>0$ for $q\in F_0(\varepsilon)$. Thus for ε small enough, $\tilde{\mu}_i$ is also negative. So, for $\tilde{\mu}_i$, it is sufficient to ensure: $\frac{1}{l}\cdot\sqrt{\left\|\sum q_j\right\|^2-l\cdot\sum\left\|q_j\right\|^2+l\cdot\left(\sum(r+r_j)^2+\varepsilon\right)}<$ $<\left\|l\cdot q_d-\sum q_j\right\|\Rightarrow\varepsilon<l\cdot\left\|l\cdot q_d-\sum q_j\right\|^2+\sum\left\|q_j\right\|^2-$ $-\frac{1}{l}\cdot\left\|\sum q_j\right\|^2-\sum(r+r_j)^2\equiv\varepsilon_0$

Another constraint arises from the fact that $\varepsilon>0$. So for a valid workspace it will be: $l\cdot\left\|l\cdot q_d-\sum q_j\right\|^2+\sum\left\|q_j\right\|^2-\frac{1}{l}\cdot\left\|\sum q_j\right\|^2>\sum(r+r_j)^2$ ♢

A.5 Proof of Lemma 6

From the proof of Lemma 5, we have at a critical point

$$\frac{G^2}{(\gamma_d+f)^{k-1}}\left(\nabla^2\hat{\varphi}\right)=kG\nabla^2(\gamma_d+f)+\\\left(1-\frac{1}{k}\right)\frac{\gamma_d+f}{G}\nabla G\nabla G^T-(\gamma_d+f)\nabla^2 G$$

We also have $\nabla f=\left(\underbrace{\sum\limits_{j=1}^{3}ja_jG_i^{j-1}}_{\sigma(G)}\right)\nabla G$ and $\nabla^2 f=\sigma\nabla^2 G+\sigma^*\nabla G\nabla G^T$, $\sigma^*=\sum\limits_{j=2}^{3}j(j-1)a_jG^{j-2}$. Hence

$$kG\nabla(\gamma_d+f)=(\gamma_d+f)\nabla G\Rightarrow\ldots\\\Rightarrow G\nabla\gamma_d=\left\{\underbrace{\frac{\gamma_d+f}{k}-G\sigma(G)}_{-\sigma_i}\right\}\nabla G$$

Taking the magnitude from both sides we have $2kG=\frac{k|\sigma_i|^2}{2G\gamma_d}\left\|\nabla G\right\|^2$. Choosing $\tilde{u}=\widehat{\nabla b_i}$ as a test direction and after some manipulation we have

$$\frac{G^2}{k(\gamma_d+f)^{k-1}}\tilde{u}^T\left(\nabla^2\hat{\varphi}\right)\tilde{u}=\underbrace{\frac{|\sigma_i|^2}{2G\gamma_d}\left\|\nabla G\right\|^2}_{L}+\\\underbrace{\xi\tilde{u}^T\nabla G\nabla G^T\tilde{u}}_{M}+\underbrace{\sigma_i\tilde{u}^T\nabla^2 G\tilde{u}}_{N}$$

where

$$\xi = \frac{k-1}{k} \cdot \frac{\gamma_d + Y}{kG} + \sum_{j=2}^{3} \left\{ j(j-1) + \frac{k-1}{k^2} \right\} a_j G^{j-1}$$

After some manipulation, we have

$$L + M + N \geq \tfrac{|\sigma_i|^2}{2G\gamma_d} \left\{ \begin{array}{l} g_i^2 \left\|\nabla \bar{g}_i\right\|^2 + \bar{g}_i^2 \left\|\nabla g_i\right\|^2 - \\ 2G \left\|\nabla \bar{g}_i\right\| \left\|\nabla g_i - 2\left(\tilde{u}^T \nabla g_i\right)\tilde{u}\right\| \end{array} \right\}$$
$$+2\left(\tfrac{|\sigma_i|^2}{\gamma_d} + \xi G + \sigma_i\right)\left(\tilde{u}^T \nabla g_i\right)\left(\nabla \bar{g}_i \tilde{u}\right)$$
$$+\xi \bar{g}_i^2 \left(\tilde{u}^T \nabla g_i\right)^2 + \sigma_i \tilde{u}^T \left(g_i \nabla^2 \bar{g}_i + \bar{g}_i \nabla^2 g_i\right) u$$

