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SEMI-SELF-SIMILAR EXTREMAL PROCESSES*

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Let g be the distribution function (d.f.) of an extremal process Y. If g is invariant with respect to a continuous one-parameter group of time-space changes $\{\eta_\alpha = (\tau_\alpha, L_\alpha)\colon\ \alpha > 0\}$, i.e., $g \circ \eta_\alpha = g\ \forall\alpha > 0$, then g is self-similar. If g is invariant w.r.t. the cyclic group $\{\eta^{\circ(n)},\ n \in \mathbf{Z}\}$ of a time-space change η, then g is semi-self-similar. The semi-self-similar extremal processes are limiting for sequences of extremal processes $Y_n(t) = L_n^{-1} \circ Y \circ \tau_n(t)$ if going along a geometrically increasing subsequence $k_n \sim \varphi^n$, $\varphi > 1$, $n \to \infty$. The main properties of multivariate semi-self-similar extremal processes and some examples are discussed in the paper. The results presented are an analog of the theory of semi-self-similar processes with additive increments developed by Maejima and Sato in 1997.

1. Introduction and Background

Extremal processes are stochastic processes with increasing right-continuous sample paths and independent max-increments. Without loss of generality, we consider extremal processes $Y\colon\ [0,\infty) \to [0,\infty)^d$ whose time space is the positive axis and whose state space is the positive orthant in $\mathbf{R}^d$. The independence of the max-increments means that for any finite sequence of time points $0 = t_0 < t_1 < \cdots < t_k$ there exist independent random vectors $U_0, \dots, U_k$ in $[0,\infty)^d$ such that

$$(Y(t_0), \dots, Y(t_k)) \stackrel{d}{=} (U_0, U_0 \vee U_1, \dots, U_0 \vee \cdots \vee U_k).$$

The extremal processes in this general setting are studied in [2, 3]. We use the following characteristics of an extremal process:

1. The *distribution function* (d.f.) $f(t,x) = \mathbf{P}(Y(t) < x)$. We observe that

- $f\colon\ (0,\infty)^{d+1} \to [0,1]$ is lower semi-continuous, so $f(t,x) = f(t+0, x-0)$;
- for any fixed t, the function $f_t(x) := f(t,x)$ is a d.f. on $[0,\infty)^d$;
- for $0 < s < t$, $f_s \mid f_t$, i.e., there exists a d.f. $H_{(s,t)}$ such that $f_t = f_s . H_{(s,t)}$.

Conversely, any function f with the three properties above is a d.f. of an extremal process. The family of univariate marginals $\{f_t :\ t \geq 0\}$ of the extremal process determines all finite-dimensional distributions (f.d.d.), because for $t_0 < \cdots < t_k$, $x_0 < \cdots < x_k$

$$F_{t_0,\dots,t_k}(x_0, \dots, x_k) = f_{t_0}(x_0) \frac{f_{t_1}(x_1)}{f_{t_0}(x_1)} \cdots \frac{f_{t_k}(x_k)}{f_{t_{k-1}}(x_k)}.$$

2. The *lower curve* $C\colon\ [0,\infty) \to [0,\infty)^d$ of an extremal process Y is defined coordinatewise by $C^{(i)}(t) = \inf\{f_t^{(i)} > 0\}$, $i = 1, \dots, d$. It is a uniquely determined increasing right-continuous curve below which the sample paths of Y cannot pass.

3. The max-increments of an extremal process are not uniquely determined by the process. This interesting phenomenon observed in the multivariate extreme value theory is called blotting, and it is discussed in [2]. However, we can always choose and fix a consistent family of max-increments $U(s,t]$, $0 \leq s < t$, such that a.s.

- $U(s,t] \geq C(t)$;
- $Y(t) = Y(s) \vee U(s,t]$;
- for any $0 = t_0 < t_1 < \cdots < t_k$, the vectors $Y(0), U(t_0,t_1], \dots, U(t_{k-1},t_k]$ are independent.

This states the *structure theorem* proved in [2].

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4. Every extremal process Y with lower curve C is generated by an *associated point process* $N = \{(T_k, X_k)\colon\ k = 0,1,2,\dots\}$ on the open set $[0,C]^c = ([0,\infty) \times [0,\infty]^d) \setminus [0,C]$ by

$$Y(t) = C(t) \vee \sup\{X_k\colon\ T_k \le t\}. \tag{1.1}$$

Almost all realizations of N are Radon measures on $[0,C]^c$, i.e., a.s.

$$N([0,t] \times [0,x)^c) < \infty, \quad \forall t \ge 0, \quad x > C(t),$$

hence the supremum on the RHS in (1.1) is, in fact, the maximum of finite many space points. Further, N is simple in time and the restrictions $N(B_1),\dots,N(B_k)$ to disjoint time slices $B_1,\dots,B_k$ are independent. Such point processes we call *Bernoulli*, and they are discussed in [3].

The structure theorem answers the question: how far does a given family $\{f_t\colon\ t \ge 0\}$ of d.f.'s on $[0,\infty)^d$ determine an extremal process Y? In the max-infinitely divisible (max-i.d.) case, the set $\operatorname{int}\{f_t > 0\}$ is an open block, $C(t) = \inf\{f_t > 0\}$, and the quotient f_t/f_s for $0 \le s < t$ determines the d.f. of the max-increments $U(s,t]$ uniquely on the set $A_t = (C(t),\infty)$ above the lower curve.

In the present paper, we deal with max-i.d. extremal processes only. As is known, they are associated with Poisson point processes and there exists a simple connection between the d.f. of the extremal process and the mean measure μ of the associated point process, namely,

$$f(t,x) = \exp\{-\mu([0,t] \times [0,x)^c)\}.$$

We are interested in characterizing a special type of extremal process, satisfying the characteristic equation

$$Y \circ \sigma(t) \overset{d}{=} L \circ Y(t), \quad \forall t \ge 0. \tag{1.2}$$

We call this type a semi-self-similar extremal process and denote it by semi-ss. The time-space change $\eta(t,x) = (\sigma(t), L(x))$, $t \in (0,\infty)$, $x \in (0,\infty)^d$, we choose continuous and strictly increasing in each coordinate, hence it is a max-automorphism of $(0,\infty)^{d+1}$. In Sec. 2, we study some of the main properties of time-space changes that we need further, e.g., the fact that the cyclic group of a time-space change can be embedded in a continuous one-parameter group. Section 3 gathers the direct consequences of the characteristic equation (1.2). There the main result states that a semi-self-similar extremal process is a max-i.d. process, either stochastically continuous everywhere or having infinitely many fixed discontinuities.

