Global Consensus for Discrete-time Multi-Agent Systems with Input Saturation Constraints

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Abstract

In this paper, we consider the global consensus problem for discrete-time multi-agent systems with input saturation constraints under fixed undirected topologies. We first give necessary conditions for achieving global consensus via a distributed protocol based on relative state measurements of the agent itself and its neighboring agents. We then focus on two special cases, where the agent model is either neutrally stable or a double integrator. For the neutrally stable case, any linear protocol of a particular form, which solves the consensus problem for the case without input saturation constraints, also solves the global consensus problem for the case with input saturation constraints. For the double integrator case, we show that a subset of linear protocols, which solve the consensus problem for the case without saturation constraints, also solve the global consensus problem for the case with input saturation constraints. The results are illustrated by numerical simulations.

Key words: Global Consensus; Input Saturation Constraints; Multi-Agent Systems

1 Introduction

In recent years, the distributed coordination of a multi-agent system (MAS) has received substantial attention due to its wide application areas, including consensus computation (Tsitsiklis, 1984; Jadbabaie, Lin, and Morse, 2003; Olfati-Saber and Murray, 2004; Bai, Arcak, and Wen, 2011), synchronization (Wu and Chua, 1995), distributed processing (Lynch, 1996), and network flow control (Low, Paganini, and Doyle, 2002; Wen and Arcak, 2004). When it comes to the consensus problem, each agent has to implement a distributed protocol based on the limited information about itself and its neighboring agents.

The design of consensus protocols can be generally divided into two categories depending on whether the agent models are continuous-time or discrete-time. Much attention has been devoted to the continuous-time case. The existing works here can be categorized into two directions depending whether the agent models are identical or not. The consensus problem for homogeneous networks (i.e., networks where the agent models are identical) has been considered (e.g., Olfati-Saber and Murray, 2004; Ren and Beard, 2005; Xiao and Wang, 2008; Scardovi and Sepulchre, 2009; Seo, Shim, and Back, 2009; Shi and Hong, 2009; Li, Duan, Chen, and Huang, 2010; Li, Du, and Lin, 2011a; Yu, Chen, and Cao, 2010; Yang, Roy, Wan, and Saberi, 2011; Seyboth, Dimarogonas, and Johansson, 2013), while the consensus problem for heterogenous networks (i.e., networks where the agent models are non-identical) has been a recent focus (e.g., Wieland, Sepulchre, and Allgöwer, 2011; Zhao, Hill, and Liu, 2011; Grip, Yang, Saberi, and Stoorvogel, 2012). The studies on the discrete-time case are rather limited, but some results can be found in (e.g., Jadbabaie et al., 2003; Blondel, Hendrickx, Olshevsky, and Tsitsiklis, 2005; Tuna, 2008; Moreau, 2005; Ren and Beard, 2005; Zhang and Tian, 2009; You and Xie, 2011).

Most consensus literature does not consider the case where the agents are subject to input saturation. However, in almost every physical application, the actuator has bounds on its input, and thus actuator saturation is important to study. The protocol design for achieving consensus for the case with input saturation constraints is a challenging problem, and only few results are available for continuous-time agent models (e.g., Cortés, 2006; Li, Xiang, and Wei, 2011b; Meng, Zhao, and Lin, 2013; Du, Li, and Ding, 2013; Yang, Stoorvogel, Grip, and Saberi, 2012a). For the single integrator case, Li et al. (2011b) showed that any linear protocol based on the relative state information, which solves the consensus prob-

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lem for the case without input saturation constraints under fixed directed network topologies, also solves the global consensus problem in the presence of input saturation constraints. Meng et al. (2013) proposed a linear protocol based on the relative state information to solve the global consensus problem for a MAS with input saturation constraints under fixed undirected network topologies and time varying network topologies. Yang et al. (2012a) studied semi-global regulation of output synchronization for heterogeneous networks under fixed directed network topologies.

To the best of the authors' knowledge, all the existing works on the consensus problem for a MAS with input saturation constraints are restricted to continuous-time agent models. This motivates us to consider the consensus problem for the case where the agents models are discrete-time, as such models are relevant for many practical sampled-data systems. As a first step, in this paper, we assume that the network topology is fixed and undirected. This paper may be seen as a continuation of the work of Meng et al. (2013). We extend their continuous-time results for fixed topologies to a discrete-time setting. The extension is considerable. First, Meng et al. (2013) considered the leader-follower case while we consider the leaderless case. Second, we use a completely new set Lyapunov stability theory argument.

The remainder of the paper is organized as follows. In Section 2, some preliminaries and notations are introduced. In Section 3, we first formulate the global consensus problem with input saturation constraints, and then give necessary conditions for achieving global consensus under fixed undirected topologies. In Sections 4 and 5, we consider the case where the agent model is neutrally stable and a double integrator, respectively. Simulation examples are presented in Section 6 followed by conclusions.

2 Preliminaries and Notations

In this paper, we assume that the network topology among the agents is described by a fixed undirected weighted graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$, with the set of agents $\mathscr{V} = \{1, \ldots, N\}$, the set of undirected edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and a weighted adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ij} > 0$ if and only if $(j,i) \in \mathscr{E}$ and $a_{ij} = 0$ otherwise. In this paper, we also assume that $a_{ij} = a_{ji}$ for all $i, j \in \mathscr{V}$, and that there are no self-loops, i.e., $a_{ii} = 0$ for $i \in \mathscr{V}$. The set of neighboring agents of agent i is defined as $\mathscr{N}_i = \{j \in \mathscr{V} | a_{ij} > 0\}$. A path from node i_1 to i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathscr{E}$ for $j = 1, \ldots, k-1$ in the undirected graph. An undirected graph is said to be connected if there exists a path between any pair of distinct nodes.

