On the Equivalence between Prescribed Performance Control and Control Barrier Functions

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***Abstract*— In this paper, we show that Prescribed Performance Control (PPC) is a model-free Control Barrier Function (CBF)-based control approach. Specifically, we establish that a function utilized in the PPC design is a Time-Varying Reciprocal Control Barrier Function (TVRCBF). We demonstrate that PPC satisfies the same gradient condition that is well-known in the CBF literature, ensuring forward invariance. As a result, the control inputs generated by the PPC law belong to the input set characterized by the TVRCBF. Apart from assuming a certain controllability property, no further knowledge on the system dynamics is required. Our theoretical findings improve the understanding of the relationship between PPC and other CBF-based controllers. The theoretical results are validated through numerical simulations.**

I. INTRODUCTION

Safety is a top priority in today's society. When a safety specification is provided, it is crucial to guarantee its satisfaction. With the rapid advancement of technology in areas such as computation, communication, and artificial intelligence, autonomous devices like cars, drones, and healthcare robots are becoming more prevalent in our daily lives. While these technological advancements have brought numerous benefits, ensuring safety remains a critical concern. Consequently, controlling *safety-critical systems* has become a prominent focus in the field of control engineering [1]–[5]. To address safety concerns, the concept of safety is mathematically handled to create a set that is forward control invariant, and a safety controller is designed based on this.

Prescribed performance control (PPC) is a promising method initially used to address tracking problems [6]–[11], but it can also handle the forward invariance problem. This technique involves using a nonlinear mapping to generate control inputs that keep the output within a predefined set, resulting in a low-complexity adaptive high-gain controller. One notable advantage of PPC is that it does not require a detailed knowledge of the model dynamics [10], hence it is a model-free approach. All knowledge that is needed on the system dynamics is a minor assumption on their controllability [7]. In the PPC design, performance and safety specifications are defined using a time-varying set, particularly through one or multiple funnel constraints. PPC guarantees that the controlled system's outputs remain within these funnels, ensuring safety.

Control Barrier Functions (CBFs) are a powerful tool used to verify the safety of controlled dynamical systems [2]. There are two types of CBFs prevalent in the literature: Reciprocal Control Barrier Functions (RCBFs) and Zeroing Control Barrier Functions (ZCBFs) [3]. CBFs are typically considered for time-invariant constrained sets, but there are also works that address time-varying constrained sets [12]–[15]. The synthesis of CBF controllers has been extensively studied, and successful results have been obtained using quadratic programming (QP) [3]. However, these CBF-QPs require knowledge on the dynamical model, and modeling uncertainties can potentially compromise safety.

Although commonly formulated for tracking problems, PPC renders a predefined set forward invariant just like CBF-based controllers, e.g. the CBF-QP. This similarity raises the question of whether PPC can be considered a CBF-based controller. This question is particularly interesting because PPC does not require knowledge of the model, unlike the CBF-QP. To the best of our knowledge, no previous research has examined the theoretical similarities between PPC and CBF-based control. Identifying these similarities and differences can be valuable for enhancing current control frameworks for safety-critical systems.

Our contributions are summarized as follows. At first, we introduce a generalized definition of time-varying RCBF that allows for handling time-varying safety-related specifications on state vectors. We then demonstrate the validity of the proposed time-varying RCBF. Secondly, we show that the PPC ensures that the resulting closed-loop system renders the set forward invariant. Additionally, we identify a function in the PPC design as a time-varying RCBF. Thirdly, we demonstrate that the control input provided by the PPC law belongs to the input set that satisfies a CBF-based gradient condition for some class $\mathcal{K}$ function. This is achieved even without the need for specific knowledge on the system dynamics.

II. PRELIMINARIES

In this section, we state the forward invariance problem for a class of time-varying constraints. Then, we introduce a notion of generalized Time-Varying Reciprocal Control Barrier Functions (TVRCBFs) as a standard tool to handle the forward invariance problem, and PPC as an alternative.

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A. Problem Setting

Consider the following first-order input affine system

$$\dot{x} = f(x) + G(x)u, \qquad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^n$ is the control input, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, and f and G are assumed to be locally Lipschitz. We assume that state x is known. Moreover, we impose the following standard assumption on $G(x)$, which can be viewed as a controllability condition:

Assumption 1 ([7], [9]): Matrix $G_{\text{sym}}(x) := \frac{G(x)+G(x)^\top}{2}$ is either uniformly positive or uniformly negative definite for all $x \in \mathbb{R}^n$. Without loss of generality, we assume that matrix $G_{\text{sym}}(x)$ is uniformly positive definite, that is, there exists some positive constant $\underline{g}$ such that $0 < \underline{g} \le \lambda_{\min}\left(G_{\text{sym}}(x)\right)$ for all $x \in \mathbb{R}^n$.