In Sec. 4, we obtain a semi-self-similar extremal process as limiting in a triangular array with asymptotic negligibility condition when going along a geometrically increasing subsequence. Theorem 4.1 states that the univariate marginals of the limiting semi-ss process are semi-self-decomposable with respect to (w.r.t.) the max-operation, briefly semi-MSD. Theorem 4.2 is an analog of Theorem 5.3 in [7] and shows that a semi-MSD random vector X can be embedded in a stochastically continuous semi-self-similar extremal process Y so that a.s. $Y(0) = C(0)$, $Y(1) = X$, $Y(\infty) = \infty$.

Section 5 is a brief account of semi-self-similar extremal processes with stationary max-increments. An extremal process $Y\colon\ [0,\infty) \to [0,\infty)$ with stationary max-increments is semi-ss if and only if its d.f. at $t = 1$ has the explicit form

$$\left\{e^{e^{-\gamma h(x)}}\right\}^{p_\varphi(h(x))},$$

where $h\colon\ (0,\infty) \to (-\infty,\infty)$ is a continuous homomorphism and p_φ is a periodic function with period $T = \log\varphi$, positive and bounded, and $\gamma > 0$. Then $f(t,x) = f_1^t(x)$. We end with several examples of semi-self-similar extremal processes.

2. Time-Space Changes

Let the time-state space S be the open block $(0,\infty)^{d+1}$. A time-space change of S is an increasing homeomorphism $\xi\colon\ S \leftrightarrow S$ with

$$\xi(t,x) = (\xi_0(t), \xi_1(x_1), \dots, \xi_d(x_d)),$$

where the one-to-one mapping $\xi_i\colon\ (0,\infty) \leftrightarrow (0,\infty)$, $i = 0,1,\dots,d$, is strictly increasing, hence continuous. Such mappings preserve the max-operation, i.e., $\xi(z_1 \vee z_2) = (\xi(z_1) \vee \xi(z_2)$, $z_1, z_2 \in S$, so they are max-automorphisms of the time-state space S. The max-automorphisms of S form a group w.r.t. the composition, $\xi \circ \eta(z) = \xi(\eta(z))$, $z \in S$, and we denote it by $\mathrm{MA}(S)$. We are interested in studying them, since they are proper norming mappings in the extreme value theory (cf. [9]).

Let $\Gamma = \{\xi_\alpha : \alpha > 0\} \subset \text{MA}(S)$ be a continuous one-parameter group (c.o.g.), i.e.,
- $\xi_1 = \text{id}$ (here id stays for the identical mapping);
- $\xi_\alpha \circ \xi_\beta = \xi_{\alpha\beta}$, $\alpha, \beta > 0$;
- $\xi_\alpha \to \xi_\beta$ if $\alpha \to \beta$.

We call Γ a *norming group* of time-space changes of S if the following boundary condition is met:

(BC) $$\xi_\alpha(z) \longrightarrow \vec{0}, \quad \alpha \downarrow 0, \qquad \xi_\alpha \longrightarrow \vec{\infty}, \quad \alpha \uparrow \infty$$

for every $z \in S$. One can check that for a norming c.o.g. Γ:
- the correspondence $\alpha \to \xi_\alpha$ is strictly increasing and continuous;
- for $\alpha > 1$, ξ_α is expanding, i.e., $\xi_\alpha(z) > z$, and for $\alpha < 1$, ξ_α is contracting, i.e., $\xi_\alpha(z) < z$;
- $\vec{0}$ and $\vec{\infty}$ are the only fixed points of ξ_α.

Proposition 2.1. *Any norming c.o.g. $\Gamma \subset \text{MA}(S)$ has the form*

$$\xi_\alpha(z) = h^{-1}\{h(z) + ec\log\alpha\}, \quad \alpha > 0, \tag{2.1}$$

where $h\colon S \leftrightarrow \mathbf{R}^{d+1}$ is an increasing homeomorphism, e is the unit vector, and c is a positive constant.

The proof of this statement (e.g., [10]) consists of giving a solution of the functional equation

$$\xi[\alpha, \xi[\beta, z]] = \xi[\alpha\beta, z],$$

and it is carried out in a way analogous to Theorem 20 in [1].

Representation (2.1) can be written briefly as $\xi_\alpha = h^{-1} \circ D_{c\log\alpha} \circ h$, where $D_r(z) = z + er$, $r \in \mathbf{R}$. This means that there is a change h of the coordinates, $z' = h(z)$, so that $y'_\alpha = h(\xi_\alpha)$ in the new coordinates is just a translation $y'_\alpha = z' + ec\log\alpha$ along the diagonal.

Below, we denote the n-time composition $\xi \circ \cdots \circ \xi$ by $\xi^{\circ(n)}$, $\xi^{-1}, \circ \cdots \circ, \xi^{-1}$ by $\xi^{\circ(-n)}$, and $\xi^0 = \text{id}$.

Proposition 2.2. *Let η be a time-space change of S. Suppose that the cyclic group $\Gamma(\eta) = \{\eta^{\circ(n)},\ n \in \mathbf{Z}\}$ satisfies the boundary condition* (BC), *i.e.,*

$$\eta^{\circ(-n)}(z) \longrightarrow \vec{0}, \quad n \to \infty,$$

$$n^{\circ(n)}(z) \longrightarrow \vec{\infty}, \quad n \to \infty.$$

Then $\Gamma(\eta)$ can he embedded in a c.o.g. $\{\xi_t : t > 0\}$ such that $\eta = \xi_\varphi$ for some $\varphi > 1$ and $\eta^{\circ(n)} = \xi_{\varphi^n}$.

Proof. Let $\varphi > 1$ be fixed. Latter we shall determine the value of φ uniquely. Define the subset in $\mathbf{R}$

$$\mathcal{S} = \{\log\varphi^n : n \in \mathbf{Z}\}$$

and set $\eta_s(\cdot) := n^{\circ(n)}$ for $s = \log\varphi^n \in \mathcal{S}$. We have

$$\eta_s \circ \eta_v(z) = \eta_{s+v}(z), \quad s, v \in \mathcal{S}.$$

Further, the assumption that $\Gamma(\eta)$ satisfies the boundary condition (BC) for $n \to \pm\infty$ implies that the correspondence $s \to \eta_s$ is strictly increasing. Indeed, assume there are $s, v \in \mathcal{S}$ with $s = \log\varphi^m < v = \log\varphi^n$, $m < n$, and such that $\eta_v \le \eta_s$. Then

$$z \ge \eta_s^{-1} \circ \eta_v(z) = \eta^{\circ(-m)} \circ \eta^{\circ(n)}(z) = \eta^{\circ(n-m)}(z),$$

and this is a contradiction to $\eta^{\circ(k)}(z) \to \vec{\infty}$ for $k \to \infty$. Moreover, $\{\eta_s : s \in \mathcal{S}\}$ satisfies the boundary condition

$$\eta_s(z) \to \vec{\infty}, \quad s \to \infty,$$

$$\eta_s(z) \to \vec{0}, \quad s \to -\infty.$$

For $z \in S$, we call the set

$$\mathcal{T}_z = \{\eta^{\circ(n)}(z) : n \in \mathbf{Z}\} = \{\eta_s : s \in \mathcal{S}\}$$

the *track* of η through the point z. Every $z \in S$ has a track that starts at $\vec{0}$ and goes up to $\vec{\infty}$.

