Robust Tube-based Model Predictive Control for Time-constrained Robot Navigation

Alexandros Nikou and Dimos V. Dimarogonas

***Abstract*— This paper deals with the problem of time-constrained navigation of a robot modeled by uncertain non-linear non-affine dynamics in a bounded workspace of $\mathbb{R}^n$. Initially, we provide a novel class of robust feedback controllers that drive the robot between Regions of Interest (RoI) of the workspace. The control laws consists of two parts: an on-line controller which is the outcome of a Finite Horizon Optimal Control Problem (FHOCP); and a backstepping feedback law which is tuned off-line and guarantees that the real trajectory always remains in a bounded hyper-tube centered along the nominal trajectory of the robot. The proposed controller falls within the so-called tube-based Nonlinear Model Predictive control (NMPC) methodology. Then, given a desired high-level specification for the robot in Metric Interval Temporal Logic (MITL), by utilizing the aforementioned controllers, a framework that provably guarantees the satisfaction of the formula is provided. The proposed framework can handle the rich expressiveness of MITL in both safety and reachability specifications. Finally, the proposed framework is validated by numerical simulations.**

I. Introduction

Navigation is an important field in both the robotics and the control communities, due to the need for autonomous control. In another line of research, the problem of controlling systems under *high-level specifications* has been gaining significant research attention. The qualitative specification language that has primarily been used to express the high-level tasks that are required to be completed within *certain time bounds* is *Metric Interval Temporal Logic (MITL)*. MITL has been originally proposed in [1] and has been used for controller synthesis in [2], [3]. Given robot dynamics and an MITL formula, the controller synthesis procedure is as follows: first, the robot dynamics are abstracted into a discrete representation, the so-called Weighted Transition System (WTS), in which the time duration for navigating between states is modeled by a weight in WTS (abstraction); second, a product between the WTS and an automaton that accepts the runs that satisfy the given formula is computed; and third, once an accepting run in the product found, it maps into a sequence of feedback controllers of the robot dynamics.

The main focus of this paper is the first part of the aforementioned procedure. In particular, we aim to provide an *abstraction of robot dynamics* into WTS in such a way that the rich MITL expressiveness in both reachability and safety specifications is exploited. In order address this problem, and due to the fact that the robot dynamics are highly nonlinear, under state/input constraints as well as under the presence of external disturbances/unmodeled dynamics, a NMPC framework is used [4]–[6].

One of the main challenges in NMPC is the efficient handling of the external disturbances/uncertainties. A promising robust strategy, originally proposed for discrete-time linear systems in [7], is the so called tube-based approach. Tube-based approaches for affine in the control continuous-time nonlinear systems with constant matrices multiplying the control input vectors have been proposed in [8], which we aim to extend here in order to cover a larger class of nonlinear systems. Preliminary results in tube-based NMPC for nonlinear non-affine systems that state and input space have the same dimension can be found in our earlier work [9]. In the current paper, these results are extended to under-actuated systems, which arise in robotic applications with Lagrangian kinematics/dynamics models and their controller design constitutes a challenging task.

By taking into consideration the aforementioned, the contribution of this paper is twofold: given a robot modeled by uncertain nonlinear non-affine dynamics and a workspace with RoI, under standard NMPC and controllability assumptions, a systematic control design methodology for tube-based NMPC which guarantees robust navigation between RoI under safety constraints is developed; we exploit the aforementioned control design in order to abstract the dynamics of the robot into a WTS. Then, given an MITL formula that the robot needs to satisfy for all times, by performing an MITL control synthesis procedure, a sequence of control laws that guarantees the satisfaction of the formula is provided.

Due to space constraints we have omitted the proofs or the theorems and lemma. A more detailed version of the paper can be found in [12].

II. Notation and Preliminaries

Given a vector $y \in \mathbb{R}^n$ denote by $\|y\|_2 := \sqrt{y^\top y}$ and $\|y\|_B := \sqrt{y^\top B y}$, $B \geq 0$ its Euclidean and weighted norm, respectively; $\lambda_{\min}(B)$ stands for the minimum absolute value of the real part of the eigenvalues of $B \in \mathbb{R}^{n\times n}$; $0_{m\times n} \in \mathbb{R}^{m\times n}$ and $I_n \in \mathbb{R}^{n\times n}$ stand for the $m \times n$ matrix with all entries zeros and the identity matrix, respectively. The notation $\mathrm{diag}\{B_1, \ldots, B_n\}$ stands for the block diagonal matrix with the matrices $B_1, \ldots, B_n$ in the main diagonal; $\mathcal{B}(\chi, r) := \{y \in \mathbb{R}^n : \|y-\chi\|_2 \leq r\}$ stands for the n-th dimensional ball with center and radius $\chi \in \mathbb{R}^n$, $r > 0$, respectively. Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, $\frac{\partial f_i}{\partial x_j}$ denotes the element of row i and column j of the Jacobian matrix of f, with $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Given two vectors $x, y \in \mathbb{R}^n$ their *convex hull* is defined by: $\mathcal{C}(x, y) := \{\theta x + (1-\theta)y : \theta \in (0, 1)\}$. A vector of a canonical basis

The authors are with the School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Stockholm, Sweden. E-mail: {anikou,dimos}@kth.se. This work was supported by the H2020 ERC Grant BUCOPHSYS, the EU H2020 Co4Robots project, the Swedish Foundation for Strategic Research (SSF), the Swedish Research Council (VR) and the Knut och Alice Wallenberg Foundation (KAW).

of $\mathbb{R}^n$ is defined by:

$$\mathfrak{e}_n^i := \Big[0, \dots, 0, \underbrace{1}_{i-\text{th element}}, 0, \dots, 0\Big]^\top. \quad (1)$$

Given sets $\mathcal{S}_1$, $\mathcal{S}_2 \subseteq \mathbb{R}^n$ and matrix $B \in \mathbb{R}^{n\times m}$, the *Minkowski addition*, the *Pontryagin difference* and the *matrix-set multiplication* are respectively defined by: $\mathcal{S}_1 \oplus \mathcal{S}_2 := \{s_1 + s_2 : s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\}$, $\mathcal{S}_1 \ominus \mathcal{S}_2 := \{s_1 : s_1 + s_2 \in \mathcal{S}_1, \forall s_2 \in \mathcal{S}_2\}$, and $B \circ \mathcal{S} := \{b : \exists s \in \mathcal{S}, b = Bs\}$.

