Multi-agent average consensus control with prescribed performance guarantees

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Abstract— **This work proposes a distributed control scheme for the state agreement problem which can guarantee prescribed performance for the system transient. In particular, i) we consider a set of agents that can exchange information according to a static communication graph, ii) we a priori define time-dependent constraints at the edge's space (errors between agents that exchange information) and iii) we design a distributed controller to guarantee that the errors between the neighboring agents do not violate the constraints. Following this technique the contributions are twofold: a) the convergence rate of the system and the communication structure of the agents' network which are strictly connected can be decoupled, and b) the connectivity properties of the initially formed communication graph are rendered invariant by appropriately designing the prescribed performance bounds. It is also shown how the structure and the parameters of the prescribed performance controller can be chosen in case of connected tree graphs and connected graphs with cycles. Simulation results validate the theoretically proven findings while enlightening the merit of the proposed prescribed performance agreement protocol as compared to the linear one.**

I. Introduction

Distributed multi-agent control is a popular research domain owing to the current increase of computational resources and the broad domain of applications including cooperative control of unmanned air vehicles, formation control of mobile robots, distributed sensor networks, power networks, attitude alignment of a group of satellites, etc. Several results have appeared recently involving consensus algorithms [1], [2] and formation control [3]–[5]. The agreement protocol is the underlying framework in the majority of the proposed works enabling the control of a group of agents that do not have access to global information (e.g. absolute positions) but they can only obtain local information (e.g. relative positions).

In this paper we consider the case of additional specifications during the transient response of the multi-agent system. Transient response specifications in multi-agent systems have been dealt with in the context of collision avoidance [6], and connectivity maintenance [2], [7] where constraints on the state space evolution of pairs of agents are imposed in the distributed control design. Recently, a distributed algorithm for consensus of multiple agents in presence of convex state constraints on individual agent state has been considered in [8]; in particular, the consensus protocol has been enriched by an auxiliary variable which utilizes a logarithmic barrier function to form a convex potential. In contrast to the previous works, the transient specifications considered in this work involve time-dependent constraints on the edge's space and enable the automated tuning of the agents' convergence rate while ensuring connectivity maintenance. Notice that the agreement protocol convergence rate is mainly dependent on the connectivity structure of the system; in particular, the convergence rate is dictated by the smallest positive eigenvalue of the graph Laplacian. This dependance implies that the agents cannot a priori be aware of the time required to reach consensus, since individual agents cannot have access to the centralized information of the structure of the underlying network.

In particular, we introduce prescribed transient specifications within the agreement protocol by following the framework of prescribed performance control introduced in [9]. The result is a nonlinear time-dependent agreement protocol which is designed to preserve the basic properties of the linear agreement protocol such as the convergence to the invariant centroid but enhanced with additional important properties: i) time-dependent transient bounds for the edges' responses, and ii) a convergence rate which is independent of the underlying communication graph. The analysis of this work considers different type of communication graphs such as spanning trees and connected graphs with cycles and examines how the structure of the communication graph affects the prescribed performance specifications. The paper is organized as follows: Section II provides preliminary material about graph theory and prescribed performance control framework. In Section II.B the stability and the asymptotic convergence properties of a simple first order prescribed performance driven system are studied in order to be used in Section III for solving the problem of prescribed performance consensus control for a multi-agent system. Section IV consists of simulation examples while in Section V, the final outcome of this work as well as future research directions are discussed.

II. Preliminaries

A. Graph Theory

We consider N agents and that agent i can communicate only with agents that belong to its communication set denoted by $\mathcal{N}_i \in \mathcal{N}$. Inter-agent communication can be represented by a communication graph:

Definition 1: The communication graph $G = (V, E)$ is an undirected graph that consists of a set of vertices $V = \mathcal{N} = \{1, \dots, N\}$ indexed by the team members, and a set

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of edges, $E = \{(i,j) \in V \times V | i \in \mathcal{N}_j\}$ containing pairs of vertices which can exchange information.

For a graph G with N vertices the *adjacency matrix* $A = A(G) = [a_{ij}] \in \Re^{N\times N}$ is a matrix with unit and zero entries. If there is an edge connecting two vertices i, j i.e. $(i,j) \in E$ then i, j are called adjacent and $a_{ij} = a_{ji} = 1$; otherwise $a_{ij} = a_{ji} = 0$. A path of length r from a vertex i to a vertex j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of the graph G, then G is called *connected*; otherwise, it is disconnected. The degree d_i of a vertex i is defined as the number of its neighboring vertices; it can be easily calculated by $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$. The Laplacian of $G \in \Re^{N\times N}$ is the symmetric matrix $L = \Delta - A$, with $\Delta = \text{diag}\{d_i\} \in \Re^{N\times N}$ being the degree matrix.

For a connected graph, L is positive semidefinite and has a single zero eigenvalue with the corresponding eigenvector $\mathbf{1} = [1, \cdots, 1]^\top \in \Re^N$. In particular, the eigenvalues of L when G is connected are: $0 = \lambda_1(L) \le \lambda_2(L) \le \cdots \le \lambda_N(L)$ where $\lambda_2(L) = \min_{x\perp\mathbf{1}, \|x\|=1} x^\top Lx$ and $\lambda_N(L) = \max_{\|x\|=1} x^\top Lx$. By assigning an orientation on the direction to each edge of G we can define the incidence matrix of G denoted by $B(G)$. In particular, $B = B(G) = [b_{ij}] \in \Re^{N\times M}$ with $M = |E|$ being the number of edges is the matrix with entries $\{0, \pm 1\}$ with rows and columns indexed by the vertices and edges of G respectively. Specifically, $b_{ij} = 1$ if the vertex i is the head of the edge j, $b_{ij} = -1$ if the vertex i is the tail of the edge j, and $b_{ij} = 0$ otherwise. We have $L = BB^\top$ independently of the choice of the orientation. Notice also that the nullspace of $B^\top$ is spanned by the vector $\mathbf{1}$.

