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Power limits for central order statistics: I. Continuous limit laws

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Abstract This paper lists the continuous limit distributions for central order statistics normalized by power transformations, and describes their domains of attraction. One may argue that power transformations are the natural normalizations to use if one wants to study the asymptotic behaviour of central order statistics. Power transformations preserve the origin, which may be assumed to be the quantile to which the order statistics converge. Our theory gives a nice extension of the theory developed by Smirnov more than sixty year ago. For the continuous power limits treated below the resemblance with the limit theory for extremes under linear transformations is striking.

Keywords Central order statistics · Limit law · Power transformations

AMS 2000 Subject Classifications 60F05 · 60G70 · 62G30

1 Introduction

Order statistics are used in many situations. It is important to understand their asymptotic behaviour as the sample size increases.

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We consider a fixed value $p \in (0, 1)$ and a sequence of order statistics $X_{k:n}$ with $k/n \to p$. Recall that the order statistics for a random variable X with distribution function (df) F are the elements

$$X_{1:n} \leq \cdots \leq X_{n:n},$$

of a sample of n independent observations from F arranged in increasing order. Of particular interest is the case when the origin is a p-quantile,

$$F(0-) \leq p \leq F(0). \tag{1.1}$$

We use the setting of Smirnov (1949) and call a sequence k with $k_n \in \{1, \dots, n\}$ a *Smirnov sequence* if

$$k/n \to p \in (0, 1) \quad \text{and} \quad \sqrt{n}(k/n - p) \to \mu \in \mathbb{R}. \tag{1.2}$$

In a more general approach one might consider two sequences $k = (k_i)$ and $n = (n_i)$. If $n_i \to \infty$ and $n_{i+1}/n_i \to 1$ one obtains the same results as the ones below. We shall therefore stick to the classic Smirnov setting where $n = 1, 2, \dots$ and $k = (k_n)$.

The transformations which we use to normalize the sequence $X_{k:n}$ are strictly increasing continuous mappings from the real line onto itself. *Power transformations* are defined as

$$A : x \mapsto cx^{\wedge a} = c|x|^a \operatorname{sign}(x) = \begin{cases} cx^a & x \geq 0 \\ -c|x|^a & x < 0 \end{cases} \qquad a, c > 0. \tag{1.3}$$

We write $\mathcal{P}$ for the set of all power transformations. The set $\mathcal{P}$ is a group. It contains the one-dimensional group $\mathcal{S}$ of scale transformations $x \mapsto cx$, $c > 0$.

The asymptotic behaviour of extremes is related to the tail behaviour of the underlying distribution; the asymptotic behaviour of central order statistics to the behaviour of the distribution at the corresponding quantile. For extremes, translations and affine normalizations are appropriate normalizations. For central order statistics, we use normalizations which preserve the p-quantile. Choose coordinates such that the origin is a p-quantile. The appropriate normalizations then are scalings or power transformations.

A non-constant variable W is a *power limit* of the sequence $X_{k:n}$ if there exist transformations $A_n : x \mapsto c_n x^{\wedge a_n}$ such that

$$W_n := A_n^{-1}(X_{k:n}) \Rightarrow W, \tag{1.4}$$

where $\Rightarrow$ denotes convergence in distribution. This is the basic limit relation of the paper. If it holds we say that *F lies in the domain of attraction of W*, or more briefly in the *domain of W*. If $X_{k:n}$ has df $F_{k:n}$ the power normed variable W_n has df $G_n(x) = F_{k:n}(c_n x^{\wedge a_n})$ and (1.4) states that the dfs G_n converge weakly to the df of W. The power limit W is *continuous* if its df is continuous.

A basic result in Smirnov (1949) is:

Theorem 1.1 ***(Smirnov)*** *If $X_{k:n}$ are order statistics from the uniform distribution on* $(0, 1)$ *and k is a sequence such that $k \to \infty$, $n - k \to \infty$ then*

$$(X_{k:n} - b_n)/a_n \Rightarrow N \qquad b_n = k/n,\ a_n = \sqrt{b_n(1 - b_n)/n},$$

where N is standard normal.

For a Smirnov sequence, see Eq. 1.2, and order statistics $U_{k:n}$ from the uniform distribution on $(-p, 1-p)$ this limit relation simplifies:

$$M_n := \sqrt{n}U_{k:n} \Rightarrow M = \mu + \sigma N \qquad \sigma = \sqrt{p(1-p)}, \tag{1.5}$$

where as above N is standard normal. The variable M will crop up in all our results.

If the underlying df F has a continuous density which is positive at the origin and $F(0) = p$, the order statistics $X_{k:n}$ for a Smirnov sequence k are asymptotically normal. If the density has a jump at the origin the order statistics may be scaled to converge to a variable which has a Gaussian density (with different scale constants) on the positive and negative half line. The jump in the density of F is mirrored in a jump in the density of the limit variable. For the normal variable M in Eq. 1.5 define

$$M_\eta = aM1_{\{M>0\}} + qM1_{\{M<0\}} \qquad a, q \geq 0, a \vee q = 1, \eta = a - q. \tag{1.6}$$

The parameter $\eta \in [-1, 1]$ measures the *imbalance* of the limit variable. Thus $M_0 = M$ and $M_1 = M \vee 0$. Smirnov (1949) showed that under affine normalization the continuous limit variables, properly normalized, are $(M_\eta)^{\wedge\rho}$, with $\eta \in (-1, 1)$ and $\rho > 0$. Up to a translation the limit variables have the form $A(M_\eta)$ with $A \in \mathcal{P}$ and $\eta \in (-1, 1)$. For power transformations the set of limit variables is larger. The continuous power limits may be written as $A(W_{\eta,\tau})$, $A \in \mathcal{P}$, $(\eta, \tau) \in (-1, 1) \times \mathbb{R}$, where $W_{\eta,\tau} = \chi_\tau(M_\eta)$ for a strictly increasing function χ_τ on $\mathbb{R}$ which satisfies $\chi_\tau(-u) = -\chi_\tau(u)$ and:

$$\chi_\tau(u) = e^{u^\tau} \quad \tau > 0; \qquad \chi_0(u) = u; \qquad \chi_\tau(u) = e^{-u^\tau} \quad \tau < 0; \qquad u > 0. \tag{1.7}$$

The balance parameter η and the exponent τ determine the shape of the limit distribution. Because of symmetry one may restrict attention to $\eta \geq 0$. Then $W_{\eta,\tau} = \chi_\tau(M)$ on the set where M is positive and $W_{\eta,\tau} = \chi_\tau(qM)$ for $q = 1 - \eta$ where M is negative. In particular for extreme imbalance, $\eta = 1$, the power limit W is non-negative and its distribution has a jump of size $\mathbb{P}\{M < 0\}$ at the origin. Restriction to power limits of moderate imbalance, $|\eta| < 1$, ensures continuity of the limit.

Note the similarity between the three kinds of power limits in Eq. 1.7 and the limit variables for maxima normalized by affine transformations, where the three kinds of limit variables, Weibull, Gumbel and Fréchet may be expressed in terms of the standard exponential variable E as

$$V_\tau = -E^\tau \quad \tau > 0; \qquad V_0 = -\log E; \qquad V_\tau = E^\tau \quad \tau < 0. \tag{1.8}$$

The resemblance is not fortuitous. For a df F in the domain of a power limit $W_{\eta,\tau}$ the balance parameter η is determined by the value of the quotient $(p - F(-x))/(F(x) - p)$ for $F(x) \to p$, the exponent τ by the rate at which $F(x) - p$ tends to zero. The vanishing function $F(x) - p$ for $x \to 0$ is related to the left tail of a df in the domain of attraction for minima.

Power transformations yield more limit distributions then scaling. Domains are also larger. The order statistics $X_{k:n}$ from F may be scaled to converge to the Gaussian variable M if and only if there is strict balance: $(p - F(-x))/(F(x) - p) \to 1$ for $x \to 0+$, and $F(x) - p$ varies regularly for $x \to 0+$ with exponent one. The power domain of M contains dfs F for which $F - p$ vanishes on a neighbourhood of the origin.

The group $\mathcal{P}$ of power transformations $A : x \mapsto cx^{\wedge a}$ was introduced by Pancheva in (1984) as an example of a finite-dimensional group of non-linear normalizations. Mohan, Ravi and Subramanya developed extreme value theory in Mohan and Subramanya (1991) and Mohan and Ravi (1992) using these power transformations rather than affine transformations to normalize the maxima. They observe:

- for positive variables one may use affine normalization of the logarithm instead of power transformations of the original variable;
- there are different classes of limit laws for maxima depending on the value of the upper endpoint of the df: negative, zero, positive or $+\infty$;
- the domain of the central power limit (with parameter $\tau = 0$) is rich. It contains all dfs with regularly varying tail, and more.