For an undirected weighted graph $\mathscr{G}$, a matrix $L = [\ell]_{ij} \in \mathbb{R}^{N \times N}$ with $\ell_{ii} = \sum_{j=1}^{N} a_{ij}$ and $\ell_{ij} = -a_{ij}$ for $j \neq i$, is called Laplacian matrix associated with graph $\mathscr{G}$. It is well known that the Laplacian matrix has the property that all the row sums are zero. If the undirected weighted graph $\mathscr{G}$ is connected, then L has a simple eigenvalue at zero with corresponding right eigenvector $\mathbf{1}$ and all other eigenvalues are strictly positive. All the eigenvalues can be ordered as $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N \leq 2\Delta$, where $\Delta = \max_{i \in \mathscr{V}} \ell_{ii}$.

Given a matrix A, A^{T} denotes its transpose and $\|A\|$ denotes its induced norm. A symmetric matrix A is positive (negative) definite if and only if all its eigenvalues are positive (negative), and is positive (negative) semi-definite if and only if all its eigenvalues are non-negative (non-positive). We denote by $A \otimes B$ the Kronecker product between matrices A and B. For two column vectors a and b of the same dimensions, $a < (\leq) b$ means that each entry of $a - b$ is negative (non-positive), while $a > (\geq) b$ means that each entry of $a - b$ is positive (non-negative). I_N denotes the identity matrix of dimension $N \times N$. $\mathbf{1}_N$ denotes the column vector with each entry being 1. For column vectors $x_1, \ldots, x_N$, the stacked column vector of $x_1, \ldots, x_N$ is denoted by $[x_1; \ldots; x_N]$.

3 Problem Formulation

We consider a MAS of N identical discrete-time agents

$$x_i(k+1) = Ax_i(k) + B\sigma(u_i(k)), \quad i \in \mathscr{V}, \tag{1}$$

where $x_i(k) \in \mathbb{R}^n$, $u_i(k) \in \mathbb{R}^m$,

$$\sigma(u_i(k)) = [\sigma_1(u_{i,1}(k)); \sigma_1(u_{i,2}(k)); \ldots; \sigma_1(u_{i,m}(k))],$$

and each $\sigma_1(u)$ is the standard saturation function

$$\sigma_1(u) = \begin{cases} 1 & \text{if } u > 1, \\ u & \text{if } |u| \leq 1, \\ -1 & \text{if } u < -1. \end{cases}$$

The only information available for agent i comes from the network. In particular, agent i receives a linear combination of its own state relative to that of neighboring agents, i.e.,

$$\zeta_i(k) = \sum_{j \in \mathscr{N}_i} a_{ij}(x_i(k) - x_j(k)).$$

Our goal is to design distributed protocols $u_i(k)$ for $i \in \mathscr{V}$ by using $\zeta_i(k)$ to solve the *global* consensus problem, i.e., for any initial conditions $x_i(0)$, where $i \in \mathscr{V}$, $\lim_{k \to \infty}(x_i(k) - x_j(k)) = 0$ for all $i, j \in \mathscr{V}$.

Each agent is subject to the input saturation constraints. These nonlinearities make the protocol design for achieving consensus difficult since we have to guarantee that consensus is achieved for all initial conditions.

3.1 Necessary Conditions

Assumption 1 *The agent model* (1) *is asymptotically null controllable with bounded controls (ANCBC), i.e., the pair* (A,B) *is stabilizable and all the eigenvalues of the matrix* A *are within or on the unit circle.*

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Based on the results in Yang, Sontag, and Sussmann (1997) and Saberi, Stoorvogel, and Sannuti (2012, Section 4.2), we have the following result.

Proposition 1 *A MAS of N agents* (1) *achieves global consensus via distributed protocols $u_i(k) = f_i(\zeta_i(k), k)$, $i \in \mathscr{V}$, only if Assumption 1 is satisfied.*

From the saturation literature (e.g., Sussmann, Sontag, and Yang, 1994; Teel, 1992), it is evident that, in general, we need to design a nonlinear protocol to solve the global consensus problem. In this paper, we shall concentrate on a linear protocol

$$u_i(k) = K\zeta_i(k) = K \sum_{j\in\mathscr{N}_i} a_{ij}(x_i(k) - x_j(k)), \quad i \in \mathscr{V}, \quad (2)$$

as such a protocol may suffice in some cases.

Given a fixed undirected graph and Assumption 1, it follows from You and Xie (2011, Theorem 3.1) that a network of N agents (1) in the absence of input saturation achieves consensus via the protocol (2) if and only if the following assumption is satisfied.

Assumption 2 *The graph $\mathscr{G}$ is connected.*

This together with Proposition 1 yields the following result.

Proposition 2 *A MAS of N agents* (1) *achieves global consensus via distributed protocols* (2) *only if Assumptions 1 and 2 are satisfied.*

There is limited knowledge regarding which discrete-time linear time-invariant systems subject to actuator saturation allow for global stabilization via linear state feedback control laws. It is known that for some special discrete-time cases, such as open-loop neutrally stable systems [1] (Bao, Lin, and Sontag, 2000), and a double integrator (Yang, Stoorvogel, and Saberi, 2012b), there exist saturated globally stabilizing linear state feedback control laws. Hence, in the following sections, we consider the global consensus problem for such special cases. We show that Assumptions 1 and 2 are also sufficient for achieving global consensus for these cases by explicitly specifying the matrix K for (2).