If G contains cycles, the edges of each cycle have a direction, where each edge is directed towards its successor according to the cyclic order. A cycle C is represented by a vector v_C with $M = |E|$ elements. For each edge, the corresponding element of v_C is equal to 1 if the direction of the edge with respect to C coincides with the orientation assigned to the graph for defining B, and -1, if the direction with respect to C is opposite to the orientation. The elements corresponding to edges not in C are zero. The cycle space of G is the subspace spanned by vectors representing cycles in G. For a connected graph, the cycle space of the graph coincides with the null space of B. For tree graphs the cycle space and subsequently the null space of B are empty implying that the edge Laplacian $L_E \triangleq B^\top B$ is a positive definite matrix.

Let $x = [x_1, \cdots, x_N]^\top$, where x_i is a real scalar variable assigned to vertex i of G. Denote by $\bar{x}$ the M-dimensional stack vector of relative differences of pairs of agents that form an edge in G, compatible with the assigned orientation. In particular, the elements of vector $\bar{x}$ are defined by $\bar{x}_k \triangleq x_{ij} = x_i - x_j \in E, k = 1, \cdots, M$, where x_i, x_j is the head and the tail of $\bar{x}_k$ respectively. It is easy to verify that $Lx = B\bar{x}$ and $\bar{x} = B^\top x$. For $\bar{x} = 0$ we have that $Lx = 0$.

B. Prescribed Performance Control

1) Definitions: This section summarizes preliminary knowledge on prescribed performance control [9]. Prescribed performance is achieved if each element y_i, $i = 1, \cdots, n$, of the system output $y \in \Re^n$ (which can also be the tracking error) evolves within a predefined region being mathematically described $\forall t \ge 0$ by the following inequalities:

$$-M_i\rho_i(t) < y_i(t) < \rho_i(t) \quad \text{in case of} \quad y_{0i} \ge 0 \tag{1}$$
$$-\rho_i(t) < y_i(t) < M_i\rho_i(t) \quad \text{in case of} \quad y_{0i} \le 0 \tag{2}$$

for $i = 1, \cdots, n$ where $0 \le M_i \le 1$, $i = 1, \cdots, n$, $y_{0i} \triangleq y_i(0)$, $i = 1, \cdots, n$, and $\rho_i(t)$, $i = 1, \cdots, n$ are smooth, bounded, strictly positive functions of time satisfying $\limsup_{t\to+\infty} \rho_i(t) > 0$, $i = 1, \cdots, n$ called performance functions. Notice that $\rho_i(t)$ and $M_i\rho_i(t)$ define the performance bounds within which $y_i(t)$ should ideally evolve. According to (1) or (2) the set of performance bounds which will be employed for each $y_i(t)$ is associated with the sign of y_{0i}; either (1) or (2) can be used in case of $y_{0i} = 0$.

By normalizing $y_i(t)$ with respect to the performance function $\rho_i(t)$ we can define the modulated output $\widehat{y}(t) \in \Re^n$ with elements $\widehat{y}_i(t)$, as well as the corresponding prescribed performance regions D_{yi}:

$$\widehat{y}_i(t) = \frac{y_i(t)}{\rho_i(t)} \tag{3}$$

$$D_{yi} \triangleq \{\widehat{y}_i(t) : \widehat{y}_i(t) \in (-M_i, 1)\} \quad \text{for} \quad y_{0i} \ge 0 \tag{4}$$
$$D_{yi} \triangleq \{\widehat{y}_i(t) : \widehat{y}_i(t) \in (-1, M_i)\} \quad \text{for} \quad y_{0i} \le 0 \tag{5}$$

Notice that the regions D_{yi} given by (4) and (5) are equivalent to the time-dependent prescribed performance bounds (1) and (2) respectively. For notation convenience, the argument t is dropped out from $y_i(t)$, $\widehat{y}_i(t)$ and the corresponding vectors. Subsequently, the components $\varepsilon_i(\widehat{y}_i)$ of the transformed error $\varepsilon(\widehat{y}) \in \Re^n$ are defined as follows:

$$\varepsilon_i(\widehat{y}_i) \triangleq T_i(\widehat{y}_i) \tag{6}$$

where the transformations $T_i(\cdot)$, $i = 1, \cdots, n$ define increasing bijective mappings of the performance domain:

$$T_i : D_{yi} \to \Re \tag{7}$$

for $i = 1, \cdots, n$, where D_{yi} is the domain of the transformation function T_i and is defined by (4) or (5). Differentiating (6) with respect to time we obtain:

$$\dot{\varepsilon}_i(\widehat{y}_i) = \mathcal{J}_{Ti}(\widehat{y}_i, t)[\dot{y}_i + \alpha_i(t)y_i] \tag{8}$$

where $\mathcal{J}_{Ti}(\widehat{y}_i, t)$ and $\alpha_i(t)$ are given by:

$$\mathcal{J}_{Ti}(\widehat{y}_i, t) \triangleq \frac{\partial T_i(\widehat{y}_i)}{\partial \widehat{y}_i}\frac{1}{\rho_i(t)} > 0 \tag{9}$$

$$\alpha_i(t) \triangleq -\frac{\dot{\rho}_i(t)}{\rho_i(t)} \tag{10}$$

Clearly, the domain of the vector function $\varepsilon(\widehat{y})$ can be defined by the Cartesian product of the open sets D_{yi},

$i = 1, \cdots, n$, i.e.

$$D_y \triangleq D_{y1} \times D_{y2} \times \cdots \times D_{yn} \quad (11)$$

with boundary ∂D_y.