These three observations also apply to central order statistics as we shall see below.

The present paper is a companion to Balkema (2013) which treats the theory of power limits for extreme and intermediate order statistics. The theory for intermediate order statistics also applies to central order statistics with $\mu = \pm\infty$ in Eq. 1.2. Nigm (2006) developed the theory of power limits for central order statistics by using affine transformations on the logarithm of $X_{k:n}$. He obtains power limits of the form $W = e^Z$ and $W = -e^{-Z}$, where Z is a limit variable of order statistics under affine normalization. Nigm assumes the p-quantile to be non-zero. Barakat and Omar (2011) realize that for central order statistics from a df for which the origin is a p-quantile a different approach is needed. They show that for two-valued variables with mass p in the left point there are seven power types and argue that these may all occur as power limits. They conjecture that the power limits agree with Smirnov's affine limits in (1949). They use Smirnov's approach via dfs and first develop a limit theory for non-linear normalization. This approach is also used by Pancheva and Gacovska-Barndovska who obtain all power limits whose df G is continuous and strictly increasing on the interior of the interval $\{0 < G < 1\}$ in their paper (Pancheva and Gacovska-Barndovska 2015). Domains of attraction have been investigated by Azzat in (2013), but only for the power limits $W = e^Z$ and $W = -e^{-Z}$ in Nigm (2006).

The paper is organized as follows. Section 2 contains definitions and a basic result, Proposition 1.1. The next section treats the continuous power limits. In Section 4 we drop the condition that the origin is a p-quantile. We then present our conclusions. The Appendix contains a limit theorem for point processes which sheds new light on the relation between the limit theory for central order statistics presented below and extreme value theory.

2 Basics

In this paper dfs are right-continuous but in general we do not distinguish between two increasing functions which agree in their continuity points. Increasing functions need not be strictly increasing.

For any rv X there is an increasing function f on $(-p, 1-p)$ such that X has the same df as $f(U)$ where U is uniformly distributed on $(-p, 1-p)$. The function f is a shifted quantile function, the inverse of $F - p$. The variables $f(U_{k:n})$ are the order statistics of $f(U)$. This observation allows us to link convergence of normalized order statistics to weak convergence of increasing functions on $\mathbb{R}$, i.e. convergence in all continuity points of the limit function.

Proposition 2.1 *Let $A_n \in \mathcal{P}$. Let f be an increasing function on $(-p, 1-p)$ and χ an increasing function on $\mathbb{R}$. Let $k = (k_n)$ be a Smirnov sequence and M the associated Gaussian variable in* Eq. 1.5. *Suppose U is uniformly distributed on $(-p, 1-p)$. Set $X = f(U)$. Then*

$$W_n := A_n^{-1}(X_{k:n}) \Rightarrow W \quad \textit{iff} \quad \chi_n(w) := A_n^{-1} f(w/\sqrt{n}) \to \chi(w) \textit{ weakly on } \mathbb{R}. \tag{2.1}$$

The variables $\chi(M)$ and W have the same distribution.

Proof Since the normal variable M has a continuous distribution weak convergence on the right implies $A_n^{-1} f(M_n/\sqrt{n}) \Rightarrow \chi(M)$ for the variables $M_n = \sqrt{n}U_{k:n}$. This is the left side with $W = \chi(M)$. Now assume the convergence on the left and choose χ right-continuous and increasing such that W is distributed like $\chi(M)$. The function χ is unique. The sequence of increasing functions χ_n on the right has a subsequence which converges weakly to a right-continuous increasing function $\tilde{\chi}$ from $\mathbb{R}$ to $[-\infty, \infty]$. By the first part of the proof it follows that $W_n \Rightarrow \tilde{\chi}(M)$. Hence $\tilde{\chi} = \chi$ and $\chi_n \to \chi$ weakly on $\mathbb{R}$ and χ is finite. □

The transformations A_n may come from any group of increasing homeomorphisms of $\mathbb{R}$. In Smirnov's analysis they are affine transformations. We consider power transformations.

We are interested in the limit relation on the left of Eq. 2.1 for a given Smirnov sequence k and the associated variables p, μ and M in Eq. 1.5. What are the possible limit laws? For a given power limit W for what dfs F can the sequence of order statistics $X_{k:n}$ be normalized by elements of $\mathcal{P}$ to converge in distribution to W? Introduce the two sets of dfs

$$\mathcal{L}(\mathcal{P}, k, 0) \qquad \mathcal{D}(W, \mathcal{P}, k, 0) \tag{2.2}$$

to describe the limit laws, and for a power limit W the domain of W, for the Smirnov sequence k and for dfs which satisfy (1.1): $F(0-) \leq p \leq F(0)$. Note that the df of the power limit $W = \chi(M)$ depends on the sequence k, since it depends on M, but χ does not. The domain of $W = \chi(M)$ depends only on χ. The domain is determined by the behaviour of the shifted quantile function $f = (F - p)^{\leftarrow}$ at the origin.

Lemma 2.2 *If $\chi(M)$ is a limit variable (for the sequence A_n) then $\chi(cM)$ for $c > 0$ is a limit variable (for the sequence $B_n = A_{[nc^2]}$).*

Proof $c_n^2 = [nc^2]/n \to c^2$ implies $B_n^{-1} f(u/\sqrt{n}) = A_{[nc^2]}^{-1} f(c_n u/\sqrt{[nc^2]}) \to \chi(cu)$ weakly on $\mathbb{R}$. □

An advantage of the approach via the function f on the right side of Eq. 2.1 over the approach via dfs is that the power transformations act on the function values rather than the argument.

Let us show how Proposition 1.1 allows one to obtain several existing results.

Replace the group $\mathcal{P}$ by the group $\mathcal{S}$ of scalings, $x \mapsto cx$, $c > 0$. One obtains the limit relation $f(u/\sqrt{n})/c_n \to \chi(u)$. The function $f(u)$ varies regularly for $u \to 0+$ or $u \to 0-$ with exponent $\tau \geq 0$ and satisfies a balance condition $f(-u)/f(u) \to r = q/a \in [0, \infty]$ for $u \to 0+$. This characterizes the domains $\mathcal{D}(W, \mathcal{S}, k, 0)$. The limit variables have the form

$$W = aM^\tau 1_{\{M>0\}} - q|M|^\tau 1_{\{M<0\}} \qquad a, q, \tau \geq 0. \tag{2.3}$$

We have thus recovered Smirnov's affine limit variables and their domains up to a translation, see Smirnov (1949). If we replace $\mathcal{P}$ by the group {id} the limit relation (2.1) becomes trivial. The limit variables are determined by $x^* = \inf\{F > p\} = f(0+)$ and $x_* = \sup\{F < p\} = f(0-)$:

$$W = x_* 1_{\{M<0\}} + x^* 1_{\{M>0\}}. \tag{2.4}$$

The inclusion $\mathcal{L}(\{\text{id}\}, k) \subset \mathcal{L}(\mathcal{S}, k) \subset \mathcal{L}(\mathcal{P}, k)$ shows that Barakat's seven two-valued power types

$$i + 2\,\text{sign}(M) \qquad i = -3, \ldots, 3, \tag{2.5}$$

all occur as power limits of central order statistics.

2.1 Two limit relations

Now return to the limit relation (2.1) for power transformations A_n and power limits $W = \chi(M)$ with continuous dfs.

The origin is a p-quantile of F, $F(0-) \leq p \leq F(0)$, if and only if f vanishes in the origin, $f(0-) \leq 0 \leq f(0)$. If f vanishes on a neighbourhood of the origin then only the constant limit $\chi \equiv 0$ is possible. If f vanishes on an interval $(0, \epsilon)$ for some $\epsilon > 0$ the limit function χ, if it exists, vanishes on $(0, \infty)$. If f vanishes on $(-\epsilon, 0)$ then so does χ on $(-\infty, 0)$. In either case the limit distribution has a jump at the origin. For a continuous limit distribution the df F has to be continuous in the origin. If F is continuous in the origin then f is positive on $(0, \infty)$ and negative on $(-\infty, 0)$ and one may introduce the finite functions

$$\hat{g}(u) = \log f(u) \qquad \bar{g}(u) = -\log f(-u) \qquad u > 0. \tag{2.6}$$

