Correct-by-Design Control Barrier Functions for Euler-Lagrange Systems with Input Constraints

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***Abstract*— Control barrier functions are valuable for satisfying system constraints for general nonlinear systems. However a main drawback to existing techniques is the proper construction of these barrier functions to satisfy system *and* input constraints. In this paper, we propose a methodology to construct control barrier functions for Euler-Lagrange systems subject to input constraints. The proposed approach is validated in simulation on a 2-DOF planar manipulator.**

I. INTRODUCTION

Recent technological advancements have increased the presence of autonomous systems in human settings. Euler-Lagrange systems are representative of real-world autonomous systems (e.g autonomous vehicles, robotic manipulators) for which safety around humans is critical. To define safety, we specify position and velocity constraints, e.g. do not leave a pre-defined region or exceed this speed, that must be respected at all times. A critical property of real-world systems is that they have limited actuation to uphold system constraints. The problem addressed here is how to satisfy state *and* input constraints for Euler-Lagrange systems.

Control barrier functions have attracted attention for constraint satisfaction of nonlinear systems [1]. Existing barrier function methods have been applied to nonlinear continuous/hybrid systems [2] and used to satisfy constraints while providing stability [3]. Recently, the distinction between reciprocal control barrier functions (RCBFs) and zeroing control barrier functions (ZCBFs) has been established [8], in which RCBFs are undefined at the constraint boundary while ZCBFs are zero at the boundary and well-defined outside of the constraint set. Aside from practical implementations, ZCBFs are advantageous in that they are robust to perturbations [9]. Those methods have been applied to bi-pedal walking, adaptive cruise control, and robotic applications [4]–[7]. A review of ZCBFs can be found in [1].

A well known set-back of barrier function methods is the difficulty in constructing them. This issue is further exacerbated when also considering input constraints, which are characteristic of real-world systems. One existing method to construct ZCBFs includes sum-of-squares programming [1], [2], however that approach is only applicable to polynomial systems, and not to the Euler-Lagrange systems considered here. One method that can be applied to nonlinear affine systems (and thus Euler-Lagrange systems) requires a pre-defined evasive maneouver [10]. However the closed-form design of that evasive maneouver is not straight forward in general. Furthermore, that approach is restricted to a finite time horizon and requires forward simulation of the system dynamics which is not tractable for general Euler-Lagrange systems. Thus despite recent advancements, there is no existing approach to construct ZCBFs in closed-form for Euler-Lagrange systems.

In this paper, we present a methodology to construct ZCBFs for Euler-Lagrange systems that respect input constraints. The proposed approach is designed off-line by exploiting properties of Euler-Lagrange systems, and guarantees safe workspace constraint satisfaction (position *and* velocity). The approach is validated in numerical simulation on the 2-DOF planar manipulator. We note that the approach extends upon [4] by providing formal guarantees regarding the construction of ZCBFs.

Notation: The term $\boldsymbol{e}_j \in \mathbb{R}^r$ denotes the jth column of the identity matrix $I_{r\times r}$. The Lie derivatives of a function $h(\boldsymbol{x})$ for the system $\dot{\boldsymbol{x}} = \mathfrak{f}(\boldsymbol{x}) + \mathfrak{g}(\boldsymbol{x})\boldsymbol{u}$ are denoted by $L_{\mathfrak{f}}h$ and $L_{\mathfrak{g}}h$, respectively. The terms $\preceq$ and $\succeq$ are used to denote element-wise vector inequalities. The matrix inequality $A < B$ for square matrices A and B means that the matrix $B - A$ is positive-definite. The interior and boundary of a set $\mathscr{A}$ are denoted Int$(\mathscr{A})$ and $\partial\mathscr{A}$, respectively. For brevity, the equation $\bar{\underline{a}} = \bar{\underline{b}} \pm c$ denotes $\bar{a} = \bar{b} + c$ and $\underline{b} = \underline{b} - c$.

II. BACKGROUND

A. Control Barrier Functions

Here we introduce the existing work regarding control barrier functions for nonlinear affine systems: $\dot{\boldsymbol{x}} = \mathfrak{f}(\boldsymbol{x}) + \mathfrak{g}(\boldsymbol{x})\boldsymbol{u}$ where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ is the control input. We will consider the case where $\mathfrak{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathfrak{g} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz functions, and the system is forward complete.

Let us first define the extended class-$\mathcal{K}$ function:

Definition 1. *[8]: A continuous function, $\alpha : (-b, a) \to (-\infty, \infty)$ for $a, b \in \mathbb{R}_{>0}$ is an extended class-$\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$.*

Note for simplicity, the extended class-K functions addressed here will be smooth and defined for $a, b = \infty$.

Let $h(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and let the associated constraint set be defined by:

$$\mathscr{C} = \{\boldsymbol{x} \in \mathbb{R}^n : h(\boldsymbol{x}) \geq 0\} \quad (1)$$

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where $\partial\mathscr{C} = \{\boldsymbol{x} \in \mathbb{R}^n : h(\boldsymbol{x}) = 0\}$ and $\text{Int}(\mathscr{C}) = \{\boldsymbol{x} \in \mathbb{R}^n : h(\boldsymbol{x}) > 0\}$.

Constraint satisfaction is ensured via Nagumo's Theorem by showing that the system states are directed into the constraint set (see Theorem 4.7 of [11]). In the context of barrier functions, this condition is written as: $\dot{h}(\boldsymbol{x}) \geq -\alpha(h(\boldsymbol{x}))$ for all $\boldsymbol{x} \in \mathscr{C}$, for a continuously differentiable, extended class-$\mathcal{K}$ function α. Here h is considered the ZCBF and formerly defined as:

Definition 2. *[8]: Given a set $\mathscr{C}$ defined by* (1) *for a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, the function h is called a ZCBF defined on an open set $\mathscr{E}$ with $\mathscr{C} \subset \mathscr{E} \subset \mathbb{R}^n$ if there exists an extended class-$\mathcal{K}$ function α such that: $\sup_{\boldsymbol{u}\in\mathscr{U}} [L_{\mathfrak{f}}h(\boldsymbol{x}) + L_{\mathfrak{g}}h(\boldsymbol{x})\boldsymbol{u} + \alpha(h(\boldsymbol{x}))] \geq 0, \forall \boldsymbol{x} \in \mathscr{E}$.*

If h is a ZCBF, the condition $\dot{h}(\boldsymbol{x}) \geq -\alpha(h(\boldsymbol{x}))$ is then enforced in the control by re-writing it as: $L_{\mathfrak{f}}h + L_{\mathfrak{g}}h\boldsymbol{u} \geq -\alpha(h(\boldsymbol{x}))$, which is linear with respect to $\boldsymbol{u}$. Resulting methods for ZCBFs then implement this condition as a constraint in a quadratic program to define $\boldsymbol{u}$. One example of such a controller is [4]:

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x})^* = &\underset{\boldsymbol{u}\in\mathbb{R}^m}{\text{argmin}} \quad ||\boldsymbol{u} - \boldsymbol{u}_{\text{nom}}(\boldsymbol{x})||_2^2 \\ &\text{s.t.} \qquad L_{\mathfrak{f}}h(\boldsymbol{x}) + L_{\mathfrak{g}}h(\boldsymbol{x})\boldsymbol{u} \geq -\alpha(h(\boldsymbol{x})) \end{aligned} \tag{2}$$

where $\boldsymbol{u}_{\text{nom}} : \mathbb{R}^n \to \mathbb{R}^m$ is a state-feedback control law designed to stabilize the system.