4 Neutrally Stable Agent Model

In this section, we consider the case where the agent model (1) is open-loop neutrally stable.

[1] A discrete-time system is said to be open-loop neutrally stable if all its open-loop poles are within or on the unit circle with those on the unit circle being simple.

Under Assumption 1, there exists a non-singular state transformation T^{-1}, such that

$$A = T^{-1}\begin{bmatrix} A_c & 0 \\ 0 & A_s \end{bmatrix} T, \quad B = T^{-1}\begin{bmatrix} B_c \\ B_s \end{bmatrix},$$

where $A_c^{\mathrm{T}}A_c = I$, A_s is Schur stable (i.e., all its eigenvalues are within the unit circle), and the pair (A_c, B_c) is controllable.

As shown in You and Xie (2011), the asymptotically stable modes can be ignored since we can set the corresponding gain matrix to zero. Thus, without loss of generality, we make the following assumption in this section.

Assumption 3 *$A^TA = I_n$ and the pair (A,B) is controllable.*

Under Assumption 3, controllability of the pair (A,B) is equivalent to stabilizability of the pair (A,B).

Consider the following control law

$$u_i(k) = -\varepsilon B^{\mathrm{T}}A \sum_{j\in\mathscr{N}_i} a_{ij}(x_i(k) - x_j(k)), \quad i \in \mathscr{V}. \quad (3)$$

Note that (3) is of the form (2) with $K = -\varepsilon B^{\mathrm{T}}A$, where ε is a designed parameter. The following lemma shows that the protocol (3) with a properly chosen ε solves the consensus problem for a MAS without input saturation.

Lemma 1 *Consider a MAS of N agents* (1) *in the absence of input saturation constraints. Assume that Assumptions 2 and 3 are satisfied. Then any protocol* (3) *with $\varepsilon \in (0, \frac{2}{\lambda_N\|B^TB\|})$, where λ_N is the largest eigenvalue of the corresponding Laplacian matrix, solves the consensus problem.*

Proof: It is well known that (e.g. Seo et al., 2009; Zhang and Tian, 2009) the consensus problem for a network of N identical agents is equivalent to the simultaneous stabilization problem of $N-1$ systems. Hence, it can be verified that consensus is achieved via (3) if all the matrices $A - \varepsilon\lambda_i BB^{\mathrm{T}}A$, where λ_i, $i \in \{2,\dots,N\}$ (i.e., the nonzero eigenvalues of the Laplacian matrix) are Schur stable. It then follows from Shi, Saberi, and Stoorvogel (2003, Lemma 4.2) that all these matrices are Schur stable if $\varepsilon \in (0, \frac{2}{\lambda_N\|B^{\mathrm{T}}B\|})$. ■

The following theorem shows that (3) with $\varepsilon \in (0, \frac{2}{\lambda_N\|B^{\mathrm{T}}B\|})$ also solves the global consensus problem for a MAS.

Theorem 1 *Consider a MAS of N agents* (1)*. Assume that Assumptions 2 and 3 are satisfied. Then any protocol* (3) *with $\varepsilon \in (0, \frac{2}{\lambda_N\|B^TB\|})$ solves the global consensus problem.*

Proof: Define $x(k) = [x_1(k);\dots;x_N(k)]$ and $u(k) = [u_1(k);\dots;u_N(k)]$. To simplify the notation, sometimes x or u without explicitly indicating the time instant will refer to

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$x(k)$ or $u(k)$ respectively. With these definitions, we obtain the following dynamics

$$x(k+1) = (I_N \otimes A)x(k) + (I_N \otimes B)\sigma(u(k)), \tag{4a}$$
$$u(k) = -\varepsilon(L \otimes B^{\mathrm{T}}A)x(k). \tag{4b}$$

Motivated by the Lyapunov function in (Cortés, 2006; Zhang, Lewis, and Qu, 2012), we consider the Lyapunov candidate

$$V(x(k)) = \frac{1}{2}x^{\mathrm{T}}(k)(L \otimes I_n)x(k).$$

Define a manifold where all the agent states are identical

$$M := \{x \in \mathbb{R}^{Nn} | x_1 = x_2 = \ldots = x_N\}.$$

Note that $V(x) \geq 0$ and $V(x) = 0$ if and only if $x \in M$. Let us now evaluate $\Delta V(x(k)) = V(x(k+1)) - V(x(k))$. We sometimes drop the dependency of $V(x(k))$ and $\Delta V(x(k))$ on $x(k)$ for notational simplification when it is clear from the context. From the dynamics of (4), we obtain

$$\begin{aligned}
\Delta V &= \frac{1}{2}\sigma^{\mathrm{T}}(u)(L \otimes B^{\mathrm{T}}A)x + \frac{1}{2}x^{\mathrm{T}}(L \otimes A^{\mathrm{T}}B)\sigma(u) \\
&\quad + \frac{1}{2}\sigma^{\mathrm{T}}(u)(L \otimes B^{\mathrm{T}}B)\sigma(u) \\
&= -\frac{1}{\varepsilon}\sigma^{\mathrm{T}}(u)u + \frac{1}{2}\sigma^{\mathrm{T}}(u)(L \otimes B^{\mathrm{T}}B)\sigma(u) \\
&\leq -\sigma^{\mathrm{T}}(u)(\frac{1}{\varepsilon}I_{Nm} - \frac{1}{2}L \otimes B^{\mathrm{T}}B)\sigma(u),
\end{aligned}$$

where we have used that $L = L^{\mathrm{T}}$ for undirected graphs and that $z^{\mathrm{T}}\sigma(z) \geq \sigma^{\mathrm{T}}(z)\sigma(z)$ for any column vector z.