Let us define $\rho^L : \mathbb{R}_{\ge 0} \to \mathbb{R}^n$ and $\rho^U : \mathbb{R}_{\ge 0} \to \mathbb{R}^n$ as time-varying vector-valued functions with continuously differentiable and bounded entries ρ_i^L and ρ_i^U, respectively. Moreover, we suppose that the derivatives of ρ_i^L and ρ_i^U are bounded and there exists a positive scalar δ such that for each entry it holds $\rho_i^U(t) - \rho_i^L(t) \ge \delta > 0$ for all times $t \ge 0$ and all $i = 1, 2, \ldots, n$. In this paper, we consider the two-sided (time-varying) state constraints in the following form

$$\rho^L(t) < x < \rho^U(t), \tag{2}$$

which shall be satisfied for all times $t \ge 0$. These constraints define multiple funnel constraints on the states (each entry of vector x has its own time-varying upper and lower bounds). If we define a time-varying compact set $\mathcal{C}(t)$ as

$$\mathcal{C}(t) := \left\{x \mid \rho^L(t) \le x(t) \le \rho^U(t)\right\} \tag{3a}$$

$$\text{Int}(\mathcal{C}(t)) := \left\{x \mid \rho^L(t) < x(t) < \rho^U(t)\right\} \tag{3b}$$

$$\partial\mathcal{C}(t) := \mathcal{C}(t) \setminus \text{Int}(\mathcal{C}(t)) \tag{3c}$$

ensuring that the set $\mathcal{C}(t)$ remains forward invariant is equivalent to satisfying constraint (2). In particular, forward invariance is defined as follows.

Definition 1: Let $u(x) : \mathbb{R}^n \to \mathbb{R}^n$ be some given state-feedback controller. Then we call $\mathcal{C}(t)$ *forward invariant* with respect to (1) if for any $x(0) = x_0 \in \mathcal{C}(0)$, it holds $x(t) \in \mathcal{C}(t)$ for all $t \ge 0$.

In this paper, we compare PPC to RCBF-based control approaches. In particular, we show that the PPC design is based on an RCBF, and PPC can be therefore considered to be a model-free RCBF-based controller. Before we proceed, we review RCBFs and PPC.

B. Time-Varying Reciprocal Control Barrier Functions

TVRCBFs are used for the verification of the forward invariance of $\mathcal{C}(t)$. Once an TVRCBF has been determined, control inputs that render $\mathcal{C}(t)$ forward invariant can be computed.

Definition 2: Consider control system (1) and the set $\mathcal{C}(t)$ as defined in (3a). A continuously differentiable function $B : \mathbb{R}_{\ge 0} \times \text{Int}(\mathcal{C}(t)) \to \mathbb{R}_{\ge 0}$ is called a *Time-Varying Reciprocal Control Barrier Function (TVRCBF)* if there exist class $\mathcal{K}$ functions α_1 and α_2 such that for all $x \in \text{Int}(\mathcal{C}(t))$ and all $t \ge 0$ it holds

$$\frac{1}{\alpha_1\left(d_H(x, \partial\mathcal{C}(t))\right)} \le B(t,x) \tag{4a}$$

$$\inf_u \Big[L_f B(t,x) + L_G B(t,x)u + \tfrac{\partial B(t,x)}{\partial t}\Big] \le \alpha_2\Big(\tfrac{1}{B(t,x)}\Big), \tag{4b}$$

where $d_H(x, \partial\mathcal{C}(t)) := \inf_{y\in\partial\mathcal{C}(t)} \|x - y\|$ denotes the Hausdorff distance.

Theorem 1: Let $\mathcal{C}(t)$ be defined as in (3a). If there exists a TVRCBF $B : \mathbb{R}_{\ge 0} \times \text{Int}(\mathcal{C}(t)) \to \mathbb{R}_{\ge 0}$, then any locally Lipschitz-continuous control input $u(\cdot,\cdot)$ with $u(t,x) \in K_{\text{rcbf}}(t,x)$, where $K_{\text{rcbf}}(t,x) := \Big\{u \mid L_f B(t,x) + L_G B(t,x)u + \frac{\partial B(t,x)}{\partial t} \le \alpha_2\left(\frac{1}{B(t,x)}\right)\Big\}$, for all $(t,x) \in \mathbb{R}_{\ge 0} \times \text{Int}(\mathcal{C}(t))$, renders $\text{Int}(\mathcal{C}(t))$ forward invariant under dynamics (1).

Proof: As B is a TVRCBF, there exist inputs $u(t,x) \in K_{\text{rcbf}}(t,x)$ for all $(t,x) \in \mathbb{R}_{\ge 0} \times \text{Int}(\mathcal{C}(t))$ such that

$$\dot{B}(t,x(t)) \le \alpha_2\left(\tfrac{1}{B(t,x(t))}\right) \tag{5}$$

holds. Assume that $x(t)$ is a solution to (1) with control input $u(\cdot,\cdot)$ and initial condition $x(0) \in \mathcal{C}(t)$. Analogously to the proof of [3, Thm. 1], we observe that the Comparison Lemma [16, Lem. 3.4] still holds if the right-hand side of (5) is assumed to be continuous and non-increasing in its argument instead of being Lipschitz-continuous; see [3] for a more detailed discussion. Then, by applying the Comparison Lemma and [3, Thm. 1], we obtain from (5) that $B(t,x(t)) \le \frac{1}{\sigma\left(\frac{1}{B(0,x_0)},t\right)}$, where σ is a class $\mathcal{KL}$ function. Then using (4a), we obtain $d_H(x, \partial\mathcal{C}(t)) \ge \alpha_1^{-1}\left(\frac{1}{B(t,x)}\right) \ge \alpha_1^{-1}\left(\frac{1}{\sigma\left(\frac{1}{B(0,x_0)},t\right)}\right) > 0$ ensuring $0 < d_H(x, \partial\mathcal{C}(t))$ for all $t \in \mathcal{I}$. Note that the last inequality holds from the fact that $B(0,x_0) > 0$ if $x_0 \in \text{Int}(\mathcal{C}(t))$ together with the property of $\mathcal{KL}$ function σ. ■

Remark 1: Definition 2 is a generalized version of [3, Def. 4], where the latter defines a subclass of RCBFs in the sense of Definition 2. The advantage of the generalized definition is that it defines RCBFs directly in terms of $\mathcal{C}(t)$ and avoids the definition via an additional function $h(x)$, that is used in [3]. Thereby, we also omit the upper bound of B in (4a) compared to [3]. This allows us to omit some technicalities in the sequel.

