A hybrid systems framework for multi agent task planning and control*

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***Abstract*— In this paper, we investigate the coordination of a multi-agent system to achieve a high level plan. In, particular, we assume that this planning is decomposed in a sequence of simple control objectives. The coordination is carried out through event-based communications between the agents, and distributed control laws are synthesized to achieve each control objective according to a time-constraint. All the elements (planning, motion, and communication) are integrated in a hybrid systems framework, which also allows to conclude about the achievement of the high level plan.**

I. INTRODUCTION

In the recent years, cooperative control for multi-agent systems has attracted the attention of both the robotics and control communities, due to its application to consensus [1], formation or reference-tracking [2]. In these scenarios, each agent serves to accomplish a global objective or fulfill a simple local goal such as reachability. Moreover, the need of communication between the agents to achieve such objectives has motivated the application of event-triggered [3] and self-triggered [4] control policies to multi-agent systems. Relevant works on the topic applied to multi-agent systems are [5]–[8], to cite a few. Alternatively, when the desired behavior of the team of agents is complex, the automated synthesis of controllers has recently found support in formal verification methods and temporal logics. The suggested solutions abstract the dynamics of the system into a finite, discrete transition system, and a discrete plan that meets the specifications is synthesized and next translated into a controller for the original system [9]. If a high resolution of the state-space is required, the number of discrete states blows up even for partially decentralized solutions.

In this paper, we propose a novel strategy for event-triggered control of multi-agent systems in which the aim of the system is to achieve a sequence of simple control objectives. This sequence constitutes the *high level plan* for the team of agents. Whereas the dynamics of the system and the synthesized distributed controllers are continuous, the occurrence of certain discrete events, such as the switch of task or the transmission of information from one agent to another, suggests a hybrid approach to the problem. In this regard, a framework for the representation of hybrid systems recently proposed and surveyed in [10], [11] provides the tools to describe this problem and useful Lyapunov-like results to be applied. There are some recent applications of this framework to event-triggered control for single loop systems [12]–[14] and for multi-agent systems [15]. In [14], different event-based control techniques are reviewed from the hybrid formulation. In [15], the rendez-vous problem of agents is solved by the design of local triggering rules for the edges and the use of clock variables. This strategy requires the continuous measurement of relative positions, and to avoid that, a self-triggered rule is also proposed.

The contributions of the paper in hand can be summarized as: (1) the high level plan, the multi-agent system, the control and the communication laws are integrated in a unified hybrid systems framework; (2) the design of distributed controllers and event-triggered policies for the individual agents that guarantee the achievement of each control objective that corresponds to an individual task in the high level plan, according to some time constraints specifications; (3) the proof of the accomplishment of the high level plan by making use of hybrid stability tools. To do so, we start from a Lyapunov function used in classical problems of multi-agent systems, and we add some extra terms to include both the progress in the global objective of the system and the errors due to the communication policy. The analysis also gives as a result design rules for the trigger functions, which do not require the continuous measurement of relative states.

The rest of the paper is organized as follows. In Section II we state the preliminaries. The problem is formulated in Section III. In Section IV we describe the proposed solution and the main result of the paper is presented. In Section V a simulation example illustrates the results. Finally, the main conclusions of the paper are given in Section VI.

II. PRELIMINARIES

A. The hybrid system framework

The modeling framework used in this paper is based on the formalism of [10], [11]. A hybrid system $\mathcal{H}$ is a tuple $(\mathcal{C}, \mathcal{D}, F, G)$, where $\mathcal{C} \subseteq \mathbb{R}^n$ and $\mathcal{D} \subseteq \mathbb{R}^n$ are the flow and jump set, respectively, while $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are set-valued mappings, called flow and jump map, respectively. Then, a hybrid system is represented by

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in \mathcal{C} \\ x^+ \in G(x) & x \in \mathcal{D}, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state. Roughly speaking, the state flows through $\mathcal{C}$ following the dynamics given by F, and it jumps when it enters the jump set according to G.

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For a hybrid system, we define the concept of hybrid time represented by a pair $(t, k) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, where t is the continuous time and k is the discrete time, which represents the number of jumps exhibited by the dynamics of the system. Formally, we define a *hybrid time domain* [11] as a subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ as the union of infinitely many intervals of the form $[t_k, t_{k+1}] \times k$, where $0 = t_0 \leq t_1 \leq \dots$, or of finitely many such intervals, with the last one possibly of the form $[t_k, t_{k+1}] \times k$, $[t_k, t_{k+1}) \times k$, or $[t_k, \infty) \times k$.

Definition 1. A hybrid trajectory is a pair $(\text{dom } \xi, \xi)$ consisting of hybrid time domain $\text{dom } \xi$ and a function ξ defined on $\text{dom } \xi$ that is continuously differentiable in t on $(\text{dom } \xi) \cap (\mathbb{R}_{\geq 0} \times \{k\})$ for each $k \in \mathbb{Z}_{\geq 0}$.

Definition 2. Let ξ be a solution to (1). Then, the solutions to (1) have a uniform semiglobal average dwell-time if for any $\Delta \geq 0$, there exist $\tau(\Delta)$ and $n_0(\Delta) \in \mathbb{Z}_{>0}$ such that for any solution ξ to (1) with $\|\xi(0,0)\| \leq \Delta$ it holds that $k_2 - k_1 \leq \frac{1}{\tau(\Delta)}(t_2 - t_1) + n_0(\Delta)$, for any $(t_1, k_1), (t_2, k_2) \in \text{dom } \xi$ with $t_1 + k_1 \leq t_2 + k_2$. We say that the solutions to (1) have a uniform global average dwell-time when τ and n_0 are independent of Δ.

