FAMILIES OF ZERO-CYCLES AND DIVIDED POWERS: I. REPRESENTABILITY

DAVID RYDH

Abstract. Let X/S be a separated algebraic space. We construct an algebraic space $\Gamma^d(X/S)$, *the space of divided powers*, which parameterizes zero cycles of degree d on X. When X/S is affine, this space is affine and given by the spectrum of the ring of divided powers. In characteristic zero or when X/S is flat, the constructed space coincides with the symmetric product $\mathrm{Sym}^d(X/S)$. We also prove several fundamental results on the kernels of multiplicative polynomial laws necessary for the construction of $\Gamma^d(X/S)$.

Introduction

Chow varieties, parameterizing families of cycles of a certain dimension and degree, are classically constructed using explicit projective methods [CW37, Sam55]. Moreover, Chow varieties are defined as *reduced* schemes and in positive characteristic the classical construction has the unpleasant property that it depends on a given projective embedding [Nag55].

Many attempts to give a nice functorial description of Chow varieties have been made and some successful steps towards this goal have been taken. For families parameterized by *seminormal* schemes, Kollár, Suslin and Voevodsky, have given a functorial description [Kol96, SV00]. In characteristic zero, Barlet [Bar75] has given an analytic description over *reduced* $\mathbb{C}$-schemes and Angéniol [Ang80] has given an algebraic description over, not necessarily reduced, $\mathbb{Q}$-schemes. The situation in characteristic zero is simplified by the fact that for a finite extension $A \hookrightarrow B$ such that the determinant $B \to A$ is defined, the determinant is determined by the trace.

In this article we will restrict our attention to Chow varieties of zero cycles, that is, families of cycles of relative dimension zero. We will construct an algebraic space $\Gamma^d(X/S)$, parameterizing zero-cycles, which coincides with Angéniol's Chow space in characteristic zero. As with Angéniol's Chow space, the algebraic space $\Gamma^d(X/S)$ is not always reduced but its reduction coincides with the classical Chow variety if we use a *sufficiently good* projective embedding. The relation with the Chow variety will be discussed in a subsequent article [**III**]. A good understanding of families of zero-cycles is crucial for the understanding of families of higher-dimensional cycles.

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In fact, a family of higher-dimensional cycles is defined by giving zero-dimensional families on "smooth projections" [Bar75, **IV**].

A natural space parameterizing zero-cycles is the symmetric product $\mathrm{Sym}^d(X/S)$. This is the correct choice, in the sense that it coincides with $\Gamma^d(X/S)$, when X is of characteristic zero or when X/S is flat. In general, however, $\mathrm{Sym}^d(X/S)$ is not functorially well-behaved and should be replaced with the "scheme of divided powers". In the affine case, this is the spectrum of the algebra of divided power $\Gamma^d_A(B)$ and it coincides with the symmetric product when $d!$ is invertible in A or when B is a flat A-algebra.

Although the ring of divided powers $\Gamma^d_A(B)$ and multiplicative polynomial laws have been studied by many authors [Rob63, Rob80, Ber65, Zip86, Fer98], there are some important results missing. We provide these missing parts, giving a full treatment of the *kernel* of a multiplicative law. Somewhat surprisingly, the kernel does not commute with flat base change, except in characteristic zero. We will show that the kernel does commute with étale base change.

After this preliminary study of $\Gamma^d_A(B)$ we define, for any separated algebraic space X/S, a functor $\underline{\Gamma}^d_{X/S}$ which parameterizes families of zero-cycles. From the definition of $\underline{\Gamma}^d_{X/S}$ and the results on the kernel of a multiplicative law, it will be obvious that $\underline{\Gamma}^d_{X/S}$ is represented by $\mathrm{Spec}(\Gamma^d_A(B))$ in the affine case. If X/S is a scheme such that for every $s \in S$, every finite subset of the fiber X_s is contained in an *affine* open subset of X, then we say that X/S is an AF-scheme, cf. Appendix A.1. In particular, this is the case if X/S is quasi-projective. For an AF-scheme X/S it is easy to show that $\underline{\Gamma}^d_{X/S}$ is representable by a scheme.