Lemma 1. *[9] For any constant $\rho > 0$, vectors y_1, $y_2 \in \mathbb{R}^n$ and matrix $B \in \mathbb{R}^{n\times n}$, $B > 0$ it holds that $y_1 B y_2 \leq \frac{1}{4\rho} y_1^\top B y_1 + \rho y_2^\top B y_2$.*

Proposition 1. *(Mean value theorem for vector valued functions [10]) Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which is differentiable on an open set $\mathcal{S} \subseteq \mathbb{R}^n$. Let x, y two points of $\mathcal{S}$ such that $\mathcal{C}(x,y) \subseteq \mathcal{S}$. Then, there exist constant vectors $\varpi_1, \dots, \varpi_m \in \mathcal{C}(x,y)$ such that:*

$$f(x) - f(y) = \left[\sum_{k=1}^{m}\sum_{j=1}^{n} \mathfrak{e}_m^k (\mathfrak{e}_n^j)^\top \frac{\partial f_k(\varpi_k)}{\partial x_j}\right](x - y). \quad (2)$$

Definition 1. [9] Consider a dynamical system $\dot{\chi} = f(\chi, u, d)$ where: $\chi \in \mathcal{X}$, $u \in \mathcal{U}$, $d \in \mathcal{D}$ with initial condition $\chi(0) \in \mathcal{X}$. A set $\mathcal{X}' \subseteq \mathcal{X}$ is a *Robust Control Invariant (RCI) set* for the system, if there exists a feedback control law $u := \kappa(\chi) \in \mathcal{U}$, such that for all $\chi(0) \in \mathcal{X}'$ and for all $d(t) \in \mathcal{D}$ it holds that $\chi(t) \in \mathcal{X}'$ for all $t \geq 0$, along every solution $\chi(t)$.

Definition 2. [2] A *Weighted Transition System* (WTS) is a tuple $(S, S_0, \mathrm{Act}, \longrightarrow, \mathfrak{t}, \Sigma, L)$ where S is a finite set of states; $S_0 \subseteq S$ is a set of initial states; Act is a set of actions; $\longrightarrow \subseteq S \times \mathrm{Act} \times S$ is a transition relation; $\mathfrak{t} : \longrightarrow \to \mathbb{Q}_+$ is a map that assigns a positive weight to each transition; Σ is a finite set of atomic propositions (an *atomic proposition* $\sigma \in \Sigma$ is a statement that is either true or false); and $L : S \to 2^\Sigma$ is a labeling function.

Definition 3. [2] A *timed run* of a WTS is an infinite sequence $r^t = (r(0), \tau(0))(r(1), \tau(1))\dots$, such that $r(0) \in S_0$, and for all $l \geq 0$, it holds that $r(l) \in S$ and $(r(l), u(l), r(l+1)) \in \longrightarrow$ for a sequence of actions $u(0)u(1)u(2)\dots$ with $u(l) \in \mathrm{Act}$, $\forall l \geq 0$. The *time stamps* $\tau(l)$, $l \geq 0$ are inductively defined as: 1) $\tau(0) = 0$; 2) $\tau(l+1) := \tau(l) + \mathfrak{t}(r(l), r(l+1))$, $\forall l \geq 0$.

The syntax of MITL (see [1]) over a set of atomic propositions Σ is defined by the grammar:

$$\varphi := \sigma \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \bigcirc_I \varphi \mid \Diamond_I \varphi \mid \Box_I \varphi \mid \varphi_1 \,\mathcal{U}_I\, \varphi_2,$$

where $\sigma \in \Sigma$, and $\bigcirc$, $\Diamond$, $\Box$ and $\mathcal{U}$ are the next, eventually, always and until temporal operator, respectively; $\neg$, $\wedge$ are the negation and conjunction operators, respectively; $I = [a, b] \subseteq \mathbb{Q}_+$ where $a, b \in [0, \infty]$ with $a < b$ is a non-empty timed interval. MITL formulas are interpreted over timed runs like the ones produced by a WTS which is given in Definition 3. For the semantics of MITL see [1]. Any MITL formula φ over Σ can be algorithmically translated into a Timed Büchi Automaton (TBA) with the alphabet 2^Σ, such that the language of timed words that satisfy φ is the language of timed words produced by the TBA [11].

III. Problem Formulation

A. System Model

Consider a robot operating in a workspace $\mathcal{W} \subseteq \mathbb{R}^n$ governed by the following *uncertain kinematics-dynamics model*:

$$\dot{\chi} = v, \quad (3a)$$
$$\dot{v} = f(\chi, v, u) + d, \quad (3b)$$

where $\chi \in \mathcal{W}$ denotes the state of the robot in the workspace (position, orientation); $v \in \mathbb{R}^n$ stands for the velocity; $u \in \mathbb{R}^n$ is the control input; $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear function; and $d \in \mathbb{R}^n$ stands for external disturbances, uncertainties and unmodeled dynamics. The velocity is constrained in a connected set $\mathcal{V} \subseteq \mathbb{R}^n$ which contains the origin. The control inputs need to satisfy $u \in \mathcal{U}$, where $\mathcal{U}$ is a convex set containing the origin. Consider also bounded disturbances $d \in \mathcal{D} := \{d \in \mathbb{R}^n : \|d\|_2 \leq \widetilde{d}\}$, where $\widetilde{d} > 0$. Define the corresponding *nominal dynamics* for (3a)-(3b) by:

$$\dot{\overline{\chi}} = \overline{v}, \quad (4a)$$
$$\dot{\overline{v}} = f(\overline{\chi}, \overline{v}, \overline{u}), \quad (4b)$$

where $w \equiv 0$, $\overline{\chi} \in \mathcal{W}$, $\overline{v} \in \mathcal{V}$ and $\overline{u} \in \mathcal{U}$. Define $J : \mathcal{W} \times \mathcal{V} \times \mathcal{U} \to \mathbb{R}^n \times \mathbb{R}^n$ by:

$$J(\chi, v, u) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathfrak{e}_n^i (\mathfrak{e}_n^j)^\top \frac{\partial f_i(\chi, v, u)}{\partial u_j}, \quad (5)$$

where f_i is the i-th component of the vector-valued function f, and $\mathfrak{e}_n^i$, $\mathfrak{e}_n^j$ as given in (1).

Assumption 1. f is *continuously differentiable* with respect to x, v and u in $\mathcal{W} \times \mathcal{V} \times \mathcal{U}$ with $f(0,0,0) = 0$.

