Today’s lecture

- Connectivity
- Connectivity maintenance for static interaction graphs
- Connectivity maintenance for dynamic interaction graphs
- Robust connectivity maintenance under bounded controls
Connectivity

- Motivation: Mobile robots with limited sensing range (e.g., omnidirectional sensors)
- $\Delta$-proximity graph: $\{v_i, v_j\} \in \mathcal{E} \iff |x_i - x_j| \leq \Delta$
  
  Notational convention $|\cdot| := \|\cdot\|_2$

- $\Delta$-proximity graph is a dynamic interaction graph
- Static interaction graph (SIG): Communication links are assumed fixed between the agents
Cooperative robots

- Single integrator dynamics: $\dot{x}_i = u_i$
- SIG $(\mathcal{V}, \mathcal{E})$: $\{i, j\} \in \mathcal{E} \iff j \in \mathcal{N}_i$
- Decentralized relative position control laws:
  $$u_i = \sum_{j \in \mathcal{N}_i} f(x_i - x_j)$$
- Antisymmetric $f$: $f(x) = -f(-x) \implies f(x_i - x_j) = -f(x_j - x_i)$
- Consensus special case:
  $$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j), \ x_i \in \mathbb{R}^n$$
Cooperative robots

• Apply component operator to reduce to \( n \) consensu
problems in \( \mathbb{R}^N \)

\[
c_l(x) := (x_{1,l}, \ldots, x_{N,l}), \ l = 1, \ldots, n, \ x_i = (x_{i,1}, \ldots, x_{i,n})
\]

• We then get

\[
\dot{x}_{i,l} = - \sum_{j \in \mathcal{N}_i} (x_{i,l} - x_{j,l}), \ l = 1, \ldots, n, \ i = 1, \ldots, N
\]

• and thus

\[
c_l(\dot{x}) = -L(G)c_l(x), \ l = 1, \ldots, n
\]

• If we apply this control law to the dynamic \( \Delta \)-proximity graph
with \( \{i, j\} \in \mathcal{E}(t) \iff |x_i(t) - x_j(t)| \leq \Delta \) we might loose
connectivity.
Weighted graph based feedback

- Feedback components $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form
  
  \[ f_i(x_i - x_j) := -w(x_i - x_j)(x_i - x_j) \]

- with nonlinear weight function $w : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ positive and symmetric

- We obtain the decentralized control law

  \[ \dot{x}_i = - \sum_{j \in \mathcal{N}_i} w(x_i - x_j)(x_i - x_j), \quad i = 1, \ldots, N \]

- and componentwise

  \[ c_l(\dot{x}) = -D(G)W(x)D(G)^T c_l(x), \quad l = 1, \ldots, n \]

- State dependent weighted Laplacian

  \[ L_w(x) = D(G)W(x)D(G)^T \]
Weighted graph based feedback

- State dependent weighted Laplacian

\[ L_w(x) = D(G)W(x)D(G)^T \]

- Properties of \( L_w(x) \) (for each \( x \))
  - symmetric
  - positive semidefinite
  - assuming that \( G \) is connected the only zero eigenvalue corresponds to \( \text{span}(1) \)

- Critical edge distance \( \delta \) with initial tolerance \( \epsilon < \delta \)

- \( \epsilon \) shrinking of a \( \delta \) constrained realization of the SIG \( G \)

\[ D_\delta^\epsilon = \{ x \in \mathbb{R}^{Nn} : |\ell_{ij}| \leq \delta - \epsilon \text{ for all } \{i, j\} \in \mathcal{E} \}, \ell_{ij}(x) = x_i - x_j \]
Weighted graph based feedback

• Edge tension function

\[ V_{ij}(x) = \begin{cases} \frac{|\ell_{ij}(x)|^2}{\delta - |\ell_{ij}(x)|}, & \text{if } \{i, j\} \in \mathcal{E}, \\ 0, & \text{otherwise}. \end{cases} \]