To treat the general case — when X/S is any separated scheme or separated algebraic space — we use the fact that $\underline{\Gamma}^d_{X/S}$ is functorial in X: For any morphism $f : U \to X$ there is an induced *push-forward* $f_* : \underline{\Gamma}^d_{U/S} \to \underline{\Gamma}^d_{X/S}$. We show that when f is étale, then f_* is étale over a certain open subset corresponding to families of cycles which are *regular* with respect to f. We then show that $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space $\Gamma^d(X/S)$ giving an explicit étale covering.

In the last part of the article we introduce "addition of cycles" and investigate the relation between the symmetric product $\mathrm{Sym}^d(X/S)$ and the algebraic space $\Gamma^d(X/S)$. Intuitively, the universal family of $\Gamma^d(X/S)$ should be related to the addition of cycles morphism $\Phi_{X/S} : \Gamma^{d-1}(X/S) \times_S X \to \Gamma^d(X/S)$. In the special case when $\Phi_{X/S}$ is *flat*, e.g., when X/S is a smooth curve, Iversen has shown that the universal family is given by the *norm* of $\Phi_{X/S}$ [Ive70]. In general, there is a similar but more subtle description. The universal family and some other properties of $\Gamma^d(X/S)$ are treated in [**II**].

We now discuss the results and methods in more detail:

Multiplicative polynomial laws. In §1 we recall the basic properties of the algebra of divided powers $\Gamma_A(B)$ and the algebra $\Gamma^d_A(B)$. We also mention the *universal multiplication* of laws which later on will be described geometrically as *addition of cycles.*

Kernel of a multiplicative polynomial law. Let B be an A-algebra. In §2 the basic properties of the *kernel* $\ker(F)$ of a multiplicative law $F : B \to A$ is established. First we show that $B/\ker(F)$ is integral over A using Cayley-Hamilton's theorem. We then show that the kernel commutes with limits, localization and smooth base change. As mentioned above, the kernel does not commute with flat base change in general and showing that the kernel commutes with smooth base change takes some effort. Finally, we show some topological properties of the kernel: The radical of the kernel commutes with arbitrary base change, the fibers of $\operatorname{Spec}(B/\ker(F)) \to \operatorname{Spec}(A)$ are finite sets, and $\operatorname{Spec}(B/\ker(F)) \to \operatorname{Spec}(A)$ is universally open.

The functor $\underline{\Gamma}^d_{X/S}$. Guided by the knowledge that $\Gamma^d_A(B)$ is what we want in the affine case, we define in §3.1 a well-behaved functor $\underline{\Gamma}^d_{X/S}$ parameterizing families of zero-cycles of degree d as follows. A family over an affine S-scheme $T = \operatorname{Spec}(A)$ is given by the following data

(i) A closed subspace $Z \hookrightarrow X \times_S T$ such that $Z \to T$ is *integral.* In particular $Z = \operatorname{Spec}(B)$ is *affine.*
(ii) A family α on Z, i.e., a morphism $T \to \Gamma^d(Z/T) := \operatorname{Spec}(\Gamma^d_A(B))$.

Moreover, two families are equivalent if they are both induced by a family for some common smaller subspace Z. We often suppress the subspace Z and talk about the family α. The smallest subspace $Z \hookrightarrow X \times_S T$ in the equivalence class containing α is the *image* of the family α and the reduction Z_{red} of the image is the *support* of the family. The image of α is given by the kernel of the multiplicative law corresponding to α. Since the kernel commutes with étale base change, as shown in §2, so does the image of a family. This is the key result needed to show that $\underline{\Gamma}^d_{X/S}$ is a sheaf in the étale topology.

In contrast to the Hilbert functor, for which families over T are determined by a subspace $Z \hookrightarrow X \times_S T$, a family of zero-cycles is not determined by its image Z. If T is *reduced*, then the image Z of a family parameterized by T is reduced and the family is determined by an effective cycle supported on Z. In positive characteristic, over non-perfect fields, this cycle may have rational coefficients. This is discussed in [**II**].