B. Multi-agent systems

Consider a set of N agents. The topology of the multi-agent system can be modeled as a static undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ is the set of edges, which denotes the communication capability between the respective agents. For each agent i, $\mathcal{N}_i$ represents the neighborhood of i, i.e., $\mathcal{N}_i = \{j \in \mathcal{V} : \ (i,j) \in \mathcal{E}\}$. The Laplacian matrix $L(\mathcal{G}) \in \mathbb{R}^{N \times N}$ of a network of agents is defined as $L(\mathcal{G}) = D(\mathcal{G})D(\mathcal{G})^T$, where $D(\mathcal{G})$ is the incidence matrix. The Laplacian matrix $L(\mathcal{G})$ is positive semidefinite, and if $\mathcal{G}$ is connected and undirected, then $0 = \lambda_1(\mathcal{G}) < \lambda_2(\mathcal{G}) \leq \dots \leq \lambda_N(\mathcal{G})$, where $\{\lambda_i(\mathcal{G})\}$ are the eigenvalues of $L(\mathcal{G})$.

III. PROBLEM FORMULATION

A. Planning description

For a multi-agent system, we assume the existence of a high level plan ϕ given and known by the agents that consists of the consecutive achievement of some cooperative tasks. Then, we can denote ϕ as a sequence of $p \in \mathbb{N}_{>0}$ objectives $\phi_1, \phi_2, ..., \phi_p$. Once ϕ_ℓ, $\ell = 1, \dots, p-1$, is completed, the agents proceed to the next task $\phi_{\ell+1}$. To be more specific, we consider that the set of agents have to visit consecutively a set of regions $\{\mathcal{B}_\ell\}$ defined as

$$\mathcal{B}_\ell = \{y \in \mathbb{R}^n : \ \|y - c_\ell\| \leq r_\ell\}, \tag{2}$$

and denoted as $\mathcal{B}_\ell = (c_\ell, r_\ell)$. The task ϕ_ℓ is completed once all the agents lie inside the region $\mathcal{B}_\ell$ within some time constraints, i.e., before some deadline $T_\ell = t_\ell - t_{0,\ell}$, where $t_{0,\ell}$ is the starting time of the task ϕ_ℓ.

This problem can be formulated in the hybrid framework introduced in Section II-A. The details will be provided in the corresponding section.

Remark 1. While the choice of the task ϕ may seem restrictive, the approach can handle a more general set of cooperative tasks, as long as local Lyapunov based controllers exist for each sub-task ϕ_ℓ.

B. Agent dynamics, control and communication

Let the N agents obey the single-integrator dynamics:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, \dots, N, \tag{3}$$

where $x_i(t), u_i(t) \in \mathbb{R}^n$ are the state and the control inputs of agent i, respectively. The state of the overall system is defined as $x = (x_1^T, \dots, x_N^T)^T$ and the control law $u = (u_1, \dots, u_N^T)^T$. For each agent of the form (3), let us consider the control law

$$u_i(t) = \kappa_i(x_i(t), \bigcup_{j \in \mathcal{N}_i} x_j(t), \pi_i), \quad i = 1, \dots, N, \tag{4}$$

where each $\kappa_i(\cdot)$ is locally Lypschitz and π_i represents a set of additional parameters. The control law (4) requires the continuous measurement of the state of the neighbors. We will refer in the future to it as *the nominal control law*.

The design of (4) has to be such that the sequence of objectives $\phi_1, \phi_2, ..., \phi_p$ is accomplished. Then, a reasonable approach is to have a switching rule for κ_i, so that the control law is updated once an objective is achieved.

To include the communication and computation constraints of a networked control system, the agents only receive measurements from their neighbors at some given time instants to be determined. Thus, (4) is transformed into $u_i(t) = \kappa_i(\hat{x}_i(t), \bigcup_{j \in \mathcal{N}_i} \hat{x}_j(t), \pi_i)$ for $i = 1, \dots, N$, where $\hat{x}_i$ and $\hat{x}_j$ denote the sampled version of x_i and x_j, respectively. The events that update $u_i(t)$ are determined by agent i from local information only. These trigger functions are denoted by f_i and an event is triggered when

$$f_i(x_i(t), \hat{x}_i(t), t) \geq 0, \quad i = 1, \dots, N. \tag{5}$$

We assume that $f_i(x_i(t), x_i(t), t) < 0$, i.e., when $\hat{x}_i$ is set to x_i. Moreover, the definition of f_i should guarantee that there exists a lower bound for the local inter-event times.

Let us define the local error e_i as

$$e_i(t) = \hat{x}_i(t) - x_i(t), \tag{6}$$

and the error for the multi-agent system as $e^T = (e_1^T, \dots, e_N^T)$. Thus, at the event-times the error e_i is reset to zero, whereas in the inter-event times $\dot{e}_i = -\dot{x}_i = -u_i$. This dynamics can be formulated as jump and flow maps (see (1)), respectively, if the jump and flow sets are adequately defined according to the trigger function (5).

C. Problem statement

Given a team of N agents subject to dynamics (3) interconnected over a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$: (1) synthesize for each agent $i \in \mathcal{V}$ control laws u_i such that the high level plan ϕ is completed according to the time constraints, and trigger functions f_i that reduce the communication and guarantee a minimum inter-event time; (2) formulate the agent dynamics, the control objectives and laws, and the communication as a hybrid system of the form (1) such that relevant stability results ([10], [11]) can be applied to ensure the accomplishment of ϕ.

