Explicit computation of sampling period in periodic event-triggered multi-agent control

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Abstract— **This paper investigates the synchronization of nonlinear sampled-data multi-agent systems. The purpose is to obtain an explicit formula for the maximum allowable sampling period (MASP) that guarantees exponential synchronization. Two implementation scenarios are considered. We first propose an approach on finding the MASP for periodic time-triggered sampled-data control. Then, a periodic event-triggered communication and control strategy is formulated, where a communication function and a control function are designed for each agent to determine whether or not the sampled data or the control input should be transmitted at each sampling instant. It is shown that there is a tradeoff between the sampling frequency and the convergence performance. The theoretical results are illustrated in simulations.**

I. INTRODUCTION

The emerging Internet of Things technology enables the interconnection of numerous smart devices for the realization of the cyber-physical systems (CPS) vision. To improve efficiency, flexibility and reliability of such large-scale applications, the current technological trend is to integrate sensing, computation, communication, and control into different levels of machine/factory operations and information processes with shared computational, communication and control resources [1], [2]. Efficient usage of these resources is therefore a central issue in CPS design.

In such applications, the communication and control actions are carried out in digital platforms, where sampling operation is one of the indispensable steps to accomplish the digital signal. Design of controllers for sampled-data systems is often carried out by using the emulation approach, in which a continuous-time controller is designed for a continuous-time plant ignoring sampling and then the controller is discretized and implemented digitally [3]. It is obvious that this approach can be successful only if the sampling period is sufficiently small. However, a small sampling period may result in unnecessary high workloads in both Sensor-Controller (SC) communication and Controller-Actuator (CA) communication when communication resources could be more usefully assigned to some other tasks. These limitations have resulted in a recent interest on event-triggered communication (ETCm) and event-triggered control (ETCt)[1] [4], [6]–[10], [16]. However, most of the existing ETCm/ETCt paradigms require the triggering condition to be monitored continuously [5]–[8] or partially continuously [9], [10], which may result in excessive use of computational resources. Moreover, different from periodic time-triggered control (PTTC), in which the functional devices (such as sensors, controllers and actuators) are activated only at the discrete sampling instants, in ETCm/ETCt mechanism, it is necessary for all functional devices to be activated all the time, which increases the energy consumption and thus reduces the lifespan of those devices. To address these problems, periodic event-triggered control (PETC) has been proposed as a solution [11]–[14]. However, several problems are still open in the area of PETC, and one of them lies in finding the minimum inter-event time (sampling period) [15].

In the area of PETC, most of the effort has been devoted to the stabilization of a single agent system [12]–[14], while the cooperation in the multiple-agent case, the so-called multi-agent systems (MAS) has not been considered to the same extent. Besides, the explicit computation of MASP is typically not carried out in the literature (except for single-integrator MAS [4]). Motivated by the above considerations, this paper proposes a periodic event-triggered communication and control (PETCC) strategy for nonlinear MAS that guarantees exponential synchronization, where an explicit computation of MASP is provided. The organization and contributions of this paper are summarized below.

In Section III, the class of nonlinear MAS is introduced. An explicit formula of the MASP that guarantees exponential synchronization of the sampled-data nonlinear MAS is presented. Then, in Section IV, a PETCC strategy is formulated. The MASP for PETCC is obtained, and communication and control functions are designed such that exponential synchronization of the nonlinear MAS is guaranteed. It is shown that there is a tradeoff between the sampling frequency and the convergence performance. Finally, in Section V, a simulation example is provided for illustration of the theoretical developments and Section VI concludes the paper.

II. PRELIMINARIES

A. Notation

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{R}_{>0} := (0, \infty)$, $\mathbb{Z}_{>0} := \{1, 2, \ldots\}$ and $\mathbb{R}_{\geq 0} := \{0, 1, 2, \ldots\}$. Denote $\mathbb{R}^n$ as the $n-$dimension real vector space, $\mathbb{R}^{n\times m}$ as the $n \times m$ real matrix

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[1]In most literatures, both ETCm and ETCt are called ETC. In this paper, we try to distinguish between the two. By ETCm, we refer to SC communication and the agent-to-agent (A2A) communication. By ETCt, we refer to the CA communication.

space. I_n is the identity matrix of order n and 1_n is the column vector of order n with all entries equal to one. For $(x_1,x_2,\ldots,x_m)\in\mathbb{R}^{n_1+n_2+\cdots+n_m}$, the notation $(x_1,x_2,\ldots,x_m)$ stands for $[x_1^T,x_2^T,\ldots,x_m^T]^T$. Let $\|x\|$ and $\|A\|$ be the Euclidean norm of vector x and matrix A, respectively. In addition, we use $\cap$ to denote the logical operator AND and $\cup$ the logical operator OR.

For locally Lipchitz functions (that are not necessarily differentiable everywhere), we will use the Clarke derivative which is defined as follows. For a locally Lipchitz function $R:\mathbb{R}^n\to\mathbb{R}^n$ and a vector $v\in\mathbb{R}^n$, $R^\circ(x,v):=\limsup_{h\to 0^+,y\to x}\frac{R(y+hv)-R(y)}{h}$, which corresponds to the usual derivative when R is continuously differentiable. We define the generalized gradient of $f:\mathbb{R}^n\to\mathbb{R}^n$ at x along v as: $\nabla_v f(x):=\{\zeta\in\mathbb{R}^n: f^\circ(x,v)\geq\langle\zeta,v\rangle,\forall v\in\mathbb{R}^n\}$, that matches the classical notion of gradient when f is differentiable.

B. Graph Theory

Let $\mathscr{G}=\{\mathscr{V},\mathscr{E}\}$ be an undirected graph of order n with the set of nodes $\mathscr{V}=\{1,2,\ldots,N\}$, and $\mathscr{E}\subseteq\{(i,j):i,j\in\mathscr{V},j\neq i\}$ being the set of edges. If $(j,i)\in\mathscr{E}$, then node j is called a neighbor of node i. The neighboring set of node i is denoted by $\mathscr{N}_i=\{j\in\mathscr{E}|(j,i)\in\mathscr{E}\}$ and $\mathscr{N}_i^+=\mathscr{N}_i\cup\{i\}$.