Assumption 2. The linear system $\dot{\overline{\eta}} = A\overline{\eta} + B\overline{u}$, where $\overline{\eta} := [\overline{\chi}^\top, \overline{v}^\top]^\top \in \mathbb{R}^{2n}$, that is the outcome of the Jacobian linearization of the nominal dynamics (4a)-(4b) around the equilibrium state $(\chi, v) = (0, 0)$ is stabilizable.

Assumption 3. There exists a constant $\underline{J}$ such that:

$$\lambda_{\min}\left[\frac{J(\cdot) + J(\cdot)^\top}{2}\right] \geq \underline{J} > 0, \forall \chi \in \mathcal{W}, v \in \mathcal{V}, u \in \mathcal{U}. \quad (6)$$

In the given workspace, there exist $\mathfrak{m} \in \mathbb{N}$ Regions of Interest (RoI) labeled by $\mathcal{M} := \{1, \dots, \mathfrak{m}\}$. Without loss of generality, assume that the RoI are modeled by balls, i.e., $\mathcal{R}_m := \mathcal{B}(y_m, p_m)$, $m \in \mathcal{M}$, where y_m and $p_m > 0$ stands for the center and radius of RoI $\mathcal{R}_m$, respectively. Define also the the union of RoI by $\mathcal{R} := \bigcup_{m \in \mathcal{M}} \mathcal{R}_m$.

Due to the fact that we are interested in imposing safety specifications to the robot, at each time $t \geq 0$, the robot is occupying a ball $\mathcal{B}(\chi(t), \mathfrak{r})$ that covers its volume, where $\chi(t)$ and $\mathfrak{r} > 0$ are its center and radius, respectively. Assume that $\min_{m \in \mathcal{M}} \{p_m\} > \mathfrak{r}$, which means that the RoI have sufficiently larger volume than the robot.

B. Objectives

The control objective is the navigation of the robot with dynamics as in (3a)-(3b) between RoI so that it obeys a given high-level timed specification over atomic tasks. Atomic tasks are captured through a given finite set of atomic propositions Σ. Each RoI is labeled with atomic propositions that hold true there. Define the labeling function:

$$L : \mathcal{R} \to 2^{\Sigma}, \tag{7}$$

which maps each RoI with a subset of atomic propositions that hold true there. Note that some of the RoI may be assigned with labels that indicate *unsafe regions*, i.e., the robot is required to avoid visiting them (*safety specifications*).

Definition 4. A trajectory $x(t)$ is *uniquely associated with a timed run* $r^t = (r(0), \tau(0))\ (r(1), \tau(1))(r(2), \tau(2))\ldots$, where $r(l) \in \mathcal{R}$, $\forall l \in \mathbb{N}$, is a sequence of RoI that the robot crosses, if the following hold:

1) $\tau(0) = 0$, i.e., the robot starts the motion at time $t = 0$;
2) $\mathcal{B}(x(\tau(0)), \mathfrak{r}) \subsetneq r(0)$, i.e., initially, the volume of the robot is entire within the RoI $r(0) \in \mathcal{R}$;
3) $\mathcal{B}(x(\tau(l)), \mathfrak{r}) \subsetneq r(l)$, $\forall l \in \mathbb{N}$, i.e., the robot changes discrete state *only when* its entire volume is contained in the corresponding RoI;
4) $\tau(l+1) \coloneqq \tau(l) + \mathfrak{t}(r(l), r(l+1))$, $\forall l \in \mathbb{N}$, where:
$$\mathfrak{t} : \mathcal{R} \times \mathcal{R} \to \mathbb{Q}_{+}, \tag{8}$$
is a function that models the duration that the robot needs to be driven between regions $r(l)$ and $r(l+1)$.

Definition 5. A trajectory $\chi(t)$ *satisfies* an MITL formula φ over the set of atomic propositions Σ, formally written as $\chi(t) \models \varphi$, $\forall t \geq 0$, if and only if there exists a timed run r^t to which the trajectory $\chi(t)$ is uniquely associated, according to Definition 4, which satisfies φ.

C. Problem Statement

The problem considered in this paper is stated as follows:

Problem 1. Consider a robot governed by dynamics (3a)-(3b), covered by the ball $\mathcal{B}(\chi(t), \mathfrak{r})$, operating in the workspace $\mathcal{W} \subseteq \mathbb{R}^n$. The workspace contains the RoI $\mathcal{R}_m$, $m \in \mathcal{M}$ modeled also by balls. Given a task specification formula φ expressed in MITL over the set of atomic propositions Σ and labeling functions L as in (7). Then, for every $d \in \mathcal{D}$, design a feedback control law $u = \kappa(\chi, v) \in \mathcal{U}$ such that the robot trajectory in the workspace fulfills the MITL specification φ, i.e., $\chi(t) \models \varphi$, $\forall t \geq 0$, according to Definition 4. Moreover, the robot is required to remain in the workspace for all times.

Remark 1. Note that Problem 1 constitutes a general time-constrained navigation problem due to the fact that the dynamics (3a)-(3b) arise in many robotic applications. Furthermore, the rich expressiveness of MITL in both reachability and safety specifications can be exploited.

IV. Main Results

In this section, the aforementioned control design problem is addressed by taking the following steps:

1) For navigating the robot between RoI, we propose a robust NMPC feedback law that has two components: an on-line control law which is the outcome of a Finite Horizon Optimal Control Problem (FHOCP) solved at each sampling time; a state feedback law whose gain is designed off-line and guarantees that the trajectory of the closed loop system remains in a hyper-tube for all times. (Section IV-A)
2) Then, the dynamics (3a)-(3b) are abstracted into a WTS, exploiting the fact that the timed runs in the WTS project onto uniquely associated trajectories according to Definition 4. (Section IV-B)
3) By invoking ideas from our previous works [2], [3], a controller synthesis procedure that gives a sequence of control laws that serve as solution to Problem 1 is performed. (Section IV-C).