B. Euler-Lagrange Dynamics

Consider the following Euler-Lagrange system:

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= M(\boldsymbol{q})^{-1}(-C(\boldsymbol{q},\boldsymbol{v})\boldsymbol{v} - F\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{u}) \end{aligned} \tag{3}$$

where $M(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(\boldsymbol{q},\boldsymbol{v}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal matrix, $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$ is the generalized gravity on the system, $F \in \mathbb{R}^{n\times n}$ is the positive semi-definite, diagonal damping matrix, $\boldsymbol{u} \in \mathscr{U} \subseteq \mathbb{R}^n$ is the control input. Let $(\boldsymbol{q}(t,\boldsymbol{q}_0), \boldsymbol{v}(t,\boldsymbol{v}_0)) \in \mathbb{R}^{2n}$ be the solution of (3), which for ease of notation is denoted by $(\boldsymbol{q},\boldsymbol{v})$.

Here we consider the following well-known properties for Euler-Lagrange systems [12]:

Property 1. *: $M(\boldsymbol{q})$ is symmetric, positive-definite, and bounded such that there exists $\mu_{m_1}, \mu_{m_2} \in \mathbb{R}_{>0}$, $\mu_{m_2} > \mu_{m_1}$ such that $\mu_{m_1}I_{n\times n} < M(\boldsymbol{q}) < \mu_{m_2}I_{n\times n}$.*

Property 2. *: There exists $k_c \in \mathbb{R}_{>0}$ such that $C(\boldsymbol{q},\boldsymbol{v})$ satisfies: $||C(\boldsymbol{q},\boldsymbol{v})|| \leq k_c||\boldsymbol{v}||$.*

Property 3. *: There exists $k_g \in \mathbb{R}_{>0}$ such that $\boldsymbol{g}(\boldsymbol{q})$ satisfies: $k_g \geq \sup_{\boldsymbol{q}\in\mathbb{R}^n} ||\boldsymbol{g}(\boldsymbol{q})||$.*

Due to the bounded, positive-definite property of the inertia matrix $M(\boldsymbol{q})$ the following lemma follows:

Lemma 1. *Under Property 1, there exists $k_{m_1}, k_{m_2} \in \mathbb{R}_{>0}$, $k_{m_2} > k_{m_1}$ such that $k_{m_1}I_{n\times n} < M(\boldsymbol{q})^{-1} < k_{m_2}I_{n\times n}$. Furthermore, there exist $k_{d_{ij}} := \max_{\boldsymbol{q}\in\mathbb{R}^n}\{|m_{ij}(\boldsymbol{q})|\}$, where $m_{ij}(\boldsymbol{q})$ is the i,j element of $M(\boldsymbol{q})^{-1}$.*

Note that we use Properties 1-3 for simplicity. We consider safety of the system over a compact set, which due to smoothness of M, C, and $\boldsymbol{g}$ ensures that k_{m_1}, k_{m_2}, $k_{d_{ij}}$, k_c, and k_g exist for general Euler-Lagrange systems [13].

C. Problem Formulation

The goal of constraint satisfaction is to ensure the states $\boldsymbol{q},\boldsymbol{v}$ stay within a set of constraint-admissible states. We define the position constraints as:

$$\mathscr{C} = \{\boldsymbol{q} \in \mathbb{R}^n : \boldsymbol{q}_{min} \preceq \boldsymbol{q} \preceq \boldsymbol{q}_{max}\} \tag{4}$$

for $\boldsymbol{q}_{min}, \boldsymbol{q}_{max} \in \mathbb{R}^n$ and $\boldsymbol{q}_{max} \succ \boldsymbol{q}_{min}$. These types of constraints are highly popular in robotics and general automated systems.

We define the system's velocity constraints as:

$$\mathscr{D} = \{\boldsymbol{v} \in \mathbb{R}^n : \boldsymbol{v}_{min} \preceq \boldsymbol{v} \preceq \boldsymbol{v}_{max}\} \tag{5}$$

where $\boldsymbol{v}_{min}, \boldsymbol{v}_{max} \in \mathbb{R}^n$, $\boldsymbol{v}_{max} \succ 0$, and for simplicity of the presentation let $\boldsymbol{v}_{min} = -\boldsymbol{v}_{max}$.

In addition to state constraints, real-world systems have limited actuation capabilities. We define the input constraints as:

$$\mathscr{U} = \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{u}_{min} \preceq \boldsymbol{u} \preceq \boldsymbol{u}_{max}\} \tag{6}$$

where $\boldsymbol{u}_{min}, \boldsymbol{u}_{max} \in \mathbb{R}^n$, $\boldsymbol{u}_{max} \succ 0$, and for simplicity of the presentation let $\boldsymbol{u}_{min} = -\boldsymbol{u}_{max}$.

The problem addressed here is to ensure the set of state constraints is forward invariant while respecting input constraints and formally stated as follows:

Problem 1. *Consider the system* (3) *with position, velocity, and input constraints* (4), (5), (6). *Design a control law $\boldsymbol{u} \in \mathscr{U}$ such that the state constraint sets $\mathscr{C}$, $\mathscr{D}$ are rendered forward invariant.*

III. Proposed Solution

To address Problem 1, we first construct the ZCBFs with particular design parameters, then explain how the design parameters should be chosen. Let $\mathscr{N} = \{1, ..., n\}$. We re-write the constraint set $\mathscr{C}$ into individual constraints with respect to functions $\bar{h}_i(q_i), \underline{h}_i(q_i) : \mathbb{R} \to \mathbb{R}$ for each q_i, $i \in \mathscr{N}$, which are defined as:

$$\bar{h}_i(q_i) = q_{max_i} - q_i, \;\; \underline{h}_i(q_i) = q_i - q_{min_i} \tag{7}$$

Let the constraint-admissible set for state q_i be:

$$\mathscr{C}_i = \{q_i \in \mathbb{R} : \bar{h}_i(q_i) \geq 0, \underline{h}_i(q_i) \geq 0\} \tag{8}$$

Note that by definition, $\mathscr{C}_i$ is always convex, compact, non-empty, and $\bar{h}_i, \underline{h}_i$ are smooth functions of their arguments.