2) First order nonlinear system: Proof of boundedness and convergence: In this section we examine the stability and convergence properties of a first-order MIMO nonlinear system of the form $\dot{y} = \mathcal{A}(y)u$, $y, u \in \Re^n$, $\mathcal{A}(y) \in \Re^{n\times n}$ driven by a prescribed performance regulator $u = -\mathcal{J}_T(\widehat{y}, t)\varepsilon(\widehat{y})$. Without loss of generality we can consider that $\rho(t) \triangleq \rho_i(t)$ and subsequently $\alpha(t) \triangleq \alpha_i(t)$, $\forall i \in \{1, \cdots, n\}$.

Theorem 1: Consider the system:

$$\dot{y} = -\mathcal{A}(y)\mathcal{J}_T(\widehat{y}, t)\varepsilon(\widehat{y}) \quad (12)$$

where $\mathcal{A}(y)$ is a nonlinear, positive definite matrix $\forall \widehat{y} \in D_y$, $\varepsilon(\widehat{y})$ is the transformed error defined in (6) and $\mathcal{J}_T(\widehat{y}, t) = \text{diag}\left[\mathcal{J}_{Ti}(\widehat{y}_i, t)\right]$ is the normalized Jacobian of the transformed error. If $y(0)$ is within the performance bounds (1) or (2) which implies $\widehat{y}(0) \in D_y$ then a) the solution $y(t)$ will respect the performance bounds $\forall t$ and b) if additionally the prescribed performance function $\rho(t)$ is designed with the property $\lim_{t\to+\infty} \dot{\rho}(t) = 0$ then y will asymptotically converge to a constant value which is zero in case of $T_i(0) = 0$.

Proof: Adding $\alpha(t)y \triangleq -\dot{\rho}(t)\widehat{y}$ in both sides of (12) and subsequently multiplying both sides with $\mathcal{J}_T(\widehat{y}, t)$ and substituting (8) we get:

$$\dot{\varepsilon}(\widehat{y}, t) = -\mathcal{J}_T(\widehat{y}, t)\mathcal{A}(y)\mathcal{J}_T(\widehat{y}, t)\varepsilon(\widehat{y}) - \mathcal{J}_T(\widehat{y}, t)\dot{\rho}(t)\widehat{y} \quad (13)$$

Consider the potential function $V : D_y \to \Re$,

$$V(\widehat{y}) = \frac{1}{2}\varepsilon(\widehat{y})^\top \varepsilon(\widehat{y}) \quad (14)$$

By differentiating (14) along the system trajectories (13), it can be easily shown that for some $\xi < \lambda_{\min}\left[\mathcal{A}(y)\right]$, the following inequality holds:

$$\dot{V}(\widehat{y}, t) \leq -\lambda V(\widehat{y}) + \mu(t) \quad (15)$$

where

$$\lambda = 2\left\{\lambda_{\min}\left[\mathcal{A}(y)\right] - \xi\right\} \sup_{t\geq 0} \rho(t) \min_{\hat{y}\in D_x} \left(\frac{\partial\varepsilon(\widehat{y})}{\partial\widehat{y}}\right) \quad (16)$$

$$\mu(t) = \frac{|\dot{\rho}(t)|^2 \|\widehat{y}\|^2}{4\xi} \quad (17)$$

Notice that we can also define an upper bound $\bar{\mu}$ of $\mu(t)$, $\forall\widehat{y} \in D_y$ and $t \in \Re^+$, i.e. $\bar{\mu} \geq \frac{|\sup_t \dot{\rho}(t)|^2 n}{4\xi}$.

a) We can now introduce the following potential (similar to the Zubov's Theorem)

$$\mathcal{V}(\widehat{y}) = 1 - e^{-V(\widehat{y})} \quad (18)$$

having the following properties: (i) $\mathcal{V}(0) = 0$ (ii) $0 < \mathcal{V}(\widehat{y}) < 1$, $\forall\widehat{y} \in D_y - \{0\}$ (iii) $\mathcal{V}(\widehat{y}) \to 1$ as $\widehat{y} \to \partial D_y$. Differentiating (18) we get:

$$\dot{\mathcal{V}}(\widehat{y}, t) = \dot{V}(\widehat{y}, t)\left[1 - \mathcal{V}(\widehat{y})\right] \quad (19)$$

Substituting (18) and (15) in (19) we can bound $\dot{\mathcal{V}}(\widehat{y}, t)$ as follows:

$$\dot{\mathcal{V}}(\widehat{y}, t) \leq -\lambda \ln\left[\frac{e^{-\frac{\mu(t)}{\lambda}}}{1 - \mathcal{V}(\widehat{y}, t)}\right]\left[1 - \mathcal{V}(\widehat{y}, t)\right] \quad (20)$$

Next, define the region $\Omega_b = \left\{\widehat{y} \in D_y : \mathcal{V}(\widehat{y}) \leq 1 - e^{-\frac{\bar{\mu}}{\lambda}}\right\} \subset D$. Clearly, $\dot{\mathcal{V}}(\widehat{y}, t) \leq 0$ for $\widehat{y} \notin \Omega_b$.