The adjacency matrix is denoted by $A=(a_{ij})_{N\times N}$ and is given by $a_{ij}=1$, if $(j,i)\in\mathscr{E}$, otherwise $a_{ij}=0$. A graph is undirected if $(i,j)\in\mathscr{E}\Leftrightarrow(j,i)\in\mathscr{E}$, and a graph is connected if for every pair of nodes i,j, there exists a path which connects i and j, where a path is an ordered list of edges such that the head of each edge is equal to the tail of the following edge. Let $D=(d_{ij})_{N\times N}$ represent the degree matrix which is a diagonal matrix with entries $d_{ii}=\sum_{j=1,j\neq i}^N a_{ij}$. Then the Laplacian matrix of the graph $\mathscr{G}$ is defined as $L=(l_{ij})_{N\times N}=D-A$.

Assumption 1: The graph $\mathscr{G}$ is connected.

III. NONLINEAR SAMPLED-DATA MAS

A. Model description

Consider nonlinear sampled-data MAS with N agents, and the dynamics of each agent can be described in the following:

$$\begin{aligned}&\dot{x}_i(t)=f(x_i(t))+\hat{u}_i(t),\\&\hat{u}_i(t)=\sum_{j\in\mathscr{N}_i}\psi_{ij}(x_j(t_l)-x_i(t_l)),\quad t\in[t_l,t_{l+1}),\end{aligned}\tag{1}$$

where $x_i\in\mathbb{R}^n,\hat{u}_i\in\mathbb{R}^n$ are respectively the state and the control input of the ith agent, $t_l=lh,l\in\mathbb{Z}$ is the increasing sampling sequence and $h>0$ is the sampling period, which is common to all agents. Note that it is possible to consider the more general problem of an aperiodic sampling satisfying $t_{l+1}-t_l\leq h,\forall l$.

Assumption 2: The function f is Lipchitz continuous with Lipchitz constant $\rho_1>0$ and $f(0)=0$.

Definition 1: The function $\phi:\mathbb{R}^n\to\mathbb{R}^n$ is said to be in sector $[l_1,l_2]$ if for all $q,\phi(q)\in\mathbb{R}^n$, one has

$$(q^T\phi(q)-l_1q^Tq)(q^T\phi(q)-l_2q^Tq)\leq 0.$$

Assumption 3: The functions $\psi_{ij}:\mathbb{R}^n\to\mathbb{R}^n,(i,j)\in\mathscr{E}$ are required to satisfy the following conditions:

a) for any $x_i,x_j\in\mathbb{R}^n$, one has $\psi_{ij}(x_{ji})=-\psi_{ji}(x_{ij})$, where $x_{ij}=x_i-x_j$;

b) there exists a constant $K>0$ such that for any $x_{ij}\in\mathbb{R}^n$, one has $x_{ij}^T\psi_{ji}(x_{ij})\geq Kx_{ij}^Tx_{ij}$;

c) ψ_{ij} are globally Lipchitz continuous functions with Lipchitz constant $\rho_2>0$, that is, $\|\psi_{ij}(x')-\psi_{ij}(x'')\|\leq\rho_2\|x'-x''\|$ for any $x',x''\in\mathbb{R}^n$, and $\psi_{ij}(0)=0,\forall(i,j)\in\mathscr{E}$.

According to item c), one has $\|\psi_{ji}(x_{ij})\|\leq\rho_2\|x_{ij}\|$ and thus $x_{ij}^T\psi_{ji}(x_{ij})\leq\rho_2\|x_{ij}\|^2$. Combining item b), one can further have $K\|x_{ij}\|^2\leq x_{ij}^T\psi_{ji}(x_{ij})\leq\rho_2\|x_{ij}\|^2$. That is to say, the functions ψ_{ji} is in sector $[K,\rho_2]$ for $\forall(j,i)\in\mathscr{E}$.

B. Convergence result

Before proceeding, the following notations are introduced. Define $y_i(t)=x_i(t_l)+\int_{t_l}^t f(y_i(s))ds, t\in[t_l,t_{l+1})$ and $u_i(t)=\sum_{j\in\mathscr{N}_i}\psi_{ij}(y_j(t)-y_i(t))$. The sampling-induced errors of agent i are defined as

$$e_{x_i}(t)=y_i(t)-x_i(t)\quad e_{u_i}(t)=\hat{u}_i(t)-u_i(t),\quad\forall i,\tag{2}$$

then one has $\|e_{x_i}(t_l)\|=0$ and $\|e_{u_i}(t_l)\|=0$ for $\forall t_l$.

The control input can then be rewritten as

$$\hat{u}_i(t)=\sum_{j\in\mathscr{N}_i}\psi_{ij}(x_j(t)-x_i(t)+e_{x_j}(t)-e_{x_i}(t))+e_{u_i}(t).$$

Let the average state of all agents be $\bar{x}(t)=\sum_{i=1}^N x_i(t)/N$. Define the state error between agent i and the average as $\xi_i(t)=x_i(t)-\bar{x}(t),\forall i$. Since the graph $\mathscr{G}$ is undirected and $\psi_{ij}(x_{ji})=-\psi_{ji}(x_{ij})$, one has

$$\dot{\xi}_i(t)=f(x_i(t))-\frac{1}{N}\sum_{j=1}^N f(x_i(t))+\hat{u}_i(t),\quad i=1,2,\ldots,N,\tag{3}$$

where $\hat{u}_i(t)$ can be equivalently rewritten as

$$\hat{u}_i(t)=\sum_{j\in\mathscr{N}_i}\psi_{ij}(\xi_j(t)-\xi_i(t)+e_{x_j}(t)-e_{x_i}(t))+e_{u_i}(t).$$

One can see that the exponential synchronization of the nonlinear sampled-data MAS (1) is achieved if and only if the stability of the error system (3) is achieved exponentially.