To ensure forward invariance of $\mathscr{C}_i$ (and consequently $\mathscr{C}$), the following conditions must hold: $\dot{\bar{h}}_i(q_i) \geq -\alpha_1(\bar{h}_i(q_i))$, $\dot{\underline{h}}_i(q_i) \geq -\alpha_1(\underline{h}_i(q_i))$ for all $q_i \in \mathscr{C}_i$, $i \in \mathscr{N}$, for some extended class-$\mathcal{K}$ function α to then apply Nagumo's theorem [11]. Substitution of (7) with (3) yields the following requirement: $-v_i \geq -\alpha(\bar{h}_i(q_i))$ and $v_i \geq -\alpha(\underline{h}_i(q_i))$. Moreover, due to the fact that this system is of relative degree two, there is no control input to ensure the conditions hold.

Thus we treat these conditions as new constraints by defining $\bar{b}_i, \underline{b}_i : \mathbb{R}^2 \to \mathbb{R}$ [14]:

$$\bar{b}_i(q_i, v_i) = -v_i + \gamma\alpha_1(\bar{h}_i(q_i)),\ \underline{b}_i(q_i, v_i) = v_i + \gamma\alpha_1(\underline{h}_i(q_i)) \tag{9}$$

where α_1 is an extended class-$\mathcal{K}$ function, and $\gamma \in \mathbb{R}_{>0}$ is a design parameter. Note that without loss of generality, we will use γ and α_1 for all $2n$ upper and lower bound constraints. Notice that when $\bar{b}_i(q_i, v_i) \geq 0$ and $\underline{b}_i(q_i, v_i) \geq 0$, it follows that $\dot{\bar{h}}_i(q_i) \geq -\gamma\alpha_1(\bar{h}_i(q_i))$, $\dot{\underline{h}}_i(q_i) \geq -\gamma\alpha_1(\underline{h}_i(q_i))$ as required by Nagumo's theorem for forward invariance of $\mathscr{C}_i$. To properly address the set of states where $\bar{b}_i(q_i, v_i) \geq 0$ and $\underline{b}_i(q_i, v_i) \geq 0$, we define the following set:

$$\mathscr{B}_i = \{(q_i, v_i) \in \mathbb{R}^2 : \bar{b}_i(q_i, v_i) \geq 0, \underline{b}_i(q_i, v_i) \geq 0\} \tag{10}$$

with $\mathscr{B}$ denoting the Cartesian product of $\mathscr{B}_i$ over all $i \in \mathscr{N}$.

In order to ensure forward invariance of $\mathscr{B}$ (and thus $\mathscr{C}$), another application of Nagumo's Theorem requires that:

$$\dot{\bar{b}}_i(q_i, v_i) \geq -\beta\alpha_2(\bar{b}_i(q_i, v_i)), \dot{\underline{b}}_i(q_i, v_i) \geq -\beta\alpha_2(\underline{b}_i(q_i, v_i)) \tag{11}$$

for $(q_i, v_i) \in (\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i$ for $i \in \mathscr{N}$, where α_2 is an extended class-$\mathcal{K}$ function, and $\beta \in \mathbb{R}_{>0}$ is a design parameter.

The aim is to correctly construct $\bar{b}_i$ and $\underline{b}_i$ via γ and β such that a control $\boldsymbol{u} \in \mathscr{U}$ exists to enforce (11). In this context, we consider $\bar{b}_i$ and $\underline{b}_i$ as the candidate ZCBFs of the system (3). We start by stating a key property of the proposed ZCBFs on this class of constraints:

Lemma 2. *Consider the constraint sets $\mathscr{C}_i$ (8), functions $\bar{b}_i$, $\underline{b}_i$ (9), and associated set $\mathscr{B}_i$ (10), $\forall i \in \mathscr{N}$. Then $||\boldsymbol{v}||_\infty \leq \gamma a$ for all $(\boldsymbol{q}, \boldsymbol{v}) \in (\mathscr{C} \times \mathbb{R}^n) \cap \mathscr{B}$, where $a := \alpha_1(\delta_q)$, $\delta_q = ||\boldsymbol{q}_{min} - \boldsymbol{q}_{max}||_\infty$.*

Proof. From (8), (9), and (10) it follows that $-\gamma\alpha_1(\underline{h}_i(q_i)) \leq v_i \leq \gamma\alpha_1(\bar{h}_i(q_i))$ for all $i \in \mathscr{N}$. Thus $\boldsymbol{v}$ is bounded for $\boldsymbol{q} \in \mathscr{C}$. From Definition 1 and (8), the max of the upper and lower bounds of $\boldsymbol{v}$ i.e $\max\{\max_{q_i \in \mathscr{C}_i, i \in \mathscr{N}} \gamma\alpha_1(\bar{h}_i(q_i)), \max_{q_i \in \mathscr{C}_i, i \in \mathscr{N}} \gamma\alpha_1(\underline{h}_i(q_i))\}$ yields γa, which completes the proof. □

Lemma 2 provides insight into how the barrier function construction affects the system behaviour. First, by appropriately tuning γ (and thus $\mathscr{B}$), the velocity bounds can be adjusted to satisfy the state constraint $\mathscr{D}$. Second, the relation $-\gamma\alpha_1(\underline{h}_i(q_i)) \leq v_i \leq \gamma\alpha_1(\bar{h}_i(q_i))$ shows that as q_i approaches the boundary $\partial\mathscr{C}_i$, the velocity approaches zero. This is an important property because it restricts the system's inertia relative to the constraint boundary. This aligns with intuition in that if the velocity is too high near the boundary, exceedingly large control effort would be required to ensure forward invariance. While γ dictates the system's velocity, β dictates the behaviour of $\boldsymbol{u}$ as the system approaches the constraint boundary. From (11), β will dictate how soon the control acts to keep the system in the constraint set. In the remainder of this section, we will exploit this property to explicitly compute γ and β to ensure state and input constraint satisfaction.