But $\left\|\nabla g_i - 2\left(\tilde{u}^T \nabla g_i\right)\tilde{u}\right\|^2 = \left\|\nabla g_i\right\|^2$ so that

$$g_i^2 \left\|\nabla \bar{g}_i\right\|^2 + \bar{g}_i^2 \left\|\nabla g_i\right\|^2 -$$
$$2G \left\|\nabla \bar{g}_i\right\| \left\|\nabla g_i - 2\left(\tilde{u}^T \nabla g_i\right)\tilde{u}\right\| = \left(g_i \left\|\nabla \bar{g}_i\right\| - \bar{g}_i \left\|\nabla g_i\right\|\right)^2$$

so that

$$L + M + N \geq 2\left(\tfrac{|\sigma_i|^2}{\gamma_d} + \xi G + \sigma_i\right)\left(\tilde{u}^T \nabla g_i\right)\left(\nabla \bar{g}_i \tilde{u}\right)$$
$$+\xi \bar{g}_i^2 \left(\tilde{u}^T \nabla g_i\right)^2 + \sigma_i \tilde{u}^T \left(g_i \nabla^2 \bar{g}_i + \bar{g}_i \nabla^2 g_i\right) u$$

It is shown in Dimarogonas et al. (2004) that the second term, which is strictly positive, dominates the third and the first term for sufficiently small ε.

A.6 Proof of Proposition 2

In the proof sketch of Proposition 2, the terms $\nabla(\cdot)$, $\nabla^2(\cdot)$ have their usual meaning and refer to the whole state space and not a single agent, namely $\nabla(\cdot) \triangleq \left[\frac{\partial}{\partial q_1}(\cdot), \ldots, \frac{\partial}{\partial q_N}(\cdot)\right]^T$ and $\nabla^2(\cdot) \triangleq \left[\frac{\partial^2}{\partial q_i \partial q_j}(\cdot)\right]$.

The Proximity function between agents i and j is:

$$\beta_{ij}(q) = \left\|q_i - q_j\right\|^2 - (r_i + r_j)^2 = q^T D_{ij} q - (r_i + r_j)^2$$

where the $2N \times 2N$ matrix D_{ij} is defined in Loizou and Kyriakopoulos (2002):

$$D_{ij} = \begin{bmatrix} \multicolumn{5}{c}{O_{2(i-1)\times 2N}} \\ O_{2\times 2(i-1)} & I_{2\times 2} & O_{2\times 2(j-i-1)} & -I_{2\times 2} & O_{2\times 2(N-j)} \\ \multicolumn{5}{c}{O_{2(j-i-1)\times 2N}} \\ O_{2\times 2(i-1)} & -I_{2\times 2} & O_{2\times 2(j-i-1)} & I_{2\times 2} & O_{2\times 2(N-j)} \\ \multicolumn{5}{c}{O_{2(N-j)\times 2N}} \end{bmatrix}$$