The two functions $\hat{g}$ and $\bar{g}$ on $(0, \infty)$ determine the df F and allow us to replace convergence of the order statistics $X_{k:n}$ normalized by power transformations by two simple analytic limit relations. The limit relation (1.4) is equivalent to $f(u/\sqrt{n})^{\wedge 1/a_n}/c_n \to \chi(u)$ weakly on $\mathbb{R}$ and hence to

$$(\hat{g}(u/\sqrt{n}) - b_n)/a_n \to \hat{h}(u) \qquad (\bar{g}(u/\sqrt{n}) - b_n)/a_n \to \bar{h}(u) \quad \text{weakly on } (0, \infty), \tag{2.7}$$

where $c_n = e^{b_n}$ and $\chi(u) = e^{\hat{h}(u)}$, $\chi(-u) = -e^{\bar{h}(u)}$ for $u > 0$. We have transformed the basic limit relation $A_n^{-1}(X_{k:n}) \Rightarrow W$ for power limits of central order statistics into a pair of simple analytic limit relations of the form $(g(u/\sqrt{n}) - b_n)/a_n \to h(u)$. This limit relation is well-known, see Resnick (1987), Section 0.4.3 or Balkema and Embrechts (2007), Section 18.3. The only real-valued limit functions are powers and logarithms. The limit functions h assumes the value $-\infty$ if and only if the function χ vanishes in a point $u \neq 0$. By monotonicity χ vanishes on the interval between u and zero, and the df of $\chi(M)$ has a discontinuity at zero. Hence if W has a continuous df the limit functions $\hat{h}$ and $\bar{h}$ are finite. Collecting these results we obtain:

Proposition 2.3 *Suppose the df of the power limit in* Eq. 2.1 *is continuous in* $x = 0$. *Then so is* F. *The functions* $\hat{g}$ *and* $\bar{g}$ *in* Eq. 2.6 *then are well-defined and finite and convergence in* Eq. 2.1 *holds if and only if there exist finite increasing functions* $\hat{h}$ *and* $\bar{h}$ *on* $(0, \infty)$ *for which* (2.7) *holds.*

We are confronted with the following questions:

- What are the possible finite limit functions h in the limit relation

$$h_n(u) := (g(u/\sqrt{n}) - b_n)/a_n \to h(u) \quad \text{weakly on } (0, \infty), \tag{2.8}$$

  and how does one characterize the increasing functions g for a given limit h?
- If Eq. 2.8 for given affine transformations $x \mapsto a_n x + b_n$ holds for two functions $\hat{g}$ and $\bar{g}$, what is the relation between these functions, and between the limits $\hat{h}$ and $\bar{h}$?

If $h_n \to h$ weakly on $(0, \infty)$ then $h_n(c_n u) \to h(u)$ weakly for any sequence $c_n \to 1$. Hence one may define $b(r)$ and $a(r)$ as b_n and a_n on $(1/\sqrt{n}, 1/\sqrt{n-1}]$, $n \geq 1$, and replace (2.7) by

$$(g(ur) - b(r))/a(r) \to h(u) \qquad r \to 0+; \qquad u > 0. \tag{2.9}$$

A log-transform, setting $u = e^{-s}$, $r = e^{-t}$ and $\varphi(s) = -g(u)$ yields the additive form:

$$(\varphi(s+t) - d(t))/c(t) \to \psi(s) \qquad t \to \infty.$$

2.2 Limit relations and regular variation

The remainder of the section is devoted to the limit relation (2.9) and its application in extreme value theory, and may be skipped on first reading. We begin with a simple lemma.

Lemma 2.4 *The set* Φ *of increasing functions of the form* $b + au^\tau$ *for* $\tau > 0$, $b - au^\tau$ *for* $\tau < 0$ *and* $b + \log u$ *for* $\tau = 0$, *is closed in the space of increasing non-constant functions on* $(0, \infty)$ *with weak convergence. The subspace* Φ *is homeomorphic to* $\mathbb{R}^3$ *and the exponent* τ *is continuous on* Φ.

Proof The exponent τ is determined by the quotient

$$(\varphi(ru)-\varphi(u))/(\varphi(u)-\varphi(u/r))=r^\tau \qquad r,u>0, \quad \varphi\in\Phi, \tag{2.10}$$

hence continuous. Set $\psi_\tau(u)=(u^\tau-1)/\tau$ for $\tau\neq 0$ and $\psi_0(u)=\log u$ by continuity. Then $(a,b,\tau)\mapsto b+a\psi_\tau$ is a homeomorphism of $(0,\infty)\times\mathbb{R}^2$ onto Φ. Now suppose $\varphi_n\to\psi$ weakly. Let D be the countable set of discontinuities of ψ. If u_0 does not lie in the countable set $E=\bigcup_{k\in\mathbb{Z}}2^kD$ then ψ is continuous in the points $u_k=2^ku_0$, $k\in\mathbb{Z}$. The quotient in Eq. 2.10 for $u=u_k$, $r=2$ and $\varphi=\varphi_n$ converges to $(\psi(2u_k)-\psi(u_k))/(\psi(u_k)-\psi(u_k/2)) =: 2^\tau$ and $\tau_n\to\tau\in[-\infty,\infty]$. If $\tau=-\infty$ then $\psi(2u_k)=\psi(u_k)$. This holds for all k. Hence ψ is constant. Similarly $\tau=\infty$ implies $\psi(u_k)=\psi(u_k/2)$ for all k. Hence $\tau_n\to\tau\in\mathbb{R}$, and if we write $\varphi_n=b_n+a_n\psi_{\tau_n}$ then convergence $\tau_n\to\tau$ implies $b_n\to b$ and $a_n\to a$, and $\psi=b+a\psi_\tau\in\Phi$. □

Theorem 2.5 *Assume* (2.8) *holds for increasing functions* g *and* h *on* $(0,\infty)$ *with values in* $[-\infty,\infty)$*, and suppose* h *is non-constant. Then* g *is real valued. If* h *is real valued then* $h\in\Phi$*. Else* $h=b-\infty 1_{(0,c)}$.

Proof The result is well known for finite limit functions, see for instance (Balkema and Embrechts 2007), Section 18.3. For infinite values see Balkema (2013), Theorem 4.1. □

If $g(u/\sqrt{n})$ may be normalized by affine transformations to converge we write

$$g\in D(\tau) \tag{2.11}$$

where τ is the exponent of the limit function h if h is finite and $\tau=-\infty$ else. One may think of $D(\tau)$ as the domain of h. There is a relation with regular variation.

- $g\in D(-\infty)$ if and only if $g(0)=-\infty$ and g varies rapidly at zero: $g(ur)/g(r)\to 0$ for $u>1$;
- $g\in D(\tau)$ and $\tau<0$ if and only if $g(0)=-\infty$ and g varies regularly at zero with exponent τ;
- $g\in D(\tau)$ and $\tau>0$ if and only if $g(0)$ is finite and $g(u)-g(0)$ varies regularly at zero with exponent τ,
- $g\in D(0)$ if and only if the function $\tilde{g}: u\mapsto -g(1/u)$ lies in the de Haan class Π. See Resnick (1987) for an analysis of Π and its relation to extreme value theory.

For $\tau\in\mathbb{R}$ there is a simple relation with limit laws for sample minima. Let $\mathcal{D}_{\min}(V,\mathcal{A})$ denote the domain of the limit variable V for minima under affine normalization.

Proposition 2.6 *Let* Y *have df* G *and suppose the quantile function* $G^{\leftarrow}$ *agrees with* g *in* Eq. 2.9 *on an interval* $(0,\epsilon)$*. Then* $g\in D(\tau)$ *for* $\tau\in\mathbb{R}$ *if and only if* $G\in\mathcal{D}_{\min}(V_\tau,\mathcal{A})$ *where, compare* (1.8),

$$V_\tau=E^\tau \quad (\tau>0); \qquad V_0=\log(E); \qquad V_\tau=-E^\tau \quad (\tau<0) \tag{2.12}$$

for a standard exponential variable E.