We can embed the track $\mathcal{T}_z$ in a curve $\eta(t)$ continuous in t so that for $t = n$, $\eta(n, z) = \eta^{\circ(n)}(z)$, $n \in \mathbf{Z}$. Indeed, define $\eta^\alpha := \alpha\eta + (1-\alpha)\text{id}$ for $\alpha \in (0,1)$, $\eta^1 = \eta$, $\eta^0 = \text{id}$. For $n \le t < n+1$, i.e., for $t = n + \alpha$, where $n = [t]$ and $\alpha = \{t\}$, we define $\eta(t, z) := \eta^\alpha \circ \eta^{\circ(n)}(z)$. Here $[t]$ and $\{t\}$ are the integer and the fractional part of t, respectively. Obviously, the correspondence $t \to \eta(t)$ is continuous and strictly increasing, hence, one-to-one, and $\eta(n,z) = \eta^{\circ(n)}(z) = \eta_s(z)$ if $s = \log \varphi^n$.

Now the orbit $\mathcal{O}_z$ through the point z defined by $\mathcal{O}_z = \{\eta(t,z): t \in \mathbf{R}\}$ overlaps the track $\mathcal{T}_z$. If $z_1 \neq z_2$, then either $\mathcal{O}_{z_1}$ does not intersect $\mathcal{O}_{z_2}$ or both orbits coincide. In the latter case, there is $s \in \mathbf{R}$ such that $z_1 = \eta(s, z_2)$.

Next we consider $y_s = \eta_s(z)$ as a function on $\mathcal{S}$. The correspondence $\eta_s \leftrightarrow es$ is one-to-one; let h^* be this single strictly increasing homeomorphism that "bends" the diagonal $\{es: s > 0\}$ into the orbit $\mathcal{O}_z = \{y_s = h^*(es): s \in \mathbf{R}\}$ overlapping $\mathcal{T}_z$. Now the group property $\eta_s \circ \eta_v(z) = \eta_{s+v}(z)$, $s, v \in \mathcal{S}$, can be written as

$$\eta_s(y_v) = y_{v+s} = h^*(e(v+s)) = h^*(ev + es) = h^*(h(y_v) + es),$$

where h^* is the mapping inverse to h and $h: \ \mathcal{S} \leftrightarrow \mathbf{R}^d$. Since $\eta_v(z) = z$, for $v = 0$ from the abovesaid we have $\eta_s(z) = h^{-1}(h(z) + es)$ and consequently

$$\eta(z) = \eta_{\log\varphi}(z) = h^{-1}(h(z) + e\log\varphi). \tag{2.2}$$

Thus, $\Gamma(\eta)$ is embedded in the c.o.g. $\{\xi_t(\cdot) = h^{-1}(h(\cdot) + e\log t): t > 0\}$ with $\eta^{\circ(n)} = \xi_{\varphi^n}$, $n \in \mathbf{Z}$.

Let $z = (t, x_1, \dots, x_d)$ and $\eta(z) = (\tau(t), L(x))$. The mapping $h(z) = (h_0(t), h_1(x_1), \dots, h_d(x_d))$ acts coordinatewise and $\log\varphi = h_0(\tau(1)) - h_0(1) =: \Delta$, hence $\varphi = e^{\Delta}$.

Note that the pair $[\varphi, h]$ uniquely determines the time-space change η. Conversely, η determines h uniquely up to a translation, namely, if $h_2 = h_1 + e\log a$, then $\eta(\cdot) = h_1^{-1}(h_1(\cdot) + e\log\varphi) = h_2^{-1}(h_2(\cdot) + e\log\varphi)$, and h_0 determines φ uniquely. We shall denote this relation by $\eta = [\varphi, h]$.

We supply the set $\mathrm{MA}(S)$ with the topology τ of the pointwise convergence. Let g_n and η_n be sequences of d.f.'s of extremal processes and of time-space changes, respectively. If $g_n \xrightarrow{\omega} g$ and $\eta_n \xrightarrow{\tau} \eta$, then the continuity of the composition entails that

$$g_n \circ \eta_n \xrightarrow{\omega} g \circ \eta, \quad n \to \infty.$$

The convergence $\eta_n \xrightarrow{\tau} \eta$ does not imply that η is a time-space change (i.e., strictly increasing and continuous).

Let us denote $\mathcal{P}_\psi = \{\{\eta\} \subset \mathrm{MA}(S): \ (*)\ e\varepsilon \le \eta(z + e\varepsilon) - \eta(z) \le \psi(e\varepsilon) \to 0 \text{ as } \varepsilon \to 0\}$, where $\psi: \ S \to S$. Put $\mathcal{P} = \bigcup_\psi \mathcal{P}_\psi$.

The sequences $\eta_n \in \mathcal{P}$ are equicontinuous. If there exists a limit mapping η, then the RHS of condition $(*)$ implies its continuity and the LHS guarantees its strict monotonicity.

Let f and g be nondegenerate d.f.'s of extremal processes and let the sequence $\eta_n \in \mathcal{P}$ of time-space changes be τ-compact. If

$$f_n \xrightarrow{\omega} f, \qquad g_n = f_n \circ \eta_n \xrightarrow{\omega} g,$$

then there is a time-space change η such that $g = f \circ \eta$. This is stated by the convergence of type theorem (CTT) for max-automorphisms of S (cf. [10]). In the limit theorems of Sec. 4 and 5, we use the continuity of the composition rather than the CTT. We assume directly that the norming sequence η converges to a time-space change η instead of the following assumptions: $\{\eta_n\} \in \mathcal{P}$ and $\{\eta_n\}$ is τ-compact.

3. Semi-Self-Similarity

Definition. An extremal process Y: $[0,\infty) \to [0,\infty)^d$ with d.f. g is referred to as *semi-self-similar* if there exists some time-space change $\eta = (\tau, L)$ of $(0,\infty)^d$ for which the cyclic group $\Gamma(\eta)$ satisfies (BC) and such that

$$Y \circ \tau(t) \stackrel{d}{=} L \circ Y(t), \quad \forall t > 0, \tag{3.1}$$

or, equivalently,

$$g(\tau(t), x) = g(t, L^{-1}(x)).$$

Below, we give several direct consequences of the definition. The n-time iteration of (3.1) shows that

$$Y \circ \tau^{\circ(n)}(t) \stackrel{d}{=} L^{\circ(n)} \circ Y(t), \quad \forall n \in \mathbf{Z}.$$

Proposition 3.1. *The d.f. g of a semi-self-similar extremal process is invariant w.r.t. the cyclic group $\Gamma(\eta)$ of a time-space change η, i.e., $g \circ \eta^{\circ(n)} = g$, $\forall n \in \mathbf{Z}$.*

Recall that the univariate marginals $g_t(x) = g(t,x)$ are left-continuous in x.