IV. PROBLEM SOLUTION

The proposed solution follows three steps. First, the proposed design for the nominal control law (continuous communication) is presented, showing that it allows the consecutive achievement of the control objectives. Under the continuous communication assumption (*nominal system*), a description of the hybrid system is given. Next, how the communication strategy can be integrated in the description of the hybrid system is presented, and the design of the trigger function is obtained following Lyapunov arguments. Finally, we prove the accomplishment of ϕ by applying the invariance principle for hybrid systems given in [10].

A. Nominal system

The task ϕ_ℓ given in Section III-A is achieved once all agents lie inside the region $\mathcal{B}_\ell$ (2) before some deadline T_ℓ. We propose a continuous controller with two terms: a cooperative part and a feedback term to approach c_ℓ (the center of $\mathcal{B}_\ell$), regulated by a gain k_ℓ which needs to be designed for each task so that the time constraint is fulfilled.

$$u_{i,\ell}(t) = k_\ell\Big(\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t)) - (x_i(t) - c_\ell)\Big). \quad (7)$$

Remark 2. The control input (7) assumes that all agents include the feedback term and this might seem conservative, but the extension to the case where only a reduced number of agents (leaders) has access to c_ℓ is straightforward. We have considered this simplification because the eigenvalues that characterize the closed loop dynamics of the multi-agent system (the matrix $L\otimes I_n + I_{nN}$) are directly computed, whereas, in the general case, they are not, though still strictly positive. Thus, the main results will still hold but with parameters corresponding to the different eigenvalues.

Remark 3. The control law (7) can be easily extended to the case of formation control if the desired distance between the agents is included in the cooperative part.

Proposition 1. *If the feedback gains k_ℓ are chosen such that*

$$k_\ell \geq \tfrac{1}{T_\ell}\log\tfrac{\|\delta_{0,\ell}\|_2}{r_\ell}, \quad (8)$$

where $\delta_{0,\ell} = x - \mathbf{1}\otimes c_\ell$, then the team of agents (3) *reaches the region $\mathcal{B}_\ell = (c_\ell, r_\ell)$ before $T_\ell = t_\ell - t_{0,\ell}$.*

Proof: The closed loop dynamics of the multi-agent system is given by $\dot{x}(t) = -k_\ell(L\otimes I_n + I_{nN})(x(t) - \mathbf{1}_N\otimes c_\ell)$, since $(L\otimes I_n)(\mathbf{1}_N\otimes c_\ell) = 0$ ($\mathbf{1}_N$ is an eigenvector of L). If we define $\delta_i(t) = x_i(t) - c_\ell$ and $\delta(t) = x(t) - \mathbf{1}_N\otimes c_\ell$, then $\dot\delta(t) = -k_\ell(L\otimes I_n + I_{nN})\delta(t)$. According to this, each agent i will reach c_ℓ asymptotically. If we denote as $x_{0,\ell}$ ($\delta_{0,\ell} = x_{0,\ell} - \mathbf{1}_N\otimes c_\ell$) and $t_{0,\ell}$ the initial conditions and the starting time, respectively, for the task ϕ_ℓ, then it holds that $\delta(t) = e^{-k_\ell(L\otimes I_n + I_{nN})(t-t_{0,\ell})}\delta_{0,\ell}$. To obtain values of k_ℓ such that the task ϕ_ℓ is achieved, we analyze the eigenvalues of $M = L\otimes I_n + I_{nN}$. The laplacian matrix L has eigenvalues $0 = \lambda_1 < \lambda_2 \leq ... \leq \lambda_N$, and it holds that $\lambda(L + I_N) = \{1, \lambda_2 + 1, \ldots, \lambda_N + 1\}$. According to this, $\delta(t)$ can be bounded as $\|\delta(t)\|_2 \leq e^{-k_\ell(t-t_{0,\ell})}\|\delta_{0,\ell}\|_2$ and, in particular, for $t = t_\ell$ $\|\delta(t_\ell)\|_2 \leq e^{-k_\ell(t_\ell - t_{0,\ell})}\|\delta_{0,\ell}\|_2$. If we impose that at that instance of time all agents reach at least the boundary of $\mathcal{B}_\ell$ ($\|\delta_i(t_\ell)\| = r_\ell$), the proof is completed. □

Next, the nominal system is rewritten as a hybrid system of the form (1). To describe the progress in the task planning, a new variable $q\in\mathbb{R}$ is defined and initialized to the number of objectives p. It remains constant during flows and it is decremented once a task is completed until it reaches the value of 0. Furthermore, we consider the dynamics of δ instead of the dynamics of x:

$$\left.\begin{aligned} \dot\delta &= -k\big((L+I_N)\otimes I_n\big)\delta \\ \dot k &= 0 \\ \dot q &= 0 \end{aligned}\right\} \begin{array}{l}\text{if } \exists i\in\mathcal{V}:\ x_i\notin\mathcal{B}_{p-q+1} \\ \text{or } q = 0\end{array} \quad (9)$$

$$\left.\begin{aligned} \delta^+ &= \delta + \mathbf{1}_N\otimes(c_{p-q+1} - c_{p-q+2}) \\ k^+ &= k_{p-q+2} \\ q^+ &= q - 1 \end{aligned}\right\} \begin{array}{l}\text{if } x_i\in\mathcal{B}_{p-q+1}\ \forall i\in\mathcal{V} \\ \text{and } q > 0.\end{array} \quad (10)$$

At jumps, the gain k is updated for the next task according to (8). Note that the equivalence $\ell = p - q + 1$ is used.