C. Prescribed Performance Control

Usually, PPC is formulated for tracking problems [6]–[11]. However, in terms of (3a), PPC can be also viewed as a controller for ensuring the forward invariance of the time-varying set $\mathcal{C}(t)$. In this respect, we formulate PPC as a controller for forward invariance.

To start with, we normalize the entries x_i of state x as in [17] with respect to their associated time-varying upper and lower bounds $\rho_i^L(t)$ and $\rho_i^U(t)$. The normalized states are given as

$$\hat{x}_i := \frac{x_i - \frac{1}{2}(\rho_i^L(t) + \rho_i^U(t))}{\frac{1}{2}(\rho_i^U(t) - \rho_i^L(t))}, \quad \text{for } i = 1, \ldots, n. \tag{6}$$

Note that $\rho_i^L(t) < x_i < \rho_i^U(t)$ is equivalent to $-1 < \hat{x}_i < 1$, $i = 1, 2, \ldots, n$. Therefore, $x \in \text{Int}(\mathcal{C}(t))$ if and only if $-\mathbb{1}_n < \hat{x} < \mathbb{1}_n$, where $\hat{x} := \text{col}(\hat{x}_i) \in \mathbb{R}^n$ and $\mathbb{1}_n$ is the n-dimensional vector of ones. Let us introduce a coordinate transformation by a diffeomorphism $T : (-1, 1) \rightarrow (-\infty, \infty)$ defined as

$$\varepsilon_i = T(\hat{x}_i) := \ln\left(\tfrac{1+\hat{x}_i}{1-\hat{x}_i}\right), \quad \text{for } i = 1, \ldots, n. \tag{7}$$

Note that (2) is satisfied if and only if ε_i is bounded for all i. Now, let us define the scalar function $B_{\text{ppc}} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ as

$$B_{\text{ppc}}(\varepsilon) := \tfrac{1}{2}\varepsilon^\top \varepsilon, \tag{8}$$

where $\varepsilon := \text{col}(\varepsilon_i) \in \mathbb{R}^n$. Since B_{ppc} is in fact a function of state x, and due to the time-dependency of ρ^L and ρ^U, a function of time t, we can also write $B_{\text{ppc}}(t, x)$. The PPC control law is now obtained by taking the negative gradient [18] of (8) as

$$u = -k\left(\tfrac{\partial B_{\text{ppc}}(\varepsilon)}{\partial x}\right)^\top = -k\Xi\varepsilon, \tag{9}$$

where $k > 0$ is some positive scalar, and $\Xi := \text{diag}(\xi_i)$ with

$$\xi_i = \frac{\partial \varepsilon_i}{\partial \hat{x}_i}\frac{\partial \hat{x}_i}{\partial x_i} = \frac{4}{(1 - \hat{x}_i^2)(\rho_i^U - \rho_i^L)}. \tag{10}$$

Theorem 2: Consider system (1) with constraints (2), where $x_0 \in \text{Int}(\mathcal{C}(t))$. The PPC law (9) guarantees the following:

(i) All the signals in the closed-loop system are bounded
(ii) The set $\mathcal{C}(t)$ as defined in (3a) is forward invariant

Remark 2: Results analogous to Theorem 2 have been presented in the PPC literature. For instance, refer to [17] for a second-order system.

Proof: First, notice that for $x(0) = x_0 \in \text{Int}(\mathcal{C}(t))$, $B_{\text{ppc}} = \frac{1}{2}\varepsilon^\top\varepsilon$ in (8) is well defined. Substituting (1), (6) and (7) into (8) and taking the time derivative of $B_{\text{ppc}}(t, x)$ yields

$$\begin{aligned}\dot{B}_{\text{ppc}} &= \tfrac{\partial B_{\text{ppc}}}{\partial \varepsilon}\tfrac{\partial \varepsilon}{\partial \hat{x}}\left(\tfrac{\partial \hat{x}}{\partial x}\dot{x} + \tfrac{\partial \hat{x}}{\partial \rho^L}\dot{\rho}^L + \tfrac{\partial \hat{x}}{\partial \rho^U}\dot{\rho}^U\right) \\ &= \varepsilon^\top\Xi\{f(x) + G(x)u + \rho_m(x, \rho^L, \rho^U, \dot{\rho}^L, \dot{\rho}^U)\},\end{aligned} \tag{11}$$

where $\Xi := \text{diag}(\xi_i)$ with ξ_i as in (10), and

$$\rho_m(\cdot) = \text{diag}\left(\tfrac{x_i - \rho_i^U}{\rho_i^U - \rho_i^L}\right)\dot{\rho}^L + \text{diag}\left(\tfrac{-x_i + \rho_i^L}{\rho_i^U - \rho_i^L}\right)\dot{\rho}^U. \tag{12}$$