We can also write $b_r^i = q^T P_r^i q - \sum_{j \in P_r} (r_i + r_j)^2$,where $P_r^i = \sum_{j \in P_r} D_{ij}$, and P_r denotes the set of binary relations in relation r. It can easily be seen that $\nabla b_r^i = 2P_r^i q, \nabla^2 b_r^i = 2P_r^i$. We also use the following notation for the r-th relation wrt agent i: $g_r^i = b_r^i + \frac{\lambda b_r^i}{b_r^i + \left(\tilde{b}_r^i\right)^{1/h}}$, $\tilde{b}_r^i = \prod_{\substack{s \in S_r \\ s \neq r}} b_s^i$ and $\nabla \tilde{b}_r^i = \sum_{\substack{s \in S_r \\ s \neq r}} \underbrace{\prod_{\substack{t \in S_r \\ t \neq s,r}} b_i^t}_{\tilde{b}_{s,r}^i} \cdot 2P_s^i q$ where S_r denotes the set of relations in the same level with relation r. An easy calculation shows that

$$\nabla g_r^i = \ldots = 2\left[d_r^i P_r^i - w_r^i \tilde{P}_r^i\right] q \triangleq Q_r^i q, \tilde{P}_r^i \triangleq \sum_{\substack{s \in S_r \\ s \neq r}} \tilde{b}_{s,r}^i P_s^i$$

where $d_r^i = 1 + (1 - \frac{b_r^i}{b_r^i + (\tilde{b}_r^i)^{1/h}}) \frac{\lambda}{b_r^i + (\tilde{b}_r^i)^{1/h}}$, $w_r^i = \frac{\lambda b_r^i (\tilde{b}_r^i)^{\frac{1}{h}-1}}{h(b_r^i + (\tilde{b}_r^i)^{1/h})^2}$. The gradient of the G_i function is:

$$G_i = \prod_{r=1}^{N_i} g_r^i \Rightarrow \nabla G_i = \sum_{r=1}^{N_i} \underbrace{\prod_{\substack{l=1 \\ l \neq r}}^{N_i} g_l^i}_{\tilde{g}_r^i} \nabla g_r^i = \sum_{r=1}^{N_i} \tilde{g}_r^i Q_r^i q \triangleq Q_i q$$

We define $\nabla G \triangleq \begin{bmatrix} \nabla G_1 \\ \vdots \\ \nabla G_N \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix} q \triangleq Qq$

Using $u_i = -K_i \frac{\partial \varphi_i}{\partial q_i}$ and $\varphi_i = \frac{\gamma_{di} + f_i}{\left((\gamma_{di} + f_i)^k + G_i\right)^{1/k}}, f_i = \sum_{j=0}^{3} a_i G_i^j$, the closed loop dynamics of the system are:

$$\dot{q} = \begin{bmatrix} -K_1 A_1^{-(1+1/k)} \left\{ G_1 \frac{\partial \gamma_{d1}}{\partial q_1} + \sigma_1 \frac{\partial G_1}{\partial q_1} \right\} \\ \vdots \\ -K_N A_N^{-(1+1/k)} \left\{ G_N \frac{\partial \gamma_{dN}}{\partial q_N} + \sigma_N \frac{\partial G_N}{\partial q_N} \right\} \end{bmatrix} = \ldots$$
$$= -A_K G\left(\partial \gamma_d\right) - A_K \Sigma Q q$$

where $(\partial \gamma_d) = \left[\frac{\partial \gamma_{d1}}{\partial q_1} \ldots \frac{\partial \gamma_{dN}}{\partial q_N}\right]^T$, $\sigma_i = G_i \sigma(G_i) - \frac{\gamma_{di} + f_i}{k}, \sigma(G_i) = \sum_{j=1}^{3} j a_j G_i^{j-1}, A_i = (\gamma_{di} + f_i)^k + G_i$ and the matrices $G \triangleq \underbrace{diag\left(G_1, G_1, \ldots, G_N, G_N\right)}_{2N \times 2N}$,

$$A_K \triangleq \underbrace{diag\begin{pmatrix} K_1 A_1^{-(1+1/k)}, K_1 A_1^{-(1+1/k)}, \ldots, \\ K_N A_N^{-(1+1/k)}, K_N A_N^{-(1+1/k)} \end{pmatrix}}_{2N\times 2N},$$

$$\Sigma \triangleq \underbrace{\left[\underbrace{\Sigma_1}_{2N\times 2N}, \ldots, \Sigma_N \right]}_{2N\times 2N^2}, \Sigma_i = diag\left(0, \ldots, \underbrace{\sigma_i, \sigma_i}_{2i-1,2i}, \ldots, 0\right)$$