Proposition 3.2. *If Y is a semi-self-similar extremal process stochastically continuous at $t = 0$ and if g_1 is continuous at the upper boundary of the support, then $Y(0) = C(0)$ a.s.*

Proof. The assumptions made permit the chain of equalities

$$g(0+, x) = \lim_{n\to\infty} g(\tau^{\circ(-n)}(1), x) = \lim_{n\to\infty} g(1, L^{\circ(n)}x) = 1,$$

i.e., $\mathbf{P}(Y(0) < x) = 1$, $\forall x > C(0)$. Hence $Y(0) = C(0)$ a.s.

Proposition 3.3. *Let Y be a semi-self-similar extremal process with lower curve C and associated Bernoulli point process $N = \{(T_k, X_k):\ k = 0, 1, \dots\}$ on $[0, C]^c$. If the d.f.'s of X_k are continuous at the upper boundary, then Y is max-i.d.*

Proof. By the decomposition theorem (cf. [2]), any extremal process Y is the maximum of two independent extremal processes Y' and Y'' with common lower curve C. The process Y' is generated by a Poisson point process N', hence it is max-i.d. The process Y'' is generated by a point process $N'' = \{(t_k, U_k):\ k = 1, 2, \dots\}$, where $t_1 < t_2 < \dots$ are fixed discontinuities of Y and U_k is the max-increment of Y at t_k. So the point process $N = N' \oplus N''$ is the Bernoulli point process associated with Y. We have still to show that Y'' is max-i.d. too.

Let F_k be the d.f. of U_k. Then, by (3.1),

$$Y_n(t) := L^{\circ(-n)} \circ Y'' \circ \tau^{\circ(n)}(t) = C_n(t) \vee \sup\{L^{\circ(-n)} \circ U_k:\ \tau^{\circ(-n)}(t_k) \le t\} \stackrel{d}{=} Y''(t)$$

or what is the same

$$g_n(t, x) = \prod\{F_k(L^{\circ(n)}x):\ t_k \le \tau^{\circ(n)}(t)\} = g''(t, x)$$

for all $t > 0$ and $x > C(t)$.

The continuity of F_k at the upper boundary, i.e., $F_k(x) \to 1$ for $x \uparrow \partial\{F_k = 1\}$, and the boundary condition $L^{\circ(n)}(x) \to \vec{\infty}$ imply that the r.v.'s $X_{nk} := L^{\circ(-n)} \circ U_k$ are asymptotically negligible, namely the condition

$$\text{(AN)} \qquad F_k(L^{\circ(n)}x) = \mathbf{P}(U_k < L^{\circ(n)}x) \longrightarrow 1, \quad n \to \infty,$$

is met for all $x > \vec{0}$. Note that if the sequence of time points t_k is finite, then $g''(t, x)$ will be degenerate.

Denote $t_{nk} := \tau^{\circ(-n)}(t_k)$. Since $\{Y_n\}$ is a sequence of extremal processes generated by an array $\{(t_{nk}, X_{nk}):\ k \ge 1\}$, $n \ge 1$, with (AN)-condition, Y'' is max-i.d.

As a by-product of the above proof, we see that a semi-ss extremal process is either stochastically continuous at all $t \ge 0$ or there is an infinite sequence $\{t_k\}$ of fixed discontinuities.

Proposition 3.4. *A semi-ss d.f. g is either continuous everywhere or there is at least one infinite sequence $z_n \uparrow \vec{\infty}$ of discontinuity points.*

Proof. Let $z = (t, x)$ be a discontinuity point of g, i.e., $g(t - 0, x + 0) > g(t, x)$. Then for all $n \in \mathbf{Z}$

$$g(\tau^{\circ(n)}(t) - 0, L^{\circ(n)}(x) + 0) > g(\tau^{\circ(n)}(t), L^{\circ(n)}(x)).$$

Let f and g be d.f.'s of extremal processes. We say that f belongs to *type* (g) if there exists a time-space change $\xi = (\sigma, T)$ such that $f = g \circ \xi$.

Proposition 3.5. *Semi-self-similarity is a type property.*

Proof. Assume that Y is semi-ss w.r.t. $\eta = (\tau, L)$, and let $\xi = (\sigma, T)$ be a time-space change. Define the processes $X_1 = Y \circ \sigma$, $X_2 = T^{-1} \circ Y$, and $X_3 = T^{-1} \circ Y \circ \sigma$. Then X_1, X_2, and X_3 are semi-ss w.r.t. the time-space changes (τ^*, L), (τ, L^*), and (τ^*, L^*), respectively, where $\tau^* := \sigma^{-1} \circ \tau \circ \sigma$ and $L^* = T^{-1} \circ L \circ T$. Further, one can check that $\Gamma((\tau^*, L^*))$ satisfies the condition (BC).

Let us come back to representation (2.2) of the norming mapping $\eta = [\varphi, h]$. Denote $f = g \circ h^{-1}$. Then the semi-self-similarity equation $g = g \circ \eta$ implies $f(z) = f(z + es)$, $\forall z \in \mathbf{R}^{d+1}$ and $s \in \{\log \varphi^n:\ n \in \mathbf{Z}\}$. Recall that h acts coordinatewise. We denote the space change $h_* := (h_1, \dots, h_d)$, hence $h = (h_0, h_*)$. If g is a d.f. of a semi-self-similar extremal process Y w.r.t. η, then f is a d.f. of the extremal process $X(t) = h_* \circ Y \circ h_0^{-1}(t)$ and X is semi-ss w.r.t. the translation $D_s(z) = z + es$. Thus

$$X(t + s) \stackrel{d}{=} X(t) + es, \quad \forall s \in \{\log \varphi^n:\ n \in \mathbf{Z}\}. \tag{3.2}$$

From here one can guess that there is a close connection between the semi-self-similar extremal processes and the periodically stationary processes.