Since $\varepsilon \in (0, \frac{2}{\lambda_N \|B^{\mathrm{T}}B\|})$, $\Delta V \leq 0$ and $\Delta V = 0$ if and only if $(L \otimes B^{\mathrm{T}}A)x = 0$. We shall show that $(L \otimes B^{\mathrm{T}}A)x = 0$ if and only if $x \in M$, which in turn implies that $\Delta V = 0$ if and only if $x \in M$. We first note that if $x \in M$, then $(L \otimes B^{\mathrm{T}}A)x = 0$ since the graph is connected. We then show that $(L \otimes B^{\mathrm{T}}A)x = 0$ implies that $x \in M$. Note that $(L \otimes B^{\mathrm{T}}A)x = 0$ implies that $(\tilde{L} \otimes B^{\mathrm{T}}A)q = 0$, where the relative state $q = [q_2; \ldots; q_N]$, $q_i = x_i - x_1$ for $i \in \{2, \ldots, N\}$, and

$$\tilde{L} = \begin{bmatrix} \ell_{2,2} - \ell_{1,2} & \ldots & \ell_{2,N} - \ell_{1,N} \\ \vdots & \ddots & \vdots \\ \ell_{N,2} - \ell_{1,2} & \ldots & \ell_{N,N} - \ell_{1,N} \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}. \tag{5}$$

Since the graph is connected, it follows from Zhang and Tian (2009, Lemma 1) that the eigenvalues of $\tilde{L}$ are the nonzero eigenvalues of the matrix L, which are positive. Thus, the matrix $\tilde{L}$ is non-singular, i.e., $\mathrm{rank}(\tilde{L}) = N - 1$.

From the fact that $A^{\mathrm{T}}A = I_n$, we see that $(\tilde{L} \otimes B^{\mathrm{T}}A)q = 0$ implies $q^{\mathrm{T}}(\tilde{L} \otimes A^{-1}B) = 0$. Also note that $q(k+1) = (I_{N-1} \otimes A)q(k)$, since $u(k) = -\varepsilon(L \otimes B^{\mathrm{T}}A)x(k) = 0$. Therefore

$$(\tilde{L} \otimes B^{\mathrm{T}}A)q(k+1) = (\tilde{L} \otimes B^{\mathrm{T}}A^2)q(k),$$

which is equivalent to $q^{\mathrm{T}}(\tilde{L} \otimes A^{-2}B) = 0$. By iteration, we obtain $q^{\mathrm{T}}(\tilde{L} \otimes A^{-r}B) = 0$ for $r = 3, 4, \ldots, n+1$. Hence,

$$q^{\mathrm{T}}\left(\tilde{L} \otimes A^{-(n+1)}\left[A^nB \ \ldots \ AB \ B\right]\right) = 0. \tag{6}$$

Note that $\mathrm{rank}\left(\left[A^nB \ \ldots \ AB \ B\right]\right) = n$ since the pair (A, B) is controllable. This together with the fact that the matrix A is non-singular implies that $\mathrm{rank}\left(A^{-(n+1)}\left[A^nB \ \ldots \ AB \ B\right]\right) = n$. Finally, using the property of Kronecker product, we obtain

$$\begin{aligned}
&\mathrm{rank}\left(\tilde{L} \otimes A^{-(n+1)}\left[A^nB \ \ldots \ AB \ B\right]\right) \\
&= \mathrm{rank}\left(\tilde{L}\right)\mathrm{rank}\left(A^{-(n+1)}\left[A^nB \ \ldots \ AB \ B\right]\right) = (N-1)n.
\end{aligned}$$

Therefore, the only solution of (6) is $q = 0$, which is equivalent to $x_1 = \ldots = x_N$, i.e., $x \in M$. Hence, we have shown that $\Delta V(x) \leq 0$ and $\Delta V(x) = 0$ if and only if $x \in M$.

Since $\Delta V(x(k)) = V(x(k+1)) - V(x(k)) \leq 0$, we conclude that $V(x(k))$ is non-increasing in k. Thus, $\lim_{k\to\infty} V(x(k)) = V_*$ for some $V_* \geq 0$ since $V \geq 0$. This implies that $\Delta V(x(k)) \to 0$ as $k \to \infty$ and hence $x(k) \to M$ as $k \to \infty$. Hence, global consensus is achieved. ■

Remark 1 *Note that the continuous-time counterpart was considered in Meng et al. (2013, Theorem* 4.1*). They considered the leader-follower case while we consider the leaderless case. It results in a completely different analysis which relies on set Lyapunov stability theory argument.*

5 Double Integrator Agent Model

In this section, we consider the case where the agent model (1) is a double integrator.

Assumption 4 *The matrices A and B are of the form*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let us first recall the following result which gives a necessary and sufficient condition on the feedback gain parameters for achieving consensus without input saturation constraints.