B. Event-triggering design

We consider the sampled version of the control law (7) $u_{i,\ell}(t) = k_\ell\Big(\sum_{j\in\mathcal{N}_i}(\hat{x}_j(t) - \hat{x}_i(t)) - (\hat{x}_i(t) - c_\ell)\Big)$. If the sampled version of δ_i is defined as $\hat\delta_i = \hat{x}_i - c_\ell$, then it holds that $\delta_i = \hat\delta_i + e_i$, where the error e_i is defined in (6). Hence, the dynamics of the multi-agent system for any feedback gain k is given by $\dot\delta = -k\big((L+I_N)\otimes I_n\big)(\delta + e)$, and $\dot{e} = -\dot\delta$ in the inter-event times. The local trigger function is of the form $f_i(e_i(t), \delta_i(t)) = \|e_i\| - \varrho\|\delta_i\|,\ i = 1,\ldots,N$, where the feasible values for $\varrho\in\mathbb{R}_{>0}$ will be determined.

We further define a new variable τ_i for each agent that accounts for the elapsed time between the last update of $\hat{x}_i$, and $\tau^T = (\tau_1,\ldots,\tau_N)$. We enforce τ_i to be bounded by a constant T (to be defined). The reason for introducing this constraint comes from some desired properties for the Lyapunov function we will be giving later on, but the good news are that, in general, T can be chosen arbitrarily large as long as it has a finite value.

When the communications are event-driven, it is required to include new variables on the state of the system (9)-(10). Moreover, to define the flow and jump sets, the following statements are introduced: 1) s_1: $\exists i\in\mathcal{V}:\ x_i\notin\mathcal{B}_{p-q+1}$; 2) s_2: $q = 0$; 3) s_3: $\|e_i\| < \varrho\|\delta_i\|\ \forall i\in\mathcal{V}$; 4) s_4: $\tau_i < T\ \forall i\in\mathcal{V}$. As a result, the hybrid system in the event-driven case is

$$\left.\begin{aligned} \dot\delta &= -k\big((L+I_N)\otimes I_n\big)(\delta + e) \\ \dot e &= k\big((L+I_N)\otimes I_n\big)(\delta + e) \\ \dot\tau &= \mathbf{1} \\ \dot k &= 0 \\ \dot q &= 0 \end{aligned}\right\} \text{ if } (s_1\vee s_2)\wedge s_3\wedge s_4 \quad (11)$$

$$\begin{pmatrix}\delta^+\\ e^+\\ \tau^+\\ k^+\\ q^+\end{pmatrix} = \begin{cases}\begin{pmatrix}\delta + \mathbf{1}_N \otimes (c_{p-q+1} - c_{p-q+2})\\ \mathbf{0}\\ \mathbf{0}\\ k_{p-q+2}\\ q-1\end{pmatrix} & \text{if } \neg s_1 \wedge \neg s_2\\ \begin{pmatrix}\delta\\ F_e(\delta,e,\tau)\\ F_\tau(\delta,e,\tau)\\ k\\ q\end{pmatrix} & \text{if } \neg s_3 \vee \neg s_4,\end{cases} \tag{12}$$

where F_e and F_τ are defined as $F_e(\delta, e, \tau) = \{F_{e,i}(\delta_i, e_i, \tau_i) : i \in \mathcal{V}\}$, $F_\tau(\delta, e, \tau) = \{F_{\tau,i}(\delta_i, e_i, \tau_i) : i \in \mathcal{V}\}$ where $F_{e,i}(\delta_i, e_i, \tau_i) = 0$ if $\|e_i\| \geq \varrho\|\delta_i\|$ or $\tau_i \geq T$ and $F_e(\delta, e, \tau) = e_i$ otherwise, $F_{\tau,i}(\delta_i, e_i, \tau_i) = 0$ if $\tau_i \geq T$ or $\|e_i\| \geq \varrho\|\delta_i\|$ and $F_{\tau,i}(\delta_i, e_i, \tau_i) = \tau_i$ otherwise, i.e., the error and the clock are updated for the agents where a control event has been detected. In summary, the system flows until an event is detected due to two different reasons: i) all agents have reached the final set of the corresponding task ($\neg s_1$), excluding the final task (s_2) where no more transitions are required (*task events*), or ii) an event has been detected in one or some of the agents because the error has exceeded the tolerable bound or the local clock has reached the value T (*control events*). In the first case, δ, k and q are updated for the next task, while e and τ are reset to zero. This requires a round of communication to update control laws (which was already required to update k).

1) Average dwell-time solutions: The jump map of the multi-agent system (11)-(12) only allows one agent to reset the error and its clock, and then a finite number of jumps can successively occur with no flow in between them when several events are simultaneously detected. So, the solutions of the hybrid system cannot exhibit dwell-time but average dwell-time (see Definition 2) if we guarantee the exclusion of the Zeno behavior for each agent i.