First, we note in the following lemma that by construction of $\bar{b}_i$ and $\underline{b}_i$, we can ensure the conditions of Nagumo's theorem for a subset of the boundary of $(\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i$:

Lemma 3. *Consider the constraint sets $\mathscr{C}_i$ (8), functions $\bar{b}_i$, $\underline{b}_i$ (9), and associated set $\mathscr{B}_i$ (10), $\forall i \in \mathscr{N}$. Let $\bar{\mathscr{S}}_i$, $\underline{\mathscr{S}}_i \subset \partial((\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i)$ be defined by:*

$$\begin{aligned} \bar{\mathscr{S}}_i &= \{(q_i, v_i) \in \mathbb{R}^2 : \bar{h}_i(q_i) = 0, v_i \in [-\gamma\alpha_1(\delta_{q_i}), 0]\}, \\ \underline{\mathscr{S}}_i &= \{(q_i, v_i) \in \mathbb{R}^2 : \underline{h}_i(q_i) = 0, v_i \in [0, \gamma\alpha_1(\delta_{q_i})]\} \end{aligned} \tag{12}$$

for $i \in \mathscr{N}$ and $\delta_{q_i} = q_{max_i} - q_{min_i}$. Then it follows that $\dot{\bar{h}}_i(q_i) \geq 0$, and $\dot{\underline{h}}_i(q_i) \geq 0$ on $\bar{\mathscr{S}}_i, \underline{\mathscr{S}}_i$, respectively.

Proof. By (12), it follows that $v_i \leq 0$ for $(q_i, v_i) \in \bar{\mathscr{S}}_i$, and $v_i \geq 0$ for $(q_i, v_i) \in \underline{\mathscr{S}}_i$. Differentiation of (7) yields $\dot{\bar{h}}_i(q_i) = -v_i$, $\dot{\underline{h}}_i(q_i) = v_i$, and the respective substitution for v_i completes the proof. □

Lemma 3 follows intuition in that violation of the constraints defined by $\bar{h}_i(q_i) \geq 0$ (resp. $\underline{h}_i(q_i) \geq 0$) can only occur if the system is moving towards the constraint boundary i.e. $v_i > 0$ (resp. $v_i < 0$). To present the next lemma, we first define the following terms:

$$\begin{aligned} \tilde{u}_{min_i} &:= u_{min_i} + \frac{k_{m_2}}{k_{m_1}} k_g + \frac{1}{k_{m_1}} \sum_{j \neq i} k_{d_{ij}} u_{max_j} \\ \tilde{u}_{max_i} &:= u_{max_i} - \frac{k_{m_2}}{k_{m_1}} k_g + \frac{1}{k_{m_1}} \sum_{j \neq i} k_{d_{ij}} u_{min_j} \end{aligned} \tag{13}$$

$$\begin{aligned} \bar{\underline{\psi}}_i(q_i, v_i) &:= k_{m_1}\tilde{u}_{min_i} + k_{m_2}k_c a^2\gamma^2 + \sum_{j \neq i} k_{d_{ij}} f_j a\gamma \\ &\mp (k_{m_1} f_i - \gamma \frac{\partial \alpha_1}{\partial \bar{\underline{h}}_i}(q_i)) v_i \end{aligned} \tag{14}$$

where $a = \alpha_1(\delta_q)$, $\delta_q = ||\boldsymbol{q}_{max} - \boldsymbol{q}_{min}||_\infty$, f_i is the ith element of the diagonal of F, and k_g, k_{m_2}, k_{m_1}, $k_{d_{ij}}$ are from Lemma 1 and Property 3. Here $\tilde{u}_{min_i}$ and $\tilde{u}_{max_i}$ denote the available control effort to actively prevent constraint violation. Note that $\tilde{u}_{max_i} = -\tilde{u}_{min_i}$ since $\boldsymbol{u}_{max} = -\boldsymbol{u}_{min}$. We make the following assumption to ensure the system has sufficient control authority:

Assumption 1. *The system (3) has sufficient control authority such that $\tilde{u}_{min_i}$ and $\tilde{u}_{max_i}$ defined by (13) satisfy: $\tilde{u}_{min_i} < 0$, $\tilde{u}_{max_i} > 0$, $\forall i \in \mathscr{N}$.*

Next, we state a sufficient condition for (11):

Lemma 4. *Consider the system (3) with the state and input constraints defined by (4), (5), and (6). Let the functions $\bar{b}_i(q_i, v_i)$ and $\underline{b}_i(q_i, v_i)$ be defined by (9) with smooth extended class-$\mathcal{K}$ functions α_1 and α_2, and consider the sets $\mathscr{C}_i$ (8), $\mathscr{B}_i$ (10), $\forall i \in \mathscr{N}$. Suppose the following conditions hold for all $(q_i, v_i) \in (\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i$, $i \in \mathscr{N}$:*

$$\begin{aligned} &\text{if } v_i \geq 0, \text{ then } \bar{\psi}_i(q_i, v_i) - \beta\alpha_2(\bar{b}_i(q_i, v_i)) \leq 0, \\ &\text{if } v_i \leq 0, \text{ then } \underline{\psi}_i(q_i, v_i) - \beta\alpha_2(\underline{b}_i(q_i, v_i)) \leq 0 \end{aligned} \tag{15}$$

where $\bar{\psi}_i$ and $\underline{\psi}_i$ are defined in (14). Then there exists $\boldsymbol{u} \in \mathscr{U}$ that satisfies (11) for $i \in \mathscr{N}$.

Proof. Here we show how satisfaction of (15) implies (11). We re-write (11) by differentiating (9) and substituting (3), which yields:

$$\boldsymbol{e}_i^T M^{-1}(\boldsymbol{u}+\boldsymbol{g}) \le \boldsymbol{e}_i^T M^{-1}(C+F)\boldsymbol{v} - \gamma \frac{\partial \alpha_1}{\partial \bar{h}_i} v_i + \beta \alpha_2(\bar{b}_i) \tag{16a}$$

$$\boldsymbol{e}_i^T M^{-1}(\boldsymbol{u}+\boldsymbol{g}) \ge \boldsymbol{e}_i^T M^{-1}(C+F)\boldsymbol{v} - \gamma \frac{\partial \alpha_1}{\partial \underline{h}_i} v_i - \beta \alpha_2(\underline{b}_i) \tag{16b}$$