A. Feedback Control Design

Consider a robot with dynamics (3a)-(3b) occupying a RoI $\mathcal{R}_{\mathrm{s}} \in \mathcal{R}$ at time $\mathfrak{t}_{\mathrm{s}} \geq 0$. The feedback control law needs to guarantee that the robot is navigated towards a desired RoI $\mathcal{R}_{\mathrm{d}} \in \mathcal{R}$, $\mathcal{R}_{\mathrm{s}} \neq \mathcal{R}_{\mathrm{d}}$ without intersecting with any other RoI, due to the fact that safety specifications are required. Denote by $\chi_{\mathrm{d}} \in \mathcal{R}_{\mathrm{d}}$ the center of the desired RoI $\mathcal{R}_{\mathrm{d}}$. Define the error vector $e \coloneqq \chi - \chi_{\mathrm{d}} \in \mathbb{R}^n$. The *uncertain error kinematics/dynamics* are given by:

$$\dot{e} = v, \tag{9a}$$
$$\dot{v} = f(e + \chi_{\mathrm{d}}, v, u) + d, \tag{9b}$$

and the corresponding *nominal error kinematics/dynamics by:*

$$\dot{\overline{e}} = \overline{v}, \tag{10a}$$
$$\dot{\overline{v}} = f(\overline{e} + \chi_{\mathrm{d}}, \overline{v}, \overline{u}). \tag{10b}$$

By recalling that $\mathcal{B}(\chi(t), \mathfrak{r})$ stands for the volume of the robot at time t, define the set that captures the state constraints by:

$$\mathcal{X} \coloneqq \{\chi(t) \in \mathbb{R}^n : \mathcal{B}(\chi(t), \mathfrak{r}) \subsetneq \mathcal{W}, \\ \mathcal{B}(\chi(t), \mathfrak{r}) \cap \{\mathcal{R} \backslash \{\mathcal{R}_{\mathrm{s}}, \mathcal{R}_{\mathrm{d}}\}\} = \emptyset\}.$$

The two constraints refer to the fact that the robot needs to remain in the workspace for all times and the fact that it should not intersect with any other RoI except from $\mathcal{R}_{\mathrm{s}}$, $\mathcal{R}_{\mathrm{d}}$. In order to translate the aforementioned constraints for the error state e, define the set $\mathcal{E} \coloneqq \{e \in \mathbb{R}^n : e \in \mathcal{X} \oplus (-\chi_{\mathrm{d}})\}$, where $\oplus$ is the Minkowski addition operator given in Section II. Under this modification, by using basic properties of Minkowsi operator $\oplus$, it is guaranteed that $\chi \in \mathcal{X} \Leftrightarrow e \in \mathcal{E}$. Consider the *feedback control law:*

$$u \coloneqq \overline{u}(\overline{e}, \overline{v}) + \kappa(e, v, \overline{e}, \overline{v}), \tag{11}$$

which consists of a nominal control action $\overline{u}(\overline{e}, \overline{v}) \in \mathcal{U}$ and a state feedback law $\kappa : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. The control action $\overline{u}(\overline{e}, \overline{v})$ will be the outcome of a nominal FHOCP which is solved on-line at each sampling time. The feedback law $\kappa(\cdot)$ is used to guarantee that the real states e, v remain in a bounded hyper-tube centered along the nominal states $\overline{e}$, $\overline{v}$. Define by $\widetilde{e} \coloneqq e - \overline{e} \in \mathbb{R}^n$, $\widetilde{v} \coloneqq v - \overline{v} \in \mathbb{R}^n$ the deviation between the real states of the uncertain system (9a)-(9b) and the states of the nominal system (10a)-(10b) with $\widetilde{e}(0) = \widetilde{v}(0) = 0$. It will be proved hereafter that the states $\widetilde{e}$, $\widetilde{v}$ are invariant in a compact set whose volume depends on the bounds of the derivatives of f as well as the bound

$\widetilde{d}$. The dynamics of the states $\widetilde{e}$, $\widetilde{v}$ are written as:

$$\dot{\widetilde{e}} = \dot{e} - \dot{\overline{e}} = v - \overline{v} = \widetilde{v}, \tag{12a}$$

$$\begin{aligned}\dot{\widetilde{v}} &= \dot{v} - \dot{\overline{v}} = f(e+\chi_{\rm d}, v, u) - f(\overline{e}+\chi_{\rm d}, \overline{v}, \overline{u}) + d \\ &= f(e+\chi_{\rm d}, v, u) - f(\overline{e}+\chi_{\rm d}, \overline{v}, u) \\ &\quad f(\overline{e}+\chi_{\rm d}, \overline{v}, u) - f(\overline{e}+\chi_{\rm d}, \overline{v}, \overline{u}) + d \\ &= \lambda(e, \overline{e}, v, \overline{v}, u, \overline{u}) + f(\overline{e}+\chi_{\rm d}, \overline{v}, u) \\ &\qquad - f(\overline{e}+\chi_{\rm d}, \overline{v}, \overline{u}) + d. \end{aligned} \tag{12b}$$

where the function λ is defined by $\lambda(e, \overline{e}, v, \overline{v}, u, \overline{u}) := f(e+\chi_{\rm d}, v, u) - f(\overline{e}+\chi_{\rm d}, \overline{v}, u)$ and is upper bounded by:

$$\begin{aligned}\|\lambda(\cdot)\|_2 &\le \|f(e+\chi_{\rm d}, v, u) - f(\overline{e}+\chi_{\rm d}, v, u)\|_2 \\ &\quad + \|f(\overline{e}+\chi_{\rm d}, v, u) - f(\overline{e}+\chi_{\rm d}, \overline{v}, u)\|_2 \\ &\le L_1\|\widetilde{e}\|_2 + L_2\|\widetilde{v}\|_2 \le L(\|\widetilde{e}\|_2 + \|\widetilde{v}\|_2).\end{aligned}$$

The constants L_1, L_2 stand for the Lipschitz constants of function f with respect to variables χ and v, respectively, and $L := \max\{L_1, L_2\}$.

Lemma 2. *([12]) The state feedback law designed as:*

$$\kappa(e, \overline{e}, v, \overline{v}) = -k(e-\overline{e}) - k(v - \overline{v}), \tag{13}$$

where $k > 0$ is chosen such that:

$$k > \frac{1}{\underline{J}}\left[(1+2\rho)L + \frac{5}{4}\right], \ \rho > \frac{L}{2}, \tag{14}$$

renders the sets:

$$\Omega_1 := \left\{\widetilde{e}(t) \in \mathbb{R}^n : \|\widetilde{e}(t)\|_2 \le \frac{\widetilde{d}}{\sqrt{\min\{\alpha_1,\alpha_2\}}}, \forall t \ge 0\right\}, \tag{15a}$$

$$\Omega_2 := \left\{\widetilde{v}(t) \in \mathbb{R}^n : \|\widetilde{v}(t)\|_2 \le \frac{2\widetilde{d}}{\sqrt{\min\{\alpha_1,\alpha_2\}}}, \forall t \ge 0\right\}, \tag{15b}$$

RCI sets for the error dynamics (12a), (12b)*, according to Definition 1. The constants α_1, $\alpha_2 > 0$ are defined by:*

$$\alpha_1 := 1 - \frac{L}{2\rho}, \alpha_2 := k\underline{J} - (1+2\rho)L - \frac{5}{4}. \tag{16}$$

Remark 2. According to Lemma 2, the volume of the hyper-tube which is centered along the nominal trajectories $\overline{e}(t)$, $\overline{v}(t)$ of system (10a)-(10b), depends on $\widetilde{d}$, which is the upper bound of the disturbances d, and on L, $\underline{J}$, which are the bounds of the derivatives of f. By tuning the parameters k, ρ from (14) appropriately, the volume of the tube can be adjusted.