Definition 2: Synchronization of the nonlinear MAS (1) is said to be achieved exponentially, if there exist positive constants κ,α such that the error vector satisfies

$$\|\xi_i(t)\|\leq\kappa e^{-\alpha t},\quad\forall i$$

and for all $t\geq 0$. The constant α is called the convergence rate and the constant κ is called the convergence coefficient.

Let ξ,e_x,e_u be the concatenated vectors of ξ_i,e_{x_i},e_{u_i}, respectively. Define $\|z\|':=d\|z\|/dt,\forall z$. Then, we get the following Propositions.

Proposition 1: Let the function $V:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ be $V(z)=z^2$, then for all $\xi,e_x,e_u\in\mathbb{R}^n$, one has that

$$\dot{V}(\|\xi\|)\leq-\hat{L}\|\xi\|^2+\frac{1}{a_1}\gamma^2\|e_x\|^2+\frac{1}{a_1}\|e_u\|^2\tag{4}$$

holds with constants $\hat{L}=2K\lambda_2(L)-4\rho_1-2a_1,\gamma=2\rho_2\sqrt{\hat{l}},\hat{l}=\max_i\{d_{ii}\}$ for some $a_1>0$, where $\lambda_2(L)$ is the algebraic connectivity of the Laplacian matrix L.

Proof: From (3), one has

$$
\begin{aligned}
\|\xi_i\|' =& \frac{d\|\xi_i\|}{dt} = \frac{\xi_i^T \dot{\xi}_i}{\|\xi_i\|} \\
=& \frac{\xi_i^T \left(f(x_i) - \frac{1}{N}\sum_{j=1}^N f(x_i) + \hat{u}_i \right)}{\|\xi_i\|} \\
\leq& \rho_1 \|\xi_i\| + \frac{1}{N}\sum_{j=1}^N \rho_1 \|\xi_i\| + \frac{\xi_i^T \hat{u}_i}{\|\xi_i\|},
\end{aligned} \tag{5}
$$

and then $\|\xi\|' = \sum_{i=1}^N \xi_i^T \dot{\xi}_i / \|\xi\| \leq 2\rho_1 \|\xi\| + \xi^T \hat{u}/\|\xi\|$. Differentiating $V(\|\xi\|)$ along the above trajectories, one has

$$
\begin{aligned}
\dot{V}(\|\xi\|) =& 2\|\xi\| \|\xi\|' \leq 2\|\xi\| \left(2\rho_1 \|\xi\| + \frac{\xi^T \hat{u}}{\|\xi\|} \right) \\
=& 4\rho_1 \|\xi\|^2 + 2\sum_{i=1}^N \xi_i^T \left(\delta_i + \hat{\delta}_i + e_{u_i} \right),
\end{aligned} \tag{6}
$$

where $\delta_i = \sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_j - \xi_i)$ and $\hat{\delta}_i = \sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_j - \xi_i + e_{x_j} - e_{x_i}) - \delta_i$. From Assumption 3, one has $x_{ij}^T \psi_{ji}(x_{ij}) \geq K x_{ij}^T x_{ij} = K\|x_{ij}\|^2, \forall x_{ij} \in \mathbb{R}^n$. Define $\xi_{ij} = \xi_i - \xi_j$, then $\sum_{(i,j)\in\mathscr{E}} \xi_{ij}^T \psi_{ji}(\xi_{ij}) \geq 2K\xi^T (L\otimes I_n)\xi$. According to the definition of ξ, one has $1_{Nn}^T \xi = 0$, and if the graph $\mathscr{G}$ is connected, one has $\xi^T (L\otimes I_n)\xi \geq \lambda_2(L)\|\xi\|^2$ and $\lambda_2(L) > 0$ [17]. Then, $\sum_{i=1}^N \xi_i^T \delta_i = \sum_{i=1}^N \xi_i^T \sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_{ji}) = -\frac{1}{2}\sum_{i=1}^N \xi_{ij}^T \sum_{j\in\mathcal{N}_i} \psi_{ji}(\xi_{ij}) \leq -K\lambda_2(L)\|\xi\|^2$. In addition, the function $\psi_{ij}, \forall (i,j) \in \mathscr{E}$ is globally Lipchitz, thus

$$
\begin{aligned}
\sum_{i=1}^N \left\|\hat{\delta}_i\right\|^2 \leq& \sum_{i=1}^N \Big\| \sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_j - \xi_i + e_{x_j} - e_{x_i}) \\
&- \sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_j - \xi_i) \Big\|^2 \\
\leq& \sum_{i=1}^N \rho_2^2 \sum_{j\in\mathcal{N}_i} \left\| e_{x_j} - e_{x_i} \right\|^2 \\
\leq& 4\hat{l}\rho_2^2 \|e_x\|^2.
\end{aligned} \tag{7}
$$

Then, (6) can be rewritten as

$$
\begin{aligned}
\dot{V}(\|\xi\|) \leq& -\left(2K\lambda_2(L) - 4\rho_1 - 2a_1\right)\|\xi\|^2 \\
&+ \frac{1}{a_1}\left(4\hat{l}\rho_2^2 \|e_x\|^2 + \|e_u\|^2 \right)
\end{aligned} \tag{8}
$$

for some $a_1 > 0$. ■

Proposition 2: For $\forall t \in [t_l, t_{l+1})$, the inequations

$$
\|e_x\|' \leq \gamma\|\xi\| + (\rho_1 + \gamma)\|e_x\| + \|e_u\| \tag{9a}
$$

$$
\nabla_t \|e_u\| \leq \rho_1\gamma\|\xi\| + \rho_1\gamma\|e_x\| \tag{9b}
$$

hold for all $\xi, e_x, e_u \in \mathbb{R}^n$.