First, we show that there exists a $\boldsymbol{u} \in \mathscr{U}$ such that $k_{m_1}\tilde{u}_{min_i} \ge \boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}(\boldsymbol{u}+\boldsymbol{g})$ and $k_{m_1}\tilde{u}_{max_i} \le \boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}(\boldsymbol{u}+\boldsymbol{g})$. The intuition is that the available actuation to enforce (11) is limited by $M(\boldsymbol{q})^{-1}$ from (16), and the worst case scenario is when the lower bound, k_{m_1}, is reached. We address this worst case scenario for $\boldsymbol{u} \in \mathscr{U}$ as follows. Note that by Lemma 1, $k_{m_2} \ge m_{ii}(\boldsymbol{q}) \ge k_{m_1}$ where $m_{ii}(\boldsymbol{q}) \in \mathbb{R}_{>0}$ is the ith element of the diagonal of $M(\boldsymbol{q})^{-1}$. Recall that $\boldsymbol{u}_{min} = -\boldsymbol{u}_{max}$, such that multiplication by the negatively valued u_{min_i} yields $k_{m_1}u_{min_i} \ge m_{ii}(\boldsymbol{q})u_{min_i}$. Now let $\mathscr{A} = \{u_i \in \mathbb{R} : m_{ii}(\boldsymbol{q})u_i - k_{m_1}u_{min_i} \ge 0\}$. By choosing $u_i = u_{min_i}$ we see that $\mathscr{A} \cap \mathscr{U}_i \ne \emptyset$ and thus we can say that there always exists a $u_i \in \mathscr{U}_i$ such that $m_{ii}(\boldsymbol{q})u_i \le k_{m_1}u_{min_i}$. Similarly, we can say $k_{m_1}u_{max_i} \le m_{ii}(\boldsymbol{q})u_{max_i}$ such that for $\boldsymbol{u} \in \mathscr{U}$, we can find a $u_i \in \mathscr{U}_i$ such that $m_{ii}(\boldsymbol{q})u_i \ge k_{m_1}u_{max_i}$. Thus we ensure that there exists a $\boldsymbol{u} \in \mathscr{U}$ such that if $m_{ii}(\boldsymbol{q})$ reaches k_{m_1}, there is sufficient control available to enforce (11).

Next we consider the off-diagonal elements of $M(\boldsymbol{q})^{-1}$, where we must address the interference from u_j, $j \ne i$ in $\dot{\bar{b}}_i$, $\dot{\underline{b}}_i$. It is straightforward to see that for $\boldsymbol{u} \in \mathscr{U}$, $\sum_{j\ne i} k_{d_{ij}} u_{max_j} \ge \sum_{j \ne i} m_{ij}(\boldsymbol{q})u_j$. Thus we can say that there exists a $\boldsymbol{u} \in \mathscr{U}$ such that the following relations hold for all $i \in \mathscr{N}$:

$$\begin{aligned} k_{m_1}u_{min_i} + \sum_{j\ne i} k_{d_{ij}}u_{max_j} &\ge m_{ii}(\boldsymbol{q})u_i + \sum_{j\ne i} m_{ij}(\boldsymbol{q})u_j \\ k_{m_1}u_{max_i} - \sum_{j\ne i} k_{d_{ij}}u_{max_j} &\le m_{ii}(\boldsymbol{q})u_i + \sum_{j\ne i} m_{ij}(\boldsymbol{q})u_j \end{aligned} \tag{17}$$

where the right hand side is equivalently $\boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}\boldsymbol{u}$. The motivation for the terms $k_{d_{ij}}u_{max_j}$ is that to ensure that constraint i is satisfied, the control u_i must be able to handle the worst case when all other u_j ($j \ne i$) are at their maximum/minimum values.

Now we address the effect of gravity. By Lemma 1 and Property 3 it follows that $k_{m_2}k_g \ge \sum_j m_{ij}(\boldsymbol{q})g_j(\boldsymbol{q}) = \boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}\boldsymbol{g}(\boldsymbol{q})$. Thus from (17) and (13), we can say that there exists a $\boldsymbol{u} \in \mathscr{U}$ such that the following hold:

$$\begin{aligned} k_{m_1}\tilde{u}_{min_i} &\ge \boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}(\boldsymbol{u}+\boldsymbol{g}) \\ k_{m_1}\tilde{u}_{max_i} &\le \boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}(\boldsymbol{u}+\boldsymbol{g}) \end{aligned} \tag{18}$$

Second, we address the term $\boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}C(\boldsymbol{q},\boldsymbol{v})\boldsymbol{v}$ from (16). From Lemma 1 and Property 2 it follows that $k_{m_2}k_c||\boldsymbol{v}|| \ge ||M^{-1}C||$. From Lemma 2 it follows that $||\boldsymbol{v}||_\infty \le \gamma a$ such that $k_{m_2}k_c a^2\gamma^2 \ge ||\boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}C(\boldsymbol{q},\boldsymbol{v})\boldsymbol{v}||_\infty$, which yields:

$$\begin{aligned} -k_{m_2}k_c a^2\gamma^2 &\le \boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}C(\boldsymbol{q},\boldsymbol{v})\boldsymbol{v} \\ k_{m_2}k_c a^2\gamma^2 &\ge \boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}C(\boldsymbol{q},\boldsymbol{v})\boldsymbol{v} \end{aligned} \tag{19}$$

Third, we address the term $\boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}F\boldsymbol{v}$ from (16), and note that F is diagonal, positive semi-definite such that $f_i \ge 0$. Recall from Lemma 1 that $m_{ii}(\boldsymbol{q}) \ge k_{m_1}$. Now we consider the two cases when $v_i \ge 0$ and $v_i \le 0$. When $v_i \ge 0$ it follows that $k_{m_1}f_iv_i \le m_{ii}f_iv_i$. When $v_i \le 0$ it follows that $k_{m_1}f_iv_i \ge m_{ii}f_iv_i$. Furthermore from Lemma 2 it follows that $\sum_{j\ne i} k_{d_{ij}} f_j \gamma a \ge \sum_{j\ne i} m_{ij}(\boldsymbol{q})f_jv_j$, which yields:

$$\begin{aligned} k_{m_1}f_iv_i - \sum_{j\ne i} k_{d_{ij}}f_j a\gamma &\le \boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}F\boldsymbol{v} \text{ for } v_i \ge 0 \\ k_{m_1}f_iv_i + \sum_{j\ne i} k_{d_{ij}}f_j a\gamma &\ge \boldsymbol{e}_i^T M(\boldsymbol{q})^{-1}F\boldsymbol{v} \text{ for } v_i \le 0 \end{aligned} \tag{20}$$

Substitution of inequalities (18), (19), and (20) in (15) yield (16) (and thus (11)) where (16a) holds for $v_i \ge 0$ and (16b) holds for $v_i \le 0$.