Assumption 4. It holds that:

$$\inf_{\substack{m,m' \in \mathcal{M}, \\ m \neq m'}} \|p_m - p_{m'}\|_2 > 2\mathfrak{r} + \frac{2\widetilde{d}}{\sqrt{\min\{\alpha_1,\alpha_2\}}},$$

i.e., there is sufficient space between any RoI such that the robot can navigate without intersecting them.

Consider a sequence of sampling times $\{t_k\}$, $k \in \mathbb{N}$, with a constant sampling period $0 < h < T$, where T is a *finite prediction horizon* such that $t_{k+1} := t_k + h$, $\forall k \in \mathbb{N}$. At every discrete sample time, a FHOCP is solved as follows:

$$\min_{\overline{u}(\cdot)} \left\{ \|\overline{\xi}(t_k+T)\|_P^2 + \int_{t_k}^{t_k+T} \left[\|\overline{\xi}(\mathfrak{s})\|_Q^2 + \|\overline{u}(\mathfrak{s})\|_R^2\right] d\mathfrak{s} \right\} \tag{17a}$$

subject to:

$$\dot{\overline{\xi}}(\mathfrak{s}) = g(\overline{\xi}(\mathfrak{s}), \overline{u}(\mathfrak{s})), \quad \overline{\xi}(t_k) = \xi(t_k), \tag{17b}$$

$$\overline{\xi}(\mathfrak{s}) \in \overline{\mathcal{E}} \times \overline{\mathcal{V}}, \quad \overline{u}(\mathfrak{s}) \in \overline{\mathcal{U}}, \quad \forall \mathfrak{s} \in [t_k, t_k+T], \tag{17c}$$

$$\overline{\xi}(t_k+T) \in \mathcal{F}, \tag{17d}$$

where $\xi := [e^\top, v^\top]^\top \in \mathbb{R}^{2n}$, $g(\xi, u) := \begin{bmatrix} v \\ f(e+\chi_{\rm d}, v, u)\end{bmatrix}$; Q, $P \in \mathbb{R}^{2n\times 2n}$ and $R \in \mathbb{R}^{n \times n}$ are positive definite gain matrices to be appropriately tuned. We will explain hereafter the sets $\overline{\mathcal{E}}$, $\overline{\mathcal{V}}$, $\overline{\mathcal{U}}$ and $\mathcal{F}$.

In order to guarantee that while the FHOCP (17a)-(17d) is solved for the nominal dynamics (10a)-(10b), the real states e, v and control inputs u satisfy the corresponding state $\mathcal{E}$, $\mathcal{V}$ and input constraints $\mathcal{U}$, respectively, the following modification is performed:

$$\overline{\mathcal{E}} := \mathcal{E} \ominus \Omega_1, \quad \overline{\mathcal{V}} := \mathcal{V} \ominus \Omega_2, \quad \overline{\mathcal{U}} := \mathcal{U} \ominus \left[\Lambda \circ \overline{\Omega}\right],$$

with $\Lambda := \mathrm{diag}\{-kI_n, -kI_n\} \in \mathbb{R}^{2n \times 2n}$, $\overline{\Omega} := \Omega_1 \times \Omega_2$, the operators $\ominus$, $\circ$ as defined in Section II, and Ω_1, Ω_2 as given in (15a), (15b), respectively. Intuitively, the sets $\mathcal{E}$, $\mathcal{V}$ and $\mathcal{U}$ are tightened accordingly, in order to guarantee that while the nominal states $\overline{e}$, $\overline{v}$ and the nominal control input $\overline{u}$ are calculated, the corresponding real states e, v and real control input u satisfy the state and input constraints $\mathcal{E}$, $\mathcal{V}$ and $\mathcal{U}$, respectively. This constitutes a standard constraints set modification technique adopted in tube-based NMPC frameworks (for more details see [8]). Define the *terminal set* by:

$$\mathcal{F} := \left\{\overline{\xi} \in \overline{\mathcal{E}} \times \overline{\mathcal{V}} : \|\overline{\xi}\|_P \le \epsilon\right\}, \quad \epsilon > 0, \tag{18}$$

which is used to enforce the stability of the system [5]. In particular, supposing that Assumption 1, 2 hold, it can be proven that (see [5, Lemma 1, p. 4]), for a sufficiently small ϵ, there exists a *local controller* $u_{\rm loc} := K\overline{\xi} \in \overline{\mathcal{U}}$, $K > 0$ which guarantees that:

$$\frac{d}{dt}\left(\|\overline{\xi}\|_P^2\right) \le -\|\overline{\xi}\|_{\widetilde{Q}}^2, \forall \overline{\xi} \in \mathcal{F},$$

with $\widetilde{Q} := Q + K^\top R K > 0$. The following theorem guarantees the robust robot navigation from RoI $\mathcal{R}_{\rm s}$ to RoI $\mathcal{R}_{\rm d}$ without intersecting any other RoI and always remaining in the workspace $\mathcal{W}$.