Next, substituting (9) into (11) gives

$$\dot{B}_{\text{ppc}} = \varepsilon^\top\Xi(f(x) - kG(x)\Xi\varepsilon + \rho_m). \tag{13}$$

Recall that the state trajectories remain in the set $\mathcal{C}(t)$, i.e., $x(t) \in \text{Int}(\mathcal{C}(t))$, if and only if $\hat{x}(t) \in \Omega_{\hat{x}} := (-1, 1)^n$. Taking the time derivative of (6) and substituting (12) yields $\dot{\hat{x}} = \text{diag}\left(\frac{2}{\rho_i^U - \rho_i^L}\right)(\dot{x} + \rho_m) = f_{\hat{x}}(t, \hat{x})$, where one can verify that $\dot{x}$ can be written as a function of t and $\hat{x}$ from (1), (6), and (9). Note that one can also show that $f_{\hat{x}}(t, \hat{x})$ is continuous in t and locally Lipschitz over the set $\Omega_{\hat{x}}$. As a result, according to [19, Thm. 54], for $x_0 \in \text{Int}(\mathcal{C})$, or equivalently $\hat{x}(0) \in \Omega_{\hat{x}}$, there exists a unique maximal solution such that $\hat{x}(t) \in \Omega_{\hat{x}}$ for all $t \in \mathcal{I} := [0, \tau_{\max})$, where $\tau_{\max} > 0$. Since $\hat{x} \in \Omega_{\hat{x}}$ for all $t \in \mathcal{I}$, from (6), one can infer that $x(t) \in \Omega_x(t)$ for all $t \in \mathcal{I}$, where $\Omega_x(t) = (\rho_i^L(t), \rho_i^U(t))^n$ is a compact set for all $t \geq 0$. Owing to the boundedness of ρ^L, ρ^U, $\dot{\rho}^L$ and $\dot{\rho}^U$, the function ρ_m in (12) is bounded for all times. In addition, due to $x(t) \in \Omega_x(t)$ for all $t \in \mathcal{I}$ and the continuity of $f(x)$, there exists a finite constant bound γ such that

$$\|f(x) + \rho_m(x, \rho^L, \rho^U, \dot{\rho}^L, \dot{\rho}^U)\| < \gamma, \quad \forall t \in \mathcal{I}. \tag{14}$$

Owing to Assumption 1, (13) and (14), we get

$$\begin{aligned}\dot{B}_{\text{ppc}} &= -k\varepsilon^\top\Xi G(x)\Xi\varepsilon + \varepsilon^\top\Xi(f(x) + \rho_m) \\ &= -k\varepsilon^\top\Xi G_{\text{sym}}(x)\Xi\varepsilon + \varepsilon^\top\Xi(f(x) + \rho_m) \\ &< -k\underline{g}\|\Xi\varepsilon\|^2 + \|\varepsilon^\top\Xi\|\gamma, \quad \forall t \in \mathcal{I} \\ &\leq \|\Xi\|\|\varepsilon\|(-k\underline{g}\|\Xi\|\|\varepsilon\| + \gamma), \quad \forall t \in \mathcal{I} \\ &=: H_{\text{ppc}}(\|\varepsilon\|).\end{aligned} \tag{15}$$

Note that from (10) and the boundedness of ρ_i^L and ρ_i^U, one can infer that $\xi_i(t)$ is lower bounded by a positive constant for all $t \in \mathcal{I}$. Thus there exists an $\eta > 0$ such that $\xi_i(t) \geq \eta$ for all $t \in \mathcal{I}$. Due to the diagonality of Ξ, we also have $\|\Xi\| \geq \eta$. As a result, by using the above fact and (15), we furthermore obtain

$$\dot{B}_{\text{ppc}} \leq 0, \quad \forall\|\varepsilon\| \geq \tfrac{\gamma}{k\underline{g}\eta} \geq \tfrac{\gamma}{k\underline{g}\|\Xi\|} \geq 0, \forall t \in \mathcal{I}, \tag{16}$$

which shows that $\varepsilon(t)$ is *uniformly ultimately bounded* [16, Thm. 4.18] with the ultimate bound $\varepsilon_0 := \frac{\gamma}{k\underline{g}\eta}$. Therefore, $\|\varepsilon(t)\|$ is upper bounded by $\bar{\varepsilon} := \max\{\|\varepsilon(0)\|, \varepsilon_0\}$, which is independent of $\tau_{\max}$.

Next, we show that the maximal time interval $\mathcal{I}$ can be arbitrarily extended, i.e., $\tau_{\max} = \infty$. To this end, note at first that taking the inverse logarithm of ε in (7) yields

$$-1 < \frac{e^{-\bar{\varepsilon}} - 1}{e^{-\bar{\varepsilon}} + 1} = -b \leq \hat{x}_i \leq b := \frac{e^{\bar{\varepsilon}} - 1}{e^{\bar{\varepsilon}} + 1} < 1. \tag{17}$$