Assuming that the initial errors are defined within the prescribed performance bounds, i.e., $y(0) \in D_y$, then $\mathcal{V}(\widehat{y}(0)) < 1$. Let $\sigma \triangleq \mathcal{V}(\widehat{y}(0))$ and define the region $\Omega_\sigma = \{\widehat{y} \in D_y : \mathcal{V}(\widehat{y}) \leq \sigma\} \subset D_y$. We distinguish two cases (i) $\sigma < 1 - e^{-\frac{\bar{\mu}}{\lambda}}$: In this case $\widehat{y} \in \Omega_b$, $\forall t \geq 0$ since $\dot{\mathcal{V}}(\widehat{x}, y) \leq 0$ outside Ω_b. (ii) $\sigma \geq 1 - e^{-\frac{\bar{\mu}}{\lambda}}$: In this case $\Omega_\sigma \supset \Omega_b$ and $\dot{\mathcal{V}}(\widehat{y}, t) \leq 0$ in Ω_σ/Ω_b; hence $\mathcal{V}(\widehat{y}) \to \Omega_b$. Thus, starting within Ω_σ, the potential function $\mathcal{V}(\widehat{y})$ remains less than 1 in any case. Consequently, the modulated error $\widehat{y}$ evolves within a closed set $\bar{D}_y$ such that $\bar{D}_y \subset D_y$, ensuring that $\widehat{y}$ does not even approach the boundary ∂D_y and hence $\varepsilon(\widehat{y})$ is bounded. Hence each component of the state y_i evolves within the predefined region (1) or (2), solely defined by the predefined performance function $\rho(t)$ and the overshoot indices M_i. Since $\varepsilon(\widehat{y})$ is bounded and $\rho(t) \neq 0$, $\forall t$, $\mathcal{J}_T(\widehat{y}, t)$ is bounded.

b) In order to prove the asymptotic convergence of y to a constant value given that $\rho(t) > 0$ and $\lim_{t\to+\infty} \dot{\rho}(t) = 0$, we omit the argument $\widehat{y}$ from $\varepsilon(\widehat{y})$ and we consider the transformed system (13) with state ε and input $\dot{\rho}(t)$ as well as the previously used potential $V(\varepsilon) = \frac{1}{2}\|\varepsilon\|^2$. Let θ being a positive constant which satisfies $\theta < \lambda$, then the derivative of $V(\varepsilon)$ with respect to time (15) satisfies the following inequality:

$$\frac{d}{dt}\left(\tfrac{1}{2}\|\varepsilon\|^2\right) \leq -(\lambda - \theta)\|\varepsilon\|^2, \quad \forall\|\varepsilon\| \leq \tfrac{|\dot{\rho}(t)|}{2}\sqrt{\tfrac{n}{\theta\xi}} \quad (21)$$

Using Theorem 4.19 [10] it is directly proved that the system (13) with state ε and input $\dot{\rho}(t)$ is input-to-state stable and thus if $\dot{\rho}(t)$ converges to zero as $t \to +\infty$, so does ε which implies $\lim_{t\to+\infty} y_i(t) = \rho_{\infty i} T_i^{-1}(0)$. In case of $T_i(0) = 0$, $\lim_{t\to+\infty} \varepsilon(t) = 0$ implies that $\widehat{y}(t)$ and subsequently $y(t)$ converges to zero as $t \to +\infty$. ∎

Remark 1: In this section we have proved that the system output respects the performance bounds by proving the uniformly ultimate boundedness of the transformed error, like in [9], but for a simpler type of system/controller and using a very simple prescribed performance control input. Furthermore, by using input-to-state stability concept we have proved asymptotic convergence results for the system output response (which have not been considered in [9]). The analysis for a simple system controlled by a simple prescribed performance controller as well as the asymptotic convergence results are useful for the consensus control problem, tackled in this work.

III. Prescribed Performance Agreement Protocol

For simplicity, we consider only one of the coordinates for each agent dynamics. We also consider that the system obeys to the single integrator dynamics:

$$\dot{x} = u \tag{22}$$

with x_i, u_i being the ith elements of the vectors x, $u \in \Re^N$ respectively. In order to propose a distributed prescribed performance algorithm for the consensus problem, we will modulate and transform the relative errors $x_{ij} = x_i - x_j$ by the performance functions denoted by $\rho_{ij}(t)$ using the transformation functions denoted by $T_{ij}(\cdot)$ and leading to the relative transformed errors $\varepsilon_{ij}(x_{ij})$. M_{ij} denotes the corresponding overshoot index.

We assume that the communication graph is static, i.e., that $\mathcal{N}_i$ do not vary over time, and we propose the following time-dependent agreement protocol:

$$u_i = -\sum_{j\in\mathcal{N}_i} \mathcal{J}_{Tij}(\widehat{x}_{ij}, t)\varepsilon_{ij}(\widehat{x}_{ij}) \tag{23}$$

If the connectivity depends on the inter-agent distances, the prescribed performance function should be designed so that the agents are connected over time.

The average of the multi-agent system state remains invariant when the graph is connected and $\mathcal{J}_{Tij}(\widehat{x}_{ij}, t)\varepsilon_{ij}(\widehat{x}_{ij}) = -\mathcal{J}_{Tji}(\widehat{x}_{ij}, t)\varepsilon_{ji}(\widehat{x}_{ji})$. The latter is satisfied by considering some specifications regarding the performance functions, the overshoot indices and the transformation functions which are used by neighboring agents. In particular, we choose $\rho_{ij}(t) = \rho_{ji}(t)$, $M_{ij} = M_{ji}$ and $T_{ji}(\widehat{x}_{ji}) = -T_{ij}(-\widehat{x}_{ij})$; this choice practically means that neighboring agents exchange information about their prescribed performance parameters and functions. Alternatively, all the agents may be tuned initially in order to use the same prescribed performance function, overshoot index and transformation.