Push-forward of cycles. A morphism $f : X \to Y$ of separated algebraic spaces induces a natural transformation $f_* : \underline{\Gamma}^d_{X/S} \to \underline{\Gamma}^d_{Y/S}$ which we call the *push-forward.* When Y/S is locally of finite type, the existence of f_* follows from standard results. In general, we need a technical result on integral morphisms given in Appendix A.2.

We say that a family $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ is *regular* if the restriction of f_T to the image of α is an isomorphism. If $f : X \to Y$ is étale then the regular locus is an open subfunctor of $\underline{\Gamma}^d_{X/S}$. A main result is that under certain regularity constraints, push-forward commutes with products, cf. Proposition (3.3.10). Using this fact we show that the push-forward along an étale morphism is representable and étale over the regular locus. This is Proposition (3.3.15).

Representability. The representability of $\underline{\Gamma}^d_{X/S}$ when X/S is affine or AF is, as already mentioned, not difficult and given in 3.1. When X/S is any separated algebraic space, the representability is proven in Theorem (3.4.1) using the results on the push-forward.

Addition of cycles. Using the push-forward we define in §4.1 a morphism

$$\Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$$

which on points is addition of cycles. This induces a morphism $(X/S)^d \to \Gamma^d(X/S)$ which has the topological properties of a quotient of $(X/S)^d$ by the symmetric group.

Relation with the symmetric product. The morphism $(X/S)^d \to \Gamma^d(X/S)$ factors through the quotient map $(X/S)^d \to \mathrm{Sym}^d(X/S)$ and it is easily proven that $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a universal homeomorphism with trivial residue field extensions, cf. Corollary (4.2.5). It is further easy to show that $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism over the non-degeneracy locus, cf. Proposition (4.2.6).

Comparison of representability techniques. Consider the following inclusions of categories:

$$\begin{tikzcd}
 & \begin{array}{l} X/S \text{ quasi-projective} \\ \text{of finite presentation} \end{array} \arrow[r, hook] \arrow[d, hook] & \begin{array}{l} X/S \text{ separated algebraic space,} \\ \text{locally of finite presentation} \end{array} \arrow[d, hook] \\
X/S \text{ affine} \arrow[r, hook] & X/S \text{ AF-scheme} \arrow[r, hook] & X/S \text{ separated algebraic space.}
\end{tikzcd}$$

When X/S is affine, it is fairly easy to show the existence of the quotient $\mathrm{Sym}^d(X/S)$ [Bou64, Ch. V, §2, No. 2, Thm. 2], the representability of $\underline{\Gamma}^d_{X/S}$ and the representability of the Hilbert functor of points $\mathcal{H}ilb^d_{X/S}$ [Nor78, GLS07]. The existence of $\mathrm{Sym}^d(X/S)$ and the representability of $\underline{\Gamma}^d_{X/S}$ and $\mathcal{H}ilb^d_{X/S}$ in the category of AF-schemes is then a simple consequence.

When X/S is (quasi-)projective and S is noetherian, one can also show the existence and (quasi-)projectivity of $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\mathrm{Hilb}^d(X/S)$ with projective methods, cf. [**III**] and [FGA, No. 221]. The representability of the Hilbert scheme in the category of separated algebraic spaces locally of finite presentation can be established using Artin's algebraization theorem [Art69, Cor. 6.2]. We could likewise have used Artin's algebraization theorem to prove the representability of $\underline{\Gamma}^d_{X/S}$ when X/S is locally of finite presentation. The crucial criterion, that $\underline{\Gamma}^d_{X/S}$ is effectively pro-representable, is shown in §3.2.

Finally, the methods that we have used in this article to show that $\underline{\Gamma}^d_{X/S}$ is representable in the category of all separated algebraic spaces can be applied, mutatis mutandis, to the Hilbert functor of points. The proofs become significantly simpler as the difficulties encountered for $\underline{\Gamma}^d_{X/S}$ are almost trivial for the Hilbert functor.