Proof: According to (2), one has $\dot{e}_{x_i}(t) = \dot{y}_i(t) - \dot{x}_i(t) = f(y_i(t)) - f(x_i(t)) - \hat{u}_i(t)$, which jumps at $t = t_l$. Thus, for $\forall t \in [t_l, t_{l+1})$, one has that e_{x_i} is continuous. Similar to Proposition 1, one can get

$$
\begin{aligned}
\|e_x\|' =& \frac{\sum_{i=1}^N e_{x_i}^T \dot{e}_{x_i}}{\|e\|} \leq \rho_1 \|e_x\| - \frac{e_x^T \hat{u}}{\|e_x\|} \\
\leq& \rho_1 \|e_x\| + \|\hat{u}\|,
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
\|\hat{u}\| \leq& \sum_{i=1}^N \left\| \sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_j - \xi_i + e_{x_j} - e_{x_i}) + e_{u_i} \right\| \\
\leq& \gamma(\|\xi\| + \|e_x\|) + \|e_u\|.
\end{aligned} \tag{11}
$$

Substituting (11) into (10), then one can get (9a) holds.

Besides, according to (2), one has $\nabla_t e_{u_i}(t) = \nabla_t \hat{u}_i(t) - \nabla_t u_i(t)$, where $\hat{u}_i(t)$ is a constant for $\forall t \in [t_l, t_{l+1})$ and $u_i(t)$ jumps at $t = t_l$. Thus, for $\forall t \in [t_l, t_{l+1})$, one can also have that e_{u_i} is continuous and $\nabla_t e_{u_i}(t) = -\nabla_t u_i(t)$. Then,

$$
\nabla_t \|e_u\| = \frac{\sum\limits_{i=1}^N e_{u_i}^T \nabla_t e_{u_i}}{\|e_u\|} = -\frac{\sum\limits_{i=1}^N e_{u_i}^T \nabla_t u_i}{\|e_u\|} \leq \|\nabla_t u\|. \tag{12}
$$

According to Assumption 3, the function ψ_{ij} is locally Lipchitz, and then one can further derive

$$
\begin{aligned}
\|\nabla_t u\| \leq& \sum_{i=1}^N \left\| \nabla_t \sum_{j\in\mathcal{N}_i} \psi_{ij}(y_j - y_i) \right\| \\
\leq& \sum_{i=1}^N \sum_{j\in N_i} \left\| \nabla_{y_j - y_i} \psi_{ij}(y_j - y_i)^T (\dot{y}_j - \dot{y}_i) \right\| \\
\leq& \rho_2\rho_1 \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} \left\| y_i - y_j \right\| \\
\leq& \rho_1\gamma(\|\xi\| + \|e_x\|),
\end{aligned} \tag{13}
$$

which corresponds to (9b). ■

Now, we provide an explicit computation of the MASP h^* for the nonlinear sampled-data MAS (1) such that synchronization can be achieved exponentially. Firstly, we introduce two auxiliary functions ϕ_1, ϕ_2. Let $\phi_1(\tau) : [0, h^*] \to \mathbb{R}$ and $\phi_2(\tau) : [0, h^*] \to \mathbb{R}$ be such that

$$
\begin{aligned}
\frac{d\phi_1(\tau)}{d\tau} =& -\left(\frac{\gamma^3}{{a_1}^2} + a_2 \right)\phi_1^2(\tau) - \big(2\rho_1 + 2\gamma \\
&+ (\hat{L} - 2a_1) \big)\phi_1(\tau) - \gamma - a_2 \quad \phi_1(0) = b_1
\end{aligned} \tag{14a}
$$

$$
\begin{aligned}
\frac{d\phi_2(\tau)}{d\tau} =& -\left(\frac{\gamma^2\rho_1^2}{a_1^2} + \frac{\gamma\rho_1^2}{a_2} \right)\phi_2^2(\tau) - (\hat{L} - 2a_1)\phi_2(\tau) \\
&- \frac{\gamma}{a_2} - 1 \qquad \phi_2(0) = b_2
\end{aligned} \tag{14b}
$$

where $b_1 > 0$ and $b_2 > 0$ are the initial value of ϕ_1 and ϕ_2, respectively, $a_2 > 0$ and the constants $\hat{L}, a_1, \gamma$ were given in Proposition 1. Then, the MASP h^* is selected as $h^* = \min\{\phi_1^{-1}(0), \phi_2^{-1}(0)\}$.

Remark 1: If $K > (2\rho_1 + 2a_1)/\lambda_2(L)$, one has $\hat{L} - 2a_1 > 0$. Then, for any given $b_1 > 0$ and $b_2 > 0$, it can be computed that

$$
\begin{aligned}
h^* \leq& \frac{b_1}{\left(\frac{\gamma^3}{{a_1}^2} + a_2 \right) b_1^2 + (2\rho_1 + 2\gamma + (\hat{L} - 2a_1))b_1 + \gamma + a_2} \\
\leq& \frac{1}{2\sqrt{(\gamma^3/{a_1}^2 + a_2)(\gamma + a_2)} + 2\rho_1 + 2\gamma + (\hat{L} - 2a_1)} := h_1
\end{aligned}
$$

and

$$h^* \leq \frac{b_2}{(\frac{\gamma^2\rho_1^2}{a_1^2}+\frac{\gamma\rho_1^2}{a_2})b_2^2+(\hat{L}-2a_1)b_2+\frac{\gamma}{a_2}+1}$$
$$\leq \frac{1}{2\sqrt{\left(\gamma^2\rho_1^2/a_1{}^2+\gamma\rho_1^2/a_2\right)\left(\gamma/a_2+1\right)}+(\hat{L}-2a_1)} := h_2.$$

Thus, a lower bound on the MASP h^* is given by $\min\{h_1,h_2\}$.

Theorem 1: Consider the nonlinear MAS (1), where the sampling period h is chosen from $h \in (0,h^*]$. Suppose Assumptions 1-3 hold with $K > (2\rho_1+2a_1)/\lambda_2(L)$. Then, the synchronization of the nonlinear MAS (1) is achieved exponentially with convergence rate $K\lambda_2(L)-2\rho_1-2a_1$ and convergence coefficient $\|\xi(0)\|$.