Proof Let U be uniform on $(0, 1)$ and let $g = G^{\leftarrow}$. The minimum of n independent copies of Y is distributed like $g(U_{1:n})$. Observe that $E_n := nU_{1:n} \Rightarrow E$, where E is standard exponential. As for Proposition 1.1 one proves that $(Y_{1:n} - b_n)/a_n \Rightarrow W$ if and only if $(g(w/n) - b_n)/a_n \to h(w)$ weakly on $(0, \infty)$, and W is distributed like $h(E)$. □

Lemma 2.7 *Let f be an increasing function on $(0, c)$ with $f(0) > 0$. Then $f \in D(0)$ with scale function $r \mapsto a(r)$ if and only if*

$$(f(ur)/f(r))^{f(0)/a(r)} \to u \qquad r \to 0+; \qquad u > 0.$$

Proof Let $r_n \to 0+$ and $u > 0$. Set $a_n = a(r_n)$. Take logarithms to find

$$\frac{f(0)}{a_n} \log \frac{f(r_n) + a_n \log u_n}{f(r_n)} = \frac{f(0)}{a_n} \log \left(1 + \frac{a_n \log u_n}{f(r_n)}\right) \to \log u$$

if and only if $f(r_n u) - f(r_n) = a_n \log u_n$ for $u_n \to u$. □

3 The continuous limit distributions

Recall the assumptions: We look at the asymptotic behaviour of central order statistics $X_{k:n}$ from a df F where $k = k_n$ satisfies the Smirnov conditions $k/n \to p \in (0, 1)$ and $\sqrt{n}(k/n - p) \to \mu \in \mathbb{R}$, and $F(0-) \leq p \leq F(0)$. Write $X = f(U)$ with f increasing and U uniformly distributed on $(-p, 1 - p)$. By Smirnov's Theorem $M_n = \sqrt{n}U_{k:n} \Rightarrow M = \mu + \sigma N$, see Eq. 1.5, with N standard normal and $\sigma = \sqrt{p(1 - p)}$. In this section we assume that there are sequences $a_n > 0$ and $c_n > 0$ such that

$$(X_{k:n}/c_n)^{\wedge 1/a_n} \Rightarrow W \tag{3.1}$$

for a random variable W with a *continuous* df. By Proposition 1.1 this means that there exists a *strictly* increasing function χ on $\mathbb{R}$ such that W is distributed like $\chi(M)$ and moreover that $(f(x/\sqrt{n})/c_n)^{\wedge 1/a_n} \to \chi(x)$ weakly on $\mathbb{R}$. Since $\mathbb{P}\{W = 0\} = 0$ one may take logarithms and (3.1) is equivalent to two limit relations of the form

$$(g(u/\sqrt{n}) - b_n)/a_n \to h(u) \qquad u \in (0, \infty),$$

for the functions $\hat{g}(u) = \log f(u)$ and $\bar{g}(u) = \log(-f(-u))$ where $\hat{h}$ and $\bar{h}$ are finite and determine χ, see Eq. 2.7. The normalization constants a_n and b_n should be the same for $\hat{g}$ and $\bar{g}$. This will be the case if $\hat{g} \equiv \bar{g}$ on an interval $(0, u_0)$.

Definition 1 For increasing functions f on $\mathbb{R}$ define

$$f^*(x) = -f(-x) \qquad x \in \mathbb{R}. \tag{3.2}$$

The function f is $*$-*symmetric* if $f^* = f$.

3.1 The continuous power limits

We claim that the continuous power limits have the form $A(\chi_\tau(M_\eta))$ with $A \in \mathcal{P}$, M_η as in Eq. 1.6 for the normal variable M in Eq. 1.5 and χ_τ a $*$-symmetric function whose restriction to $(0, \infty)$ has the simple form $\chi_\tau = \exp(h_\tau)$ where

$$h_\tau(x) = x^\tau \quad \tau > 0; \qquad h_0(x) = \log x; \qquad h_\tau(x) = -x^\tau \quad \tau < 0. \tag{3.3}$$

Conditions on g_0 which ensure convergence of $(g_0(u/t) - b(t))/a(t)$ for $t \to \infty$ are related to regular variation and the de Haan theory for maxima and have been treated in Section 2.2. The question which we have to address now is: What conditions should g_1 satisfy in order that the limit relation also holds for g_1 with the same normalizations? We saw above that it suffices that g_1 and g_0 agree on an interval $(0, u_0)$, but will asymptotic equality also suffice? And what is the relation between the limit functions h_0 and h_1?

Lemma 3.1 *Let g_0 and g_1 be increasing functions on* $(0, \infty)$. *Suppose A_t are affine transformations such that $A_t^{-1} g_0(x/\sqrt{t}) \to h_0(x)$ weakly on* $(0, \infty)$ *for a strictly increasing function h_0. If $A_t^{-1} g_1(1/\sqrt{t}) \to h_0(r)$ for some $r > 0$ then for any $r_1 < r < r_2$ there exists $\delta > 0$ such that*

$$g_0(r_1 x) < g_1(x) < g_0(r_2 x) \qquad x \in (0, \delta). \tag{3.4}$$

Proof The conditions imply that $A_t^{-1} g_0(r_1/\sqrt{t}) < A_t^{-1} g_1(1/\sqrt{t}) < A_t^{-1} g_0(r_2/\sqrt{t})$ for $t > t_0$. This yields the desired relation for $\delta = 1/\sqrt{t_0}$. □

Corollary 3.2 *The conditions of the lemma imply convergence $A_t^{-1} g_1(x/\sqrt{t}) \to h_0(rx)$.*

We conclude that $\bar{h}(x) = \hat{h}(rx)$ for some $r > 0$ and that $\bar{g}$ and $\hat{g}$ are related by the inequalities (3.4). By Theorem 1.5 any non constant real valued limit function h of $A_n^{-1} g(x/\sqrt{n})$ on $(0, \infty)$ has the form $h(x) = b + a h_\tau(x)$ with $h_0(x) = \log x$ and h_τ a power for $\tau \neq 0$, see Eq. 3.3. This yields a description of the set $\mathcal{L}_c(\mathcal{P}, k, 0)$ of continuous limit laws.

Theorem 3.3 *Let $X_{k:n}$ be central order statistics from a df F which satisfies the quantile condition* (1.1). *Suppose the Smirnov setting holds, see* Eq. 1.2. *If there exist power transformations $A_n(y) = c_n y^{\wedge a_n}$ such that $A_n^{-1}(X_{k:n}) \Rightarrow W$ for a random variable W with a continuous df then W has one of the following three forms:*

$$\begin{aligned} W &= c e^{(aM)^\tau} 1_{\{M>0\}} - c e^{|qM|^\tau} 1_{\{M<0\}} && a, q, c > 0;\ \tau > 0. \\ W &= (aM)^c 1_{\{M>0\}} - |qM|^c 1_{\{M<0\}} && a, q, c > 0;\ \tau = 0. \\ W &= c e^{-(aM)^\tau} 1_{\{M>0\}} - c e^{-|qM|^\tau} 1_{\{M<0\}} && a, q, c > 0;\ \tau < 0. \end{aligned}$$

Here $M = \mu + \sigma N$ is the normal random variable defined in Eq. 1.5. *The balance parameter for these continuous power limits W is defined by $\eta = (a - q)/(a \vee q)$.*

The structure of $\mathcal{L}_c(\mathcal{P}, k, 0)$ is clear. Power limits have the form $W = \chi(M)$, $\chi \in X$, where the set X does not depend on the sequence k. Alternatively one may write the power limits as $W = \chi(\mu/\sigma + N)$, $\chi \in X$, where N is standard normal.

Corollary 3.4 *Every df* $G \in \mathcal{L}_c(\mathcal{P}, k, 0)$ *satisfies* $G(0) = \Phi(-\mu/\sigma)$ *with* Φ *the standard normal df.*

Corollary 3.5 *If* k *and* k' *are Smirnov sequences the sets* $\mathcal{L}_c(\mathcal{P}, k', 0)$ *and* $\mathcal{L}_c(\mathcal{P}, k, 0)$ *are equal if* $p'(1-p')/\mu'^2 = p(1-p)/\mu^2$ *and else disjoint.*

If $\sqrt{n}(k/n - p) \to 0$ then F is said to belong to the normal domain of p-attraction of the power limit, see Smirnov (1949). In that case $\mu = 0$, $G(0) = 1/2$ and the value of p does not affect the limit.

Proposition 3.6 *Suppose* F *is continuous at zero,* $0 < F(0) < 1$ *and* $\sqrt{n}(k/n - F(0)) \to 0$*. If there exist positive constants* a_n *and* c_n *such that* $F_{k:n}(c_n x^{\wedge a_n}) \to G(x)$ *weakly for a continuous df* G *then there are constants* $a, q, c > 0$ *and* $\tau \in \mathbb{R}$ *such that* $G = G_\tau$*, where*

$$G_0(x) = \begin{cases} \Phi(-|x|^c/q) & x < 0 \\ \Phi(|x|^c/a) & x > 0, \end{cases}$$

and, writing $\kappa(x) = |\log|x/c||^{1/\tau}$ *for* $\tau \neq 0$*,*

$$G_\tau(x) = \begin{cases} \Phi(-\kappa(x)/q) & x < -c \\ 1/2 & -c < x < 0 \\ 1/2 & 0 < x < c \\ \Phi(\kappa(x)/a) & x > c \end{cases} \quad \tau > 0; \qquad G_\tau(x) = \begin{cases} 0 & x < -c \\ \Phi(-\kappa(x)/q) & -c < x < 0 \\ \Phi(\kappa(x)/a) & 0 < x < c \\ 1 & x > c \end{cases} \quad \tau < 0.$$

Proof In the expressions above for the power limit W one may replace $M = \sigma N$ by N and write $W = \chi(N)$ adapting the positive constants a and q. Then $G(x) = \Phi(\chi^{\leftarrow}(x))$. □

The dfs G_0 were shown to be limit distributions in Barakat and Omar (2011), the dfs G_τ, $\tau \leq 0$, in Pancheva and Gacovska-Barndovska (2015).