More generally, these methods apply to the Hilbert stack of points [Ryd08b]. The existence of $\mathrm{Sym}^d(X/S)$ can also be proven in the same vein and this is done in [Ryd07].

Notation and conventions. We denote a *closed* immersion of schemes or algebraic spaces with $X \hookrightarrow Y$. When A and B are rings or modules we use $A \hookrightarrow B$ for an injective homomorphism. We let $\mathbb{N}$ denote the set of non-negative integers $0, 1, 2, \dots$ and use the notation $((a, b)) = \binom{a+b}{a}$ for binomial coefficients.

Contents

1. The algebra of divided powers

We begin this section by briefly recalling the definition of polynomial laws in §1.1, the algebra of divided powers $\Gamma_A(M)$ in §1.2 and the multiplicative structure of $\Gamma^d_A(B)$ in §1.3.

1.1. **Polynomial laws and symmetric tensors.** We recall the definition of a polynomial law [Rob63, Rob80].

Definition (1.1.1). Let M and N be A-modules. We denote by $\mathcal{F}_M$ the functor

$$\mathcal{F}_M \,:\, A\text{–}\mathbf{Alg} \to \mathbf{Sets}, \qquad A' \mapsto M \otimes_A A'$$

A *polynomial law* from M to N is a natural transformation $F : \mathcal{F}_M \to \mathcal{F}_N$. More concretely, a polynomial law is a set of *maps* $F_{A'} : M \otimes_A A' \to N \otimes_A A'$ for every A-algebra A' such that for any homomorphism of A-algebras $g : A' \to A''$ the diagram

$$\begin{tikzcd}
M \otimes_A A' \arrow[r, "F_{A'}"] \arrow[d, "\mathrm{id}_M \otimes g"'] \arrow[dr, phantom, "\circ"] & N \otimes_A A' \arrow[d, "\mathrm{id}_N \otimes g"] \\
M \otimes_A A'' \arrow[r, "F_{A''}"] & N \otimes_A A''
\end{tikzcd}$$

commutes. The polynomial law F is *homogeneous of degree* d if for any A-algebra A', the corresponding map $F_{A'} : M \otimes_A A' \to N \otimes_A A'$ is such that $F_{A'}(ax) = a^d F_{A'}(x)$ for any $a \in A'$ and $x \in M \otimes_A A'$. If B and C are A-algebras then a polynomial law from B to C is *multiplicative* if for any A-algebra A', the corresponding map $F_{A'} : B \otimes_A A' \to C \otimes_A A'$ is such that $F_{A'}(1) = 1$ and $F_{A'}(xy) = F_{A'}(x) F_{A'}(y)$ for any $x, y \in B \otimes_A A'$.

Notation (1.1.2). Let A be a ring and M and N be A-modules (resp. A-algebras). We let $\mathrm{Pol}^d(M, N)$ (resp. $\mathrm{Pol}^d_{\mathrm{mult}}(M, N)$) denote the polynomial laws (resp. multiplicative polynomial laws) $M \to N$ which are homogeneous of degree d.

Notation (1.1.3). Let A be a ring and M an A-algebra. We denote the d^{th} tensor product of M over A by $\mathrm{T}^d_A(M)$. We have an action of the symmetric group $\mathfrak{S}_d$ on $\mathrm{T}^d_A(M)$ permuting the factors. The invariant ring of this action is the symmetric tensors and is denoted $\mathrm{TS}^d_A(M)$. By $\mathrm{T}_A(M)$ and $\mathrm{TS}_A(M)$ we denote the graded A-modules $\bigoplus_{d \geq 0} \mathrm{T}^d_A(M)$ and $\bigoplus_{d \geq 0} \mathrm{TS}^d_A(M)$ respectively.