Lemma 2 *Xie and Wang (2012) Consider a MAS of N agents described by*

$$\begin{bmatrix} x_i(k+1) \\ v_i(k+1) \end{bmatrix} = A\begin{bmatrix} x_i(k) \\ v_i(k) \end{bmatrix} + Bu_i(k), \quad i \in \mathscr{V}. \tag{7}$$

Assume that Assumptions 2 and 4 are satisfied. Then the protocol (2) *with* $K = -\left[\alpha \ \beta\right]$:

$$u_i(k) = -\alpha \sum_{j\in\mathscr{N}_i} a_{ij}(x_i(k) - x_j(k)) - \beta \sum_{j\in\mathscr{N}_i} a_{ij}(v_i(k) - v_j(k)), \tag{8}$$

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solves the consensus problem if and only if

$$0 < \alpha < \beta < \frac{\alpha}{2} + \frac{2}{\lambda_N}. \tag{9}$$

The following theorem shows that a subset of the protocols (8), which solve the consensus problem for a MAS without input saturation constraints, also solve the global consensus problem for a MAS with input saturation constraints.

Theorem 2 *Consider a MAS of N agents* (1). *Assume that Assumptions 2 and 4 are satisfied. Then the protocol* (8) *with*

$$0 < \sqrt{3}\alpha < \beta < \frac{3}{2\lambda_N}, \tag{10}$$

solves the global consensus problem.

Proof: Let $x(k) = [x_1(k);\ldots;x_N(k)]$, $v(k) = [v_1(k);\ldots;v_N(k)]$, $u(k) = [u_1(k);\ldots;u_N(k)]$, $y_i(k) = [x_i(k);v_i(k)]$, and $y(k) = [y_1(k);\ldots;y_N(k)]$. To simplify the notation, sometimes x, v, u or y without explicitly indicating the time instant will refer to $x(k)$, $v(k)$, $u(k)$ or $y(k)$ respectively. With these definitions, we obtain the following dynamics

$$\begin{aligned} y(k+1) &= (I_N \otimes A)y(k) + (I_N \otimes B)\sigma(u(k)), \\ u(k) &= (L \otimes [\,-\alpha \; -\beta\,])\, y(k). \end{aligned}$$

We also obtain that $x(k+1) = x(k) + v(k)$ and $v(k+1) = v(k) + \sigma(u(k))$. Note that $u(k)$ can be written in terms of $x(k)$ and $v(k)$ as

$$u(k) = -\alpha L x(k) - \beta L v(k). \tag{11}$$

Hence, we obtain

$$u(k+1) = u(k) - \alpha L v(k) - \beta L \sigma(u(k)). \tag{12}$$

Motivated by the Lyapunov function in Yang et al. (2012b), we consider the following Lyapunov candidate

$$V(y) = -\sigma^{\mathrm{T}}(u)\sigma(u) + 2\sigma^{\mathrm{T}}(u)u + 2\alpha\sigma^{\mathrm{T}}(u)Lv + \alpha v^{\mathrm{T}}Lv.$$

We sometimes drop the dependency of $V(y(k))$ on $y(k)$ for notation simplification when it is clear from the context. Similar to the proof of Theorem 1, we define a manifold where all the agent states are identical

$$M := \{y \in \mathbb{R}^{2N} | y_1 = y_2 = \ldots = y_N\}.$$

Note that $V(y) = 0$ if $y \in M$. We will show that $V(y) \geq 0$ and $V(y) = 0$ if only if $y \in M$. Since $\sigma^{\mathrm{T}}(z)z \geq \sigma^{\mathrm{T}}(z)\sigma(z)$ for any column vector z, where the equality holds if and only if $-\mathbf{1} \leq z \leq \mathbf{1}$, we obtain

$$V \geq \sigma^{\mathrm{T}}(u)\sigma(u) + 2\alpha\sigma^{\mathrm{T}}(u)Lv + \alpha v^{\mathrm{T}}Lv \tag{13}$$

$$= \begin{bmatrix} \sigma(u) \\ Lv \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & \alpha \\ \alpha & \frac{2}{3}\alpha\beta \end{bmatrix} \begin{bmatrix} \sigma(u) \\ Lv \end{bmatrix} + v^{\mathrm{T}}(\alpha L - \frac{2}{3}\alpha\beta L^{\mathrm{T}}L)v, \tag{14}$$

where the equality of (13) holds if and only if $-\mathbf{1} \leq u \leq \mathbf{1}$. Since $\beta > \sqrt{3}\alpha > \frac{3}{2}\alpha > 0$, the first term of (14) is non-negative, and is equal to zero if and only if $\sigma(u) = 0$ and $Lv = 0$. Note that from (11), we see that $u = -\alpha Lx - \beta Lv = -\alpha Lx$, therefore, $Lx = 0$ since $\alpha \neq 0$. Thus, the first term is equal to zero if and only if $y \in M$. We next show that the second term is also non-negative. Since $L = L^{\mathrm{T}}$, we see that the eigenvalues of the matrix $\alpha L - \frac{2}{3}\alpha\beta L^{\mathrm{T}}L$ are $\alpha\lambda_i(1 - \frac{2}{3}\beta\lambda_i)$, where λ_i, $i \in \{1,\ldots,N\}$ are the eigenvalues of the Laplacian matrix L. Since $\beta\lambda_N < \frac{3}{2}$, the second term is non-negative and equal to zero if and only if $Lv = 0$. Therefore, $V(y) \geq 0$ and $V(y) = 0$ if and only if $y \in M$.