3.2 Domains of attraction

The two limit relations in Eq. 2.7 describe the domain of attraction. There are two conditions:

i) The function $\hat{g}$ belongs to $D(\tau)$ where τ is the exponent of the limit function $\hat{h}$, and
ii) there should be a certain balance between the functions $\hat{g}$ and $\bar{g}$ at zero.

The condition $\hat{g} \in D(\tau)$ is only a reformulation of the limit relation for $\hat{g}$ in Eq. 2.7. For $\tau < 0$ it states that $\hat{g}$ varies regularly at zero with exponent τ and that one may take $b_n \equiv 0$ in Eq. 2.7; for $\tau > 0$ it states that one may take $b_n \equiv \hat{g}(0)$ and that

$\hat{g}(u) - \hat{g}(0)$ varies regularly for $u \to 0+$ with exponent τ; for $\tau = 0$ the function $\hat{g}$ varies regularly with exponent $\tau = 0$ and satisfies an additional condition, $u \mapsto -\hat{g}(1/u)$ belongs to Π. The theory in Section 2.2 introduces a certain simplification for the limit relations in Eq. 2.7 for $\tau \neq 0$. The reason for including that section is to embed the limit relations (2.7) in a theory - regular variation - which has been well investigated in the past century.

There is a nice relation with extreme value theory. The limit variables for minima are

$$V_\tau = e^{\tau V_0}, \tau > 0 \quad V_0 \quad \text{and} \quad V_\tau = -e^{\tau V_0}, \tau < 0,$$

where $-V_0$ has a standard Gumbel distribution, $\mathbb{P}\{V_0 > v\} = \exp(-e^v)$. A df G lies in the domain $\mathcal{D}_{\min}(\tau)$ of V_τ if and only if the quantile function $g = G^{\leftarrow}$ lies in $D(\tau)$. The inverse of $\hat{g}$ is the function $t \mapsto \hat{G}(t) = F(e^t) - p$. The function $\hat{G} = \hat{g}^{\leftarrow}$ is a df except for the fact that $\hat{G}(\infty) = 1 - p$. So we see that $\hat{g}$ satisfies the limit relation in Eq. 2.7 with limit $\hat{h} = \log \chi_\tau$ if and only if the df $t \mapsto (F(e^t) - p)/(1 - p)$, or any df G which agrees with $F(e^t) - p$ on a right neighbourhood of its lower endpoint $\inf\{G > 0\}$, lies in $\mathcal{D}_{\min}(\tau)$.

The convergence condition i) has been translated into a condition on the behaviour of $F(x) - p$ for $x \downarrow x_* = \inf\{F > p\}$. Let us now consider condition ii) linking the behaviour of $\hat{g}$ and $\bar{g}$ at the origin, or, alternatively, linking the behaviour of $F(x) - F(0)$ and $F(0) - F(-x)$ for $F(x) \to F(0)$. The imbalance η was introduced initially to handle a jump at zero in the density of X. We shall say that the *balance condition* holds and write $F \in \mathcal{B}(\eta)$ if $\{F = F(0)\} = [-x_*, x_*]$ for some $x_* \geq 0$ and

$$\frac{F(0) - F(-x)}{F(x) - F(0)} \to R \in (0, \infty) \qquad x \downarrow x_* = \inf\{F > F(0)\} \qquad \eta = \frac{R - 1}{R \vee 1}. \tag{3.5}$$

To belong to the domain $\mathcal{D}(W, \mathcal{P}, k, 0)$ of a continuous power limit the df F has to be continuous in zero with value $p \in (0, 1)$. The quantile set has to be closed and symmetric, $\{F = p\} = [-x_*, x_*]$ for some $x_* \geq 0$, and F has to be continuous in the endpoints $\pm x_*$. Convergence of the order statistics depends on the behaviour of F on a neighbourhood of the quantile set. These statements are obvious in terms of the functions $\hat{g}$ and $\bar{g}$. In terms of $\hat{g}$ and $\bar{g}$ the domain is determined by the condition $\hat{g} \in D(\tau)$ and the balance condition (3.4) linking the asymptotic behaviour of $\hat{g} - c_*$ and $\bar{g} - c_*$ at the origin. This balance condition then also holds for f: For positive r_1 and r_2 with $r_1 < r < r_2$ there exists $\delta > 0$ such that $f(r_1 u) < -f(-u) < f(r_2 u)$ holds for $u \in (0, \delta)$. In terms of the inverse $f^{\leftarrow} = F - F(0)$ the condition simplifies to Eq. 3.5 with $R = 1/r$.

Let $\mathcal{D}_0(\eta, \tau)$ denote the power domain of $W_{\eta,\tau}$ and write $F \in \mathcal{D}^+(\tau)$ if $F(0) = p$ and if there exists a df $G \in \mathcal{D}_{\min}(\tau)$ such that $G(t) = F(e^t) - F(0)$ on a neighbourhood of $t_* = \inf\{t \mid F(e^t) > F(0)\}$. Then $F \in \mathcal{D}^+(\tau)$ if and only if $\hat{g} \in D(\tau)$. Define $\mathcal{D}^-(\tau)$ similarly as the set of dfs F with $F(0-) = p$ such that $t \mapsto G(t) = (F(-e^{-t}) - p)/p$ lies in the domain of attraction $\mathcal{D}_{\max}(\tau)$ of the extreme value limit variable V_τ in Eq. 1.8. The result of the arguments above may now be recapitulated as:

$$\mathcal{D}_0(\eta, \tau) = \mathcal{B}(\eta) \cap \mathcal{D}^+(\tau) = \mathcal{B}(\eta) \cap \mathcal{D}^-(\tau) \tag{3.6}$$

This is the fundamental result of the paper. Here is the proof:

Theorem 3.7 *Let $A_n^{-1}(X_{k:n}) \Rightarrow W$ for a sequence of power transformations A_n and a sequence of order statistics from a df F. Assume $p \in (0,1)$, $F(0-) \le p \le F(0)$, k is a Smirnov sequence, see Eq. 1.2, and W has a continuous df. Then F is continuous at zero. The power limit W has parameters τ (exponent) and η (imbalance), see Theorem 2.3 above, if and only if $F \in \mathcal{B}(\eta) \cap \mathcal{D}^+(\tau)$ if and only if $F \in \mathcal{B}(\eta) \cap \mathcal{D}^-(\tau)$.*

Proof Convergence $A_n^{-1}(X_{k:n}) \Rightarrow W$ under the conditions above is equivalent to convergence of $\hat{g}$ and $\bar{g}$ in Eq. 2.7. Since the normalizations are the same Lemma 2.1 applies and $\hat{h}(u) = \bar{h}(ru)$ for some $r > 0$ by Corollary 2.2. Hence $\hat{g}$ and $\bar{g}$ belong to the same set $D(\tau)$ in Eq. 2.11. Relation (3.4) in Lemma 2.1 holds for $\hat{g}$ and $\bar{g}$. Equivalently the balance condition $\mathcal{B}(\eta)$ holds for F with $\eta = (r-1)/(r \vee 1)$ as in Eq. 3.5. Since $\hat{g} \in D(\tau)$ is equivalent to $F \in \mathcal{D}^+(\tau)$ and by symmetry $\bar{g} \in D(\tau)$ to $F \in \mathcal{D}^-(\tau)$ convergence of $\hat{g}$ and $\bar{g}$ in Eq. 2.7 is equivalent to $F \in \mathcal{B}(\tau) \cap \mathcal{D}^+(\tau)$ and to $F \in \mathcal{B}(\tau) \cap \mathcal{D}^-(\tau)$. □

The special cases below are presented to flesh out the cryptic description of $\mathcal{D}_0(\eta, \tau)$ in Eq. 3.6.

For $\tau > 0$ there are simple regular variation criteria for the domain $\mathcal{D}_0(\eta, \tau)$: The quantile set $\{F = p\}$ is a closed symmetric interval $[-q_*, q_*]$ for some $q_* > 0$, $F(q_* + x) - p$ varies regularly for $x \to 0+$ with exponent $1/\tau$ and $\mathcal{B}(\eta)$ holds. An example should make this clear.

Example 1 Let F be the uniform distribution on $(-1-p, -1) \cup (1, 2-p)$. Then F lies in $\mathcal{D}_0(0,1)$, the domain of the limit variable $e^M 1_{\{M>0\}} - e^{|M|} 1_{\{M<0\}}$. The balance condition $\mathcal{B}(0)$ holds by symmetry and $\hat{g}(u) = \log(1+u)$ on $(0, 1-p)$. The asymptotic equality $\log(1+u) \sim u$ for $u \to 0+$ implies that $\hat{g}$ varies regularly at $0+$ with exponent $\tau = 1$.