Definition. An $\overline{\mathbf{R}}^d$-valued stochastic process $X\colon (-\infty,\infty) \to [-\infty,\infty)^d$ is said to be *periodically stationary* with period $s > 0$ if

$$X(t+s) \stackrel{d}{=} X(t), \quad \forall t \in \mathbf{R}. \tag{3.3}$$

Proposition 3.6. *Let $Y\colon [0,\infty) \to [0,\infty)^d$ be a semi-self-similar extremal process w.r.t. $\eta = [\varphi, h]$. Then the stochastic process $X^*\colon (-\infty,\infty) \to \overline{\mathbf{R}}^d$ defined by $X^*(t) := h_* \circ Y \circ h_0^{-1}(t) - et$ is periodically stationary with period $s = \log\varphi$.*

Proof. We have, by (3.2),

$$X^*(t+s) = h_* \circ Y \circ h_0^{-1}(t+s) - e(t+s) = h_* \circ Y \circ h_0^{-1}(t) - et = X^*(t).$$

Note that the process X^* is not an extremal process, since the relation $f_{t_1} \mid f_{t_2}$ for $t_1 < t_2$ is violated. The mapping $\eta(t,x) = (t, x - et)$ is not a time-space change.

4. Semi-Self-Similar Extremal Processes as Limiting

In [11], the following stochastic model is considered: assume we are given an extremal process $X\colon [0,\infty) \to [0,\infty)^d$ with lower curve C, d.f. f, and associated point process $\{(t_k, X_k)\colon k = 0,1,\dots\}$ whose time points $0 = t_0 < t_1 < t_2 < \dots$ form an increasing to ∞ sequence and X_k are i.r.v.'s in $[0,\infty)^d$, i.e.,

$$X(t) = C(t) \vee \sup\{X_k\colon t_k \le t\}.$$

Assume further that there is a sequence $\xi_n(t,x) = (\tau_n(t), L_n(x))$ of time-space changes such that:

(i) $\xi_n \to \vec{\infty}$ and $\forall \alpha > 0$ $\xi_n^{-1} \circ \xi_{[\alpha n]} \to \eta_\alpha$, where $\{\eta_\alpha\colon \alpha > 0\}$ is a norming group. Such sequences are called *regular*;

(ii) there exists a nonconstant extremal process Y with d.f. g continuous at $t = 0$ with

$$Y_n(t) := L_n^{-1} \circ X \circ \tau_n(t) = C_n(t) \vee \sup\{L_n^{-1} \circ X_k\colon \tau_n^{-1}(t_k) \le t\} \Longrightarrow Y. \tag{4.1}$$

Then the limiting extremal process is max-i.d. and, moreover, it is self-similar w.r.t. the c.o.g. $\{\eta_\alpha = (\sigma_\alpha, \mathrm{L}_\alpha)\colon \alpha > 0\}$, i.e.,

$$Y \circ \sigma_\alpha(t) \stackrel{d}{=} \mathrm{L}_\alpha \circ Y(t), \quad \forall \alpha > 0,$$

or, equivalently,

$$g \circ \eta_\alpha = g, \quad \forall \alpha > 0,$$

and the univariate marginals g_t are max-self-decomposable (MSD), namely, $\forall \alpha \in (0,1)$ there is a max-i.d. d.f. Q_α such that

$$g_t(x) = g_t(\mathrm{L}_\alpha^{-1} x) Q_{\alpha,t}(x).$$

In this section, we consider a similar stochastic model as above with the only difference that $Y_n = L_n^{-1} \circ X \circ \tau_n \not\Rightarrow Y$ for $n \to \infty$, but there exists a geometrically increasing subsequence $m_n \sim \varphi^n$, $\varphi > 1$, such that $Y_{m_n} \Rightarrow Y$. To characterize the limit class of extremal processes we need a weaker condition (4.2) than the regularity of $\{\xi_n\}$. Indeed, replace ξ_{m_n} by ξ_n and Y_{m_n} by Y_n in the new model. Then the condition $\xi_{m_n}^{-1} \circ \xi_{m_{n+1}} \to \eta_\varphi$ can be rewritten as

$$\xi_n^{-1} \circ \xi_{n+1} \longrightarrow \eta = (\sigma, L) \tag{4.2}$$

and one gets the following characterizing theorem.

Theorem 4.1. *Let $X\colon [0,\infty) \to [0,\infty)^d$ be an extremal process with nondegenerate d.f. f, and let $\xi_n = (\tau_n, L_n)$ be a sequence of time-space changes of $(0,\infty)^{d+1}$ such that*

(a) $\xi_n \to \vec{\infty}$, $\xi_n^{-1} \circ \xi_{n+1} \to \eta = (\sigma, L)$, and $\Gamma(\eta)$ is a norming group;

(b) $Y_n = L_n^{-1} \circ X \circ \tau_n \Rightarrow Y$, where Y is a nondegenerate extremal process stochastically continuous at $t = 0$ with d.f. g and lower curve C with $C(0) = 0$.

Then

(1) the limiting process Y is semi-self-similar w.r.t. $\Gamma(\eta)$;

(2) the associated point process is Poisson;

(3) the univariate marginals g_t of Y are semi-MSD, i.e.,

$$g_t(x) = g_t(L_x) Q_t(x),$$

where Q_t is a max-i.d. d.f., $x > C(t)$, and $L(x) > x$.

Conversely, if Y is a nondegenerate semi-self-similar extremal process stochastically continuous at $t = 0$, then Y is such a limit.

Proof. Statement (2) is a consequence of (a) and the continuity of g at $t = 0$. For (1) let us express $L_n^{-1} \circ X \circ \tau_{n+1}(t)$ in two different ways:

$$L_n^{-1} \circ X \circ \tau_{n+1}(t) = L_n^{-1} \circ L_{n+1}(L_{n+1}^{-1} \circ X \circ \tau_{n+1})(t) = L_n^{-1} \circ X \circ \tau_n(\tau_n^{-1} \circ \tau_{n+1})(t).$$

Then assumptions (a) and (b) imply for $n \to \infty$ the semi-self-similarity of Y, i.e.,

$$L \circ Y(t) \stackrel{d}{=} Y \circ \sigma(t), \quad t \geq 0.$$

Here $\sigma(t) > t$. By the structure theorem, there is a random vector $U(\sigma^{-1}(t), t] \geq C(t)$ a.s., independent of $Y(\sigma^{-1}(t))$ so that

$$Y(t) = Y(\sigma^{-1}(t)) \vee U(\sigma^{-1}(t), t].$$

Let Q_t be the d.f. of the max-increment $U_{(\sigma^{-1}(t),t]}$ of Y. It is max-i.d., since Y is max-i.d. Now, using the semi-self-similarity of Y on the RHS of the last equation, we get (3).

Conversely, suppose Y is a semi-ss w.r.t. $\eta = (\tau, L)$ Poisson extremal process. Define $L_n := L^{\circ(n)}$, $\tau_n := \tau^{\circ(n)}$. Then the semi-self-similarity implies

$$L_n^{-1} \circ Y \circ \tau_n \stackrel{d}{=} Y,$$

i.e., Y is limiting in a model described by (a) and (b).