Proof. Define $\tau:\mathbb{R}_{\geq 0}\to[0,h]$ as i)$\tau(0)=0$, ii) $\dot{\tau}(t)=1, \tau\in[0,h)$, iii) $\tau(t^+)=0, \tau=h$, where $h\in(0,h^*]$ and $\tau(t^+)$ corresponds to a jump when $\tau=h$.

Let $z:=(\xi,e_x,e_u)$. Define the Lyapunov function candidate as

$$R_1(z) = V(\|\xi\|)+\frac{1}{a_1}\gamma\phi_1(\tau)V(\|e_x\|)+\frac{1}{a_1}\phi_2(\tau)V(\|e_u\|). \tag{15}$$

Since $K>(2\rho_1+2a_1)/\lambda_2(L)$, one has $\hat{L}-2a_1>0$. One can see from (14a) and (14b) that both ϕ_1 and ϕ_2 are strictly decreasing and thus the MASP h^* is unique. Since $h\leq h^*$, one has $\phi_1(\tau)\geq 0$ and $\phi_2(\tau)\geq 0$ for all $\tau\in[0,h]$. Therefore, the Lyapunov function $R_1(z)$ is positive definite.

Taking the derivative of $R_1(z)$ on $\forall t\in[t_l,t_{l+1})$, one has

$$\begin{aligned}\dot{R}_1(z) =&\dot{V}(\|\xi\|)+\frac{1}{a_1}\gamma\dot{\phi}_1(\tau)\|e_x\|^2+2\frac{1}{a_1}\gamma\phi_1(\tau)\,\|e_x\|\,\|e_x\|' \\ &+\frac{1}{a_1}\dot{\phi}_2(\tau)\|e_u\|^2+2\frac{1}{a_1}\phi_2(\tau)\,\|e_u\|\,\nabla_t\,\|e_u\|\,.\end{aligned} \tag{16}$$

Substituting (4), (9a), (9b) and (14a), (14b) into (16), one can further have

$$\begin{aligned}\dot{R}_1(z) \leq& -\hat{L}\|\xi\|^2+\frac{1}{a_1}\gamma^2\|e_x\|^2+\frac{1}{a_1}\|e_u\|^2\\
&+2\frac{1}{a_1}\gamma\phi_1(\tau)\,\|e_x\|\left(\gamma\|\xi\|+(\rho_1+\gamma)\,\|e_x\|+\|e_u\|\right)\\
&+\frac{1}{a_1}\gamma\Big(-\left(\frac{\gamma^3}{a_1{}^2}+a_2\right)\phi_1^2(\tau)-(2\rho_1+2\gamma)\phi_1(\tau)\\
&+(\hat{L}-2a_1)\phi_1(\tau)-\gamma-a_2\Big)\|e_x\|^2\\
&+2\frac{1}{a_1}\phi_2(\tau)\,\|e_u\|\left(\rho_1\gamma\|\xi\|+\rho_1\gamma\|e_x\|\right)\\
&+\frac{1}{a_1}\Big(-(\frac{\gamma^2\rho_1^2}{a_1^2}+\frac{\gamma\rho_1^2}{a_2})\phi_2^2(\tau)-(\hat{L}-2a_1)\phi_2(\tau)\\
&-\frac{\gamma}{a_2}-1\Big)\|e_u\|^2\\
=&-(\hat{L}-2a_1)\|\xi\|^2-\frac{1}{a_1}(\hat{L}-2a_1)\gamma\phi_1(\tau)\|e_x\|^2\\
&-\frac{1}{a_1}(\hat{L}-2a_1)\phi_2(\tau)\|e_u\|^2\\
\leq&-(\hat{L}-2a_1)R_1(z),\end{aligned} \tag{17}$$

which implies $\dot{R}_1(z)<0$ for $\forall t\in[t_l,t_{l+1})$ when $\|\xi\|\neq 0$. In addition, during the jump, i.e., $t=t_l^+$, one has $\|e_x(t_l^+)\|=0\leq\|e_x(t_l)\|$ and $\|e_u(t_l^+)\|=0\leq\|e_u(t_l)\|$. That is to say, $R_1(z)$ is nonincreasing during the jump. Then, based on the comparison theorem and (17), one can get that the solution of $R_1(z)$ satisfies

$$R_1(z(t))\leq e^{-(\hat{L}-2a_1)t}R_1(z(0))=e^{-(\hat{L}-2a_1)t}\|\xi(0)\|^2.$$

Moreover, one has $\|\xi(t)\|\leq\sqrt{R_1(z(t))}\leq\|\xi(0)\|e^{-(\hat{L}-2a_1)t/2}$ from (15). That is, synchronization of the nonlinear MAS (1) is achieved exponentially with convergence rate $(\hat{L}-2a_1)/2$ and convergence coefficient $\|\xi(0)\|$. ■

Remark 2: In this paper, the communication graph is assumed to be undirected. Nevertheless, it is worth to point out that the results obtained in this paper are applicable to the case of a directed graph with a spanning tree when a linear system model is considered.

Remark 3: It can be seen that both the MASP h^* and the convergence rate are related to the constant a_1. When the constants b_1 and b_2 of (14a) and (14b) are given, a small value of a_1 means faster convergence, however, the MASP h^* will be smaller. Therefore, a_1 can be used as a tradeoff between the sampling frequency and the convergence rate. Besides, from (14a) and (14b), one can see that a small value of a_2 means a bigger $\phi_1^{-1}(0)$ while a smaller $\phi_1^{-2}(0)$, and a big value of a_2 means smaller $\phi_1^{-1}(0)$ and bigger $\phi_1^{-2}(0)$. Therefore, a_2 can be used as a tradeoff between $\phi_1^{-1}(0)$ and $\phi_2^{-1}(0)$ such that the maximum h^* can be achieved.