• with partial derivatives

\[ \frac{\partial V_{ij}(x)}{\partial x_i} = \begin{cases} \frac{2\delta - |\ell_{ij}(x)|}{(\delta - |\ell_{ij}(x)|)^2} (x_i - x_j)^T, & \text{if } \{i, j\} \in \mathcal{E}, \\ 0, & \text{otherwise}. \end{cases} \]

• Total energy of \( \mathcal{G} \)

\[ V(x) = \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} V_{ij}(x) \]
Weighted graph based feedback

LEMMA
Given an initial position $x_0 \in D_\delta^\epsilon$, for a given $\epsilon \in (0, \delta)$, if the SIG $\mathcal{G}$ is connected then the set $\Omega(\delta, x_0) = \{x : V(x) \leq V(x_0)\}$ is invariant under the control law

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} \frac{2\delta - |\ell_{ij}(x)|}{(\delta - |\ell_{ij}(x)|)^2} (x_i - x_j)$$

(1)
THEOREM
Consider the connected SIG $\mathcal{G}$ with initial condition $x_0 \in D_\delta^\epsilon$ and a given $\epsilon \in (0, \delta)$. Then the multiagent system under the control law (1) converges asymptotically to the static centroid $\overline{x}$. 
Dynamic graphs

- Dynamic $\Delta$-proximity graph with
  \[
  \{i, j\} \in \mathcal{E}(t) \iff |x_i(t) - x_j(t)| \leq \Delta
  \]

- Add new edge $\{i, j\}$ when crossing the switching threshold
  \[
  |l_{ij}| \leq \Delta - \epsilon
  \]

- Switching protocol
  \[
  \sigma(i, j)[t] = \begin{cases} 
    0, & \text{if } \sigma(i, j)(s) = 0, \forall s \in [0, t) \text{ and } |l_{ij}| > \Delta - \epsilon \\
    1, & \text{otherwise.}
  \end{cases}
  \]
Dynamic graphs

THEOREM
Consider the an initial position \( x_0 \in D^\epsilon_\Delta(G_0) \) where \( \epsilon \in (0, \Delta) \) is the switching threshold and \( G_0 \) is the initial \( \Delta \)-disk DIG. Assume that the graph \( G_\sigma \) induced by the indicator function is initially connected. with initial condition \( x_0 \in D^\epsilon_\delta \) and a given \( \epsilon \). Then the control law

\[
\dot{x}_i = - \sum_{j \in N_i^\sigma} \frac{2\Delta - |\ell_{ij}(x)|}{(\Delta - |\ell_{ij}(x)|)^2} (x_i - x_j)
\]

with the switching protocol \( \sigma(i, j) \) as previously defined asymptotically converges to \( \text{span}(1) \).
Systems Description

- Consider the **single integrator** multi-agent system

\[ \dot{x}_i = u_i, \ x_i \in \mathbb{R}^n, \ i = \{1, \ldots, N\} := \mathcal{N} \]

- Network graph \( \mathcal{G} := (\mathcal{V}, \mathcal{E}) \) undirected & connected

- Network graph \( \mathcal{G} := (\mathcal{V}, \mathcal{E}) \) undirected & connected

- \( \mathcal{V} := \mathcal{N} \quad j \in \mathcal{N}_i \iff \{i,j\} \in \mathcal{E} \)

- Design **decentralized** control laws

\[ u_i = f_i(x_i, x_{j_1}, \ldots, x_{j_{|\mathcal{N}_i|}}) + v_i, \ i \in \mathcal{N} \]

with free input terms \( v_i \).
Motivation

- Design **bounded feedback laws** which guarantee
  - **connectivity maintenance** of the multi-robot network
  - **invariance** of systems solutions inside a **bounded domain**
  - **robustly** wrt free input terms

- Exploit bounds on dynamics and acceptable bound on free input terms to extract a **discretized model of the continuous time system**\(^1\)

- Exploit invariance to extract a **finite transition system**

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\(^1\)D. B. and D. V. Dimarogonas, Decentralized Abstractions for Feedback Interconnected Multi-Agent Systems, CDC 2015
Problem Description

ASSUMPTIONS

• Multi agent network with static interaction graph.

• Network initially connected.