By using $\varphi = \sum_i \varphi_i$ as a candidate Lyapunov function we have $\varphi = \sum_i \varphi_i \Rightarrow \dot{\varphi} = \left\{ \sum_i (\nabla \varphi_i)^T \right\} \dot{q}$, with $\nabla \varphi_i = A_i^{-(1+1/k)} \{G_i \nabla \gamma_{di} + \sigma_i \nabla G_i\}$. Thus,

$$\sum_i (\nabla \varphi_i)^T = \ldots = (\partial \gamma_d)^T A_G + q^T Q^T A_\Sigma$$

where $A_G = \underbrace{diag\begin{pmatrix} G_1 A_1^{-(1+1/k)}, G_1 A_1^{-(1+1/k)}, \ldots, \\ G_N A_N^{-(1+1/k)}, G_N A_N^{-(1+1/k)} \end{pmatrix}}_{2N\times 2N}$

$$A_\Sigma = \underbrace{\begin{bmatrix} \underbrace{A_{\Sigma_1}}_{2N\times 2N} \\ \vdots \\ \underbrace{A_{\Sigma_N}}_{2N\times 2N} \end{bmatrix}}_{2N^2\times 2N}, A_{\Sigma_i} = \underbrace{diag\begin{pmatrix} A_i^{-(1+1/k)}\sigma_i, \ldots, \\ A_i^{-(1+1/k)}\sigma_i \end{pmatrix}}_{2N\times 2N}$$

So we have $\dot{\varphi} = \left\{ \sum_i (\nabla \varphi_i)^T \right\} \dot{q} = \ldots$

$= -\begin{bmatrix} (\partial \gamma_d)^T & q^T \end{bmatrix} \underbrace{\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}}_{M} \begin{bmatrix} \partial \gamma_d \\ q \end{bmatrix}$ where $M_1 = A_G A_K G$, $M_2 = A_G A_K \Sigma Q$, $M_3 = Q^T A_\Sigma A_K G$, $M_4 = Q^T A_\Sigma A_K \Sigma Q$. In Dimarogonas et al. (2004), we provide an analytic expression for the elements of the matrix Q. The positive definiteness of M is checked using the next theorems:

Theorem 2.1 (Gersgorin) Horn and Johnson (1996): *Given a matrix $A \in \Re^{n\times n}$ then all its eigenvalues lie in the union of n discs:*

$$\bigcup_{i=1}^{n} \left\{ z : |z - a_{ii}| \le \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}| \right\} \triangleq \bigcup_{i=1}^{n} R_i(A) \triangleq R(A)$$

Each of these discs is called a Gersgorin disc of A.

Corollary 2.2 Horn and Johnson (1996): *Given a matrix $A \in \Re^{n\times n}$ and n positive real numbers $p_1, \ldots, p_n$ then all its eigenvalues of A lie in the union of n discs:*

$$\bigcup_{i=1}^{n} \left\{ z : |z - a_{ii}| \le \frac{1}{p_i} \sum_{\substack{j=1 \\ j\ne i}}^{n} p_j |a_{ij}| \right\}$$

We also make use of the following lemma:

Lemma 2.3: *The following bounds hold for the terms $Q_{ii}^i, Q_{ii}^j, \sigma_i$*

$$\sigma_i(\varepsilon) \in \begin{cases} \left[-Y\left(\frac{1}{k} + \frac{8}{9}\right) - \frac{\gamma_{di}}{k}, \underbrace{-\frac{Y}{k} - \frac{\gamma_{di}}{k}}_{\sigma_i(0)} \right], 0 \le \varepsilon \le \varepsilon^* \\ \left[-Y\left(\frac{1}{k} + \frac{8}{9}\right) - \frac{\gamma_{di}}{k}, \underbrace{-\frac{\gamma_{di}}{k}}_{\sigma_i(X)} \right], X \ge \varepsilon \ge \varepsilon^* \end{cases}$$