Substituting the control input (23) in to the linear dynamic model (22) we get:

$$\dot{x} = -B\mathcal{J}_T(\widehat{\bar{x}}, t)\varepsilon(\widehat{\bar{x}}) \tag{24}$$

The closed loop system dynamics can be expressed in the edges-space by using $\bar{x} = B^\top x$ as follows:

$$\dot{\bar{x}} = -B^\top B\mathcal{J}_T(\widehat{\bar{x}}, t)\varepsilon(\widehat{\bar{x}}) \tag{25}$$

Theorem 2: If the graph describing the agents communication is a spanning tree, then i) the prescribed performance agreement protocol (23) applied to the single integrator dynamics (22) ensures that the relative errors x_{ij} will evolve within the prescribed performance bounds defined by $\rho_{ij}(t)$ and M_{ij}, and ii) if additionally the prescribed performance functions $\rho_{ij}(t)$ are designed with the property $\lim_{t\to+\infty} \dot{\rho}_{ij}(t) = 0$, the relative errors x_{ij} will asymptotically converge to constant values which are zero in case of $T_{ij}(0) = 0$.

Proof: When the graph describing the agents communication is a spanning tree then the matrix $B^\top B$ is positive definite. Hence, by directly applying Theorem 1 we can prove the results of Theorem 2. ■

It is clear that we cannot use the potential function $V(\widehat{\bar{x}})$ given by (14) if the communication graph contains cycles, since $B^\top B$ is not positive definite. However we can use the following potential function in order to extend the results in case of communication graphs with cycles:

$$V_e(\widehat{\bar{x}}, x) = \frac{1}{2}\varepsilon(\widehat{\bar{x}})^\top \varepsilon(\widehat{\bar{x}}) + \frac{\gamma}{2}x^\top x \tag{26}$$

with γ being an appropriately chosen constant, and to state the following theorem:

Theorem 3: If the graph describing the agents communication is connected, then the prescribed performance agreement protocol (23) with $T_{ij}(0) = 0$ applied at the single integrator dynamics (22) ensures that the relative errors x_{ij} will evolve within the prescribed performance bounds defined by $\rho_{ij}(t)$ and $M_{ij} \neq 0$, and they converge to zero for any strictly positive prescribed performance function $\rho_{ij}(t)$ with bounded derivative.

Proof: Differentiating (26) and substituting (25) and (24), we get:

$$\begin{aligned} \dot{V}_e(\widehat{\bar{x}}, x) = &- \varepsilon(\widehat{\bar{x}})^\top \mathcal{J}_T(\widehat{\bar{x}}, t)B^\top B\mathcal{J}_T(\widehat{\bar{x}}, t)\varepsilon(\widehat{\bar{x}}) \\ &- \varepsilon(\widehat{\bar{x}})^\top \mathcal{J}_T(\widehat{\bar{x}}, t)\dot{\mathrm{P}}(t)\widehat{\bar{x}} \\ &- \gamma x^\top B\mathcal{J}_T(\widehat{\bar{x}}, t)\varepsilon(\widehat{\bar{x}}) \end{aligned} \tag{27}$$

Substituting $\widehat{\bar{x}} = \mathrm{P}(t)^{-1}B^\top x$ with $\mathrm{P}(t) \in \Re^{M\times M}$ being a diagonal matrix with diagonal entries $\rho_{ij}(t)$ and taking into consideration that for connected graphs $B^\top B$ is positive semidefinite we arrive to the following inequality:

$$\dot{V}_e(\widehat{\bar{x}}, x) \leq -\widehat{\bar{x}}^\top [\gamma I_M - \mathrm{A}(t)]\frac{\partial\varepsilon\left(\widehat{\bar{x}}\right)}{\partial\widehat{\bar{x}}}\varepsilon(\widehat{\bar{x}}) \tag{28}$$

with $\mathrm{A}(t)$ being diagonal matrix with bounded entries $-\frac{\dot{\rho}_{ij}(t)}{\rho_{ij}(t)}$ (since $\rho_{ij}(t)$ is strictly positive with bounded derivative) i.e. $\sup_t[|\mathrm{A}(t)|] \leq \bar{\alpha}$, for some constant $\bar{\alpha}$. By setting $\gamma := \theta + \bar{\alpha}$, with θ being arbitrarily positive constant we get:

$$\dot{V}_e(\widehat{\bar{x}}, x) \leq -\theta\widehat{\bar{x}}^\top \frac{\partial\varepsilon\left(\widehat{\bar{x}}\right)}{\partial\widehat{\bar{x}}}\varepsilon(\widehat{\bar{x}}) \tag{29}$$

Clearly, (29) implies that $V_e(\widehat{\bar{x}}, x) < V_e(\widehat{\bar{x}}(0), x(0))$ which in turns implies that $x, \varepsilon(\widehat{\bar{x}}) \in \mathcal{L}_\infty$ given that $V_e(\widehat{\bar{x}}(0), x(0))$ is finite. It is obvious that if $\bar{x}(0)$ is chosen within the performance bounds then $V_e(\widehat{\bar{x}}(0), x(0))$ is finite, $\varepsilon(\widehat{\bar{x}}) \in \mathcal{L}_\infty$ and subsequently $\bar{x}(t)$ evolves within the prescribed performance bounds $\forall t$. By applying Barbalat's Lemma we can also prove that $\bar{x}(t) \to 0$. In particular we calculate the derivative of $\dot{V}_e(\widehat{\bar{x}}, x)$ and we prove that it is bounded based on the boundedness of $\varepsilon(\widehat{\bar{x}})$ and $\dot{\varepsilon}(\widehat{\bar{x}})$. The boundedness of $\ddot{V}_e(\widehat{\bar{x}}, x)$ implies the uniform continuity of $\dot{V}_e(\widehat{\bar{x}}, x)$ which in turn implies that $\dot{V}_e(\widehat{\bar{x}}, x) \to 0$. For $T_{ij}(0) = 0$, $\dot{V}_e(\widehat{\bar{x}}, x) \to 0$ implies $\bar{x} \to 0$. ■