(1.1.4) The covariant functor $\mathrm{TS}^d_A(\cdot)$ commutes with filtered direct limits. In fact, denoting the group ring of $\mathfrak{S}_d$ by $\mathbb{Z}[\mathfrak{S}_d]$ we have that

$$\mathrm{TS}^d_A(\cdot) = \mathrm{T}^d_A(\cdot)^{\mathfrak{S}_d} = \mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_d]}\big(\mathbb{Z}, \mathrm{T}^d_A(\cdot)\big)$$

where $\mathfrak{S}_d$ acts trivially on $\mathbb{Z}$. As tensor products, being left adjoints, commute with any (small) direct limit so does T^d. Reasoning as in [EGA$_{\mathrm{I}}$, Prop. 0.6.3.2] it follows that $\mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_d]}(\mathbb{Z}, \cdot)$ commutes with filtered direct limits. In fact, $\mathbb{Z}$ is a $\mathbb{Z}[\mathfrak{S}_d]$-module of finite presentation and that $\mathbb{Z}[\mathfrak{S}_d]$ is non-commutative is not a problem here.

(1.1.5) *Shuffle product* — When B is an A-algebra, then $\mathrm{TS}^d_A(B)$ has a natural A-algebra structure induced from the A-algebra structure of $\mathrm{T}^d_A(B)$. The multiplication on $\mathrm{TS}^d_A(B)$ will be written as juxtaposition. For any A-module M, we can equip $\mathrm{T}_A(M)$ and $\mathrm{TS}_A(M)$ with A-algebra structures. The multiplication on $\mathrm{T}_A(M)$ is the ordinary tensor product and the multiplication on $\mathrm{TS}_A(M)$ is called the *shuffle product* and is denoted by ×. If $x \in \mathrm{TS}^d_A(M)$ and $y \in \mathrm{TS}^e_A(M)$ then

$$x \times y = \sum_{\sigma \in \mathfrak{S}_{d,e}} \sigma\,(x \otimes_A y)$$

where $\mathfrak{S}_{d,e}$ is the subset of $\mathfrak{S}_{d+e}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(d)$ and $\sigma(d+1) < \sigma(d+2) < \dots \sigma(d+e)$.

1.2. **Divided powers.** Most of the material in this section can be found in [Rob63] and [Fer98].

(1.2.1) Let A be a ring and M an A-module. Then there exists a graded A-algebra, the algebra of divided powers, denoted $\Gamma_A(M) = \bigoplus_{d\geq 0} \Gamma^d_A(M)$ equipped with maps $\gamma^d : M \to \Gamma^d_A(M)$ such that, denoting the multiplication with × as in [Fer98], we have that for every $x, y \in M$, $a \in A$ and $d, e \in \mathbb{N}$

$$\Gamma^0_A(M) = A, \quad \text{and} \quad \gamma^0(x) = 1 \tag{1.2.1.1}$$

$$\Gamma^1_A(M) = M, \quad \text{and} \quad \gamma^1(x) = x \tag{1.2.1.2}$$

$$\gamma^d(ax) = a^d \gamma^d(x) \tag{1.2.1.3}$$

$$\gamma^d(x+y) = \textstyle\sum_{d_1+d_2=d} \gamma^{d_1}(x) \times \gamma^{d_2}(y) \tag{1.2.1.4}$$

$$\gamma^d(x) \times \gamma^e(x) = ((d,e))\gamma^{d+e}(x) \tag{1.2.1.5}$$

Using (1.2.1.1) and (1.2.1.2) we will identify A with $\Gamma^0_A(M)$ and M with $\Gamma^1_A(M)$. If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a family of elements of M and $\nu \in \mathbb{N}^{(\mathcal{I})}$ then we let

$$\gamma^\nu(x) = \mathop{\times}_{\alpha \in \mathcal{I}} \gamma^{\nu_\alpha}(x_\alpha)$$

which is an element of $\Gamma^d_A(M)$ with $d = |\nu| = \sum_{\alpha \in \mathcal{I}} \nu_\alpha$.

(1.2.2) *Functoriality* — $\Gamma_A(\cdot)$ is a covariant functor from the category of A-modules to the category of graded A-algebras [Rob63, Ch. III §4, p. 251].