Proposition 2. *The system* (11)-(12) *has uniform global average dwell-time solutions with parameters $\tau = N^{-1} min_{\ell=1,\ldots,p}\eta_\ell$ and $n_0 = N + p$, where η_ℓ is a lower bound of the transmission events for the objective ϕ_ℓ.*

Proof: From the definition of the error e_i, in the inter-event times, it holds that $\dot{e}_i = -\dot{\delta}_i$. Then $\|\dot{e}_i\| = \|\dot{\delta}_i\| \leq \|\dot{\delta}\| = \|k((L+I_N)\otimes I_n)(\delta+e)\| \leq k(\lambda_N+1)(\|\delta\|+\|e\|)$. To find a lower bound η_i of the inter-event times, one can estimate the time derivative of $\|e_i\|/\|\delta\|$: $\frac{d}{dt}\frac{\|e_i\|}{\|\delta\|} = \frac{e_i^T\dot{e}_i}{\|e_i\|\|\delta\|} - \frac{\|e_i\|\delta^T\dot{\delta}}{\|\delta\|^3} \leq \frac{\|\dot{e}_i\|}{\|\delta\|} + \frac{\|e_i\|}{\|\delta\|}\cdot\frac{\|\dot{\delta}\|}{\|\delta\|} \leq \frac{\|\dot{e}_i\|}{\|\delta\|} + \frac{\|e\|}{\|\delta\|}\cdot\frac{\|\dot{\delta}\|}{\|\delta\|} \leq \big(1 + \frac{\|e\|}{\|\delta\|}\big)\frac{\|k((L+I_N)\otimes I_n)(\delta+e)\|}{\|\delta\|} \leq k(\lambda_N+1)\big(1+\frac{\|e\|}{\|\delta\|}\big)^2$. Then, $\frac{d}{dt}\frac{\|e_i\|}{\|\delta\|} \leq k(\lambda_N+1)\big(1+\frac{\|e\|}{\|\delta\|}\big)^2$. Similar derivations yield $\frac{d}{dt}\frac{\|e\|}{\|\delta\|} \leq k(\lambda_N+1)\big(1+\frac{\|e\|}{\|\delta\|}\big)^2$ in the inter-event times. It is noted that $\|e_i\|/\|\delta\|$ is always upper bounded by $\|e\|/\|\delta\|$ and both of them are nonnegative. We can conclude that $\|e_i\|/\|\delta\|$ satisfies $\|e_i\|/\|\delta\| < y(t, y_0)$, where $y(t, y_0)$ is the solution of $\dot{y}(t) = k(\lambda_N+1)(1+y)^2,\ y_0 = 0$. Then the evolution of $\|e_i\|/\|\delta\|$ in $[0, \varrho)$ is lower bounded by

$$\eta_i = \frac{\varrho}{k(\lambda_N+1)(1+\varrho)}. \tag{13}$$

We can denote the solution of (13) as η_ℓ since it does not depend on the agent but on the control gain k, so it takes a different value for each task ℓ.

According to Definition 2, let us consider two hybrid times $(t_1, k_1), (t_2, k_2)$ with $t_1+k_1 < t_2+k_2$. The number of events associated with the agent i between (t_1, k_1) and (t_2, k_2) satisfies $n_i(t_1, t_2) \leq \frac{t_2-t_1}{\min_{\ell=1,\ldots,p}\eta_\ell} + 1$. Note also $k_2 - k_1$ is the number of control events, i.e., $\sum_{i\in\mathcal{V}} n_i(t_1, t_2)$, plus the number of tasks completed, that is at most p. Then $k_2 - k_1 \leq \sum_{i\in\mathcal{V}}\big(\frac{t_2-t_1}{\min\limits_{\ell=1,\ldots,p}\eta_\ell}+1\big)+p = \frac{N}{\min\limits_{\ell=1,\ldots,p}\eta_\ell}(t_2-t_1)+N+p$. Hence, from Definition 2, the system has an uniform global average dwell-time with $\tau = N^{-1}\min_{\ell=1,\ldots,p}\eta_\ell$ and $n_0 = N + p$. □

C. Accomplishment of the high level plan

Lemma 1. *If the compact set*

$$\mathcal{A} = \{0\}\times\{0\}\times([0,T]\otimes\mathbf{1}_N)\times k_p\times\{0\}, \tag{14}$$

$\mathcal{A} \subset \mathbb{R}^{nN}\times\mathbb{R}^{nN}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}$, is globally asymptotically stable (GAS) for the multi-agent system described as the hybrid system (11)-(12)*, then the high level plan ϕ is successfully completed.*

Proof: On the one hand, the variable q is 0 in the compact set (14). On the other hand, according to (11)-(12), the variable q (initialized to p) decreases anytime the system enters the objective regions $\mathcal{B}_\ell$, $\ell = 1,\ldots,p$. When q reaches the value of 0, it means that the system has already visited all the regions $\mathcal{B}_\ell$, $\ell = 1,\ldots,p-1$, and is already inside the region $\mathcal{B}_p$. Once this occurs, no more jumps due to task events can take place, and ϕ has been completed. □

Theorem 1. *Define $\sigma > \frac{\lambda_N+2}{2}$, $\lambda_N = \lambda_{max}(L(\mathcal{G}))$, and $\beta > 0$. If the communication policy is such that any agent i transmits a new measure when $\|e_i\| \geq \varrho\|\delta_i\|$ or $\tau_i \geq T$ with*

$$\varrho = \gamma_1\sqrt{\frac{k(1-\frac{\lambda_N+2}{2\sigma})}{k(\lambda_N+1)(\frac{\sigma}{2}+1)+\frac{k\sigma}{2}-\frac{\beta e^{-\beta T}}{2}}},\ 0<\gamma_1<1 \tag{15}$$

$$T > \frac{\log\Big(2\beta\Big(k(\lambda_N+1)(\frac{\sigma}{2}+1)+\frac{k\sigma}{2}\Big)\Big)}{\beta}, \tag{16}$$

then the compact set (14) *is GAS for the multi-agent system described as the hybrid system* (11)-(12)*.*

Proof: The proof is inspired by [10], [12]. A Lyapunov function U is defined such as U is non-increasing at jumps in (12) and non-increasing during flows in (11). Combining these results with the invariance principle for hybrid systems in [10], global asymptotically stability is concluded.