Gaussian power limits may occur even when the density vanishes on the interval $[-1, 1]$.

Proposition 3.8 *If $\{F = p\} = [-q, q]$ for some $q > 0$ and F satisfies the balance condition $\mathcal{B}(\eta)$ for some $\eta \in (-1,1)$ then $F \in \mathcal{D}_0(\eta, 0)$ if and only if $F(q+x) - F(q) \sim e^{-\psi(x)}$ for a C^2 function which satisfies the von Mises condition*

$$\psi'(x) \to \infty \qquad (1/\psi')'(x) \to 0 \qquad x \to 0+\,. \tag{3.7}$$

Proof See Balkema and Embrechts (2007) Theorem 6.1 and Section 6.6 where it is shown that Eq. 3.7 means that a df G which agrees with $F - p$ on $(0, q+\delta)$ belongs to $\mathcal{D}_{\min}(-V, \mathcal{A})$ for the Gumbel variable V. By Proposition 1.6 this is equivalent to $f 1_{(0,\infty)} \in D(0)$, and by Lemma 1.7 to $\hat{g} \in D(0)$ since q is positive. □

From the theory of power normalization for maxima it is known that the domain of the limit distribution for $\tau = 0$ is large. A good way to describe the difference between the domain of M under scaling and under power transformations is via the loglog transform. We assume $\{F = p\} = \{0\}$ and $F \in \mathcal{B}(0)$. Define the increasing function $L(t) = -\log(F(e^{-t}) - p)$. The df F lies in $\mathcal{D}(M, \mathcal{S}, k, 0)$ if and only if $F(x) - p$ varies regularly with exponent one for $u \to 0+$. In terms of L this means that there exists a C^2 function L_0 such that $L(x) - L_0(x)$ vanishes for $x \to \infty$ and $1/L_0' \to 1$. For $F \in \mathcal{D}(M, \mathcal{P}, k, 0)$ the condition is weaker: $(1/L_0')'(x) \to 0$ if $x_* = 0$. All non-constant monic polynomials satisfy the latter condition, but only those of degree one will satisfy the first condition.

It should be stressed that asymptotic equality of $\bar{g}(x)$ and $\hat{g}(x)$ or of $f(x)$ and $-f(-x)$ for $x \to 0$ does not imply exact balance, $\mathcal{B}(0)$, even when $f(0) = 0$. In the example below $\hat{g}(0) = \bar{g}(0) = -\infty$ and $d(x) = \hat{g}(x) - \bar{g}(x) \to 0$. The function $\hat{g}$ may be normalized to converge to $\log x$ and hence $F \in \mathcal{D}^+(0)$. The order statistics may be normalized to converge, but the limit is not normal. There are two different power limits with overlapping domains.

Example 2 Let $\hat{g}(x) = -\log|\log x|$ and $\bar{g} = \hat{g} - d$ where $d(x) = 1/\sqrt{|\log x|}$. Then $d(x)$ vanishes for $x \to 0+$ which implies that $-f(-x)/f(x) \to 1$. Set $c_n = \log\sqrt{n}$, $a_n = 1/c_n$, $b_n = \log a_n$. Then

$$\hat{g}(x/\sqrt{n}) = b_n + a_n \log x + o(a_n) \qquad d(x/\sqrt{n}) = \sqrt{a_n} + o(a_n). \tag{3.8}$$

It follows that $(\hat{g}(x/\sqrt{n}) - b_n)/a_n \to \log x$ and $\bar{g}(x/\sqrt{n}) - b_n)/a_n \to -\infty$. If a_n is replaced by $\sqrt{a_n}$ one obtains the limits 0 and -1. The associated order statistics $X_{n:k}$ satisfy the two limit relations:

$$(c_n X_{n:k})^{\wedge c_n} \Rightarrow M \vee 0 \qquad (c_n X_{n:k})^{\wedge\sqrt{c_n}} \Rightarrow 1_{\{M>0\}} - 1_{\{M<0\}}/e. \tag{3.9}$$

How should we interpret this surprising result: the limit variable is $M \vee 0$, the positive part of the Gaussian variable M, but may also be a two-valued function of M (Fig. 1).

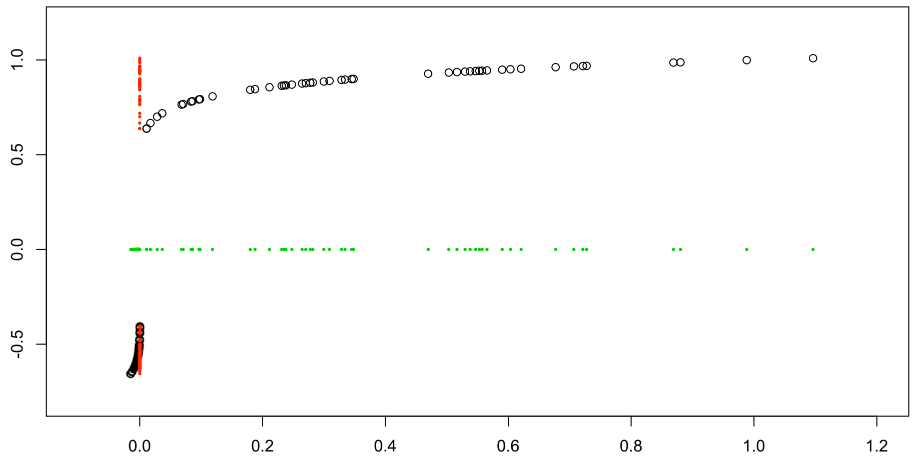

Fig. 1 Sample points from $M1_{\{M>0\}} + 0.01M1_{\{M<0\}}$ and their transforms

The figure above depicts a hundred points on the graph of the power transformation $A(x) = x^{\wedge 0.1}$. The x_i are sample points from the continuous power limit $W = M1_{\{M>0\}} + 0.01M1_{\{M<0\}}$ with $p = 1/2$ and $\mu = 0$. The y_i are sample points from $A(W)$, a continuous power limit with the same parameter $(\eta, \tau) = (0.99, 0)$ as W. The y_i cluster around the two values 0.2 ± 0.7. For $\eta \to 1$ one obtains the mysterious conjugacy between the power limit $M \vee 0$ and the two-valued power limit in Eq. 3.9 which is of the power type of $1 + 2\,\mathrm{sign}(M)$.

The example shows how our intuition for convergence of normalized probability distributions breaks down for discontinuous power limits. There actually are two more power limits in addition to Eq. 3.9. The sequence $(c_n X_{k:n})^{\wedge c_n^\theta}$ converges to $\mathrm{sign}(M)$ for $\theta \in (0, 1/2)$ and to $1_{\{M>0\}}$ for $\theta \in (1/2, 1)$. This follows from Eq. 3.8. The two new power limits exhibit a certain degeneracy. They are invariant under pure power transformations, $W^{\wedge a} = W$ for $a > 0$. In the companion paper on discontinuous limit distributions it will be shown that the domain of $W = \mathrm{sign}(M)$ contains all dfs F which are continuous at the origin and satisfy $F(0) = p$, and that the domain of $W = 1_{\{M>0\}}$ contains the domains of all non-negative power limits. It should be clear that there is no convergence of types for power limits.

Theorem 3.9 ***(Convergence of power types)*** *Let $X_n \Rightarrow X_0$ and let $Y_n = c_n X_n^{\wedge a_n} \Rightarrow Y_0$ for positive c_n, a_n. Suppose X_0 has a continuous df and Y_0 is not a.s. zero. Then $c_n \to c > 0$ and $a_n \to a \geq 0$, and $Y_0 = cX_0^{\wedge a}$ in distribution if a is positive and $Y_0 = c\,\mathrm{sign}(X_0)$ if $a = 0$.*

Proof Write $X_n = f_n(M_0)$ for an increasing function f_n where M_0 is a normal variable such that $\mathbb{P}\{M_0 < 0\} = \mathbb{P}\{X_0 < 0\}$ if X_0 assumes both positive and negative values. Then f_0 is strictly increasing and $f_n \to f_0$ weakly as in Proposition 1.1. Let $Y_0 = g_0(M_0)$ with g_0 increasing and let $u_0 < u_1$ be non-zero continuity points of g_0 and f_0. By symmetry we may assume that g_0 is positive in these two points. Then $g_n(u_i)$ is positive eventually and hence so is $f_n(u_i)$. We may set $y_{ni} = \log f_n(u_i)$ and $z_{ni} = \log g_n(u_i) = b_n + a_n y_{ni}$ for $b_n = \log c_n$. Then

$$a_n = (z_{n1} - z_{n0})/(y_{n1} - y_{n0}) \to (z_{01} - z_{00})/(y_{01} - y_{00}) =: a$$

and $b_n = z_{n0} - a_n y_{n0} \to z_{00} - a y_{00} =: b$. This yields the relation above between X_0 and Y_0. □

3.3 Parametrization

The set of continuous power limits $W = A(\chi_\tau(M_\eta))$ may be parametrized by $(\eta, \tau, a, c) \in (-1, 1) \times \mathbb{R} \times (0, \infty)^2$ where (a, c) describes the power transformation $A(x) = cx^{\wedge a}$. The continuous power limits form a subspace $\mathcal{L}(\mathcal{P}, k, 0)$ of the space of dfs on $\mathbb{R}$ with the topology of weak convergence. This subspace is homeomorphic to $(-1, 1) \times \mathbb{R} \times (0, \infty)^2$, and hence to $\mathbb{R}^4$. This does *not* follow from the parametrization above, which is not continuous, see Eq. 3.3. For a good parametrization one would like to select a power limit with df G_θ for each $\theta = (\eta, \tau) \in \Theta_0 = (-1, 1) \times \mathbb{R}$ with disjoint domains $\mathcal{D}_0(\theta)$ such that $\theta_n \to \theta \in \Theta_0$ if and only if $G_{\theta_n} \to G_\theta$ weakly.