Theorem 1. *([12]) Suppose that Assumptions 1-4 hold. Let $\mathfrak{t}_{\rm s} \ge 0$ be the time at which the robot occupies RoI $\mathcal{R}_{\rm s}$ and $\chi_{\rm d}$ be the center of a desired RoI $\mathcal{R}_{\rm d}$. Suppose also that the FHOCP* (17a)-(17d) *is feasible at time $\mathfrak{t}_{\rm s}$. Then, the feedback control law* (11) *applied to the system* (9a)-(9b) *guarantees that there exists a finite time $\mathfrak{t}_{\rm d} > \mathfrak{t}_{\rm s}$ such that $\forall t \ge \mathfrak{t}_{\rm d}$ it holds that:*

$$\|\chi(t) - \chi_{\rm d}\|_2 \le \frac{\epsilon}{\sqrt{\lambda_{\min}(P)}} + \frac{\widetilde{d}}{\sqrt{\min\{\alpha_1,\alpha_2\}}}, \tag{19a}$$

$$\|v(t)\|_2 \le \frac{\epsilon}{\sqrt{\lambda_{\min}(P)}} + \frac{2\widetilde{d}}{\sqrt{\min\{\alpha_1,\alpha_2\}}}. \tag{19b}$$

By observing (19a) it holds that at time $\mathfrak{t}_{\rm d}$ the error $\|\chi(t) - \chi_{\rm d}\|_2$ has reached the steady-state, i.e., the robot has been navigated to the desired RoI $\mathcal{R}_{\rm d}$ at time $\mathfrak{t}_{\rm d}$. Recalling (8) and taking into consideration the aforementioned discussion, $\mathfrak{t}_{\rm d}$ models the time that the robot needs to be driven from $\mathcal{R}_{\rm s}$ to $\mathcal{R}_{\rm d}$, i.e., $\mathfrak{t}_{\rm d} = \mathfrak{t}(\mathcal{R}_{\rm s}, \mathcal{R}_{\rm d})$, and it can be computed by Algorithm 1. Intuitively, as time evolves, the norm of the states of the robot is being monitored at each sampling time t_k until the inequalities of line 7 of Algorithm 1 are satisfied, i.e., when the trajectory of the robot has reached the steady-state. When they are satisfied, the robot is within RoI $\mathcal{R}_{\rm d}$ and the time constant $\mathfrak{t}_{\rm d} = \mathfrak{t}(\mathcal{R}_{\rm s}, \mathcal{R}_{\rm d})$ has been computed.

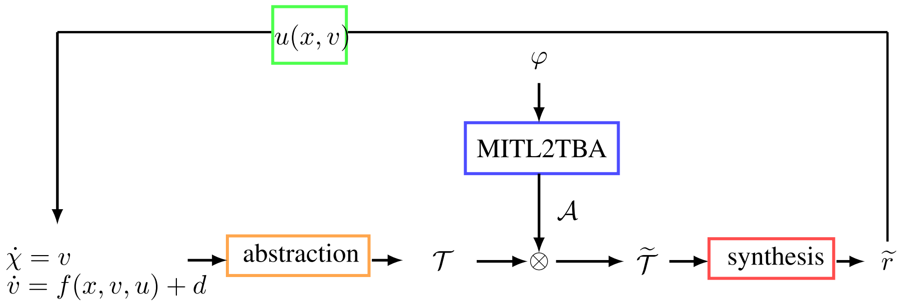

Fig. 1: A graphical illustration of the combined abstraction and controller synthesis framework.

Algorithm 1 Computation of $\mathfrak{t}_{\mathrm{d}} := \mathfrak{t}(\mathcal{R}_{\mathrm{s}}, \mathcal{R}_{\mathrm{d}})$

1: **Input**: $\mathfrak{t}_{\mathrm{s}}$, $\overline{\chi}(t_k)$, $\overline{v}(t_k)$, $k \in \mathbb{N}$;
2: **Output**: $\mathfrak{t}_{\mathrm{d}}$;
3: $t_k \leftarrow \mathfrak{t}_{\mathrm{s}}$;
4: flag $\leftarrow 1$
5: **while** flag $= 1$ **do**
6: measure $\overline{\chi}(t_k)$, $\overline{v}(t_k)$;
7: **if** $\|\overline{\chi}(t_k) - \chi_d\|_2 \leq \frac{\epsilon}{\sqrt{\lambda_{\min}(P)}}$ **and** $\|\overline{v}(t_k)\|_2 \leq \frac{\epsilon}{\sqrt{\lambda_{\min}(P)}}$ **then**
8: flag $\leftarrow 0$; break;
9: **Go to** "line 12"
10: **end if**
11: $t_k \leftarrow t_k + h$;
12: **end while**
13: $\mathfrak{t}_{\mathrm{d}} \leftarrow t_k$;

B. Discrete System Abstraction

We have provided so far a feedback control law that drives the robot with dynamics as in (3a)-(3b) from RoI $\mathcal{R}_{\mathrm{s}}$ to RoI $\mathcal{R}_{\mathrm{d}}$ within time $\mathfrak{t}(\mathcal{R}_{\mathrm{s}}, \mathcal{R}_{\mathrm{d}})$. The abstraction that captures the dynamics of the robot into a WTS is given through he following definition:

Definition 6. The motion of the robot in the workspace $\mathcal{W}$ is modeled by the WTS $\mathcal{T} = (S, S_0, \text{Act}, \longrightarrow, \mathfrak{t}, \Sigma, L)$ where:

- $S = \mathcal{R} = \bigcup_{m \in \mathcal{M}} \mathcal{R}_m$ is the set of states of the robot that contains all the RoI of the workspace $\mathcal{W}$;
- $S_0 \subseteq S$ is a set of initial states defined by the robot' s initial position $\chi(0)$ in the workspace;
- Act is the set of actions containing the union of all feedback controllers (11) which can navigate the robot between RoI;
- $\longrightarrow \subseteq S \times \text{Act} \times \text{S}$ is the transition relation. We say that $(\mathcal{R}_{\mathrm{s}}, u, \mathcal{R}_{\mathrm{d}}) \in \longrightarrow$, with $\mathcal{R}_{s}$, $\mathcal{R}_{s} \in \mathcal{R}$ with $\mathcal{R}_{s} \neq \mathcal{R}_{d}$ if there exist feedback control law $u \in Act$ as in (11) which can drive the robot from the region $\mathcal{R}_{\mathrm{s}}$ to the region $\mathcal{R}_{\mathrm{d}}$ without intersecting with any other RoI of the workspace;
- $\mathfrak{t}$ and L is the time weight and the labeling function as given in (7) and (8), respectively; Σ is the set of atomic propositions imposed by Problem 1.

The aforementioned WTS will allow us to work directly at the discrete level and design a sequence of feedback controllers as in (11) that solve Problem 1. By construction, each timed run produced by the WTS $\mathcal{T}$, as given in Definition 3, is uniquely associated with the trajectory $\chi(t)$ of the system (3a)-(3b), as given in Definition 4. Hence, if we find a timed run of $\mathcal{T}$ satisfying the given MITL formula φ, we also find a desired timed word of the original system, and hence a trajectory $\chi(t)$ that is a solution to Problem 1.