1) Lyapunov analysis: Let us define

$$U(X) = \tfrac{1}{2}\sum_{i\in\mathcal{V}}\big(\sum_{j\in\mathcal{N}_i}\|\delta_{ij}\|^2 + \|\delta_i\|^2 + e^{-\beta\tau_i}\|e_i\|^2\big) + \tfrac{\varrho}{2}q^2 \tag{17}$$

$$\rho = \gamma_2 N\max_{\ell=1,\ldots,p}\|c_\ell - c_{\ell+1}\|^2,\ \gamma_2 > 1 \tag{18}$$

where $\beta > 0$, $X^T = (\delta^T e^T \tau^T k\ q)$, and $\delta_{ij} = \delta_i - \delta_j$. Note that $U(X) = 0$ at $\mathcal{A}$ and $U(X) > 0\ \forall X \neq 0$. Additionally, (17) has the following properties:

a) It is radially unbounded: Let us define $\underline{\alpha} = \min\{\frac{1}{2}e^{-\beta T}, \frac{\rho}{2}\}$ and $\bar{\alpha} = \max\{\frac{\rho}{2}, \frac{1}{2}+\lambda_N\}$. Then, $\underline{\alpha}\|X\|^2_{\mathcal{A}} \leq U(X) \leq \bar{\alpha}\|X\|^2_{\mathcal{A}}$, where $\|X\|_{\mathcal{A}} = \inf_{y\in\mathcal{A}}\|y - X\|$. This is derived from $\frac{1}{2}\|\delta\|^2 \leq \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\|\delta_i - \delta_j\|^2 + \frac{1}{2}\sum_{i\in\mathcal{V}}\|\delta_i\|^2 \leq (\frac{1}{2}+\lambda_N)\|\delta\|^2$ and $\frac{e^{-\beta T}}{2}\|e\|^2 \leq \frac{1}{2}\sum_{i\in\mathcal{V}} e^{-\beta\tau_i}\|e_i\|^2 \leq \frac{1}{2}\|e\|^2$.

b) Behavior at jumps: We distinguish the two types of events of the system. If a task has been completed, then $U(X^+) - U(X) = \frac{1}{2}\sum_{i\in\mathcal{V}}\|\delta_i - c_{p-q+2} + c_{p-q+1}\|^2 + \frac{\rho}{2}(q-1)^2 - \frac{1}{2}\sum_{i\in\mathcal{V}}\|\delta_i\|^2 - \frac{1}{2}\sum_{i\in\mathcal{V}} e^{-\beta\tau_i}\|e_i\|^2 - \frac{\rho}{2}q^2 \leq \frac{N}{2}\|c_{p-q+1} - c_{p-q+2}\|^2 + \frac{\rho}{2} - \rho q$. If ρ fulfills $\rho \geq \frac{N\|c_{p-q+1}-c_{p-q+2}\|^2}{2q-1}$, then $U(X^+)-U(X)$ is non-increasing. Indeed, because ρ is defined to be independent of the task, and $q = 0$ only when all tasks are completed and is positive otherwise, then defining ρ as in (18) guarantees that the Lyapunov function decreases at jumps:

$$U(X^+) - U(X) \leq \frac{N}{2}\max_{\ell=1,\dots,p}\|c_\ell - c_{\ell+1}\|^2(1+\gamma_2 - 2\gamma_2 q). \tag{19}$$

If the jump is due to a control event, for example, for an agent i_1, then $U(X^+)-U(X) = \frac{1}{2}\sum_{i\in\mathcal{V}:i\neq i_1} e^{-\beta\tau_i}\|e_i\|^2 - \frac{1}{2}\sum_{i\in\mathcal{V}} e^{-\beta\tau_i}\|e_i\|^2 = -\frac{e^{-\beta\tau_{i_1}}}{2}\|e_{i_1}\|^2 \leq -\frac{e^{-\beta T}}{2}\|e_{i_1}\|^2$, which is negative for $e_{i_1} \neq 0$.

c) Behavior on flows: Note that $\dot{q} = 0$, and then to compute $\dot{U}$, only the first three terms in (17) are considered. We denote them as U_1, U_2 and U_3, respectively.

U_1 can be rewritten in the following form $U_1 = \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\|\delta_{ij}\|^2 = \sum_{l=1}^{M}\|z_l\|^2$, where z_l denotes the state of the edge l and M is the number of edges of the graph. More specifically, $z_l = \delta_{ij}$ if j is the positive end of the edge l and δ_{ji} otherwise. Thus $\dot{U}_1 = 2\sum_{l=1}^{M} z_l^T\cdot\dot{z}_l = 2\sum_{l=1}^{M} z_l^T\Delta u_l = 2\delta^T(D\otimes I_n)(D^T\otimes I_n)u$, where $\Delta u_l = u_j - u_i$ when j positive end of the edge l and $\Delta u_l = u_i - u_j$ otherwise. The matrix D is the incidence matrix. It holds that $z = (D^T\otimes I_n)\delta$ and $\Delta u = (D^T\otimes I_n)u$ (both of these expressions are used in the last equality above). This yields $\dot{U}_1 = 2\delta^T(L\otimes I_n)u = -2k\delta^T\big(L\otimes I_n\big)\big((L+I_N)\otimes I_n\big)(\delta+e)$.