Theorems 2 and 3 consider only the edge space and thus they do not state any result regarding the convergence of the agents' absolute position. Our study and the corresponding results regarding the absolute position of the agents are summarized in the following Lemma:

Lemma 1: The prescribed performance agreement protocol (23) with the following characteristics: $\rho_{ij}(t) = \rho_{ji}(t)$, $M_{ij} = M_{ji}$ and $T_{ji}(\widehat{x}_{ji}) = -T_{ij}(-\widehat{x}_{ij})$ applied to the single integrator dynamics (22) ensures i) the convergence of the agents' absolute position to some constant points around the agents' centroid, for $T_{ji}(0) \neq 0$ and $\lim_{t\to+\infty} \dot{\rho}_{ij}(t) = 0$ ii) the convergence of the agents to the centroid for $T_{ij}(0) = 0$.

Proof: The prescribed performance agreement protocol with symmetric prescribed performance characteristics implies that the centroid of the system states evaluated for any $t \geq 0$, remains constant during the motion of the agreement dynamics i.e. $\frac{d}{dt}\left(\sum_{i\in\mathcal{N}} x_i\right) = 0$ or equivalently $\mathbf{1}^\top x = \mathbf{1}^\top x(0)$. Given that $\lim_{t\to+\infty} \dot{\rho}_{ij}(t) = 0$, then the part (ii) of Theorem 2 for $T_{ij}(0) \neq 0$ implies $\lim_{t\to+\infty} x_{ij}(t) = \rho_{ij\infty} T_{ij}^{-1}(0)$. Hence the error between the system equilibrium x_∞ and the invariant centroid of the system for a spanning tree communication graph is given by:

$$x_\infty - \frac{1}{N}\sum_{i\in\mathcal{N}} x_i(0)\mathbf{1} = BL_E^{-1}\bar{x}_\infty \qquad (30)$$

where $\bar{x}_\infty$ is the stuck vector of $\rho_{ij\infty}T_{ij}^{-1}(0)$. Since $T_{ij}(0) \neq 0$, $\bar{x}_\infty$ can be reduced by reducing $\rho_{ij\infty}$.

For $T_{ij}(0) = 0$, Theorem 3 implies that the relative states converge to zero for any strictly positive $\rho_{ij}(t)$ with bounded derivative. By combining the aforementioned convergence result with the invariance of system states centroid we can prove that the system states converge to the centroid of the multi-agent system as follows:

$$\begin{bmatrix} I_N & -\mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}\begin{bmatrix} x \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{1}^\top x(0) \end{bmatrix} \qquad (31)$$

Equation (31) clearly implies $c = \frac{1}{N}\sum_{i\in\mathcal{N}} x_i(0)$ which subsequently yields $x_i \to \frac{1}{N}\sum_{i\in\mathcal{N}} x_i(0), \forall i \in \mathcal{N}$. ∎

IV. Simulation Examples

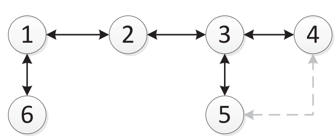

Fig. 1: Spanning tree communication graph: neighboring agents are connected with solid arrows (CS-2). Connected communication graph with one cycle: neighboring agents are connected with solid and dashed arrows (CS-3)

We consider six 2 dofs agents on a planar surface. Let $p_i = [\mathrm{x}_i \;\; \mathrm{y}_i]^\top,\ i \in \{1, \cdots, 6\}$ describe the position of each agent, v_{av} denote the average velocity of the agents, i.e.

$$v_{\text{av}} \triangleq \frac{1}{6}\left(\sum_{i=1}^{i=6} \|\dot{p}_i\|\right)$$

and d denote the sum of distances between the agents and their centroid i.e.

$$d \triangleq \sum_{i=1}^{i=6} \|p_i - p_c\|$$

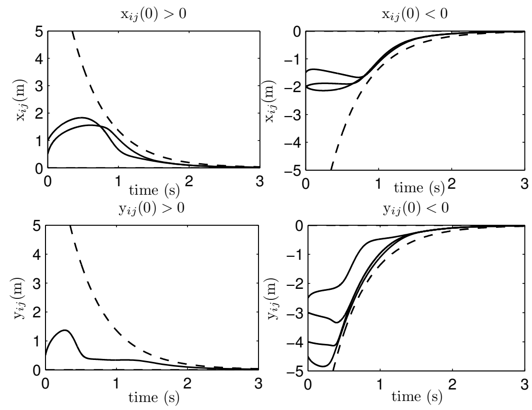

Fig. 2: Position errors (solid lines) and performance bounds (dashed lines) in case of CS-2 and PPC-1

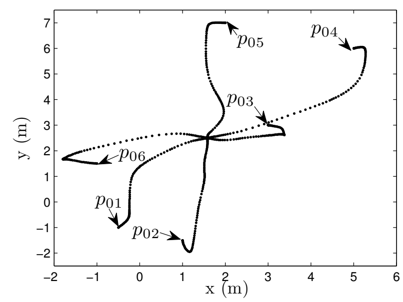

Fig. 3: Trace of the robots on $\mathrm{x}-\mathrm{y}$ plane in case of CS-2 and PPC-1

where $p_c \triangleq \frac{1}{6}\sum_{i=1}^{i=6} p_i$. The initial position p_{0i} of the agents are: $p_{01} = [-0.5 \;\; -1]^\top$, $p_{02} = [1 \;\; -1.5]^\top$, $p_{03} = [3 \;\; 3]^\top$, $p_{04} = [5 \;\; 6]^\top$, $p_{05} = [2 \;\; 7]^\top$, $p_{06} = [-1 \;\; 1.5]^\top$. We will consider three different communication scenarios:

- **CS-1** complete communication graph,
- **CS-2** spanning tree (Fig. 1 - solid arrows),
- **CS-3** connected graph with one cycle (Fig. 1 - solid and dashed arrows)

and two different types of prescribed performance controllers:

PPC-1 $M_i = 0$ and $T_i(0) \neq 0$ in particular:

$$T_i(\widehat{y}) = \begin{cases} \ln\left(\frac{M_i+\widehat{y}}{1-\widehat{y}}\right), & y(0) > 0 \\ \ln\left(\frac{1+\widehat{y}}{M_i-\widehat{y}}\right), & y(0) < 0 \end{cases}$$

PPC-2 $M_i = 0.1$ and $T_i(0) = 0$ in particular:

$$T_i(\widehat{y}) = \begin{cases} \ln\left(\frac{M_i+\widehat{y}}{M_i(1-\widehat{y})}\right), & y(0) > 0 \\ \ln\left(\frac{M_i(1+\widehat{y})}{M_i-\widehat{y}}\right), & y(0) < 0 \end{cases}$$

where $\widehat{y} = y/\rho_i(t)$ with $y \in \{\mathrm{x}_{ij}, \mathrm{y}_{ij}\}$, $i \in \{1, \cdots, 6\}$, $j \in \mathcal{N}_i$ and $\rho_{ij}(t)$ being chosen as follows:

$$\rho_{ij}(t) = (10 - 10^{-2})\exp(-2t) + 10^{-2},\ \ \forall(i,j) \in E$$

We have proved theoretically that PPC-1 can be used in case of spanning tree communication graphs and cannot achieve asymptotic convergence of the agents' positions to

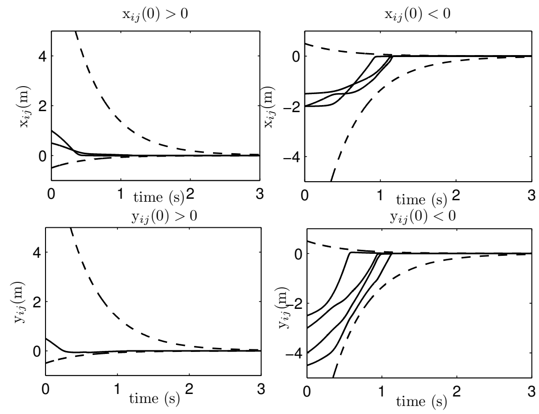

Fig. 4: Position errors (solid lines) and performance bounds (dashed lines) in case of CS-2 and PPC-2

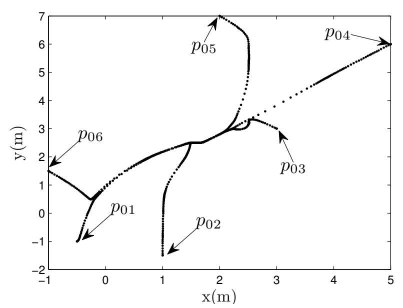

Fig. 5: Trace of the robots on $\mathrm{x}-\mathrm{y}$ plane in case of controller in case of CS-2 and PPC-2

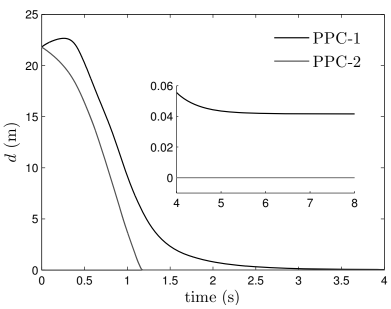

Fig. 6: Total distance between the robotic agents and the centroid in case of CS-2

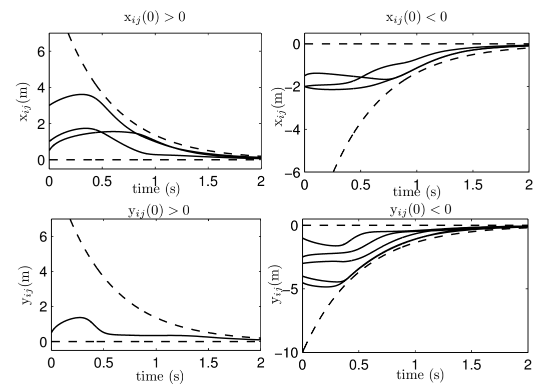

Fig. 7: Position errors (solid lines) and performance bounds (dashed lines) in case of CS-3 and PPC-1

their centroid.

We consider first the case of a spanning tree communication graph (CS-2, Fig. 1) for prescribed controllers PPC-1 and PPC-2 in order to validate the theoretical results. Simulation results are shown in Figs. 2, 3 and Figs. 4, 5 for PPC-1 and PPC-2, while Fig. 6 depicts the sum of distances from the centroid with solid black and grey lines for PPC-1 and PPC-2 respectively. In Figs. 2 and 4, the edge errors are depicted with the performance bounds (dashed lines); the first and the second rows correspond to x and y coordinates respectively while the first and the second columns correspond to positive or negative initial edge errors. In Figs. 3 and 5, the traces of the agents on the $\mathrm{x}-\mathrm{y}$ plane are shown. Notice that, the use of the zero overshoot controller PPC-1 allows us to ensure the agents will not collide during the procedure, which is crucial in robotic applications. On the other hand, the use of the controller PPC-2 enables the agents to converge asymptotically to the centroid as shown in Fig. 6.