$0 < Q_{ii}^i < \left|Q_{ii}^i\right|_{\max} < \infty$ *and* $0 < Q_{ii}^j < \left|Q_{ii}^j\right|_{\max} < \infty$

Proof: See Dimarogonas et al. (2004).

Let us examine the Gersgorin discs of the first half rows of the matrix M. For $1 \le i \le 2N$ we have:

$$|z - M_{ii}| \le \tfrac{1}{p_i} \sum_{j\ne i} p_j |M_{ij}|, 1 \le i \le 2N \Rightarrow$$
$$\left| z - A_i^{-2(1+1/k)} K_i G_i^2 \right| \le \tfrac{p_{2N+i}}{p_i} \left| A_i^{-2(1+1/k)} \sigma_i K_i G_i Q_{ii}^i \right| \Rightarrow$$
$$\Rightarrow z \ge A_i^{-2(1+1/k)} K_i G_i^2 - \tfrac{p_{2N+i}}{p_i} \left| A_i^{-2(1+1/k)} \sigma_i K_i G_i Q_{ii}^i \right|$$

We examine the following three cases:

- $G_i < \varepsilon$ At a critical point in this region, the corresponding eigenvalue tends to zero, so that the derivative of the Lyapunov function could achieve zero values. However, the result of Lemma 6 indicates that φ_i is a Morse function, hence its critical points are isolated (Koditschek and Rimon (1990)). Thus the set of initial conditions that lead to saddle points are sets of measure zero (Milnor (1963)).
- $G_i > X$ The corresponding eigenvalue is guaranteed to be positive as long as:

$$z > 0 \Leftarrow A_i^{-2(1+1/k)} K_i \left(G_i - \tfrac{p_{2N+i}}{p_i} \left| \sigma_i Q_{ii}^i \right| \right) > 0 \Leftarrow$$
$$G_i \ge X > \tfrac{p_{2N+i}}{p_i} \left| \sigma_i Q_{ii}^i \right| = \tfrac{\gamma_{di}}{k} \tfrac{p_{2N+i}}{p_i} \left| Q_{ii}^i \right| \Leftarrow$$
$$\Leftarrow k > \tfrac{(\gamma_{di})_{\max}}{X} \tfrac{p_{2N+i}}{p_i} \left| Q_{ii}^i \right|_{\max}$$

- $0<\varepsilon\le G_i\le X$

$$\begin{aligned}&z>0\Leftarrow\varepsilon>\left\{Y\left|\tfrac{1}{k}+\tfrac{8}{9}\right|+\left|\tfrac{\gamma_{di}}{k}\right|\right\}\tfrac{p_{2N+i}}{p_i}\left|Q^i_{ii}\right|_{\max}\Leftarrow\\&\Leftarrow\varepsilon>2\max\left\{\begin{array}{l}2\max\left\{\frac{Y}{k},\frac{8Y}{9}\right\},\\\left|\frac{(\gamma_{di})_{\max}}{k}\right|\end{array}\right\}\tfrac{p_{2N+i}}{p_i}\left|Q^i_{ii}\right|_{\max}\\&\overset{Y\le\frac{\Theta_1}{k}}{\Leftarrow}k>2\max\left\{\begin{array}{l}2\sqrt{\frac{\Theta_1}{\varepsilon}},\frac{16\Theta_1}{9\varepsilon},\\\frac{(\gamma_{di})_{\max}}{\varepsilon}\end{array}\right\}\tfrac{p_{2N+i}}{p_i}\left|Q^i_{ii}\right|_{\max}\end{aligned}$$