For U_2, it follows that $\dot{U}_2 = \sum_{i\in\mathcal{V}}\delta_i^T\dot{\delta}_i = \delta^T\dot{\delta} = -k\delta^T\big((L+I_N)\otimes I_n\big)(\delta+e)$. Note that $\dot{U}_1 + \dot{U}_2 \leq -k(\delta^T\delta + \delta^T e) = -k\sum_{i\in\mathcal{V}}\|\delta_i\|^2 + \delta_i^T e_i$. Exploiting the inequality $-a^Tb \leq \frac{1}{2\sigma}\|a\|^2 + \frac{\sigma}{2}\|b\|^2$ where $\sigma > 0$, it follows $\dot{U}_1 + \dot{U}_2 \leq k\sum_{i\in\mathcal{V}}(-\|\delta_i\|^2 + \frac{1}{2\sigma_i}\|\delta_i\|^2 + \frac{\sigma_i}{2}\|e_i\|^2)$.

Finally, the time derivative of U_3 is $\dot{U}_3 = \frac{-\beta}{2}\sum_{i\in\mathcal{V}} e^{-\beta\tau_i}\|e_i\|^2 + \sum_{i\in\mathcal{V}} e^{-\beta\tau_i}e_i^T\dot{e}_i$. The first term is upper bounded by $\frac{-\beta e^{-\beta T}}{2}\sum_{i\in\mathcal{V}}\|e_i\|^2$. The second term can be rewritten as $e^T M(\tau)\dot{e}$, where $M(\tau) = \mathrm{diag}(e^{-\beta\tau_1},\dots,e^{-\beta\tau_N})$, and then $\sum_{i\in\mathcal{V}} e^{-\beta\tau_i}e_i^T\dot{e}_i = ke^T M(\tau)((L+I_N)\otimes I_n)(\delta+e) \leq k\|M(\tau)\|\|(L+I_N)\otimes I_n\|(\|e\|\|\delta\|+\|e\|^2) \leq k(\lambda_N+1)\sum_{i\in\mathcal{V}}(\|e_i\|\|\delta_i\|+\|e_i\|^2) \leq k(\lambda_N+1)\sum_{i\in\mathcal{V}}(\frac{1}{2\sigma_i}\|\delta_i\|^2 + (\frac{\sigma_i}{2}+1)\|e_i\|^2)$, Then, including and grouping all terms: $\dot{U} \leq \sum_{i\in\mathcal{V}}\big(\|\delta_i\|^2 k(\frac{\lambda_N+2}{2\sigma_i} - 1) + \|e_i\|^2(k(\lambda_N+1)(\frac{\sigma_i}{2}+1) + \frac{k\sigma_i}{2} - \frac{\beta e^{-\beta T}}{2})\big)$. Taking $\sigma_i = \sigma$ $\forall i\in\mathcal{V}$ and because $\|e_i\| < \varrho\|\delta_i\|$ with ϱ as in (15):

$$\dot{U} \leq -k(1 - \tfrac{\lambda_N+2}{2\sigma})\sum_{i\in\mathcal{V}}\|\delta_i\|^2(1-\gamma_1^2), \tag{20}$$

which is non-increasing if $\sigma > \frac{\lambda_N+1}{2}$, and strictly negative if, additionally, $\|\delta_i\| \neq 0$ for some $i\in\mathcal{V}$.

2) GAS of the set A by invariance principle: We apply the invariance principle of Theorem 23 in [10] to illustrate that the system cannot remain in a set with $U(X) = \mu$, for any given $\mu > 0$. Define $l(\mu) = \{X :\ U(X) = \mu\}$. Furthermore, let us denote the jump set in (12) as $\bar{\mathcal{D}} = \bar{\mathcal{D}}_1 \cup \bar{\mathcal{D}}_2$, where $\bar{\mathcal{D}}_1$ corresponds to the set where a task event can take place and $\bar{\mathcal{D}}_2$ has an equivalent meaning for the control events.

a) Task events: If $X(\delta = 0, e = 0) \in l(\mu)\cap\bar{\mathcal{D}}_1$, then it must hold that $q > 0$ otherwise $X \notin \bar{\mathcal{D}}_1$. Thus, $U(X)$ decreases according to (19).

b) Control events: Suppose that $X(0,0) \in l(\mu)\cap\bar{\mathcal{D}}_2$ with some $e_i(0,0) \neq 0$. Then, $U(X)$ decreases according to (20). Assume now that $X(0,0) \in l(\mu)\cap\bar{\mathcal{D}}_2$ with $e_i(0,0) = 0$, $\forall i\in\mathcal{V}$. Then, from the definition of the trigger function, $\delta_i = 0\ \forall i\in\mathcal{V}$. This situation can only occur if $q = 0$, otherwise as soon as $\|\delta_i\| < r_\ell$, with $r_\ell > 0$, $X \in \bar{\mathcal{D}}_1$, there is a switch of task, δ_i is updated according to (12) taking a value different from zero, and the Lyapunov function decreases according to the discussion above. Then if $e_i = 0$, $\delta_i = 0$, $\forall i\in\mathcal{V}$, and $q = 0$, $U(X) = 0$.

c) Flows: At flows, for each $X(0,0) \in l(\mu)\cap\bar{\mathcal{C}}$ and some $\delta_i \neq 0$, the Lyapunov function decreases according to (20). If $X(0,0) \in l(\mu)\cap\bar{\mathcal{C}}$ and $\delta_i = 0$, $\forall i\in\mathcal{V}$, this implies necessarily that $q = 0$ (otherwise $X(0,0) \notin \bar{\mathcal{C}}$ but $X(0,0) \in \bar{\mathcal{D}}_1$). Furthermore, by the definition of the trigger function, then $e_i = 0$, $\forall i\in\mathcal{V}$ and $U(X) = 0$.