Recall that self-similar extremal processes are stochastically continuous and can also be expressed as

$$Y(t) = L_{\alpha(t)} \circ Y(1), \quad \forall t > 0,$$

where $\alpha(t)$ is the unique solution of $\sigma_\alpha(1) = t$. This means that we know the process Y if we know the d.f. $G(\cdot) = g_1(\cdot) \in$ MSD and the space-change family $\{L_{\alpha(t)}\colon\ t > 0\}$. The following theorem is a counterpart of this fact in the semi-ss model. Here, by max-support of G we mean the smallest rectangle containing the support of G. Note that $G \in$ semi-MSD w.r.t. a space-change L means

$$G(x) = G(Lx)Q_1(x) = \cdots = G(L^{\circ(n)}x)Q_n(x), \tag{4.3}$$

where

$$Q_n(x) = \prod_{k=1}^{n-1} Q_1(L^{\circ(n)}x),$$

i.e., G is semi-MSD w.r.t. the semi-group $\{L^{\circ(n)}\colon\ n \geq 1\}$. Further, if G does not have mass at $+\infty$, then

$$\lim_{n\to\infty} G(L^{\circ(n)}x) = \lim_{n\to\infty} Q_1(L^{\circ(n)}x) = 1.$$

Hence

$$G(x) = \lim_{n\to\infty} \prod_{k=1}^{n} Q_1(L^{\circ(k)}x),$$

i.e., G is a max-i.d. d.f.

THEOREM 4.2. *Suppose that G is a nondegenerate d.f. with max-support $[0,\infty)^d$ and continuous at the upper boundary, and suppose that L is a space-change for which cyclic group satisfies the boundary condition* (BC). *Then G is semi-MSD w.r.t. L if and only if there exists a Poisson extremal process $Y\colon\ [0,\infty) \to [0,\infty)^d$ with d.f. g and a time-change $\tau\colon\ (0,\infty) \to (0,\infty)$, such that Y is semi-ss w.r.t. (τ, L) and $g_1 = G$.*

Proof. We still have to show the "only if" part. So, assume G is a d.f. of a max-i.d. r.v. X in $[0,\infty)^d$ and $G \in$ semi-MSD w.r.t. the space-change L. We shall construct an extremal process Y (more precisely, a family $\{g_t\colon\ t > 0\}$ of univariate d.f.'s determining Y) such that

(i) Y is stochastically continuous (hence Poisson);

(ii) $Y(1) \stackrel{d}{=} X$;

(iii) there exists a time-change $\tau\colon\ (0,\infty) \to (0,\infty)$ so that Y is semi-ss w.r.t. (τ, L).

Denote by $\Gamma(L)$ the cyclic group of L. By the embedding Proposition 2.2, there exists a homeomorphism $h: (0,\infty)^d \leftrightarrow (-\infty,\infty)^d$ and a constant $\varphi > 1$ such that $L(x) = h^{-1}(h(x) + e\log\varphi) > x$. So, we start by defining g_t at

$$t = 1, \quad g_1(x) := G(x),$$

$$t = \varphi, \quad g_\varphi(x) := G(L^{-1}x).$$

Next we determine g_t uniquely in the interval

$$t \in (1,\varphi), \quad g_t(x) := [G(x)]^{(\varphi-t)/(\varphi-1)}[G(L^{-1}x)]^{(t-1)/(\varphi-1)}.$$

It is a d.f. on $[0,\infty)^d$ and has the following properties:

- g_t is continuous in $t \in [1,\varphi]$;
- $g_t \in$ semi-MSD w.r.t. L, i.e.,

$$g_t(x) = g_t(Lx)q_t(x),$$

where $q_t(x) = [Q_1(x)]^{(\varphi-t)/(\varphi-1)}[Q_1(L^{-1}x)]^{(t-1)/(\varphi-1)}$ is max-i.d.;

- for any s, t, $1 \le s < t \le \varphi$, $g_s \mid g_t$, i.e., the quotient g_t/g_s is a d.f.

Indeed,

$$\frac{g_t(x)}{g_s(x)} = \frac{[G(x)]^{(\varphi-t)/(\varphi-1)}[G(L^{-1}x)]^{(s-1)/(\varphi-1)}[G(L^{-1}x)]^{(t-s)/(\varphi-1)}}{[G(x)]^{(\varphi-t)/(\varphi-1)}[G(L^{-1}x)]^{(s-1)/(\varphi-1)}[G(x)]^{(t-s)/(\varphi-1)}} = [Q_1(L^{-1}x)]^{(t-s)/(\varphi-1)}.$$

Now for any $t > 0$, $t \notin [1,\varphi]$, there is $n \in \mathbf{Z}$ such that $\varphi^n \le t < \varphi^{n+1}$, so $1 \le \varphi^{-n}t < \varphi$, and we define

$$g_t(x) := g_{\varphi^{-n}t}(L^{\circ(-n)}x). \tag{4.4}$$

At $t = 0$, we define g_t by the right-continuity $g(0,x) = \lim_{n\to\infty} g(t_n,x)$ with $t_n \downarrow 0$. So g_t is defined for all $t \ge 0$. The family $\{g_t\colon\ t \ge 0\}$ has the following properties:

- $g_1 = G$;
- g_t is continuous in t;
- $g_t \in$ semi-ss w.r.t. (τ, L) with $\tau(t) = t\varphi$.

Indeed, for arbitrary $t > 0$ choose $n \in \mathbf{Z}$ satisfying $\varphi^n \le t < \varphi^{n+1}$. We have

$$g_{t\varphi}(x) = g_{t\varphi^{-n}}(L^{\circ(-n-1)}x) = g_t(L^{-1}x).$$

We still have to check that $g_{s'} \mid g_{t'}$ for arbitrary $0 < s' < t'$. There are several possible cases:

(a) $1 \le s' < t' < \varphi$. This case has already been discussed, and we get $g_{t'}(x) = g_{s'}(x)[Q_1(L^{-1}x)]^{(t'-s')/(\varphi-1)}$.