To continue, define $\Omega_b := [-b, b]^n$, which is a nonempty and compact subset of $\Omega_{\hat{x}}$, and note that $\hat{x} \in \Omega_b \subset \Omega_{\hat{x}}$ for all $t \in \mathcal{I}$. From [19, Prop. C.3.6], assuming $\tau_{\max} < \infty$ reveals that there must exists a time instant $t' \in \mathcal{I}$ such that $\hat{x}(t) \notin \Omega_b$, which is a contradiction. Therefore, all closed-loop signals remain bounded for all $t \geq 0$. Consequently, since $\hat{x} \in \Omega_b$ for all $t \geq 0$ then the specifications in (2) are satisfied for all times, i.e., the PPC law in (9) ensures the forward invariance of set $\mathcal{C}(t)$ in (3a), and $\tau_{\max} = \infty$ follows. ■

As shown in the proof above, PPC ensures the forward invariance of a time-invariant compact set $\Omega_{\hat{x}}$. To achieve this, the PPC renders Ω_b forward invariant, which is a subset of $\Omega_{\hat{x}}$. Therefore, applying the PPC methodology implies that the resulting system will stay within a more conservative level set than originally intended as a safety specification. Although the performance might be conservative, PPC is implemented in a model-free manner, without relying on specific model knowledge.

III. PRESCRIBED PERFORMANCE CONTROL AS A MODEL-FREE TIME-VARYING RCBF BASED-CONTROL

In this section, we show that $B_{\text{ppc}}(\varepsilon)$ as defined in (8) is a TVRCBF. Moreover, it turns out that the PPC control law (9) satisfies the TVRCBF condition for any $k > 0$, and the forward invariance of $\text{Int}(\mathcal{C}(t))$ follows.

A. Main results

To begin with, let us introduce a lemma used later in the proof of Theorem 3.

Lemma 1: Let $\mathcal{F} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function with a maximizer $y^* = \text{argmax}_{y \in [0,y_0]} \mathcal{F}(y)$. Moreover, let α be a class $\mathcal{K}$ function. If $\mathcal{F}(y^*) \leq \alpha\left(2/y_0^2\right)$, then it holds that $\mathcal{F}(y) \leq \alpha\left(2/y^2\right)$ for all $y \in [0, y_0]$.

Proof: If $0 \leq y \leq y_0$, it holds that $\alpha\left(2/y^2\right) \geq \alpha\left(2/y_0^2\right) \geq \mathcal{F}(y^*) \geq \mathcal{F}(y)$. ■

The following two theorems, namely Theorem 3 and Theorem 4, capture the relationship between PPC and TVRCBFs.

Theorem 3 (B_{ppc} is a TVRCBF): Suppose that Assumption 1 holds for control system (1), then B_{ppc} defined in (8) is a TVRCBF according to Definition 2.

Proof: Our goal is to show that (9) with $k > 0$ along with B_{ppc} satisfies (4) where $B = B_{\text{ppc}}$. Therefore, in this proof, we investigate the properties of B_{ppc} and show that (4) holds for $B = B_{\text{ppc}}$ for some class $\mathcal{K}$ functions α_1 and α_2. Then it follows that B_{ppc} is a TVRCBF.

Step1: Checking if $B_{ppc}(\varepsilon)$ satisfies (4a)

Without loss of generality, for simplicity, consider a scalar system ($x \in \mathbb{R}$). If $B_{\text{ppc}}(\varepsilon)$ satisfies (4a), we must have that

$$B_{\text{ppc}}(t,x) - \tfrac{1}{\alpha_1(d_H(x,\partial\mathcal{C}(t)))} \geq 0, \;\text{ for } x \in \text{Int}(\mathcal{C}(t)), \quad (18)$$

i.e., $\rho^L < x < \rho^U$. From (6)-(8), we have that

$$B_{\text{ppc}} = \tfrac{1}{2}\ln^2\left(\tfrac{x-\rho^L}{\rho^U - x}\right). \quad (19)$$

Here, let us take the following class $\mathcal{K}$ function α_1

$$\alpha_1\left(d_H(x,\partial\mathcal{C}(t))\right) := \frac{2}{\ln^2\left(\frac{2}{\rho^U-\rho^L}d_H(x,\partial\mathcal{C}(t))\right)}, \quad (20)$$

where

$$d_H = \begin{cases} x - \rho^L & \text{if } \rho^L < x < \frac{\rho^U+\rho^L}{2} \\ \rho^U - x & \text{if } \frac{\rho^U+\rho^L}{2} \leq x < \rho^U \end{cases}. \quad (21)$$

When $x \in \text{Int}(\mathcal{C}(t))$, according to (21) as x varies between ρ^L and ρ^U, d_H changes from zero to $(\rho^U - \rho^L)/2$, thus insider the logarithm term varies from 0 to 1 which makes α_1 varies from zero to infinity. Due to the symmetry of functions, B_{ppc} and α_1, it suffices to consider only for an interval $\frac{\rho^U+\rho^L}{2} \leq x < \rho^U$. Then, from (19)-(21), (18) becomes $\frac{1}{2}\ln^2\left(\frac{x-\rho^L}{\rho^U-x}\right) + \frac{1}{2}\ln^2\left(\frac{2(\rho^U-x)}{\rho^U-\rho^L}\right)$. Since $\frac{x-\rho^L}{\rho^U-x} \geq 1$ and $\frac{2(\rho^U-x)}{\rho^U+\rho^L} \leq 1$ for the interval, it is reduced to consider the sign of $\ln\left(\frac{x-\rho^L}{\rho^U-x}\right) + \ln\left(\frac{2(\rho^U-x)}{\rho^U-\rho^L}\right)$, or equivalently, $2(\rho^U - x)(x-\rho^L) - (\rho^U - x)(\rho^U - \rho^L)$, which becomes zero as its unique minimum at $x = \frac{\rho^L+\rho^U}{2}$, ensuring non-negativeness of (18). Therefore, we conclude that B_{ppc} defined in (8) satisfies (4a) for a class $\mathcal{K}$ function α_1 defined in (20). It should be noted that the results presented in this step can be extended to a nonscalar system.