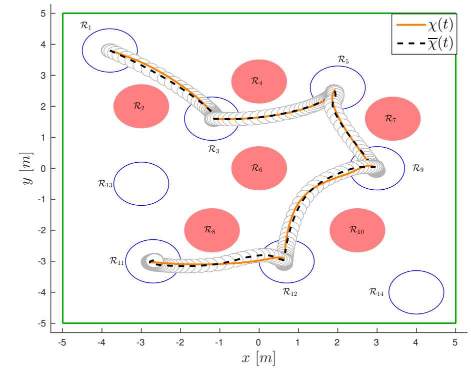

Fig. 2: The evolution of the trajectory robot in the workspace $\mathcal{W}$. RoI and unsafe regions are depicted with blue and red color, respectively. The tube of the robot is depicted with light gray color. The real and the nominal trajectories $\chi(t)$ and $\overline{\chi}(t)$, respectively, are depicted with orange and dashed black color. The robot successfully satisfies the task φ given in (20).

C. Controller Synthesis

Fig. 1 depicts a framework under which a sequence of feedback control laws $u(\chi, v)$ that guarantee the satisfaction of the MITL formula φ can be computed. First, a TBA $\mathcal{A}$ that accepts all the timed runs satisfying the specification formula φ is constructed. Second, a product between the WTS $\mathcal{T}$ given in Definition 6 and the TBA $\mathcal{A}$ is computed, which gives the product WTS $\widetilde{\mathcal{T}}$. By performing graph search to the product WTS $\widetilde{\mathcal{T}}$, a timed run that satisfies the MITL formula φ can be found. For more details regarding the control synthesis procedure we refer to our previous work [2], [3]

V. Simulation Results

For a simulation example, consider a robot operating in a workspace $\mathcal{W} = \{x, y \in \mathbb{R} : -5 \leq x, y \leq 5\} \subseteq \mathbb{R}^2$ with dynamics: $\dot{x} = v_1$, $\dot{y} = v_2$, $\dot{v}_1 = 0.25x^2 + u_1 + 0.25\cos(t)$, $\dot{v}_2 = \dfrac{0.1 - 0.1e^{-x}}{1 + e^{-x}} + 0.25y^2 + u_2 + 0.1u_2^3 + 0.25\sin(t)$, where $\chi = [x, y]^\top \in \mathbb{R}^2$, $v = [v_1, v_2]^\top \in \mathbb{R}^2$, $u = [u_1, u_2]^\top \in \mathbb{R}^2$, $d = [0.25\cos(t), 0.25\sin(t)]^\top$ and $\widetilde{d} = 0.25$. From (6), we get $J(\chi, v, u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 + 0.3u_2^2 \end{bmatrix}$, which results in $\lambda_{\min}\left[\frac{J(\cdot) + J(\cdot)^\top}{2}\right] \geq \underline{J} = 1$. The velocity and input constraints are $\mathcal{V} = \{v \in \mathbb{R}^2 : -5 \leq v_1, v_2 \leq 5\}$ and $\mathcal{U} = \{u \in \mathbb{R}^2 : -2.125 \leq u_1, u_2 \leq 2.125\}$, respectively. The Lipschitz constant is $L = 2.5$. The initial states of the robot are $\chi(0) = [-3.8, 3.8]^\top$ and $v(0) = [0, 0]^\top$.

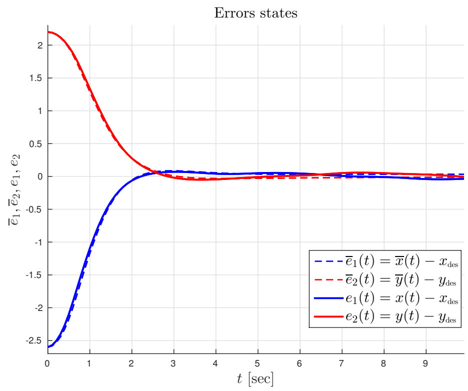

Fig. 3: The evolution of the real errors $e_1(t) = x(t) - x_{\mathrm{des}}$, $e_2(t) = y(t) - y_{\mathrm{des}}$ and the corresponding nominal errors $\overline{e}_1(t) = \overline{x}(t) - x_{\mathrm{des}}$, $\overline{e}_2(t) = y(t) - x_{\mathrm{des}}$ for the transition between $\mathcal{R}_1$-$\mathcal{R}_3$.

The sampling time and the prediction horizon are set to $h = 0.1\,\mathrm{sec}$, $T = 1.2\,\mathrm{sec}$, respectively. The NMPC gains are set to $Q = P = I_4$, $R = 0.5 I_2$.

In the workspace we have $\mathfrak{m} = 14$ RoI with radius $p_m = 0.7$, $\forall m \in \mathcal{M}$ from which 6 of them stand for unsafe regions that the robot is required not to visit (depicted with red color in Fig. 2). The desired MITL formula is set as:

$$\varphi = \Box_{[0,\infty)}\{\neg \mathrm{obs}\} \wedge \Diamond_{[6,12]}\{\mathrm{goal}_1\} \wedge \Diamond_{[20,30]}\{\mathrm{goal}_2\}, \quad (20)$$

over the set of atomic propositions $\Sigma = \{\mathrm{obs}, \mathrm{goal}_1, \mathrm{goal}_2\}$ and labeling function $L(\mathcal{R}_5) = \{\mathrm{goal}_1\}$, $L(\mathcal{R}_{11}) = \{\mathrm{goal}_2\}$, $L(\mathcal{R}_i) = \{\mathrm{obs}\}$, $i \in \{2,4,6,7,8,10\}$ and $L(\mathcal{R}_i) = \emptyset$, $i \in \{1,3,9,12,13,14\}$. Fig. 2 depicts the workspace with RoI, unsafe regions, the nominal trajectory of the robot (orange color), the real trajectory of the robot (black color) and the tube centered along the nominal trajectory. By using Algorithm 1 the time duration of the transitions between RoI $\mathcal{R}_1$-$\mathcal{R}_3$, $\mathcal{R}_3$-$\mathcal{R}_5$, $\mathcal{R}_5$-$\mathcal{R}_9$, $\mathcal{R}_9$-$\mathcal{R}_{12}$ and $\mathcal{R}_{12}$-$\mathcal{R}_{11}$ are $\mathfrak{t}(\mathcal{R}_1, \mathcal{R}_3) = 5.2\,\mathrm{sec}$, $\mathfrak{t}(\mathcal{R}_3, \mathcal{R}_5) = 6.1\,\mathrm{sec}$, $\mathfrak{t}(\mathcal{R}_5, \mathcal{R}_9) = 4.5\,\mathrm{sec}$, and $\mathfrak{t}(\mathcal{R}_9, \mathcal{R}_{12}) = 7.1\,\mathrm{sec}$ and $\mathfrak{t}(\mathcal{R}_{12}, \mathcal{R}_{11}) = 4.8\,\mathrm{sec}$, respectively. According to Fig. 2, the robot never visits the unsafe RoI. Furthermore, it navigates to goal RoI $\mathcal{R}_5$ and $\mathcal{R}_{11}$ at time $11.3\,\mathrm{sec}$, $27.7\,\mathrm{sec}$, respectively, which results in a successful satisfaction of φ. Thus, $\chi(t) \models \varphi$, $\forall t \geq 0$. The error signals for the transition between RoI $\mathcal{R}_1$-$\mathcal{R}_3$ are depicted in Fig. 3. The evolution of the velocities $v_1(t)$ and $v_2(t)$ is presented in Fig. 4. The control effort for the transition from $\mathcal{R}_1$ to $\mathcal{R}_3$ is presented in Fig. 5. Finally, Fig. 6 shows a more detailed zoom in the tube of the trajectory of the robot.