Next, we show that $\Delta V(y(k)) = V(y(k+1)) - V(y(k)) \leq 0$. We sometimes drop the dependency $\Delta V(y(k))$ on $y(k)$ for notational simplification when it is clear from the context. With some algebra, we obtain

$$\begin{aligned} V(y(k+1)) = &-t^{\mathrm{T}}t + 2t^{\mathrm{T}}u + 2(\alpha-\beta)t^{\mathrm{T}}L\sigma(u) + \alpha v^{\mathrm{T}}Lv \\ &+ 2\alpha v^{\mathrm{T}}L\sigma(u) + \alpha\sigma^{\mathrm{T}}(u)L\sigma(u), \end{aligned}$$

where to simplify notation we have used $t = \sigma(u(k+1))$. Note that $-\mathbf{1} \leq t \leq \mathbf{1}$ by the definition of the saturation function. Thus,

$$\begin{aligned} \Delta V(y(k)) &= V(y(k+1)) - V(y(k)) \\ &= -t^{\mathrm{T}}t + 2t^{\mathrm{T}}u + 2(\alpha-\beta)t^{\mathrm{T}}L\sigma(u) \\ &\quad + \sigma^{\mathrm{T}}(u)(\alpha L + I)\sigma(u) - 2\sigma^{\mathrm{T}}(u)u. \end{aligned}$$

Without loss of generality, we assume that $u_i > 1$ for $i \in \{1,\ldots,N_1\} := S_p$, $|u_i| \leq 1$ for $i \in \{N_1+1,\ldots,N_2\} := S_m$, and $u_i < -1$ for $i \in \{N_2+1,\ldots,N\} := S_q$, since if this is not the case, we can always relabel the nodes to achieve this. Note that the sets S_p, S_m, and S_q may be empty. We then define the partition $t = [t_p;t_m;t_q]$, $u = [u_p;u_m;u_q]$, where $t_p, u_p \in \mathbb{R}^{N_1}$, $t_m, u_m \in \mathbb{R}^{N_2-N_1}$, and $t_q, u_q \in \mathbb{R}^{N-N_2}$ are defined accordingly. We partition the Laplacian matrix L accordingly

$$L = \begin{bmatrix} L_{pp} & L_{pm} & L_{pq} \\ L_{pm}^{\mathrm{T}} & L_{mm} & L_{mq} \\ L_{pq}^{\mathrm{T}} & L_{mq}^{\mathrm{T}} & L_{qq} \end{bmatrix},$$

where L_{pp}, L_{pm}, L_{pq}, L_{mm}, L_{mq} and L_{qq} are real matrices of appropriate dimensions. With some algebra, we obtain

$$\begin{aligned} \Delta V = &-t_p^{\mathrm{T}}t_p - t_m^{\mathrm{T}}t_m - t_q^{\mathrm{T}}t_q + 2t_p^{\mathrm{T}}u_p + 2t_m^{\mathrm{T}}u_m + 2t_q^{\mathrm{T}}u_q \\ &+ 2(\alpha-\beta)\,[\,t_p^{\mathrm{T}}\; t_m^{\mathrm{T}}\; t_q^{\mathrm{T}}\,] \begin{bmatrix} L_{pp} & L_{pm} & L_{pq} \\ L_{pm}^{\mathrm{T}} & L_{mm} & L_{mq} \\ L_{pq}^{\mathrm{T}} & L_{mq}^{\mathrm{T}} & L_{qq} \end{bmatrix} \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} \\ &+ \alpha\,[\,\mathbf{1}_p^{\mathrm{T}}\; u_m^{\mathrm{T}}\; -\mathbf{1}_q^{\mathrm{T}}\,]\, L \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} + \mathbf{1}_p^{\mathrm{T}}\mathbf{1}_p + u_m^{\mathrm{T}}u_m + \mathbf{1}_q^{\mathrm{T}}\mathbf{1}_q \end{aligned}$$

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$$
\begin{aligned}
&-2\left[\mathbf{1}_p^{\mathrm T}\ u_m^{\mathrm T}\ -\mathbf{1}_q^{\mathrm T}\right]\begin{bmatrix}u_p\\u_m\\u_q\end{bmatrix}\\
&= 2(t_p-\mathbf{1}_p)^{\mathrm T}(u_p-\mathbf{1}_p)+2t_p^{\mathrm T}\mathbf{1}_p-2\mathbf{1}_p^{\mathrm T}\mathbf{1}_p\\
&\quad +2(t_q+\mathbf{1}_q)^{\mathrm T}(u_q+\mathbf{1}_q)-2t_q^{\mathrm T}\mathbf{1}_q-2\mathbf{1}_q^{\mathrm T}\mathbf{1}_q\\
&-t_p^{\mathrm T}t_p+2t_p^{\mathrm T}\left[(\alpha-\beta)\left[L_{pp}\ L_{pm}\ L_{pq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]-2t_p^{\mathrm T}\mathbf{1}_p\\
&-t_m^{\mathrm T}t_m+2t_m^{\mathrm T}\left[(\alpha-\beta)\left[L_{pm}^{\mathrm T}\ L_{mm}\ L_{mq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+u_m\right]\\
&-t_q^{\mathrm T}t_q+2t_q^{\mathrm T}\left[(\alpha-\beta)\left[L_{pq}^{\mathrm T}\ L_{mq}^{\mathrm T}\ L_{qq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}-\mathbf{1}_q\right]+2t_q^{\mathrm T}\mathbf{1}_q\\
&\quad +\alpha\left[\mathbf{1}_p^{\mathrm T}\ u_m^{\mathrm T}\ -\mathbf{1}_q^{\mathrm T}\right]L\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p^{\mathrm T}\mathbf{1}_p+\mathbf{1}_q^{\mathrm T}\mathbf{1}_q-u_m^{\mathrm T}u_m.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&-t_p^{\mathrm T}t_p+2t_p^{\mathrm T}\left[(\alpha-\beta)\left[L_{pp}\ L_{pm}\ L_{pq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]\\
&=-\left\{t_p-\left[(\alpha-\beta)\left[L_{pp}\ L_{pm}\ L_{pq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]\right\}^{\mathrm T}\\
&\quad\times\left\{t_p-\left[(\alpha-\beta)\left[L_{pp}\ L_{pm}\ L_{pq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]\right\}\\
&\quad+\left[(\alpha-\beta)\left[L_{pp}\ L_{pm}\ L_{pq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]^{\mathrm T}\\
&\quad\times\left[(\alpha-\beta)\left[L_{pp}\ L_{pm}\ L_{pq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right].
\end{aligned}
$$