One may ask: Is $\mathcal{L}(\mathcal{P}, k, 0)$ a closed set in the space of continuous dfs? Is the closure of $\mathcal{L}(\mathcal{P}, k, 0)$ in the space of non-degenerate dfs the set of all power limits (both continuous and not)? Does the good parametrization mentioned above extend to a good parametrization of all non-degenerate power limits? Is the parameter $(\eta, \tau) \in (-1, 1) \times \mathbb{R}$ a good parameter for the domains of attraction? Some of these questions will be answered below, some in the companion paper.

Proposition 3.10 *The map $G \mapsto \theta(G)$ which associates the parameter $\theta = (\eta, \tau)$ with $\eta \in (-1, 1)$ (the balance) and $\tau \in \mathbb{R}$ (the exponent) to the df G of a continuous power limit is continuous.*

Proof Let G denote the df of $\chi(M)$. Then $G_n \to G_0$ weakly if and only if $\chi_n \to \chi_0$ weakly on $\mathbb{R}$ if and only if $\hat{h}_n \to \hat{h}_0$ weakly on $(0, \infty)$ and $\bar{h}_n \to \bar{h}_0$. The latter functions are elements of Φ, see Lemma 1.4, and τ is continuous on Φ by Eq. 2.10. Let η_n be the imbalance of G_n. By symmetry we may assume that $\eta_n \geq 0$ infinitely often and $\eta_{k_n} \to \eta \in [0, 1]$. Then $\bar{h}_{k_n}(u) = \hat{h}_{k_n}((1 - \eta_{k_n})u)$. Convergence of $\bar{h}_n$ and $\hat{h}_n$ implies $\bar{h}_0(u) = \hat{h}_0((1 - \eta)u)$. Hence $\eta < 1$ and $\eta = \eta_0$, and $\eta_n \to \eta_0$. □

Let F_θ for $\theta = (\eta, \tau) \in \Theta_0 = (-1, 1) \times \mathbb{R}$ denote the df of the power limit

$$W_\theta = \chi_\theta(M) = e^{\psi_\tau(aM)} 1_{\{M>0\}} - e^{\psi_\tau(q|M|)} 1_{\{M<0\}} \quad a, q > 0, a \vee q = 1, a - q = \eta, \tau \in \mathbb{R}, \tag{3.10}$$

where $\psi_\tau \in \Phi$, see Lemma 1.4, is $\psi_\tau(u) = \operatorname{sign}(\tau) u^\tau$ for $|\tau| \geq 1$, and

$$\psi_t(u) = \frac{u^\tau - 1}{\tau} + \tau \qquad 0 < |\tau| \leq 1; \qquad \psi_0(u) = \log u. \tag{3.11}$$

Continuity of $\tau \mapsto \psi_\tau$ implies continuity of $\theta \mapsto \chi_\theta$ in Eq. 3.10. This yields:

Theorem 3.11 *Let k be a Smirnov sequence, see* Eq. 1.2, *and M the associated Gaussian variable in* Eq. 1.5. *For $\theta = (\eta, \tau) \in \Theta_0 = (-1, 1) \times \mathbb{R}$ define F_θ as the df of the power limit W_θ in* Eq. 3.10. *Then*

- *the map $\theta \mapsto F_\theta$ is continuous;*
- *F_θ has domain $\mathcal{D}_0(\theta)$ for $\theta \in \Theta_0$ and these domains are disjoint;*
- *every continuous power limit has df $F_\theta(A(x))$ for some $\theta \in \Theta_0$, $A \in \mathcal{P}$.*

Corollary 3.12 *The space $\mathcal{L}_c(\mathcal{P}, k, 0)$ is homeomorphic to $\mathbb{R}^4$.*

The F_θ form a continuous selection of dfs, one for each domain. If $G_n \to G_0$ are continuous power limits and we choose F_{θ_n} to have the same domain as G_n, then $F_{\theta_n} \to F_{\theta_0}$.

Theorem 3.13 *The continuous power limits for central order statistics for a given Smirnov sequence k and under the quantile condition* (1.1) *form a closed subset of the set of all continuous dfs.*

Proof Suppose $G_n \to G$ with G continuous and $G_n \in \mathcal{L}_c(\mathcal{P}, k, 0)$. Then $\chi_n \to \chi$ weakly where $\chi(M)$ has df G; χ is positive on $(0, \infty)$ and negative on $(-\infty, 0)$ by continuity of G. Hence $\bar{h}_n$ and $\hat{h}_n$ converge weakly on $(0, \infty)$. By Lemma (1.4) the limit functions $\bar{h}$ and $\hat{h}$ lie in Φ. Continuity of τ implies that $\bar{h}$ and $\hat{h}$ have the same exponent. The argument in the proof of Proposition 2.10 shows that $\bar{h}(u) = \hat{h}((1-\eta)u)$, where $\eta \in (-1, 1)$ is the limit of the balance η_n of χ_n. It follows that the pair $(\bar{h}, \hat{h})$ determines a function χ and that $\chi_n \to \chi$. □

4 Power limits without quantile restrictions

$F(0) < p$ implies a positive p-quantile. We may assume that $\inf\{F > p\} = 1$ by an initial scaling. The inequality $F(0) < p$ implies $F(\delta) < p$ for some $\delta > 0$ and hence $\mathbb{P}\{X_{k:n} > \delta\} \to 1$. Altering the df F on $(-\infty, \delta)$ has no influence on the limit distribution of the normalized order statistics. We may assume $F(0) = 0$. Then X is positive and one may consider convergence $(Y_{k:n} - b_n)/a_n \Rightarrow Z$ where $Y = \log X$. This is equivalent to $(X_{k:n}^{\wedge 1/a_n})/e^{b_n} \Rightarrow W$ for a positive power limit W. There may be non-negative power limits which charge zero, but we restrict attention to continuous power limits. Thus we find:

Theorem 4.1 *Let k be a Smirnov sequence and suppose $A_n^{-1}(X_{k:n}) \Rightarrow W$ for the order statistics from a df F which satisfies $F(0) < p$. Then*

$$W = ce^{(aM)^\tau} 1_{\{M>0\}} + ce^{|qM|^\tau} 1_{\{M<0\}} \qquad a, c, q, \tau > 0; \qquad \eta = (a-q)/(a \vee q). \tag{4.1}$$

The domains $\mathcal{D}_1(\eta, \tau)$ of $W = A(e^{M_\eta^{\wedge\tau}})$ have been characterized by Smirnov. His characterization is in terms of the df of $\log(X \vee \delta)$ for an appropriate $\delta > 0$. It may be translated into a condition on F. Alternatively write $X = f(U)$ where f is increasing and U uniformly distributed on $(-p, 1-p)$. The assumption $\inf\{F > p\} = 1$ implies that $f(0+) = 1$ and the assumption $F(\delta) < p$ for some $\delta > 0$ implies that f is positive on a neighbourhood of zero. Hence $g = \log f$ is well defined on a neighbourhood of zero and vanishes in $0+$. The condition that $g(u/\sqrt{n}) - b_n)/a_n \to h(u)$ on $(0, \infty)$ and that $h(0-)$ is finite imply that g varies regularly in $0+$ with exponent $\tau > 0$.