Thus, since $U(X)$ is radially unbounded and no complete solutions remain in $l(\mu)$, then the set $\mathcal{A}$ is GAS. □

Remark 4. The resulting values of the parameter ϱ (15) of the trigger function requires that $\sigma > \frac{\lambda_N+2}{2}$ and $\beta e^{-\beta T} < 2\big(k(\lambda_N+1)(\frac{\sigma}{2}+1) + \frac{k\sigma}{2}\big)$ to be feasible. The second constraint yields the condition (16) over the value of T, that T can be freely chosen by tuning β as long as $T > 0$. A desired strategy to choose the parameter ϱ can be to set its value as large as possible, since this implies that events are generated less frequently. Two comments follow on this: For a given T, the value of β that gives less conservative results on ϱ is $\beta = 1/T$. For large values of T the term $\frac{\beta e^{-\beta T}}{2}$ is not comparable to the positive terms on the denominator of (15), and (15) can be seen as a function $\varrho(\sigma)$. Then, the value of σ that maximizes ϱ is $\sigma_{max} = \frac{\lambda_N+2}{2}\big(1+\sqrt{1+\frac{4(\lambda_N+1)}{(\lambda_N+2)^2}}\big)$. Finally, note that (15) depends on the gain k, so ϱ can be updated for each task.

V. SIMULATION EXAMPLE

Let us consider a system of four agents with $n = 2$ (2D setup) with the communication graph characterized by

$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix},$$

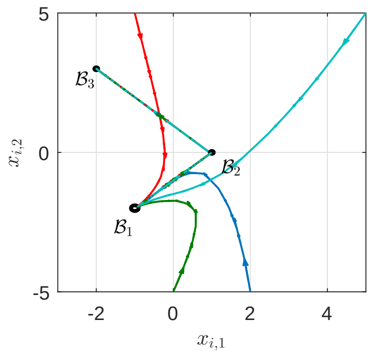

Fig. 1: Movement of the agents in the plane: agent 1 (blue), agent 2 (red), agent 3 (green), agent 4 (light blue).

Agent	1	2	3	4
Max. IeT	1.3443	1.9930	2.000	2.000
Av. IeT	0.6082	0.6106	0.5783	0.6777

TABLE I: Average and maximum inter-event times.

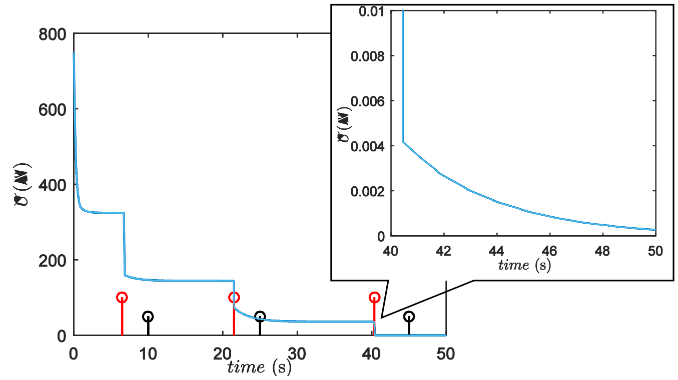

Fig. 2: Evolution of the Lyapunov function. Black lines represent the deadline times for each task and red lines are the actual switching times. On the right, zoom of the Lyapunov function when all the agents are inside $\mathcal{B}_3$.

and thus, $\lambda_N(L) = 3.4142$. We further consider the following configuration: Initial conditions $x_0^T = (2, -5, -1, 5, 0, -5, 5, 5)$; sequence of tasks, defined by the regions $\mathcal{B}_\ell = (c_\ell, r_\ell)$ and the deadline t_ℓ $\mathcal{B}_1 = ((-1, -2), 0.1)$ and $t_1 = 10$ s, $\mathcal{B}_2 = ((1, 0), 0.05)$ and $t_2 = 25$ s, $\mathcal{B}_3 = ((-2, 3), 0.05)$ and $t_3 = 45$ s; parameters of the trigger functions $T = 2$ s, $\beta = 0.5$, and $\sigma = 6.1339$. Furthermore, according to (18), the parameter ρ of the Lyapunov takes the value $\rho = 72$.

The trajectories of the agents in the state space are depicted in Figure 1. The agents evolve to agree to a common state and, at the same time, move towards the different regions of the state space. The agents start moving towards one region once they have reached the previous one. The control gains k_ℓ computed at the beginning of each task are $k_1 = 0.4846$, $k_2 = 0.2584$, and $k_3 = 0.2187$, and consequently, the parameter ϱ of the trigger function is set to 0.1638, 0.1644, and 0.1647, respectively. The average and maximum inter-event times (IeT) for each agent are summarized in Table I. Finally, the evolution of the Lyapunov function is depicted on Figure 2. Note that $U(X)$ experiments big jumps once one task is completed. Note also that the most relevant decrease of the Lyapunov during tasks is at the beginning of the simulation, where the agents are far from the consensus. The deadline times for each task are marked in black, while the actual switching times are depicted in red. Note that, with the proposed design, all the time constraints are satisfied. On the right, the convergence to zero once the last task has been completed is illustrated.

VI. CONCLUSIONS

We have presented distributed control laws and communication policies for the coordination of a multi-agent system that is requested to visit sequentially a finite number of regions of the state-space with some time constraints. The achievement of this objective is demonstrated by using the hybrid stability tools presented in [10], [11]. This work can be considered a novel viewpoint for the planning of tasks that require the coordination of multi-agent systems with time and communication constraints. In this regard, we will investigate the application of this framework to more complex and heterogeneous objectives and general coordination rules.

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