(b) $\varphi \le s' < t'$. Let $\varphi^m \le s' < \varphi^{m+1}$ and $\varphi^n \le t' < \varphi^{n+1}$. Then $m \le n$ and we have two possibilities: $\varphi^{-m}s' =: s < t := \varphi^{-n}t'$ or $t < s$. We take the first case; the other one can be handled similarly. Below we use the equalities

$$\begin{aligned} G(L^{\circ(-n)}x) &= G(L^{\circ(-n+1)}x)Q_1(L^{\circ(-n)}x) = \cdots = G(L^{\circ(-m)}x)Q_1(L^{\circ(-m-1)}x)\cdots Q_1(L^{\circ(-n)}x) \\ &= \cdots = G(x)Q_1(L^{-1}x)\cdots Q_1(L^{\circ(-n)}x). \end{aligned} \tag{4.5}$$

Thus

$$\begin{aligned} g_{t\varphi^n}(x) &= g_t(L^{\circ(-n)}x) = [G(L^{\circ(-n)}x)]^{(\varphi-t)/(\varphi-1)}[G(L^{\circ(-n-1)}x)]^{(t-1)/(\varphi-1)} \\ &= [G(L^{\circ(-n)}x)]^{(\varphi-t)/(\varphi-1)}[G(L^{\circ(-n)}x)Q_1(L^{\circ(-n-1)}x)]^{(t-1)/(\varphi-1)} = G(L^{\circ(-n)}x)[Q_1(L^{\circ(-n-1)}x)]^{(t-1)/(\varphi-1)} \\ &= G(L^{\circ(-m)}x)Q_1(L^{\circ(-m-1)}x)\cdots Q_1(L^{\circ(-n)}x)[Q_1(L^{\circ(-n-1)}x)]^{(t-1)/(\varphi-1)} = G(L^{\circ(-m)}x)[Q_1(L^{\circ(-m-1)}x)]^{(s-1)/(\varphi-1)} \\ &\quad \times [Q_1(L^{\circ(-m-1)}x)]^{(\varphi-s)/(\varphi-1)} \prod_{k=m+2}^{n} Q_1(L^{\circ(-k)}x)[Q_1(L^{\circ(-n-1)}x)]^{(t-1)/(\varphi-1)}. \end{aligned}$$

In the last equality, the product of the first two components is just $g_s(L^{\circ(-m)}x) = g_{s\varphi^m}(x)$. The product of the other components is a max-i.d. d.f. that will be denoted $H_{(s\varphi^m, t\varphi^n)}$. Hence,

$$g_{t\varphi^n} = g_{s\varphi^m}H_{(s\varphi^m, t\varphi^n)}.$$

(c) $0 < s' < t' \le 1$. Let $\varphi^{-m} < s' \le \varphi^{-m+1}$ and $\varphi^{-n} < t' \le \varphi^{-n+1}$. Then $m \ge n$ and there are again two possibilities: $\varphi^m s' =: s < t := \varphi^n t'$ or $t < s$. One handles them in the same way as above.

(d) $0 < s' < 1 \le t'$. Let $\varphi^{-m} \le s' < \varphi^{-m+1}$ and $\varphi^n \le t' < \varphi^{n+1}$. Here we decompose

$$\frac{g_{t'}}{g_{s'}} = \frac{g_{t'}}{g_1}\frac{g_1}{g_{s'}}.$$

Using $g_{t'}(x) = g_{t'\varphi^{-n}}(L^{\circ(-n)}x)$, $g_{s'} = g_{s'\varphi^m}(L^{\circ(m)}x)$, (4.5), and (4.3), we get

$$\frac{g_{t'}}{g_1} = \left[\frac{G(L^{\circ(-m)}x)}{G(x)}\right]^{(\varphi-t'\varphi^{-n})/(\varphi-1)}\left[\frac{G(L^{\circ(-n-1)}x)}{G(x)}\right]^{(t'\varphi^{-n}-1)(\varphi-1)}$$

$$= \left[\prod_{k=1}^{n} Q_1(L^{\circ(-k)}x)\right]^{(\varphi-t'\varphi^{-n})/(\varphi-1)}\left[\prod_{k=1}^{n+1} Q_1(L^{\circ(-k)}x)\right]^{(t'\varphi^{-n}-1)/(\varphi-1)}$$

$$= \prod_{k=1}^{n} Q_1(L^{\circ(-k)}x)[Q_1(L^{\circ(-n-1)}x)]^{(t'\varphi^{-n}-1)/(\varphi-1)}$$

and also

$$\frac{g_1}{g_{s'}} = \left[\frac{G(x)}{G(L^{\circ(m)}x)}\right]^{(\varphi-s'\varphi^m)/(\varphi-1)}\left[\frac{G(x)}{G(L^{\circ(m-1)}x)}\right]^{(s'\varphi^m-1)/(\varphi-1)}$$

$$= \left[\prod_{k=1}^{m-1} Q_1(L^{\circ(k)}x)\right]^{(\varphi-s'\varphi^m)/(\varphi-1)}\left[\prod_{k=1}^{m-2} Q_1(L^{\circ(k)}x)\right]^{(s'\varphi^m-1)/(\varphi-1)} = \prod_{k=1}^{m-2} Q_1(L^{\circ(k)}x)[Q_1(L^{\circ(m-1)}x)]^{(\varphi-s'\varphi^m)/(\varphi-1)}.$$

Obviously, $g_{t'}/g_{s'}$ is a max-i.d. d.f.

(e) $1 \le s' < \varphi \le t'$. Here again $g_{t'}/g_{s'}$ is a max-i.d. d.f., and one shows this in a similar way as in (d) by decomposing

$$\frac{g_{t'}}{g_{s'}} = \frac{g_{t'}}{g_\varphi}\frac{g_\varphi}{g_{s'}}.$$

Finally, let us summarize: a d.f. g_t, $t \ge 0$, is max-i.d. hence the set $\operatorname{int}\{g_t > 0\}$ is the open block $(C(t), \vec{\infty})$. Thus the quotient g_t/g_s for $0 < s < t$ uniquely determines the d.f. of the max-increment $U(s,t] \ge C(t)$ a.s. So $\{g_t\colon\ t \ge 0\}$ is the family of univariate marginals of an extremal process Y that satisfies conditions (i)–(iii). Furthermore, (4.4) implies that $\lim_{t\to\infty} \mathbf{P}(Y(t) < x) = 0$, i.e., $Y(\infty) = \vec{\infty}$ a.s.

5. Semi-Self-Similar Extremal Processes with Stationary Increments

Let us consider the same asymptotic model as in Theorem 4.1 with one additional condition: the initial extremal process X has stationary max-increments, i.e., for $0 \le s < t$

$$U_X(s,t] = C_X(t) \vee \sup\{X_k\colon\ s < t_k \le t\} \stackrel{d}{=} U_X(0, t-s).$$

Then the limit extremal process Y with d.f. g:

(a) is semi-ss w.r.t. a time-space change $\eta = (\tau, L)$, i.e.,

$$g(\tau(t), x) = g(t, L^{-1}x);$$

(b) belongs (cf. [11]) to the Resnick and Rubinovich class $\mathfrak{R}$, i.e.,

$$g(t,x) = G^t(x), \quad G \in \text{max-i.d.}$$

The extremal processes of the class $\mathfrak{R}$ are stochastically continuous processes starting at the origin with independent and stationary max-increments, hence they are the counterpart of the Lévy processes in the extreme value theory.