Finally, we will show that there exists a $\boldsymbol{u} \in \mathscr{U}$ such the system (3) upholds Nagumo's condition (11). The boundary of $(\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i$ is defined by the union of $\bar{\mathscr{S}}_i, \underline{\mathscr{S}}_i$ from (12) with $\bar{\mathscr{E}}_i = \{(q_i,v_i) \in \mathbb{R}^2 : \bar{h}_i(q_i) \ge 0, \underline{h}_i(q_i) \ge 0, \bar{b}_i = 0\}$ and $\underline{\mathscr{E}}_i = \{(q_i,v_i) \in \mathbb{R}^2 : \bar{h}_i(q_i) \ge 0, \underline{h}_i(q_i) \ge 0, \underline{b}_i = 0\}$. From (11), we see that for $(q_i,v_i) \in \bar{\mathscr{E}}_i$ (where $v_i \ge 0$) $\dot{\bar{b}}_i \ge -\alpha_2(\bar{b}_i) = 0$. Similarly for $(q_i,v_i) \in \underline{\mathscr{E}}_i$ (where $v_i \le 0$) it follows that $\dot{\underline{b}}_i \ge -\alpha_2(\underline{b}_i) = 0$. From these conditions and from Lemma 3, (11) is satisfied for $i \in \mathscr{N}$ for $\boldsymbol{u} \in \mathscr{U}$. ☐

The result from Lemma 4 effectively reduces the design of the ZCBFs to determining γ and β to satisfy (15). First, we note that for $\bar{b}_i > 0$, $\underline{b}_i > 0$, we can choose β sufficiently large to satisfy (15). We thus define $\beta^* \in \mathbb{R}$ as:

$$\beta^* := \max\Big\{ \max_{\substack{(q_i,v_i) \in (\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i \\ i \in \mathscr{N}}} \frac{\bar{\psi}_i(q_i,v_i)}{\alpha_2(\bar{b}_i(q_i,v_i))}, \max_{\substack{(q_i,v_i) \in (\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i \\ i \in \mathscr{N}}} \frac{\underline{\psi}_i(q_i,v_i)}{\alpha_2(\underline{b}_i(q_i,v_i))} \Big\} \tag{21}$$

By definition, if $\beta \ge \beta^*$ ($\beta > 0$), then (15) holds. However it appears that β^* is ill-defined at the set boundary where $\bar{b}_i = 0$ (resp. $\underline{b}_i = 0$).

Our next task is to design γ to ensure β^* is always well-defined. The solution is to choose γ such that when $\bar{b}_i = 0$ (resp. $\underline{b}_i = 0$), $\bar{\psi}_i < 0$ (resp. $\underline{\psi}_i < 0$), and so the right hand side of (21) is *upper* bounded. To do this, first we set $\bar{b}_i = 0$ (resp. $\underline{b}_i = 0$), where by definition $v_i = \gamma\alpha_1(\bar{h}_i(q_i))$ (resp. $v_i = -\gamma\alpha_1(\bar{h}_i(q_i))$) . We substitute $\bar{b}_i = 0$ (resp. $\underline{b}_i = 0$) into (14) and set $\bar{\psi}_i = 0$ (resp. $\underline{\psi}_i = 0$), which yield the following quadratic expressions:

$$\gamma^2 + \bar{d}_i(q_i)\gamma + \bar{c}_i(q_i) = 0, \;\; \gamma^2 + \underline{d}_i(q_i)\gamma + \underline{c}_i(q_i) = 0 \tag{22}$$

where $\underline{d}_i(q_i) := \dfrac{\sum_{j\ne i} k_{d_{ij}} f_j a - k_{m_1} f_i \alpha_1(\underline{h}_i(q_i))}{k_{m_2}k_c a^2 + \dfrac{\partial \alpha_1}{\partial \underline{h}_i}(q_i)\alpha_1(\underline{h}_i(q_i))}$ and

$\underline{c}_i(q_i) := \dfrac{k_{m_1}\tilde{u}_{min_i}}{k_{m_2}k_c a^2 + \dfrac{\partial \alpha_1}{\partial \underline{h}_i}(q_i)\alpha_1(\underline{h}_i(q_i))}$. We make a few notes regarding (22). First, the definition of $\underline{c}_i$ follows since $\tilde{u}_{max_i} = -\tilde{u}_{min_i}$. Second, the derivation of (22) requires division by $k_{m_2}k_c a^2 + \dfrac{\partial \alpha_1}{\partial \underline{h}_i}(q_i)\alpha_1(\bar{h}_i(q_i))$, which is always positive in $\mathscr{C}_i$. Finally, $\bar{\underline{c}}_i < 0$ from Assumption 1.

Next, we apply the standard quadratic formula to solve (22) over all $q_i \in \mathscr{C}_i$, $i \in \mathscr{N}$, which yields the following definition for $\gamma^* \in \mathbb{R}_{>0}$:

$$\gamma^* = \min\Big\{ \min_{\substack{q_i \in \mathscr{C}_i,\\ i \in \mathscr{N}}} \bar{\phi}(q_i), \; \min_{\substack{q_i \in \mathscr{C}_i,\\ i \in \mathscr{N}}} \underline{\phi}(q_i) \Big\} \tag{23}$$

where $\bar{\underline{\phi}}_i(q_i) := \frac{1}{2}(-\bar{\underline{d}}_i(q_i) + \sqrt{\bar{\underline{d}}_i(q_i)^2 - 4\bar{\underline{c}}_i(q_i)})$. We note that since $\bar{\underline{c}}_i$ is strictly negative, $\bar{\underline{\phi}}_i$ has two real roots. Since there are two solutions to (22), we choose the largest, positive value hence the "+" in $\bar{\underline{\phi}}_i$.

Now it is straightforward to see from (22), (23) that γ^* ensures $\bar{\psi}_i = 0$ (resp. $\underline{\psi}_i = 0$) when $\bar{b}_i = 0$ (resp. $\underline{b}_i = 0$). Thus by choosing $\gamma < \gamma^*$, $\bar{\psi}_i < 0$ when $\bar{b}_i = 0$ (resp. $\underline{\psi}_i < 0$ when $\underline{b}_i = 0$), and so $\beta^* \in \mathbb{R}$ is now well-defined. Consequently the choice of $\beta \geq \beta^*$ ensures (15) is satisfied. To additionally satisfy the velocity bounds of $\mathscr{D}$, we must also ensure $\gamma \leq \frac{1}{a}\min_{i\in\mathscr{N}} v_{\max_i}$, which from Lemma 2 ensures that $\boldsymbol{v} \in \mathscr{D}$. We further note that (23) and (21) can be solved off-line as will be shown in Section IV.