IV. APPLICATION TO PETCC

In the above, the PTTC is considered for nonlinear MAS. To reduce the communication frequency as well as the controller update frequency, in this section, a distributed PETCC strategy is developed. Different from PTTC, at each sampling instant, a communication function and a control function are designed for each agent to determine whether or not the sampled data or the control input should be transmitted through the network, respectively.

A. Periodic Event-triggered Communication (PETCm)

For each agent i, let $t^i_{\sigma_i}, \sigma_i\in\mathbb{Z}$ be the increasing sequence of communication time instants at which x_i is transmitted and $\{t^i_{\sigma_i}\}$ be the set of communication instants. On the sensor side, each agent implements an estimator of itself using the most recently transmitted data $x_i(t^i_{\sigma_i})$, that is:

$$\begin{aligned}&\dot{\hat{y}}_i(t)=f(\hat{y}_i(t)), \quad t\in[t^i_{\sigma_i},t^i_{\sigma_i+1})\\ &\hat{y}_i(t^i_{\sigma_i})=x_i(t^i_{\sigma_i}).\end{aligned} \tag{18}$$

For agent i, the communication error at the sampling instant t_l is defined as $e_{y_i}(t_l)=\hat{y}_i(t_l)-x_i(t_l)$, and the communication time instant $t^i_{\sigma_i+1}$ is generated by

$$t^i_{\sigma_i+1}=\inf_{t_l}\left\{t_l>t^i_{\sigma_i}:h_1\left(t_l,e_{y_i}\left(t_l\right)\right)\geq 0\right\}, \tag{19}$$

where

$$h_1\left(t_l,e_{y_i}\left(t_l\right)\right)=\left\|e_{y_i}\left(t_l\right)\right\|-c_1e^{-\alpha_1t_l} \tag{20}$$

with constants $c_1 > 0, \alpha_1 > 0$. The function $h_1(t_l, e_{y_i}(t_l)) \geq 0$ is called the communication function. Without loss of generality, we assume $t_0^i = 0, \forall i$.

B. Periodic Event-triggered Control (PETCt)

On the control side, each agent implements estimators of itself $\hat{y}_i(t)$ as well as its neighbors $\hat{y}_j(t)$ based on the received states, that is,

$$\begin{aligned}\dot{\hat{y}}_j(t) &= f(\hat{y}_j(t)), \quad t \in [t^j_{\sigma_j}, t^j_{\sigma_j+1}) \\ \hat{y}_j(t^j_{\sigma_j}) &= x_j(t^j_{\sigma_j}), \quad j \in \mathscr{N}_i^+.\end{aligned} \tag{21}$$

The distributed event-triggered controller for agent i is designed as

$$\hat{u}_i(t) = \sum_{j \in \mathscr{N}_i} \psi_{ij}(\hat{y}_j(T_k^i) - \hat{y}_i(T_k^i)), \quad t \in [T_k^i, T_{k+1}^i), \tag{22}$$

where $T_k^i, k \in \mathbb{Z}$ is the controller update time instants.

Define $q_i(t) = \sum_{j \in \mathscr{N}_i} \psi_{ij}(\hat{y}_j(t) - \hat{y}_i(t))$. Then the control error at sampling instant t_l is defined as $\hat{e}^l_{u_i}(t_l) = q_i(T_k^i) - q_i(t_l)$. Let $\{T_k^i\}$ be the set of controller update instants, in which T_{k+1}^i is generated by

$$T_{k+1}^i = \inf_{t_l} \left\{ t_l > T_k^i : h_2\left(t_l, \hat{e}^l_{u_i}(t_l)\right) \geq 0 \right\}, \tag{23}$$

where

$$h_2\left(t_l, \hat{e}^l_{u_i}(t_l)\right) = \left\| \hat{e}^l_{u_i}(t_l) \right\| - c_2 e^{-\alpha_2 t_l} \tag{24}$$

with constants $c_2 > 0, \alpha_2 > 0$. The function $h_2\left(t_l, \hat{e}^l_{u_i}(t_l)\right) \geq 0$ is called the control function. Without loss of generality, we assume $T_0^i = 0, \forall i$.

C. Convergence result

Define $\hat{e}_{u_i}(t) = q_i(t_l) - q_i(t), t \in [t_l, t_{l+1})$ and $e_{y_i}(t) = \hat{y}_i(t) - y_i(t)$. Then, one has $e_{y_i}(t_l) = \hat{y}_i(t_l) - y_i(t_l) = \hat{y}_i(t_l) - x_i(t_l)$. Combining the definition of $e_{x_i}(t)$ given in (2) and $\hat{e}^l_{u_i}(t_l), \hat{e}_{u_i}(t)$ and $e_{y_i}(t)$, (22) can be rewritten as

$$\begin{aligned}\hat{u}_i(t) = &\sum_{j \in \mathscr{N}_i} \psi_{ij}(\xi_j(t) - \xi_i(t) + e_{x_j}(t) - e_{x_i}(t) + e_{y_j}(t) - e_{y_i}(t)) \\ &+ \hat{e}^l_{u_i}(t_l) + \hat{e}_{u_i}(t), \quad t \in [t_l, t_{l+1}).\end{aligned} \tag{25}$$

Let $\xi, e_x, e_y, \hat{e}^l_u, \hat{e}_u$ be the concatenated vectors of $\xi_i, e_{x_i}, e_{y_i}, \hat{e}^l_{u_i}(t_l), \hat{e}_{u_i}$, respectively. Then, we get the following Proposition.