Let us examine the Gersgorin discs of the second half rows of the matrix M. The discs of Corollary 2.2 are:

$$\begin{aligned}&|z-M_{ii}|\le\sum_{j\ne i}\tfrac{p_j}{p_i}|M_{ij}|,2N+1\le i\le 4N,1\le j\le 4N\Rightarrow\\&\Rightarrow|z-(M_4)_{ii}|\le R_i(M_3)+R_i(M_4)\end{aligned}$$

where $(M_4)_{ii}=\sum_j K_iA_i^{-(1+1/k)}A_j^{-(1+1/k)}\sigma_j\sigma_iQ^i_{ii}Q^j_{ii}$

$$\begin{aligned}&R_i(M_3)=\sum_{j=1}^{2N}\tfrac{p_j}{p_i}\left|(M_3)_{ij}\right|=\\&=\sum_{j=1}^{2N}\tfrac{p_j}{p_i}\left|\sum_l A_l^{-(1+1/k)}\sigma_lA_j^{-(1+1/k)}K_jG_jQ^l_{ij}\right|\end{aligned}$$

$$\begin{aligned}&R_i(M_4)=\sum_{\substack{j=2N+1\\j\ne i}}^{4N}\tfrac{p_j}{p_i}\left|(M_4)_{ij}\right|=\\&=\sum_{j\ne i}\tfrac{p_j}{p_i}\left|\sum_l(A_lA_j)^{-(1+1/k)}\sigma_l\sigma_jK_jQ^l_{ij}Q^j_{jj}\right|\end{aligned}$$

A sufficient condition for the positive definiteness of the corresponding eigenvalue for raw i is then:

$$\begin{aligned}&(M_4)_{ii}>R_i(M_3)+R_i(M_4)\Leftarrow\\&\Leftarrow(M_4)_{ii}>\max\{2R_i(M_3),2R_i(M_4)\}\end{aligned}$$

We first show that we always have $R_i(M_3)\ge R_i(M_4)$. By taking into account the relations $Q^i_{jk}=Q^i_{kj}=0,Q^i_{ij}=-Q^i_{jj},j\ne i\ne k\ne j$ and expanding it is easy to see that

$$\begin{aligned}&R_i(M_3)=-\tfrac{1}{p_i}\sum_{j=1}^{2N}p_j\left\{\begin{array}{l}A_j^{-2(1+1/k)}\sigma_jK_jG_jQ^j_{ii}+\\(A_jA_i)^{-(1+1/k)}\sigma_iK_jG_jQ^i_{jj}\end{array}\right\}=\\&=-\sum_{\substack{j=1\\j\ne i}}^{2N}\tfrac{p_j}{p}\left\{\begin{array}{l}\underbrace{A_j^{-2(1+1/k)}\sigma_jK_jG_jQ^j_{ii}}_{(I)}+\\\underbrace{(A_jA_i)^{-(1+1/k)}\sigma_iK_jG_jQ^i_{jj}}_{(II)}\end{array}\right\}\\&-2\tfrac{p_i}{p}A_i^{-2(1+1/k)}\sigma_iK_iG_iQ^i_{ii}\end{aligned}$$

where without loss of generality we choose $p_i=p,2N+1\le i\le 4N$.We also have

$$R_i(M_4)=\sum_{j\ne i}\left\{\begin{array}{l}\underbrace{A_j^{-2(1+1/k)}\sigma_j^2K_jQ^j_{ii}Q^j_{jj}}_{(I)}+\\\underbrace{(A_iA_j)^{-(1+1/k)}\sigma_i\sigma_jK_jQ^i_{jj}Q^j_{jj}}_{(II)}\end{array}\right\}$$