We are now ready to state the main result of this paper:

Theorem 1. *Consider the system* (3) *with the state and input constraints defined by* (4)*,* (5)*, and* (6)*. Let the functions* $\bar{b}_i(q_i, v_i)$ *and* $\underline{b}_i(q_i, v_i)$ *be defined by* (9) *with smooth extended class-*$\mathcal{K}$ *functions* α_1 *and* α_2*, and consider the sets* $\mathscr{C}_i$ (8)*,* $\mathscr{B}_i$ (10)*,* $\forall i \in \mathscr{N}$*. Let* $\tilde{u}_{min_i}$ *and* $\tilde{u}_{max_i}$ *be defined by* (13)*. Suppose Assumption 1 holds. For* γ^* *defined by* (23) *and* β^* *defined by* (21)*, if* $\gamma < min\{\gamma^*, \frac{1}{a}\min_{i\in\mathscr{N}} v_{\max_i}\}$*,* $\gamma > 0$*, and* $\beta \geq \beta^*, \beta > 0$*, then* $((\mathscr{C} \times \mathbb{R}^n) \cap \mathscr{B}) \subset \mathscr{C} \times \mathscr{D}$*,* $((\mathscr{C} \times \mathbb{R}^n) \cap \mathscr{B}) \neq \emptyset$*, and there exists a control* $\boldsymbol{u} \in \mathscr{U}$ *to render* $(\mathscr{C} \times \mathbb{R}^n) \cap \mathscr{B}$ *forward invariant.*

Proof. Under Assumption 1, (23) has a positive, real solution such that $\gamma^* > 0$. Now by definition of $\bar{\psi}_i$ and $\underline{\psi}_i$ from (14) and Lemma 2, $\bar{\psi}_i$ and $\underline{\psi}_i$ are bounded for all $(q_i, v_i) \in (\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i$. The terms $\dfrac{1}{\alpha_2(\bar{b}_i)}$ and $\dfrac{1}{\alpha_2(\underline{b}_i)}$ are bounded over any compact subset of $\text{Int}((\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i)$. Now by choosing $\gamma \in (0, \gamma^*)$ we ensure that on the boundary $\partial\mathscr{B}$, $\dfrac{\bar{\psi}_i}{\alpha_2(\bar{b}_i)} < 0$ and $\dfrac{\underline{\psi}_i}{\alpha_2(\underline{b}_i)} < 0$. Thus $\dfrac{\bar{\psi}_i}{\alpha_2(\bar{b}_i)}$ and $\dfrac{\underline{\psi}_i}{\alpha_2(\underline{b}_i)}$ are upper bounded for all $(q_i, v_i) \in (\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i$, and so β^* from (21) is well-defined. Thus the choice of $\gamma \in (0, \gamma^*)$ and $\beta \geq \beta^*$, $\beta > 0$ satisfies (15) for all $(q_i, v_i) \in (\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i$, $i \in \mathscr{N}$. By Lemma 4, it follows that there exists a $\boldsymbol{u} \in \mathscr{U}$ that satisfies (11). Thus for a given $\boldsymbol{x}$, there exists a $\boldsymbol{u} \in \mathscr{U}$ to satisfy the conditions of Nagumo's Theorem [11] and render $(\mathscr{C} \times \mathbb{R}^n) \cap \mathscr{B}$ forward invariant. Note by ensuring $\gamma > 0$ it follows that $(\mathscr{C} \times \mathbb{R}^n) \cap \mathscr{B}$ is non-empty. Furthermore, since $\gamma < \frac{1}{a}\min_{i\in\mathscr{N}} v_{\max_i}$ it follows from Lemma 2 that $\boldsymbol{v} \in \mathscr{D}$ and $((\mathscr{C} \times \mathbb{R}^n) \cap \mathscr{B}) \subset \mathscr{C} \times \mathscr{D}$. □

Theorem 1, and all the prerequisite analysis, exploits the proposed ZCBF formulation to appropriately restrict the velocities and the required actuator effort near the constraint boundaries. We may thus define a controller that is guaranteed to satisfy position, velocity, and *input* constraints for a Euler-Lagrange system as:

$$\begin{aligned} \boldsymbol{u}^*(\boldsymbol{q}, \boldsymbol{v}) \; &= \underset{\boldsymbol{u}\in\mathscr{U}}{\text{argmin}} \quad ||\boldsymbol{u} - \boldsymbol{u}_{\text{nom}}(\boldsymbol{q}, \boldsymbol{v})||_2^2 \\ &\quad\;\; \text{s.t.} \qquad A(\boldsymbol{q}, \boldsymbol{v})\boldsymbol{u} \succeq \boldsymbol{p}(\boldsymbol{q}, \boldsymbol{v}) \end{aligned} \tag{24}$$

where A is the concatenation of all $L_{\mathfrak{g}}\bar{b}_i, L_{\mathfrak{g}}\underline{b}_i$, $\boldsymbol{p}$ is the concatenation of $-L_{\mathfrak{f}}\bar{b}_i - \beta\alpha_2(\bar{b}_i)$, $-L_{\mathfrak{f}}\underline{b}_i - \beta\alpha_2(\underline{b}_i)$ for $i \in \mathscr{N}$, where $\mathfrak{f}$ and $\mathfrak{g}$ represent the nonlinear affine dynamics of (3). Here $\boldsymbol{u}_{nom}$ can represent a stabilizing control law or a human input for the system (3). Note that the proposed ZCBFs satisfy Definition 2 and are thus robust to model uncertainties [9]. We refer the reader to [4] to address sampled-data implementations of (24) for the proposed ZCBFs.

IV. Numerical Example

Here we apply the proposed ZCBF approach to a 2-DOF planar manipulator. The manipulator consists of two identical links with a length of 1 m and mass of 1 kg, which are parallel to the ground such that $\boldsymbol{g} = 0$. The system is equipped with motors capable of $u_{max_1} = -u_{min_2} = 18$ Nm, and $u_{max_2} = -u_{min_2} = 10$ Nm of torque. The system damping is $F = 0.001I_{2\times 2}$ kg/s. Let the position/velocity safety constraints be defined by $q_{max_1} = -q_{min_1} = \pi/2$ rad, $q_{max_2} = 5\pi/6$ rad, $q_{min_2} = \pi/2$ rad, and $v_{max_{1,2}} = -v_{min_{1,2}} = 1.5$ rad/s. We choose the following extended class-$\mathcal{K}$ functions for the ZCBFs: $\alpha_1(h) = \tan(h)^{-1}$, $\alpha_2(b) = b^3$. The α_1 term was chosen to enlarge the set $\mathscr{B}$ (see Example 1 of [4]). The α_2 term was chosen because it empirically works well in practice [4], [5].