GOAL

• Specify apriori bounds on the initial distances between interconnected agents

• Design bounded (for bounded inter-agent distances) control laws

• Guarantee connectivity maintenance robustly wrt. free inputs

EXTENSION

• Guarantee invariance of solutions inside a spherical domain.
Connectivity Assumptions

- Agents $i, j$ connected iff $\{i, j\} \in \mathcal{E}$ and $|x_i - x_j| \leq R$

- Initial Connectivity Hypothesis: $\forall \{i, j\} \in \mathcal{E}: |x_i(0) - x_j(0)| \leq \tilde{R} < R$
Potential field based Controllers\textsuperscript{2}

- \( r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0} \) continuous; increasing.

- Define the potential function

\[
P(\rho) = \int_0^{\rho} r(s)sd\xi, \rho \in \mathbb{R}_{\geq 0}
\]

- Gradient of the potential function

\[
\nabla_{x_i} P(|x_i - x_j|) = r(|x_i - x_j|)(x_i - x_j)
\]

- Select the control law

\[
u_i = \sum_{j \in \mathcal{N}_i} \nabla_{x_j} P(|x_i - x_j|) + v_i
\]

\[
= - \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i
\]

\textsuperscript{2}M. Ji and M. Egerstedt, Distributed Coordination Control of Multi-Agent Systems While Preserving Connectedness, 2007
Dynamics in Compact Form

- Overall dynamics

\[ c_l(\dot{x}) = - L_w(x)c_l(x) + c_l(v), \quad l = 1, \ldots, n \]

\[ c_l(x) := (x_{1,l}, \ldots, x_{N,l}), \quad x_i = (x_{i,1}, \ldots, x_{i,n}) \]

- Weighted Laplacian

\[ L_w(x) := D(G)W(x)D(G)^T \]

- \( D(G) \): incidence matrix

- Edge weights

\[ W(x) := \text{diag}\{w_1(x), \ldots, w_M(x)\} := \text{diag}\{r(|x_i - x_j|), \{i, j\} \in \mathcal{E}\} \]
Energy Function

- For each \( \{i, j\} \in \mathcal{E} \) define
  \[
  V_{ij}(x) = P(|x_i - x_j|), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nn}
  \]

- Define the energy function
  \[
  V := \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} V_{ij}
  \]

- Partial derivatives
  \[
  \frac{\partial}{\partial x_i} V(x) = \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j)^T
  \]

- Componentwise derivative of energy function
  \[
  c_l \left( \frac{\partial}{\partial x} V(x) \right) = c_l(x)^T L_w(x), \quad l = 1, \ldots, n
  \]
Derivative along System Trajectories

- Derivative of energy function

\[
\dot{V} = -\sum_{l=1}^{n} c_l \left( \frac{\partial}{\partial x} V(x) \right) c_l(x)
\]

\[
\leq -\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) + \left| \sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v) \right|
\]

- Lower bound for term 1

\[
\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) \geq [\lambda_2(G)r(0)]^2 |x^\perp|^2
\]

- Upper bound for term 2

\[
\left| \sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v) \right| \leq \sqrt{N} |D(G)^T| \|\Delta x\| r(\|\Delta x\|_{\infty}) |v|_{\infty}
\]

\[\Delta x := \text{stack vector of } x_i - x_j, \{i, j\} \in \mathcal{E}\]

\[|\Delta x|_{\infty} := \max\{|x_i - x_j| : \{i, j\} \in \mathcal{E}\}\]
Derivative along System Trajectories

- Requirement on $\dot{V}$

$$|\Delta x|_\infty \geq \tilde{R} \Rightarrow \dot{V} \leq 0 \quad (#)$$

- A sufficient condition for (#) is that

$$|v|_\infty \leq \frac{1}{K} r(0)^2 \frac{s}{r(s)}, \forall s \geq \tilde{R}; \quad K := \frac{2 \sqrt{N(N - 1)|D(G)^T|}}{\lambda_2(G)^2}$$