Proposition 3: For all $\xi, e_x, e_y, \hat{e}^l_u, \hat{e}_u \in \mathbb{R}^n$, the inequality

$$\begin{aligned}\dot{V}(\|\xi\|) \leq &-\hat{L}\|\xi\|^2 + \frac{4}{a_1}\gamma^2\|e_x\|^2 + \frac{4}{a_1}\gamma^2\|e_y\|^2 \\ &+ \frac{5}{a_1}\left\|\hat{e}^l_u\right\|^2 + \frac{4}{a_1}\|\hat{e}_u\|^2\end{aligned} \tag{26}$$

holds for $\forall t \geq 0$, and the inequalities

$$\begin{aligned}\|e_x\|' \leq &\gamma\|\xi\| + (\rho_1 + \gamma)\|e_x\| + \gamma\|e_y\| \\ &+ \|\hat{e}_u\| + \left\|\hat{e}^l_u\right\|,\end{aligned} \tag{27a}$$

$$\nabla_t \|e_u\| \leq \rho_1\gamma\|\xi\| + \rho_1\gamma\|e_x\| + \rho_1\gamma\|e_y\| \tag{27b}$$

hold for $\forall t \in [t_l, t_{l+1})$, where $\hat{L}, \gamma, a_1$ were given in Proposition 1.

The MASP for PETCC is denoted by $\hat{h}^*$. To define $\hat{h}^*$, we firstly introduce two auxiliary functions ϕ_x, ϕ_u. Let $\phi_x(\hat{\tau}) : [0, \hat{h}^*] \to \mathbb{R}$ and $\phi_u(\hat{\tau}) : [0, \hat{h}^*] \to \mathbb{R}$ be such that

$$\begin{aligned}\frac{d\phi_x(\hat{\tau})}{d\hat{\tau}} = &-\left(\frac{4\gamma^3}{{a_1}^2} + a_2\right)\phi_x^2(\hat{\tau}) - \big(2\rho_1 + 2\gamma \\ &+ (\hat{L} - 2a_1)\big)\phi_x(\hat{\tau}) - \gamma - a_2 - 2a_3 \quad \phi_x(0) = b_1\end{aligned} \tag{28a}$$

$$\begin{aligned}\frac{d\phi_u(\hat{\tau})}{d\hat{\tau}} = &-\left(\frac{4\gamma^2\rho_1^2}{a_1^2} + \frac{\gamma\rho_1^2}{a_2}\right)\phi_u^2(\hat{\tau}) - (\hat{L} - 2a_1)\phi_u(\hat{\tau}) \\ &- \frac{\gamma}{a_2} - 1 - 2a_3 \qquad \phi_u(0) = b_2\end{aligned} \tag{28b}$$

where $a_3 > 0$ is a constant. The MASP $\hat{h}^*$ is selected as $\hat{h}^* = \min\{\phi_x^{-1}(0), \phi_u^{-1}(0)\}$. The introduction of a_3 allows a trade off between the sampling frequency and the convergence performance since the value of a_3 will affect the convergence rate and coefficient, as will become clear later.

Theorem 2: Consider the nonlinear MAS (1) with the control input (22), where the sampling period h is chosen from $h \in (0, \hat{h}^*]$. Suppose Assumptions 1-3 hold with $K > (2\rho_1 + 2a_1)/\lambda_2(L)$. The communication time sequence and the controller update time sequence are given respectively by (19) and (23) with $\alpha > K\lambda_2(L) - 2\rho_1 - 2a_1$. Then, synchronization of the nonlinear MAS (1) is achieved exponentially with convergence rate $K\lambda_2(L) - 2\rho_1 - 2a_1$ and convergence coefficient $\sqrt{d_1}$, where $d_1 = \|\xi(0)\|^2 + (\beta_1 + \beta_2)/(2\alpha - 2K\lambda_2(L) + 4\rho_1 + 4a_1)$, $\beta_1 = Nc_1^2\gamma^2\left(4/a_1 + 2(2b_1^2\gamma + b_2^2\rho_1^2)/(a_3 a_1)\right)e^{2(\rho_1+\alpha)h}$ and $\beta_2 = Nc_2^2(5/a_1 + 4\gamma b_1^2/(a_3 a_1))e^{2\alpha h}$.

Remark 4: From (28a), (28b), one can see that $\hat{h}^* < h^*, \forall a_3 > 0$. A small value of a_3 means a bigger $\hat{h}^*$ and also a bigger convergence coefficient, while a big value of a_3 means a smaller $\hat{h}^*$ and also a smaller convergence coefficient. Therefore, a_3 can be used as a trade-off between the sampling frequency and the convergence performance.

V. SIMULATION RESULTS

Consider a MAS consisting of 4 single-link robot arm, the communication graph is characterized by the adjacency matrix $A = [0,1,0,1;1,0,1,0;0,1,0,1;1,0,1,0]$. The initial state $x_i(0)$ of each robot arm i is chosen randomly from the box $[-5;5] \times [-5;5]$. The dynamics of the $i-$th robot arm is described by (1), where

$$x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix}, f(x_i(t)) = \begin{pmatrix} x_{i2} \\ -\sin(x_{i1}) \end{pmatrix}.$$

Clearly, $f(x_i)$ is Lipchitz with a Lipschitz constant $\rho_1 = 1$. The control input for agent i is designed as

$$\hat{u}_i(t) = 2\sum_{j \in \mathscr{N}_i}(x_j(t_l) - x_i(t_l)), \quad t \in [t_l, t_{l+1}).$$

One can verify that Assumption 3 holds with $K = 2$ and $\rho_2 = 2$. According to Proposition 1, one can calculate $\hat{L} = 4 - 2a_1, \gamma = 4\sqrt{2}$. Choosing $a_1 = 0.5$ such that $\hat{L} - 2a_1 = 4 - 4a_1 = 2 > 0$ and $a_2 = 0.2$, then one can get $h_1 = 6.8 \times 10^{-3}, h_2 = 7.3 \times 10^{-3}$ according to Remark 1. Then, the sampling period h is chosen as $h = 6 \times 10^{-3} < h^*$. The