Similar completion of squares for

$$-t_m^{\mathrm T}t_m+2t_m^{\mathrm T}\left[(\alpha-\beta)\left[L_{pm}^{\mathrm T}\ L_{mm}\ L_{mq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+u_m\right],$$

and

$$-t_q^{\mathrm T}t_q+2t_q^{\mathrm T}\left[(\alpha-\beta)\left[L_{pq}^{\mathrm T}\ L_{mq}^{\mathrm T}\ L_{qq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}-\mathbf{1}_q\right],$$

yields

$$
\begin{aligned}
\Delta V &= 2(t_p-\mathbf{1}_p)^{\mathrm T}(u_p-\mathbf{1}_p)+2(t_q+\mathbf{1}_p)^{\mathrm T}(u_q+\mathbf{1}_p) &(15)\\
&\quad-\left\{t_p-\left[(\alpha-\beta)\left[L_{pp}\ L_{pm}\ L_{pq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]\right\}^{\mathrm T}\\
&\quad\times\left\{t_p-\left[(\alpha-\beta)\left[L_{pp}\ L_{pm}\ L_{pq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]\right\} &(16)\\
&\quad-\left\{t_m-\left[(\alpha-\beta)\left[L_{pm}^{\mathrm T}\ L_{mm}\ L_{mq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+u_m\right]\right\}^{\mathrm T}\\
&\quad\times\left\{t_m-\left[(\alpha-\beta)\left[L_{pm}^{\mathrm T}\ L_{mm}\ L_{mq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}+u_m\right]\right\} &(17)\\
&\quad-\left\{t_q-\left[(\alpha-\beta)\left[L_{pq}^{\mathrm T}\ L_{mq}^{\mathrm T}\ L_{qq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}-\mathbf{1}_q\right]\right\}^{\mathrm T}\\
&\quad\times\left\{t_q-\left[(\alpha-\beta)\left[L_{pq}^{\mathrm T}\ L_{mq}^{\mathrm T}\ L_{qq}\right]\begin{bmatrix}\mathbf{1}_p\\u_m\\-\mathbf{1}_q\end{bmatrix}-\mathbf{1}_q\right]\right\} &(18)\\
&\quad+s^{\mathrm T}\tilde M s, &(19)
\end{aligned}
$$

where $s=[\mathbf{1}_p;u_m;-\mathbf{1}_q]$ and $\tilde M=(\alpha-\beta)^2L^2+(3\alpha-2\beta)L$ since $L=L^{\mathrm T}$. Note that the two terms in (15) are negative since $t_p-\mathbf{1}_p<0$, $u_p-\mathbf{1}_p>0$, $t_q+\mathbf{1}_q>0$, $u_q+\mathbf{1}_q<0$, and that the terms in (16), (17), (18) are all non-positive. In order to show that $\Delta V\le 0$, it is sufficient to show that the term (19) is also non-positive, i.e., to show that the matrix $\tilde M$ is negative semidefinite. It is easy to see that the eigenvalues of the matrix $\tilde M$ are $(\alpha-\beta)^2\lambda_i^2+(3\alpha-2\beta)\lambda_i$, $i\in\{1,\dots,N\}$. Hence, $\tilde M$ has one simple eigenvalue at zero with the corresponding right eigenvector $\mathbf{1}$, while all other eigenvalues are $(\alpha-\beta)^2\lambda_i^2+(3\alpha-2\beta)\lambda_i$, $i\in\{2,\dots,N\}$. We shall show that all these eigenvalues are negative. Since $\lambda_i>0$ and $\lambda_i\le\lambda_N$, it is sufficient to show that $\lambda_N<\frac{2\beta-3\alpha}{(\alpha-\beta)^2}$. We note that $\lambda_N<\frac{3}{2\beta}$ from (10). Thus, it is sufficient to show that $\frac{3}{2\beta}<\frac{2\beta-3\alpha}{(\alpha-\beta)^2}$. With some algebra, we see that this is equivalent to show that $\beta>\sqrt{3}\alpha$, which is true given (10).

Hence, we have shown that $\Delta V(y)\le 0$. We then show that $\Delta V(y)=0$ if and only if $y\in M$. To show this, we first note that $\Delta V<0$ if the first two terms (15) are not empty since they are negative. Therefore, $\Delta V=0$ only if these terms are empty. This is the case when $|u_i|\le 1$ for all the agents $i\in\{1,\dots,N\}$, i.e., when the sets S_p and S_q are empty. In this case, we have

$$
\begin{aligned}
\Delta V &= -t^{\mathrm T}t+2t^{\mathrm T}u+2(\alpha-\beta)t^{\mathrm T}Lu+u^{\mathrm T}(\alpha L-I)u\\
&= -\{t-[(\alpha-\beta)L+I_N]u\}^{\mathrm T}\{t-[(\alpha-\beta)L+I_N]u\} \quad (20)\\
&\quad+u^{\mathrm T}\tilde M u.
\end{aligned}
$$

Note that the term in (20) is non-positive and is equal to zero if and only if $t=[(\alpha-\beta)L+I_N]u$.