Step2: Deriving a condition on k satisfying (4b) for some class $\mathcal{K}$ function α_2.

Recall that the maximum of a quadratic function $-a\|\varepsilon\|^2 + b\|\varepsilon\| + c$, where a, b, and c are some scalar values with $a > 0$, is given by $\frac{b^2}{4a} + c$ and $\frac{b}{2a}$ is its corresponding maximizer. Hence, the maximum value of $H_{\text{ppc}}(\|\varepsilon\|)$ in (15), which is the upper bound of $\dot{B}_{\text{ppc}}$, and its maximizer ε^* are given as $\max_{\|\varepsilon\|} H_{\text{ppc}} = \frac{\gamma^2}{4k\underline{g}\eta}$ and $\varepsilon^* := \text{argmax}_{\|\varepsilon\|} H_{\text{ppc}} = \frac{\gamma}{2k\underline{g}\eta}$, respectively. Now note that if

$$H_{\text{ppc}}(\varepsilon^*) \leq \alpha_2\left(2/\varepsilon_0^2\right) \quad (22)$$

holds, employing Lemma 1 with $y = \|\varepsilon\|$, $y_0 = \varepsilon_0$, and $\mathcal{F} = H_{\text{ppc}}$ yields

$$\dot{B}_{\text{ppc}} \leq H_{\text{ppc}} \leq \alpha_2\left(1/B_{\text{ppc}}\right), \quad \forall\, 0 \leq \|\varepsilon\| \leq \varepsilon_0. \quad (23)$$

Next, let us define $B_0 := \frac{1}{2}\varepsilon_0^2$, then (22) can be written as $\frac{\gamma^2}{4k\underline{g}\eta} \leq \alpha_2\left(1/B_0\right)$. From this, we obtain the condition

$$k \geq \tfrac{\gamma^2}{4\underline{g}\eta}\alpha_2(B_0), \quad (24)$$

which ensures the satisfaction of (23). Since (16) and (23) hold as previously shown, the satisfaction of property (4b) for B_{ppc} follows.

At last, we note that the value of k depends on a class $\mathcal{K}$ function α_2 as it can be seen from (24). In the following step, we show that for any choice of $k > 0$, one can find a class $\mathcal{K}$ function α_2 such that (24) is satisfied.

Step3: Deriving a class $\mathcal{K}$ function α_2 for any $k > 0$

Let us consider the class $\mathcal{K}$ function $\alpha_2(x) = \kappa x$ for some scalar $\kappa > 0$. Substituting it into (24) yields $k \geq \frac{\gamma^2}{4\underline{g}\eta}\kappa B_0 = \frac{\gamma^2}{8\underline{g}\eta}\kappa\varepsilon_0^2$. Then, for any $k > 0$, we can choose a scalar $\kappa > 0$ such that $0 < \kappa \leq k\frac{8\underline{g}\eta}{\gamma^2\varepsilon_0^2}$. Thereby we have also shown, that (16) and (23) imply the satisfaction of (4b) with $B = B_{\text{ppc}}$ for any $k > 0$, which completes the proof. ■

Even though PPC is a model-free control approach, according to Theorem 3, B_{ppc} is proven to be a TVRCBF. Therefore, PPC represents a model-free CBF-based controller which contradicts the common claim that CBF-based controllers are inherently model-based. This theorem demonstrates that for certain classes of systems, it is possible to design a model-free CBF-based controller.

We showed in the proof of Theorem 3 that if $k > 0$, B_{ppc} is a TVRCBF. The following theorem shows that the converse also holds, namely, if PPC ensures forward invariance of $\mathcal{C}(t)$ by $u \in K_{\text{rcbf}}$, then $k > 0$.

Theorem 4 (PPC is a TVRCBF-based controller): Let $\rho^L, \rho^U : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be arbitrarily bounded and time-dependent functions with $\rho_i^U(t) - \rho_i^L(t) \geq \delta_i > 0$ for all

$i = 1,\dots,n$. Suppose Assumption 1 holds for control system (1), then the following are equivalent:
For the system (1),

(i) the controller (9) satisfies $k > 0$.
(ii) the controller (9) satisfies $u \in K_{\text{rcbf}}(t,x)$.

Proof: Sufficiency is established in the proof of Theorem 3. Necessity is derived by contradiction. To this end, let α_2 be a class $\mathcal{K}$ function and assume that (4b) holds for $k \le 0$. Then, according to (4b) and (9), it must hold that

$$\dot{B}_{\text{ppc}} = \varepsilon^\top \Xi(-kG_{\text{sym}}(x)\Xi\varepsilon + f(x) + \rho_m) \le \alpha_2\left(\tfrac{2}{\varepsilon^\top\varepsilon}\right) \quad (25)$$

for all $x \in \text{Int}(\mathcal{C}(t))$ and all $t \ge 0$, where $G_{\text{sym}}(x)$ is positive definite by Assumption 1 and bounded for all $x(t) \in \mathcal{C}(t)$ due to the local Lipschitz-continuity of G and the compactness of $\mathcal{C}(t)$ for all $t \ge 0$. Moreover, for every fixed x, it is known from (14), that $||f(x) + \rho_m|| \le \gamma$ for some $\gamma > 0$.