Proposition 4.2 *Suppose $F(0) < p$ and $\inf\{F > p\} = 1$. Then $F \in \mathcal{D}_1(\eta, \tau)$ if and only if the function $F(1+x) - p$ varies regularly for $x \to 0+$ with exponent $1/\tau$, and F satisfies the balance condition $(p - F(1-x))/(F(1+x) - p) \to R \in (0, \infty)$ where $(R-1)/(R \vee 1) = \eta$.*

Let F_λ for $\lambda = (\eta, \tau) \in \Lambda_0 = (-1, 1) \times (0, \infty)$ be the df of $W = \chi_\lambda(M)$ where

$$\chi_\lambda(u) = e^{(au)^\tau} 1_{(0,\infty)}(u) + e^{|qu|^\tau} 1_{(-\infty,0)}(u) \quad a, q > 0,\ a \vee q = 1,\ a - q = \eta,\ \lambda = (\eta, \tau). \tag{4.2}$$

Recall that $\chi^*(u) = -\chi(-u)$. The dfs of $W = \chi^*_\lambda(M)$ describe the power limits when $F(0-) > p$.

Now drop the quantile condition. Let Γ_0 be the disjoint union of the sets $\Theta_0 = (-1, 1) \times \mathbb{R}$, $\Lambda_0 = (-1, 1) \times (0, \infty)$ and $\Lambda^*_0 = (-1, 1) \times (0, \infty)$. Define F_γ as the df of W_θ in Eq. 3.10 for $\theta \in \Theta_0$, and as the df of $\chi_\lambda(M)$ or $\chi^*_\lambda(M)$ for $\lambda \in \Lambda$ or $\lambda \in \Lambda^*$ with χ_λ as in Eq. 4.2.

Theorem 4.3 *Let k be a Smirnov sequence and let $\mathcal{L}_c(\mathcal{P}, k)$ be the set of all continuous power limits for the sequence of order statistics $X_{k:n}$ normalized by power transformations. Then $\mathcal{L}_c(\mathcal{P}, k)$ is the set of dfs $F_\gamma(A(x))$, $\gamma \in \Gamma_0$, $A \in \mathcal{P}$. The map $\gamma \mapsto F_\gamma$ is a homeomorphism. The domains $\mathcal{D}(F_\gamma, \mathcal{P}, k)$, $\gamma \in \Gamma_0$, are disjoint.*

5 Conclusion

Power limits give a substantial extension of the limit theory for central order statistics developed by Smirnov (1949). This holds in particular for dfs F which satisfy the quantile condition (1.1). There are more limit laws and domains are larger. Even when the quantile is not unique, and the order statistics converge in distribution to a two-valued limit without any normalization, power transformations may yield a continuous limit law. This implies that a sequence of order statistics may have two unrelated power limits. Continuous limit laws have disjoint domains, but in general domains may overlap, or one domain may contain another as a proper subset.

If the order statistics converge in probability to a positive constant they may have a continuous power limit. This power limit then has the form $W = e^Z$ where Z is a limit variable in Smirnov's theory. Normalizing by power transformations does not add anything new here.

There exist finite dimensional extensions of the one-dimensional group $\mathcal{S}$ of scale transformations apart from $\mathcal{P}$. Our basic tool, Proposition 1.1, will work for any extension of $\mathcal{S}$ which preserves the origin. A paper on the topic of finite-dimensional normalization groups for extremes and order statistics is scheduled to appear next year.

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Appendix: Extremes

In multivariate extreme value theory one may be interested in extremes of the sample rather than coordinatewise maxima. It may be possible to scale samples from a random vector $\mathbf{X}$ to converge weakly to a Poisson point process outside ϵ-balls centered at the origin. What happens if one replaces scaling by power transformations?

$$\mathbf{x} = r\xi \mapsto cr^a\xi \qquad a, c > 0; \quad r = \|\mathbf{x}\| > 0.$$

We consider the univariate case below.

The transformation $x \mapsto -1/x$ is increasing apart from the discontinuity at the origin. It maps points close to the origin into points far out and vice versa. For a variable X with a df which is continuous at the origin with the value $F(0) = p \in (0, 1)$ the asymptotic behaviour of $F - p$ for $F(x) \to p$ is reflected in the tail behaviour of the df of $Y = -1/X$. Power transformations are able to handle the transformation $Y = -1/X$.

For a power transformation $A : x \mapsto cx^{\wedge a}$ define the conjugate transformation $\bar{A}$ by

$$\bar{A}(x) = x^{\wedge a}/c. \tag{5.1}$$

Conjugation is an automorphism: $\bar{AB} = \bar{A}\bar{B}$. Moreover $\bar{A}(-1/x) = -1/A(x)$. This implies:

Lemma 5.1 *Suppose $A_n \in \mathcal{P}$ and $W_n = A_n^{-1}(X_n) \Rightarrow W_0$. Assume $\mathbb{P}\{W_n = 0\} = 0$ for $n \geq 0$. Then $Y_n = -1/X_n$ and $Z_n = -1/W_n$ are well defined and $Z_n = \bar{A}_n^{-1}(Y_n) \Rightarrow Z_0$.*

Suppose $X = f(U)$ has df $F \in \mathcal{D}_0(\eta, \tau)$ in the notation of this paper. Then $\{F = p\}$ is a set $[-x^*, x^*]$ with $x^* \geq 0$ and F is continuous in $\pm x^*$. The point process $N_n^X = \{f(U_{1:n}), \ldots, f(U_{n:n})\}$ is a sample from X. The corresponding samples $N_n^U = \{U_{1:n}, \ldots, U_{n:n}\}$ from U, inflated by a factor n converge: $N_n := nN_n^U \Rightarrow N$ vaguely on $\mathbb{R}$, where N is the standard Poisson point process on $\mathbb{R}$. Let k be a Smirnov sequence and A_n power transformations such that $A_n^{-1}(X_{k:n}) \Rightarrow \chi(M)$.

Proposition 5.2 *Set $B_n = A_{n^2}$. Then $B_n^{-1}(N_n^X) \Rightarrow \chi(N)$ weakly on $[a, b]$ for any closed interval $[a, b] \subset (\chi(-\infty), \chi(\infty))$.*

Proof By Proposition 1.1 $A_n^{-1} f(u/\sqrt{n}) \to \chi(u)$ weakly on $\mathbb{R}$. Hence $\chi_n(u) := B_n^{-1} f(u/n) \to \chi(u)$ weakly on $\mathbb{R}$. By Skorohod's representation theorem one may assume that $N_n^X \to N$ a.s. vaguely on $\mathbb{R}$. It follows that $K_n := \chi_n(N_n) \to \chi(N) =: K$ a.s. vaguely on $(\chi(-\infty), \chi(\infty))$. This is equivalent to $\int \varphi dK_n \to \int \varphi dK$ for every bounded function which vanishes outside a compact interval in $(\chi(-\infty), \chi(\infty))$ and which is κ-a.e. continuous for the mean measure κ of K, the image of Lebesgue measure under χ. Since χ is strictly increasing κ is continuous and $\varphi = \psi 1_{[a,b]}$ is κ-a.e. continuous and bounded for any continuous function ψ on $[a, b]$. □

The lemma above transforms this proposition into a result on extremes:

Theorem 5.3 *Let $X_{k:n}$ be order statistics from a df F for a Smirnov sequence k, and let M be the Gaussian variable associated with k, see* Eqs. 1.2 *and* 1.5. *Assume $A_n^{-1}(X_{k:n}) \Rightarrow W$ for a continuous power limit $W = \chi(M)$ and for a sequence of power transformations $A_n : x \mapsto c_n x^{\wedge a_n}$. Define $Q_n(x) = x^{\wedge a_{n^2}}/c_{n^2}$. The df F is continuous in zero and hence $Y = -1/X$ is well-defined. Let $N_n^Y = \{Y_{1:n}, \ldots, Y_{n:n}\}$*

be a sample of size n from Y. Let K be the image of the standard Poisson point process N on $\mathbb{R}$ *under the map* $x \mapsto -1/\chi(x)$. *Then*

$$Q_n^{-1}\left(N_n^Y\right) \Rightarrow K \qquad \text{weakly on } \mathbb{R}\backslash J$$

for any open interval J which contains the two elements $-1/\chi(\infty) < 0$ *and* $-1/\chi(-\infty) > 0$.

The condition $F \in \mathcal{D}_0(\eta, \tau)$ may be formulated in terms of three conditions on the df of Y. Let V be the standard Gumbel variable and $V_\sigma = \text{sign}(\sigma)e^{\sigma V}$ for $\sigma \neq 0$ and $V_0 = V$, let y_* and y^* be the lower and upper endpoint of the df of Y and $c \in (0, y^*)$. The three conditions are:

- (symmetric domain) $y_* = -y^*$;
- (balance) $\mathbb{P}\{Y > y\}/\mathbb{P}\{Y < -y\} \to r \in (0, \infty)$ for $y \to y^* - 0$ with $\eta = (r-1)/(r \vee 1)$;
- (exponent) the df of $\log(Y \vee c)$ lies in $\mathcal{D}_{\max}(V_\sigma, \mathcal{A})$ for $\sigma = -\tau$.

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