In the second part of the simulation section we consider connected communication graphs with cycles; in particular we consider CS-1 and CS-3. In both cases, the theoretical results cannot guarantee that PPC-1 enforces the system to a stable equilibrium. Simulation shows that the system is unstable in CS-1 but stable in case of the communication graph with one cycle (CS-3); results for CS-3 are shown in Fig. 7. Figs. 8 and 9 validate also the theoretical findings for PPC-2 since the edges' errors evolve within the prescribed performance bounds and asymptotically converges to zero.

In the last part of the simulation section, the prescribed performance agreement protocol PPC-2 is compared to the simple linear agreement protocol based on the responses of the total distance between the agents and centroid (Fig. 10) and the average velocity norm of the agents (Fig. 11). In case of the linear controller with unit gains the convergence rate depends on the communication graph; the tuning of the controller in order to achieve faster convergence requires a known communication topology which is a centralized information and thus being inadequate for distributed control design. By inspecting Fig. 10 and 11, notice that the higher convergence rate to the centroid in case of PPC-2 requires speed abilities for the agents which are comparable with the maximum speed of the linear agreement protocol (Fig 10, 11: (a), (b)). Notice also that the maximum average velocity in case of the linear controller is the double of the maximum velocity in case of PPC-2 while the agents' convergence time is approximately the same.

V. Conclusions and Future Works

In this work, we proposed a nonlinear distributed agreement controller in order to guarantee a priori defined specifications such as overshoot and speed of response in the edges' space. The framework of prescribed performance control enables not only to ensure that the errors between neighboring agents will be restricted in a predefined region

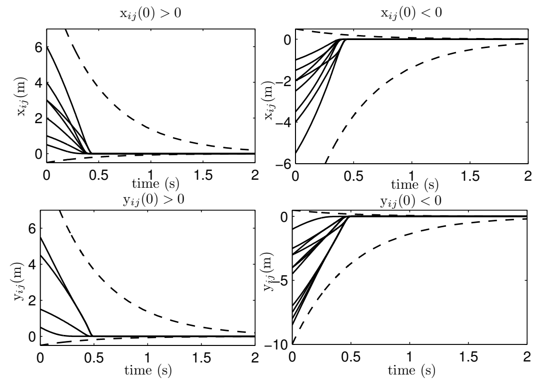

Fig. 8: Position errors (solid lines) and performance bounds (dashed lines) in case of CS-1 and PPC-2

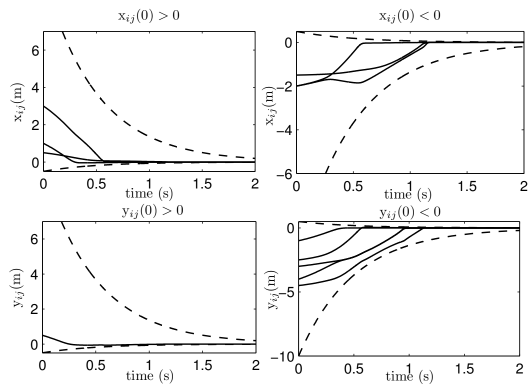

Fig. 9: Position errors (solid lines) and performance bounds (dashed lines) in case of CS-3 and PPC-2

(wherein the communication graph is invariant) but also to dissociate the convergence rate from the algebraic connectivity of the communication graph. Hence, each agent can know beforehand the minimum speed of convergence without knowing the communication topology of the overall network (typically centralized information). It was also proved that in case of a communication graph with cycles, the specification of zero overshoot cannot be set. On the other hand, the specification of zero overshoot which is a safe choice for a spanning tree communication graph does not enable the proof of asymptotic convergence to zero but allows small steady state errors which can be defined by the designer. Theoretical findings were illustrated through simulations. Simulation also compared the prescribed performance agreement protocol with the linear one and showed that the prescribed performance controller can achieve faster convergence without requiring higher maximum velocity values. Future work includes the consideration of multi-agent systems with more complex dynamics such as nonholonomic mobile robots, robotic manipulators and second order integrators.

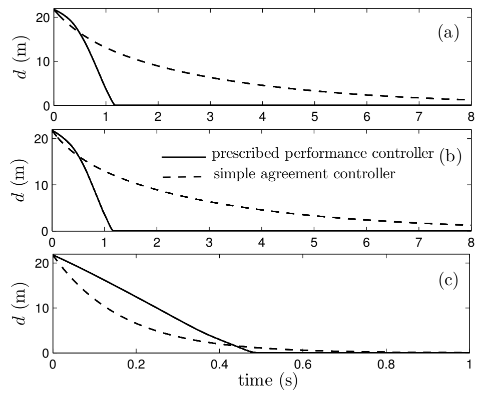

Fig. 10: Response of the total distance between the agents and the centroid: (a) CS-1, (b) CS-2, (c) CS-3

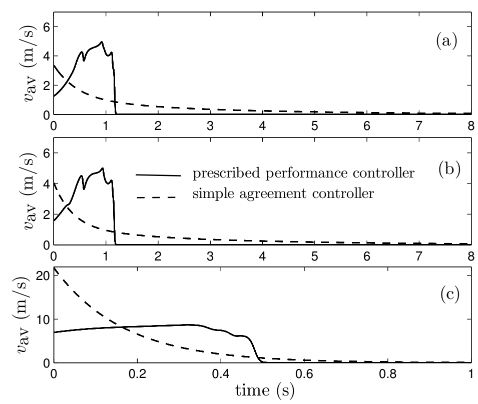

Fig. 11: Response of the average velocity norm: (a) CS-1, (b) CS-2, (c) CS-3

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