Case 1 ($k < 0$)

To start with, for $k > 0$, we determine a lower bound for $\dot{B}_{\text{ppc}}$ as

$$\begin{aligned}\dot{B}_{\text{ppc}} &= \varepsilon^\top \Xi(-kG_{\text{sym}}(x)\Xi\varepsilon + f(x) + \rho_m)\\ &\ge -k\underline{g}||\Xi||^2\,||\varepsilon||^2 - ||\varepsilon||\,||\Xi||\,\gamma =: \zeta(||\varepsilon||, ||\Xi||).\end{aligned}$$

If we choose states x arbitrarily close to the boundary of $\mathcal{C}(t)$, i.e., $x \to \partial\mathcal{C}(t)$, we obtain $\lim_{x\to\partial\mathcal{C}(t)} \dot{B}_{\text{ppc}} \ge \lim_{x\to\partial\mathcal{C}(t)} \zeta(||\varepsilon||, ||\Xi||) = +\infty$, since $x \to \partial\mathcal{C}(t)$ implies $||\varepsilon|| \to \infty$ and $||\Xi|| \to \infty$. On the other hand, it holds $\lim_{x\to\partial\mathcal{C}(t)} \left[\alpha_2\left(\frac{2}{\varepsilon^\top\varepsilon}\right)\right] = 0$, which contradicts (25).

Case 2 ($k = 0$)

Let $f(x) = x$, and $-\rho_i^L = \rho_i^U = \rho_c$ for some scalar $\rho_c > 0$. By substituting (9) with $k = 0$ into (1), we obtain the unstable system $\dot{x} = x$, and for $x(0) \in \mathcal{C}(0)$ with $x(0) \ne 0$, we have $||x(t)|| \to \infty$ as $t \to \infty$. Thus, there exists a time $\tau_{\max}$ such that $x(t) \to \partial\mathcal{C}(t)$ as $t \to \tau_{\max}$, and we derive for the left-hand side of (25)

$$\begin{aligned}\lim_{x\to\partial\mathcal{C}(t)} \dot{B}_{\text{ppc}} &= \lim_{x\to\partial\mathcal{C}(t)} \left[\varepsilon^\top \Xi x\right]\\ &= \lim_{x\to\partial\mathcal{C}(t)} \left[\sum_{i=1}^{n}\left(\ln\left(\tfrac{\rho_c + x_i}{\rho_c - x_i}\right)\right)\tfrac{2\rho_c x_i}{\rho_c^2 - x_i^2}\right] = +\infty.\end{aligned}$$

On the other hand, analogously to Case 1, it holds $\lim_{x\to\partial\mathcal{C}(t)} \left[\alpha_2\left(\frac{2}{\varepsilon^\top\varepsilon}\right)\right] = 0$ which contradicts (25).

Therefore, *Case 1* and *Case 2* altogether contradict $k \le 0$ and hence $k > 0$, which completes the proof. ■

As we already know from Theorem 3, the scalar function B_{ppc} from the PPC derivation is a TVRCBF. Compared to other CBF-based controllers such as those using quadratic programs, Assumption 1 gives rise to the alternative, model-free controller. Indeed, as Theorem 4 shows, if we are given a TVRCBF, then the PPC law (9) is the corresponding model-free TVRCBF-based controller. In this way, we strengthen the theoretical connections between PPC and CBF-based control.

Remark 3: We note that in (15), the upper bound of $\dot{B}_{\text{ppc}}$, denoted by H_{ppc}, consists of a quadratic function in $||\varepsilon||$. In contrast, the class $\mathcal{K}$ function $\alpha_2\left(1/B_{\text{ppc}}\right) = \alpha_2\left(2/||\varepsilon||^2\right)$ is a strictly decreasing function for all $||\varepsilon||$. The relationship between H_{ppc} and $\alpha_2\left(1/B_{\text{ppc}}\right)$, the right-hand side of (4b), is shown in Fig. 1. Note that Fig. 1 captures the effect of k on the curves H_{ppc}. In particular, the gap between H_{ppc} and $\alpha_2(1/B_{\text{ppc}}(t,x))$ increases as k increases. This is consistent with the intuition that a larger k makes the system safer.

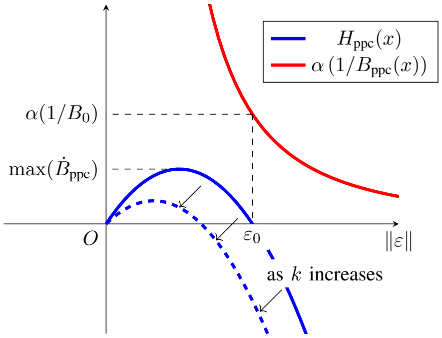


Fig. 1: $H_{\text{ppc}}(x)$ and $\alpha_2(1/B_{\text{ppc}}(x))$ for some given B_{ppc}.

