FEL 3330: Networked and Multi-Agent Control Systems
Lecture 1 compendium:
Essentials of Algebraic Graph Theory

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An undirected graph $G = \{V,E\}$ consists of a set of vertices $V = \{1,\ldots,N\}$ and a set of edges, $E = \{(i,j) \in V \times V\}$ containing pairs of vertices.

For an undirected graph $G = \{V,E\}$ with $N$ vertices $V = \{1,\ldots,N\}$ and edges $E \subset V \times V$, the adjacency matrix $A = A(G) = (a_{ij})$ is the $N \times N$ matrix given by $a_{ij} = 1$, if $(i,j) \in E$, and $a_{ij} = 0$, otherwise. If $(i,j) \in E$, then $i,j$ are adjacent. A path of length $r$ from $i$ to $j$ is a sequence of $r+1$ distinct vertices starting with $i$ and ending with $j$ such that consecutive vertices are adjacent. For $i = j$, this path is a cycle. If there is a path between any two vertices of $G$, then $G$ is connected. A connected graph is a tree if it contains no cycles. The degree $d_i$ of vertex $i$ is given by $d_i = \sum_j a_{ij}$. Let $\Delta = \text{diag}(d_1,\ldots,d_N)$. The Laplacian of $G$ is the symmetric positive semidefinite matrix

$$L = \Delta - A$$

For a connected graph, $L$ has a simple zero eigenvalue with the corresponding eigenvector $1 = [1,\ldots,1]^T$. This will be formally stated in Theorem 2 below. We denote by $0 = \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_N(G)$ the eigenvalues of $L$.

Two important relations resulting from the symmetry of $L$ and the variational characterization of the eigenvalues of symmetric matrices are as follows:

$$\lambda_2(G) = \min_{x \perp 1, \|x\| = 1} x^T L x$$

and

$$\lambda_N(G) = \max_{\|x\| = 1} x^T L x$$

An orientation on $G$ is the assignment of a direction to each edge. The incidence matrix $B = B(G) = (b_{ij})$ of an oriented graph is the $\{0, \pm 1\}$-matrix with rows and columns indexed by the vertices and edges of $G$, respectively, such that $b_{ij} = 1$ if the vertex $i$ is the head of the edge $j$, $b_{ij} = -1$ if the vertex $i$ is the tail of the edge $j$, and $b_{ij} = 0$ otherwise. It can be shown that $L = BB^T$, and this is independent of the choice of orientation.

If $G$ contains cycles, the edges of each cycle have a direction, where each edge is directed towards its successor according to the cyclic order. A cycle $C$ is represented by a vector $v_C$ with $M = |E|$ elements. For each edge, the corresponding element of $v_C$ is equal to 1 if the direction of the edge with respect to $C$ coincides with the orientation assigned to the graph for defining $B$, and $-1$, if the direction with respect to $C$ is opposite to the orientation. The elements corresponding to edges not in $C$ are zero. The cycle space of $G$ is the subspace spanned by vectors representing cycles in $G$ [2].

Let $x = [x_1,\ldots,x_N]^T$, where $x_i$ is a real scalar variable assigned to vertex $i$ of $G$. Denote by $\bar{x}$ the $M$-dimensional stack vector of relative differences of pairs of agents that form an edge in $G$, where $M = |E|$ is the number of edges, in agreement with a defined orientation. In particular, denoting by $e_i = (h_i,t_i) \in E$, $i = 1,\ldots,M$, the edges of $G$, where $h_i,t_i$ the head and tail of $e_i$ respectively, we denote $\bar{x}_{e_i} = x_{h_i} - x_{t_i}$. The vector $\bar{x}$ is given by $\bar{x} = [\bar{x}_{e_1},\ldots,\bar{x}_{e_M}]^T$. It is easy to verify that $Lx = B\bar{x}$ and $\bar{x} = B^T x$. For $\bar{x} = 0$ we have that $Lx = 0.$
Lemma 1 If $G$ is a tree, then $B^T B$ is positive definite.

Proof: For any $y \in \mathbb{R}^M$, we have $y^T B^T B y = |B y|^2$ and hence $y^T B^T B y > 0$ if and only if $B y \neq 0$, i.e., the matrix $B$ has empty null space. For a connected graph, the cycle space of the graph coincides with the null space of $B$ (Lemma 3.2 in [2]). Thus, for $G$ with no cycles, zero is not an eigenvalue of $B$. This implies that $B^T B$ is positive definite. ♦

The following theorem also holds:

Theorem 2 The graph $G$ is connected if and only if $\lambda_2(G) > 0$.

Proof: (Sketch, full proof at [1]). We can show that $B^T$ and $L$ have the same null space, so that it suffices to show that the null space of $B^T$ has dimension one, or that its rank is $n - 1$ when $G$ is connected. Suppose that $z$ is a vector such that $z^T B = 0$. This then implies that for $(i, j) \in E$, then $z_i - z_j = 0$. Since $G$ is connected, this means $z \in \text{span}(\{1\})$. Thus the rank of the null space of $B^T$ and thus, $L$, is one, which implies that the multiplicity of $\lambda_1(G) = 0$ is one. ♦
Bibliography