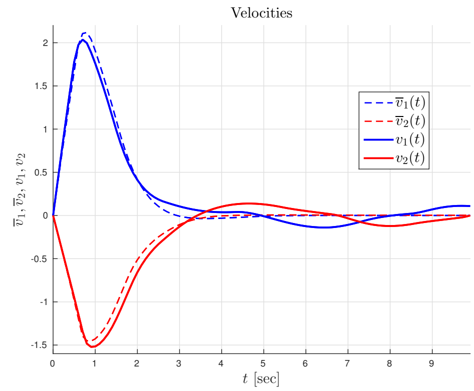

Fig. 4: The real and the nominal velocity signals $v_1(t)$, $v_2(t)$ and $\overline{v}_1(t)$, $\overline{v}_2(t)$, respectively for the transition between $\mathcal{R}_1$-$\mathcal{R}_3$. It holds that $v_i(t) \in \mathcal{V}$, $i \in \{1,2\}$.

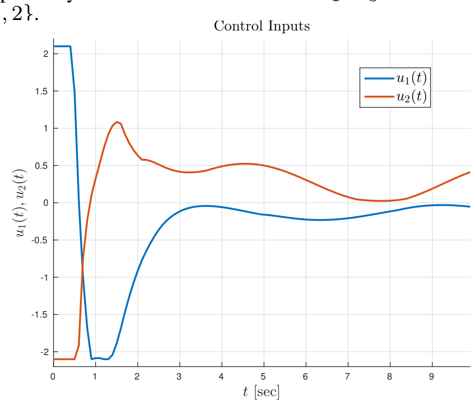

Fig. 5: The control input signals $u_1(t)$, $u_2(t)$ for navigating the robot between $\mathcal{R}_1$-$\mathcal{R}_3$. It holds that $u_i(t) \in \mathcal{U}$, $i \in \{1,2\}$.

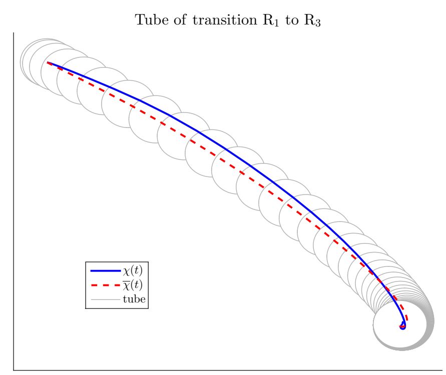

Fig. 6: The tube (gray color) from the transition from $\mathcal{R}_1$ to $\mathcal{R}_3$ centered along the nominal trajectory $\overline{\chi}(t)$ (red dashed color) and the real trajectory $\chi(t)$ (blue color). The real trajectory $\chi(t)$ remains always within the tube.

References

[1] R. Alur, T. Feder, and T. Henzinger, "The Benefits of Relaxing Punctuality," *Journal of the Association for Computing Machinery (JACM)*, vol. 43, no. 1, pp. 116–146, 1994.

[2] A. Nikou, J. Tumova, and D. Dimarogonas, "Cooperative Task Planning of Multi-Agent Systems Under Timed Temporal Specifications," *American Control Conference (ACC)*, pp. 13–19, 2016.

[3] A. Nikou, D. Boskos, J. Tumova, and D. V. Dimarogonas, "On the Timed Temporal Logic Planning of Coupled Multi-Agent Systems," *Automatica*, vol. 97, pp. 339–345, 2018.

[4] H. Michalska and D. Mayne, "Robust Receding Horizon Control of Constrained Nonlinear Systems," *IEEE Transactions on Automatic Control (TAC)*, vol. 38, no. 11, pp. 1623–1633, 1993.

[5] H. Chen and F. Allgöwer, "A Quasi-Infinite Horizon Nonlinear Model Predictive Control Scheme with Guaranteed Stability," *Automatica*, vol. 34, no. 10, pp. 1205–1217, 1998.

[6] D. Mayne, J. Rawlings, C. Rao, and P. Scokaert, "Constrained Model Predictive Control: Stability and Optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[7] W. Langson, I. Chryssochoos, S. Rakovic, and D. Mayne, "Robust Model Predictive Control Using Tubes," *Automatica*, vol. 40, no. 1, pp. 125–133, 2004.

[8] S. Yu, C. Maier, H. Chen, and F. Allgöwer, "Tube MPC Scheme Based on Robust Control Invariant Set with Application to Lipschitz Nonlinear Systems," *Systems and Control Letters*, vol. 62, no. 2, pp. 194–200, 2013.

[9] A. Nikou and D. V. Dimarogonas, "Decentralized Tube-based Model Predictive Control of Uncertain Nonlinear Multi-Agent Systems," *International Journal of Robust and Nonlinear Control*, 2019.

[10] A. Zemouche, N. Boutayeb, and G. Bara, "Observers for a Class of Lipschitz Systems with Extension to H-inifinity Performance Analysis," *Systems and Control Letters*, vol. 57, no. 1, pp. 18–27, 2008.

[11] O. Maler, D. Nickovic, and A. Pnueli, "From MITL to Timed Automata," *Formal Modeling of Timed Systems*, pp. 274–289, 2006.

[12] A. Nikou and D. V. Dimarogonas, "Robust Tube-based Model Predictive Control for Timed-constrained Robot Navigation," *American Control Conference (ACC), https://arxiv.org/pdf/1809.09825.pdf*, 2019.