- Network remains connected for all times if

$$MP(\tilde{R}) \leq P(R) \quad \& \quad (#)$$

- worst case initial energy
- minimum energy required to loose connectivity
PROPOSITION 1
Consider the control law

\[ u_i = - \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i \]

and a constant \( \delta > 0 \). Assume that \( r(\cdot) \), \( \delta \) and the maximum initial distance \( \tilde{R} \) satisfy the restrictions

\[ \delta \leq \frac{1}{K} r(0)^2 \frac{s}{r(s)}, \forall s \geq \tilde{R}; \quad K := \frac{2\sqrt{N(N-1)}|D(G)^T|}{\lambda_2(G)^2} \]

and

\[ MP(\tilde{R}) \leq P(R) \]

Then the network remains connected for all \( t \geq 0 \) provided that the inputs terms \( v_i \) satisfy the bound

\[ |v_i(t)| \leq \delta, \forall t \geq 0 \]
Illustrative Controller Selection

- Recall that
  \[ u_i = - \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i \]

- Selection of a linear and a nonlinear control law providing the same bound on $|v|_{\infty}$

- Linear case
  \[ r(s) := 1, \quad s \geq 0 \quad \& \quad \tilde{R} \leq \frac{1}{\sqrt{M}} R \]

- Nonlinear case
  \[ r(s) := \begin{cases} 
  1, & s \in [0, \tilde{R}] \\
  \frac{s}{\tilde{R}}, & s \in (\tilde{R}, R] \\
  \frac{R}{\tilde{R}}, & s \in (R, \infty) 
  \end{cases} \quad \& \quad \tilde{R} \leq \left( \frac{2}{3M - 1} \right)^{\frac{1}{3}} R \]
Invariance for a Spherical Domain
Repulsive Dynamics

- Define the vector field $g : B(\mathcal{A}) \to \mathbb{R}^n$ as

$$g(x) := \begin{cases} 
-c\delta\frac{|x| - \mathcal{A}}{\varepsilon} \frac{x}{|x|}, & \text{if } x \in N_\varepsilon \\
0, & \text{if } x \in D_\varepsilon
\end{cases}$$

- Regions

$$N_\varepsilon := \{ x \in \mathbb{R}^n : \mathcal{A} - \varepsilon \leq |x| < \mathcal{A} \} \quad D_\varepsilon := B(\mathcal{A}) \setminus N_\varepsilon$$

- Control law

$$u_i = g(x_i) - \sum_{j \in N_i} r(|x_i - x_j|)(x_i - x_j) + v_i$$
Derivative along System Trajectories

- Derivative of (same as before) energy function

\[ \dot{V} \leq \sum_{i=1}^{N} \frac{\partial}{\partial x_i} V(x) g(x_i) - \sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) + \left| \sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v) \right| \]

 extra term

- Focus on the extra term

\[
\sum_{i=1}^{N} \sum_{j \in N_i} r(|x_i - x_j|) \langle (x_i - x_j), g(x_i) \rangle \\
= \sum_{\{i \in N : x_i \in N_\varepsilon\}} \sum_{j \in N_i^{D_\varepsilon}} r(|x_i - x_j|) \langle (x_i - x_j), g(x_i) \rangle \\
+ \sum_{\{i,j\} \in E^{N_\varepsilon}} r(|x_i - x_j|) \left[ \langle (x_i - x_j), g(x_i) \rangle + \langle (x_j - x_i), g(x_j) \rangle \right]
\]
Sign of the Extra Terms
Invariance Result

THEOREM
Consider the control law

\[ u_i = g(x_i) - \sum_{j \in N_i} r(|x_i - x_j|)(x_i - x_j) + v_i \]

where \( g(x) := \begin{cases} 
-c\delta \frac{\varepsilon + |x| - \tilde{R}}{\varepsilon} \frac{x}{|x|}, & \text{if } x \in N_\varepsilon \\
0, & \text{if } x \in D_\varepsilon 
\end{cases} \)

with \( c > 1 \) and assume that \( r(\cdot) \), the maximum initial distance \( \tilde{R} \) and the bound \( \delta \) on the inputs \( v_i \) satisfy the conditions in Proposition 1.

Then the solution of the closed loop system remains in \( D \) for all \( t \geq 0 \) and the network remains connected for all \( t \geq 0 \).