Recall that $\tilde M$ has exactly one zero eigenvalue with the corresponding right eigenvector $\mathbf{1}$, while all other eigenvalues are negative. Therefore, the term $u^{\mathrm T}\tilde M u$ is also non-positive and it is equal to zero if and only if $Lu=0$.

Hence, we conclude that $\Delta V=0$ if and only if $t=[(\alpha-\beta)L+I]u$ and $Lu=0$. Since $Lu=0$, we obtain $t=u$. On the other hand, from (12), we obtain

$$t=\sigma(u(k+1))=u(k+1)=u-\alpha Lv-\beta L\sigma(u)=u-\alpha Lv.$$

Thus, we see that $Lv=0$ since $\alpha\ne 0$. Thus $v_1=\ldots=v_N$ since the graph is connected. From (11), we obtain $u=-\alpha Lx-\beta Lv=-\alpha Lx$. This together with the fact that $Lu=0$ implies that $\tilde L q=0$, where the relative state $q=[q_2;\dots;q_N]$, $q_i=x_i-x_1$ for $i\in\{2,\dots,N\}$, and $\tilde L$ is given by (5). Since the matrix $\tilde L$ is non-singular, we see that $q=0$, i.e., $x_1=\ldots=x_N$. Therefore $\Delta V(y)=0$ if and only if $y\in M$.

Hence, we have shown that $\Delta V(y)\le 0$ and $\Delta V(y)=0$ if and only if $y\in M$. It then follows from a similar analysis as in

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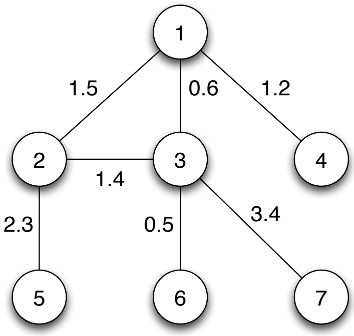


Fig. 1. Network with seven agents

the end of the proof of Theorem 1, that $y(k) \to M$ as $k \to \infty$. Hence, global consensus is achieved. ■

Remark 2 *Note that for the continuous-time case, Meng et al. (2013, Theorem* 5.1*) shows that any linear protocols, which solve the consensus problem for a MAS without input saturation constraints, also solve the global consensus problem for a MAS with input saturation constraints. In this sense, the result of Theorem 2 is different since it requires a more restricted condition given by* (10) *on the feedback gain parameters of the linear protocol* (8)*.*

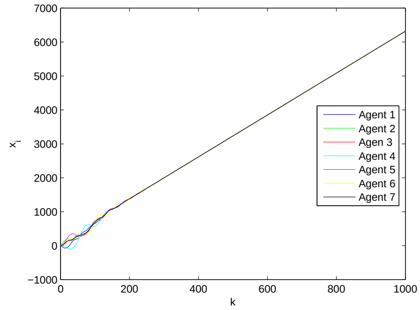
Remark 3 *The Lyapunov function* (14) *has one additional term $\alpha\sigma^T(u)Lv$ compared to the Lyapunov function used in Meng et al. (2013, Theorem* 5.1*) for the continuous-time case. The stability analysis is substantially different from the one in Meng et al. (2013, Theorem* 5.1*).*

6 Illustrative Example

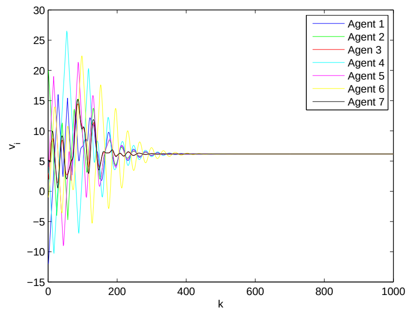
In this section, we illustrate our results on global consensus with input saturation constraints for a network with $N = 7$ double integrators, whose topology is given in Fig. 1. Choose $\alpha = 0.07$ and $\beta = 0.15$ such that the condition (10) is satisfied. The simulation results shown in Fig. 2 confirm the results of Theorem 2.

7 Conclusions and Future Work

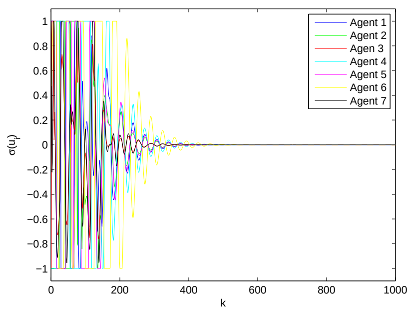
This paper considered the global consensus problem for a MAS of discrete-time identical linear agents, where the agent dynamics are either neutrally stable or a double integrator, with input saturation constraints under fixed undirected network topologies. Extensions to directed topologies and time-varying topologies are currently under investigation. Another interesting topic is to consider heterogeneous networks.



(a) The evolution of x_i



(b) The evolution of v_i



(c) Saturated input $\sigma(u_i)$

Fig. 2. Simulation results with input saturation constraint

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