From (a) and (b) we see that the d.f. $G(x) = \mathbf{P}(Y(1) < x)$ satisfies the functional equation

$$G^{\tau(1)}(x) = G(L^{-1}x). \tag{5.1}$$

Hence G is a max-semistable d.f. This class of d.f.'s are studied in [5, 6, 8].

Recall that a max-i.d. d.f. G is called *max-semistable* (briefly max-ss) if there exists a pair (α, L), $\alpha \in (0,1)$, $L(x) > x$, such that $G^\alpha(x) = G(Lx)$. Obviously, if $G \in$ max-ss, then $\forall t > 0$ G^t is max-ss w.r.t. the same pair (α, L). In $\mathbf{R}^1$, the solution of the functional equation (5.1) is given by

$$G(x) = \exp\{-e^{ch(x)} p_\alpha(h(x))\}, \tag{5.2}$$

where $L(x) = h^{-1}(h(x) + e \log \varphi)$, $\alpha = 1/\tau(1)$, $c > 0$ is the unique solution of $\alpha\varphi^c = 1$, and $p_\alpha(y)$ is a positive bounded periodic function with period $T = \log \varphi$.

Theorem 5.1. *Let $Y\colon\ [0,\infty) \to [0,\infty)^d$ be an extremal process with d.f. g and stationary max-increments. Then Y is semi-self-similar if and only if g_1 is a max-semistable d.f.*

Proof. We still have to prove the "only if" part. Let $g_1 = G$ be max-ss w.r.t. (α, L) and, without loss of generality, let us assume that $G(x) < 1$ $\forall x \in [0,\infty)^d$. Then $\forall t > 0$, $x > C(t)$

$$g(t\alpha, x) = G^{t\alpha}(x) = G^t(Lx) = g(t, Lx).$$

Further, the cyclic group of $\eta = (1/\alpha, L)$ is a norming group, since $t/\alpha^n \to \infty$, $L^{\circ(n)}(x) \to \bar{\infty}$ as $n \to \infty$. Hence Y is semi-ss w.r.t. $\Gamma(\eta)$.

Note that the lower curve of a process $Y \in \mathfrak{R}$ is always constant, namely, $C(t) = C(1) = \inf\{G > 0\}$.

Let us consider the multivariate version of (5.2). Since $Y \in \mathfrak{R}$, the mean measure μ of the associated Poisson point process N has the form

$$\mu([0,t] \times [0,x)^c) = t\nu([0,x)^c), \quad \forall x > C(1),$$

where $\nu(\cdot)$ is the exponent measure of $G = g_1$ (cf. [4]), which satisfies the semi-stability equation

$$\alpha\nu(A) = \nu(LA), \quad \forall A \in \mathcal{B}([C(1), \bar{\infty}] \setminus \{C(1)\}). \tag{5.3}$$

Recall that here $\alpha \in (0,1)$ and $L = [\varphi, h]$.

Denote $\chi = \{a \in (0,\infty)^d\colon\ \max(a_1, \dots, a_d) = 1\}$ and set $s(x,a) := \min_{1 \le i \le d} \exp\{h_i(x_i) - h_i(a_i)\}$. There exists a finite measure Q on $\mathcal{B}(\chi)$ such that the solution of (5.3) is given by (cf. [9])

$$\nu([0,x)^c) = \int\limits_{\chi} s^{-1}(x,a) \rho_\varphi(\log s(x,a))\, Q(da)$$

and such that the function

$$\rho(h_i(x_i)) = \int\limits_{\chi} \exp\{h_i(a_i)\} \rho_\varphi(\log s(s,a))\, Q(da)$$

is a positive bounded periodic function with period $T = \log\varphi$.

Let us come back to (5.2). We can rewrite it as

$$G(x) = \left[e^{-e^{-ch(x)}}\right]^{p_\alpha(h(x))}.$$

The expression in the brackets is the general form of a max-stable d.f. (cf. [9]). Hence, (5.2) says that any max-ss d.f. has the form of a max-stable d.f. to a power $p_\alpha(h(x))$. Using this and Theorem 5.1, we construct examples of semi-self-similar extremal processes.

Example 1. Let $Y\colon\ (0,\infty) \to (-\infty,\infty)$ be an extremal process with d.f.

$$g(t,x) = \exp\{-te^{-[x]}\}, \quad t > 0, \quad x \in \mathbf{R}.$$

Here $Y \in \mathfrak{R}$, $g_1(x)$ is max-ss w.r.t. $\alpha = e^{-1}$, and $L(x) = x + 1$. Then Y is semi-ss w.r.t. $\eta(t,x) = (t/\alpha, x+1)$.

Example 2. Let $g(t,x) = \exp[-(t/x)\{\log x\}]$ be the d.f. of an extremal process $Y\colon\ (0,\infty) \to (0,\infty)$. Here $p(y) = \{y\}$ is the fractional part of y and has period $T = 1$. Comparing with (5.2), we conclude that $h(x) = \log x$, $\varphi = e$, $L(x) = x\varphi$, $\alpha = \varphi^{-1}$. Then

$$g(t, x\varphi) = \exp\left[-\frac{t\{\log x + 1\}}{x\varphi}\right] = g\left(\frac{t}{\varphi}, x\right)$$

and Y is semi-ss w.r.t. $\eta(t,x) = (t\varphi, x\varphi)$.

Example 3. Let the r.v. X be uniformly distributed on the diagonal of the square $[0,1]^2$. Then its d.f. G has the form

$$G(x_1,x_2) = \begin{cases} 0, & x \in \{y > 0\}^c, \\ x_1, & x_1 \le x_2 \le 1, \\ x_2, & x_2 \le x_1 \le 1, \\ 1, & x \in \{y \ge 1\}, \end{cases}$$

and G is semi-MSD w.r.t. $L(x) = (x_1/\alpha, x_2/\alpha)$, $\alpha \in (0,1)$. Hence, by Theorem 4.2, there is a d.f. g with $g(1,x) = G(x)$ so that g is the d.f. of a semi-ss extremal process $Y\colon\ (0,\infty) \to [0,1]^2$ w.r.t. $\eta(t,x) = (t/\alpha, x_1/\alpha, x_2/\alpha)$.

Note that the d.f. G has a zero density. Such a d.f. cannot be self-decomposable in the classical model of sums of i.r.v.'s.

Example 4. The d.f. $G(x) = \exp\{-(1/x)(c - \sin(\log x))\}$, $x > 0$, $c > 1$, is max-ss w.r.t. ($\alpha = e^{-2\pi}$, $L(x) = x/\alpha$). For c large enough, the function $-\log G$ is convex, hence $G \in$ MSD. Thus, the d.f. $g(t,x) = G^t(x)$ is the d.f. of a semi-ss extremal process $Y\colon\ (0,\infty) \to (0,\infty)$ w.r.t. $\eta(t,x) = (t/\alpha, x/\alpha)$.

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