In the following steps we walk through the procedure to define γ and β. We refer to [13] for the full system dynamics. First we identify the bounds on $M(q_2)^{-1}$. Here we will use $k_{m1_i}, k_{m2_i} \in \mathbb{R}_{>0}$ as the respective upper and lower bounds of the diagonals of $M(q_2)^{-1}$ for joint i to reduce the conservatism in (13). After substitution of the model parameters, $M(q_2)^{-1}$ is computed over the constraint set $q_2 \in [\pi/2, 5\pi/6]$ for which we find the following bounds: $k_{m1_1} = 0.5455$, $k_{m2_1} = 0.7869$, $k_{m1_2} = 3.0$, $k_{m2_2} = 3.546$, and $k_{d_{1,2}} = 0.5455$. We similarly find the bound $k_c = 0.9$ for C and set $k_g = 0$. We appropriately substitute k_{m1_i}, k_{m2_i} $i \in \{1, 2\}$ for k_{m_1}, k_{m_2} terms in (13) and compute $\tilde{u}_{max_1} = -\tilde{u}_{min_1} = 8$ and $\tilde{u}_{max_2} = -\tilde{u}_{min_2} = 6.727$. Note that Assumption 1 is satisfied. Now we plot $\bar{\underline{\phi}}_i$ from (23) for $i \in \{1, 2\}$, which is shown in Figure 1. As shown $\bar{\underline{\phi}}_i$ is lower bounded by $\gamma^* = 0.8907$, and so we choose a slightly smaller value of $\gamma = 0.8906$ to ensure $\gamma < \gamma^*$. Next we check the maximum velocity via Lemma 2 $||\boldsymbol{v}|| \leq 0.8906\alpha_1(\delta_q = \pi) = 1.1245$. Thus the velocity safety constraints are satisfied, and we continue by computing β^* from (21). Figure 1 shows the plots of $\bar{\psi}_i(q_i, v_i)/\alpha_2(\bar{b}_i(q, v))$

and $\underline{\psi}_i(q_i, v_i)/\alpha_2(\underline{b}_i(q, v))$ for $(q_i, v_i) \in (\mathscr{C}_i \times \mathbb{R}) \cap \mathscr{B}_i$, $i \in \{1, 2\}$ for which $\beta^* = -0.6482$. Thus we are free to choose $\beta = 10$.

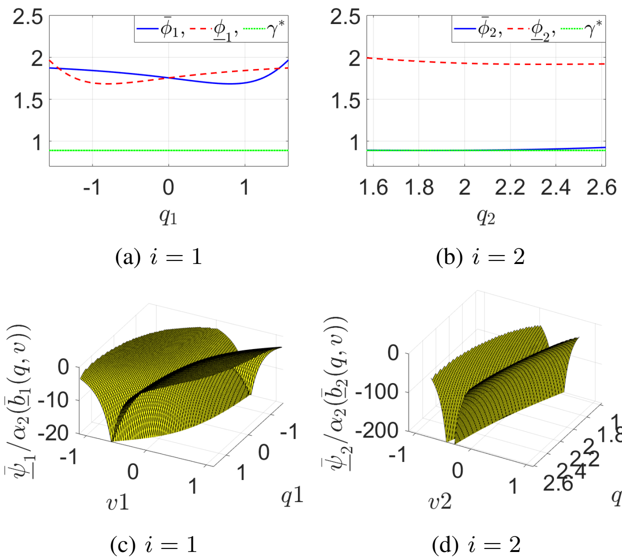

Fig. 1: Plots of $\bar{\underline{\phi}}_i$ and $\bar{\underline{\psi}}_i(q_i, v_i)\ /\alpha_2(\bar{\underline{b}}_i(q_i, v_i))$ for $i \in \{1, 2\}$

Next, we construct the ZCBFs as per (9) and implement the control (24). For this example, the nominal control is the computed torque control law: $u_{nom} = M(q_2)(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{e}} - \boldsymbol{e}) + C\boldsymbol{v}$ [13] where $\boldsymbol{e} = \boldsymbol{q} - \boldsymbol{r}$ and $\boldsymbol{r} = [3.4708 \sin(1.3t), 2.6236 \sin(1.3t) + 2.0944]^T$ is the reference that attempts to move the system outside of $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{U}$. This nominal control is used to represent a pre-defined control law or equivalently a human that is incorrectly operating the system. The implementation of the proposed control shows satisfaction of $\boldsymbol{q} \in \mathscr{C}$, $\boldsymbol{v} \in \mathscr{D}$, and $u \in \mathscr{U}$ as shown in Figure 2.

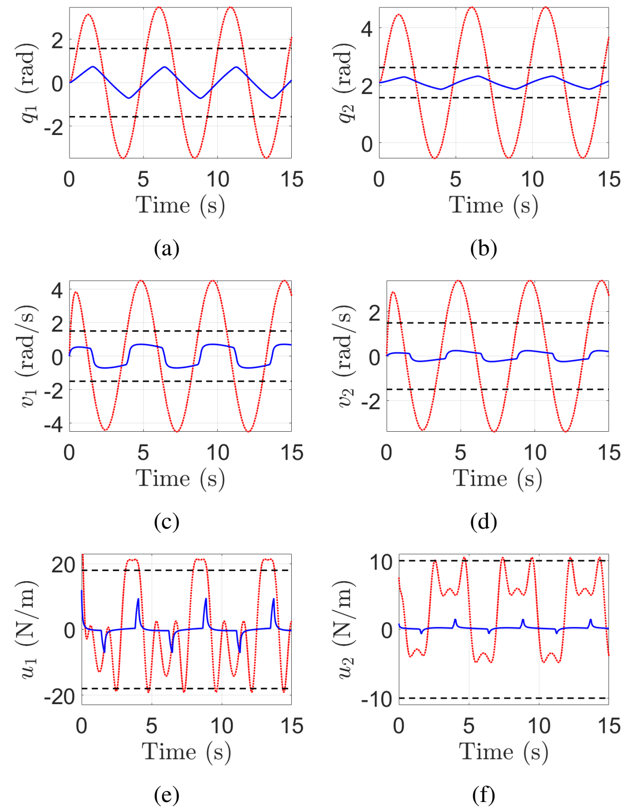

Fig. 2: Plots of $\boldsymbol{q}$, $\boldsymbol{v}$, and $\boldsymbol{u}$ for the control $\boldsymbol{u} = \boldsymbol{u}^*$ from (24) (blue-solid line) and $\boldsymbol{u} = \boldsymbol{u}_{nom}$ (red-dotted line). The black-dashed lines depict the boundaries of $\mathscr{C}_i$ in (a), (b), $\mathscr{D}_i$ in (c), (d), and $\mathscr{U}_i$ in (e), (f), respectively for $i \in \{1, 2\}$.

V. Conclusion

In this paper we design ZCBFs for Euler-Lagrange systems. The proposed approach ensures satisfaction of position, velocity, *and* input constraints. The ZCBF parameters can be computed offline as demonstrated by the numerical example. Simulation results are used to validate the proposed approach. Future work will investigate how the choice of α_1 and α_2 affect the design process for better performance of the proposed approach.

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