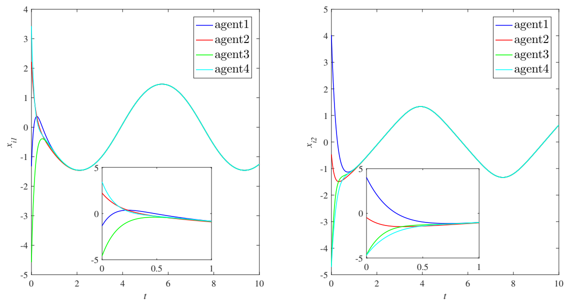

Fig. 1. The evolution of x_{i1}, x_{i2} under PTTC

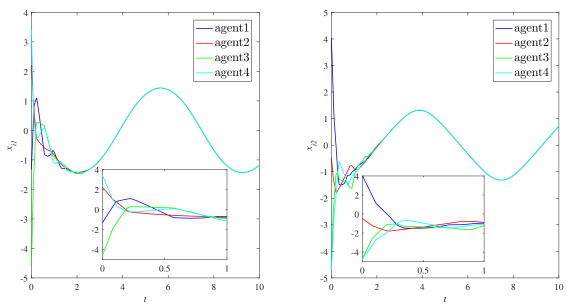

Fig. 2. The evolution of x_{i1}, x_{i2} under PETCC

simulation results for PTTC are shown in Fig. 1, where the evolution of the states x_{i1} and x_{i2} is plotted.

For PETCC, the sampling period $\hat{h}$ is chosen as $\hat{h} = 3 \times 10^{-3}$. Given $c_1 = c_2 = 4$, chose $\alpha_1 = \alpha_2 = 2.01 > \hat{L} - 2a_1$. The simulation results for the evolution of the states x_{i1} and x_{i2} under PETCC are shown in Fig. 2.

Table I summarises the communication times (CT) and controller update times (UT) for each agent under the two scenarios. One can see that the introduction of communication and control functions significantly reduces both the frequency of communication and controller update.

VI. CONCLUSION

In this paper, we have presented an approach on finding the MASP that guarantees exponential synchronization of nonlinear sampled-data MAS. Two different scenarios are considered. Firstly, PTTC of the nonlinear MAS was considered and the approach on finding the MASP was established. Then, a PETCC strategy was formulated and the explicit formula for MASP was obtained based on the approach. It was shown that there is a tradeoff between the sampling frequency and the convergence performance. Future work includes the extension to more general nonlinear systems.

TABLE I
COMMUNICATION TIMES AND CONTROLLER UPDATE TIMES

	CT/UT (PTTC)	CT/UT (PETCC)
agent1	1666/1666	29/76
agent2	1666/1666	18/47
agent3	1666/1666	30/102
agent4	1666/1666	31/85

REFERENCES

[1] F. L. Lian, J. K. Yook, D. M. Tilbury, and J. Moyne, "Network architecture and communication modules for guaranteeing acceptable control and communication performance for networked multi-agent systems", *IEEE Trans. Ind. Inf.*, vol. 2, no. 1, pp. 12-24, 2006.

[2] A. Girard, "Dynamic triggering mechanisms for event-triggered control', *IEEE Trans. Automat. Control*, vol. 60, no.7, pp. 1992-1997, 2015.

[3] D. Nesic, A. R. Teel, and D. Carnevale, "Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems", *IEEE Trans. Automat. Control*, vol. 54, no.3, pp. 619-624, 2009.

[4] X. Meng, and T. Chen, "Event based agreement protocols for multi-agent networks", *Automatica*, vol. 49, no. 7, pp. 2125-2132, 2013.

[5] C. D. Persis, and R. Postoyan, "A Lyapunov redesign of coordination algorithms for cyber-physical systems", *IEEE Trans. Automat. Control*, vol. 62, no. 2, pp. 808-823, 2017.

[6] T. Liu, and Z. P. Jiang, "Event-based control of nonlinear systems with partial state and output feedback", *Automatica*, vol. 53, no. 10-22, 2015.

[7] E. Garcia, Y. Cao, A. Giua, and D. Casbeer, "Decentralized eventtriggered consensus with general linear dynamics', *Automatica*, vol. 50, no. 10, pp. 2633-2640, 2014.

[8] D. Yang, W. Ren, X. Liu and W. Chen, "Decentralized event-triggered consensus for linear multi-agent systems under general directed graphs", *Automatica*, Vol. 69, pp. 242-249, 2016.

[9] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nesic, "Stabilization of nonlinear systems using event-triggered output feedback controllers", *IEEE Trans. Automat. Control*, vol. 61, no.9, pp. 2682-2687, 2016.

[10] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nesic, "Robust event-triggered output feedback controllers for nonlinear systems", *Automatica*, vol. 75, pp. 96-108, 2017.

[11] M.C.F. Donkers, and W. Heemels, "Output-based event-triggered control with guaranteed $\mathscr{L}_\infty$-gain and improved and decentralized event-triggering", *IEEE Trans. Automat. Control*, vol. 57, no. 6, pp 1362-1376, 2012.

[12] W. Heemels, M.C.F. Donkers, and A.R. Teel, "Periodic event-triggered control for linear systems", *IEEE Trans. Automat. Control*, vol. 58, pp 847-861, 2013.

[13] R. Postoyan, A. Anta, W. P. M. H. Heemels, P. Tabuada, and D. Nesic, "Periodic event-triggered control for nonlinear systems", *in Proc. 2013 IEEE Conference on Decision and Control (CDC)*, 2013, pp. 7397-7402.

[14] A. Fuand, and M. Mazo, "Periodic asynchronous event-triggered control", *in Proc. 2016 IEEE Conference on Decision and Control (CDC)*, Las Vegas, USA, Dec. 2016, pp. 1370-1375.

[15] M. Miskowicz (Ed.), Event-based control and signal processing, CRC press, 2015.

[16] H. Li, X. Liao, T. Huang, and W. Zhu, "Event-triggering sampling based leader-following consensus in second-order multi-agent systems", *IEEE Trans. Automat. Control*, vol. 60, no.7, pp. 998-2003, 2015.

[17] R. Olfati-Saber, and R.M. Murray, "Consensus problems in networks of agents with switching topology and time-delays", *IEEE Trans. Automat. Control*, vol. 49, no.9, pp. 1520-1533, 2004.

[18] H.K. Khalil, Nonlinear Systems, *Prentice Hall*, 3rd edn. 2002.