IV. SIMULATION RESULTS

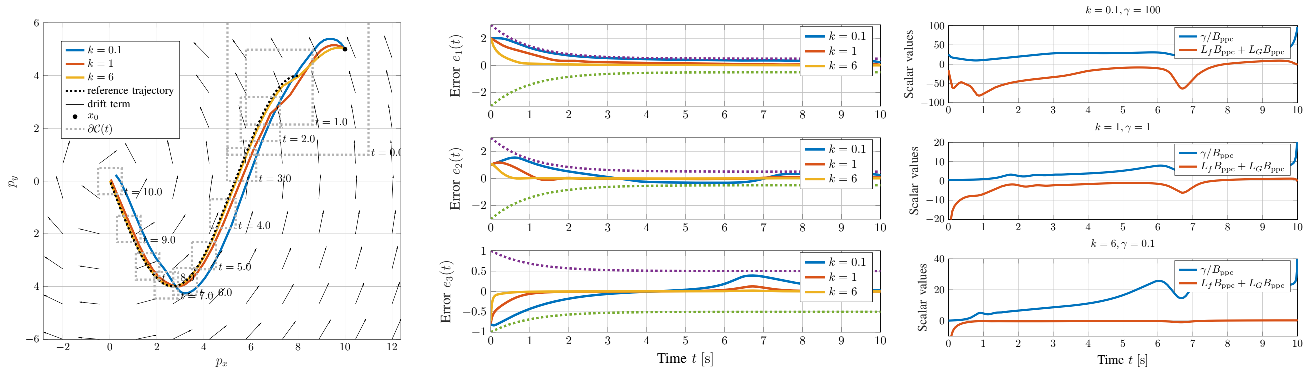
Consider the problem of controlling an unmanned surface vessel that behaves according to the kinematic model [20]

$$\dot{x} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} + \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \\ r \end{bmatrix}, \quad (26)$$

where $x = [p_x, p_y, \psi]^\top$ and $\theta = \tan^{-1}(p_y/p_x)$. Note that $f(x)$ can be thought of as the movements of a vessel resulting from waves. Suppose that the following constraint on ψ, $-1 < \psi < 1$, holds, which ensures the validity of Assumption 1 since $\lambda_{\min}(G_{\text{sym}}) = 0.5403 = \underline{g} > 0$.

The objective is to track the reference trajectories $x^{\text{ref}}(t) = \text{col}(x_i^{\text{ref}}(t))$ with $x_1^{\text{ref}}(t) = -\frac{8}{10}t + 8$, $x_2^{\text{ref}}(t) = 4\cos\left(\frac{3}{20}\pi t\right)$, $x_3^{\text{ref}}(t) = \tan^{-1}\left(\frac{3\pi}{2}\sin\left(\frac{3\pi}{20}t\right)\right)$ under a user-defined prescribed performance, which can be formulated as a set invariance problem for the tracking error signals. To this end, we set $\rho^L(t) = -\rho(t) + x^{\text{ref}}(t)$ and $\rho^U(t) = \rho(t) + x^{\text{ref}}(t)$), where $\rho(t) = \text{col}(\rho_i(t))$ with $\rho_1(t) = \rho_2(t) = (3 - 0.5)\exp(-t) + 0.5$ and $\rho_3(t) = (1 - 0.5)\exp(-t) + 0.5$, which constitute the performance functions of the tracking errors. The initial state is assumed to be $x_0 = x(0) = [10, 5, -0.8]^\top$.

The state trajectories under the PPC law for different choices of the gain $k > 0$ are shown in Fig. 2 (a). Moreover, the evolution of the tracking errors e as well as $\rho(t)$ are depicted in Fig. 2 (b). We observe that independently of the particular gain $k > 0$, the invariance of $\mathcal{C}(t)$ in (3a) is guaranteed. Note that, a larger $k > 0$ leads to a faster convergence to the reference signals. To validate Theorem 3, let us take a class $\mathcal{K}$ function in the form of $\alpha(1/B(t,x)) = \gamma/B(t,x)$, and show that (4b) holds for some appropriate choice of γ. In Fig. 2 (c), for a proper choice of γ, we verify that B_{ppc} is a valid TVRCBF candidate and $u \in K_{\text{rcbf}}$ holds for all times.



(a) (b) (c)

Fig. 2: (a) State trajectory of the USV in the earth-fixed frame with different gain settings. The safety constraints are satisfied for any chosen positive gain. (b) Element-wise tracking errors $e_i(t)$ and their corresponding $\rho_i^L(t)$ and $\rho_i^U(t)$ functions under different gain settings. All errors stay within the bounds given by $\rho^L(t)$ and $\rho^U(t)$, which implies $x(t) \in \text{Int}(\mathcal{C}(t))$ for all times. (c) $u \in K_{\text{rcbf}}$ holds for every $k > 0$ by appropriately choosing γ where $\alpha_2(1/B(t,x)) = \gamma/B_{\text{ppc}}$.

V. CONCLUSION

We showed that PPC is a time-varying CBF-based control method. At first, we presented a generalized definition of time-varying RCBF (TVRCBF). This definition allowed us to handle time-varying safety-related compact constraints on state vectors. Next, we introduce PPC as a procedure that begins with a scalar function as the candidate for TVRCBF. We carefully define the concept of TVRCBF and prove that the scalar function indeed qualifies as TVRCBF. By summarizing these findings, we can identify PPC as a feedback law based on TVRCBF. Since this proposed scheme only requires access to the state variable and its associated TVRCBF, the resulting control scheme is model-free. Our results establish a stronger theoretical relationship between PPC and CBF and provide valuable insights for further development in the design of model-free CBF-based controllers.

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