By comparing (I) and (II) in the last two equations:

$$\begin{aligned}&(I):-\tfrac{p_j}{p}A_j^{-2(1+1/k)}\sigma_jK_jG_jQ^j_{ii}\ge A_j^{-2(1+1/k)}\sigma_j^2K_jQ^j_{ii}Q^j_{jj}\\&\Leftarrow-\tfrac{p_j}{p}\sigma_jG_j\ge\sigma_j^2Q^j_{jj}\Leftrightarrow\sigma_j\left(\sigma_jQ^j_{jj}+\tfrac{p_j}{p}G_j\right)\le 0\\&\overset{\sigma_j\le 0}{\Leftrightarrow}\sigma_jQ^j_{jj}+\tfrac{p_j}{p}G_j\ge 0\\&(II):-\tfrac{p_j}{p}(A_jA_i)^{-(1+1/k)}\sigma_iK_jG_jQ^i_{jj}\ge\\&\ge(A_iA_j)^{-(1+1/k)}\sigma_i\sigma_jK_jQ^i_{jj}Q^j_{jj}\\&\Leftarrow-\tfrac{p_j}{p}\sigma_iG_j\ge\sigma_i\sigma_jQ^j_{jj}\Leftrightarrow\sigma_i\left(\sigma_jQ^j_{jj}+\tfrac{p_j}{p}G_j\right)\le 0\\&\overset{\sigma_i\le 0}{\Leftrightarrow}\sigma_jQ^j_{jj}+\tfrac{p_j}{p}G_j\ge 0\end{aligned}$$

Thus, the condition $\sigma_jQ^j_{jj}+\frac{p_j}{p}G_j\ge 0$ guarantees that $R_i(M_3)\ge R_i(M_4)\forall i$. Hence it suffices to show that $(M_4)_{ii}>2R_i(M_3)$. The fact that $\sigma_jQ^j_{jj}+\frac{p_j}{p}G_j\ge 0$ is a direct conclusion of the results of the previous calculations. For example, by the last bound on k we have:

$$\begin{aligned}&k>2\max\left\{2\sqrt{\tfrac{\Theta_1}{\varepsilon}},\tfrac{16\Theta_1}{9\varepsilon},\tfrac{(\gamma_{dj})_{\max}}{\varepsilon}\right\}\tfrac{p}{p_j}\left|Q^j_{jj}\right|_{\max}\\&\underset{G_j\ge\varepsilon}{\overset{Y\le\frac{\Theta_1}{k}}{\Rightarrow}}G_j>2\max\left\{2\max\left\{\tfrac{Y}{k},\tfrac{8Y}{9}\right\},\left|\tfrac{(\gamma_{dj})_{\max}}{k}\right|\right\}\tfrac{p}{p_j}\left|Q^j_{jj}\right|_{\max}\\&\Rightarrow G_j>\left\{Y\left|\tfrac{1}{k}+\tfrac{8}{9}\right|+\left|\tfrac{\gamma_{dj}}{k}\right|\right\}\tfrac{p}{p_j}\left|Q^j_{jj}\right|_{\max}\\&\Rightarrow\tfrac{p_j}{p}G_j>|\sigma_j|_{\max}\left|Q^j_{jj}\right|_{\max}\Rightarrow\sigma_jQ^j_{jj}+\tfrac{p_j}{p}G_j>0\end{aligned}$$

The fact that $(M_4)_{ii}>0$ is guaranteed by Lemma 2.3. This lemma also guarantees that there is always a finite upper bound on the terms

$$\left|(M_3)_{ij}\right|=\left|\sum_l A_l^{-(1+1/k)}\sigma_lA_j^{-(1+1/k)}K_jG_jQ^l_{ij}\right|$$

We have $(M_4)_{ii}>2R_i(M_3)=2\sum_{j=1}^{2N}\frac{p_j}{p}\left|(M_3)_{ij}\right|\Leftarrow p>\frac{4N}{(M_4)_{ii}}\max_j\left\{p_j\left|(M_3)_{ij}\right|\right\},2N+1\le i\le 4N,1\le j\le 2N$

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