(1.2.3) *Base change* — If A' is an A-algebra then there is a natural isomorphism $\Gamma_A(M) \otimes_A A' \to \Gamma_{A'}(M \otimes_A A')$ mapping $\gamma^d(x) \otimes_A 1$ to $\gamma^d(x \otimes_A 1)$ [Rob63, Thm. III.3, p. 262]. This shows that γ^d is a homogeneous polynomial law of degree d.

(1.2.4) *Universal property* — The map $\mathrm{Hom}_A\big(\Gamma^d_A(M), N\big) \to \mathrm{Pol}^d(M, N)$ given by $f \to f \circ \gamma^d$ is a bijection [Rob63, Thm. IV.1, p. 266].

(1.2.5) *Basis and generators* — If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a set of generators of M, then $\big(\gamma^\nu(x)\big)_{\nu \in \mathbb{N}^{(\mathcal{I})}}$ is a set of generators of $\Gamma_A(M)$ as an A-module. If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a basis of M then $\big(\gamma^\nu(x)\big)_{\nu \in \mathbb{N}^{(\mathcal{I})}}$ is a basis of $\Gamma_A(M)$ [Rob63, Thm. IV.2, p. 272]. Furthermore, if A is an algebra over an infinite field or A is an algebra over $\Lambda_d =$

$\mathbb{Z}[T]/P_d(T)$ where P_d is the unitary polynomial $P_d(T) = \prod_{0\leq i<j\leq d}(T^i - T^j) - 1$, then $\gamma^d(M)$ generates $\Gamma^d_A(M)$ [Fer98, Lemme 2.3.1]. In particular, there is always a finite *faithfully flat* base change $A \to A'$ such that $\Gamma^d_{A'}(M')$ is generated by $\gamma^d(M')$. More generally $\gamma^d(M)$ generates $\Gamma^d_A(M)$ if and only if every residue field of A has at least d elements [**III**].

(1.2.6) *Exactness* — The functor $\Gamma_A(\cdot)$ is a left adjoint [Rob63, Thm. III.1, p. 257] and thus commutes with any (small) direct limit. It is thus *right exact* [GV72, Def. 2.4.1] but note that $\Gamma_A(\cdot)$ is a functor from A–**Mod** to A–**Alg** and that the latter category is not abelian. By [GV72, Rem. 2.4.2] a functor is right exact if and only if it takes the initial object onto the initial object and commutes with finite coproducts and coequalizers. Thus $\Gamma_A(0) = A$ and given an exact diagram of A-modules

$$\begin{tikzcd} M' \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & M \arrow[r, "h"] & M'' \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \Gamma_A(M') \arrow[r, shift left, "\Gamma f"] \arrow[r, shift right, "\Gamma g"'] & \Gamma_A(M) \arrow[r, "\Gamma h"] & \Gamma_A(M'') \end{tikzcd}$$

is exact in the category of A-algebras and

$$\Gamma_A(M \oplus M') = \Gamma_A(M) \otimes_A \Gamma_A(M').$$

The latter identification can be made explicit [Rob63, Thm. III.4, p. 262] as

$$\begin{aligned} \Gamma^d_A(M \oplus M') &= \bigoplus_{a+b=d} \big(\Gamma^a_A(M) \otimes_A \Gamma^b_A(M')\big) \\ \gamma^d(x+y) &= \sum_{a+b=d} \gamma^a(x) \otimes \gamma^b(y). \end{aligned} \tag{1.2.6.1}$$

This makes $\Gamma_A(M \oplus M') = \bigoplus_{a,b\geq 0} \Gamma^{a,b}(M \oplus M')$ into a bigraded algebra where $\Gamma^{a,b}(M \oplus M') = \Gamma^a_A(M) \otimes_A \Gamma^b_A(M')$.

(1.2.7) *Surjectivity* — If $M \twoheadrightarrow N$ is a surjection then it is easily seen from the explicit generators of $\Gamma(N)$ in (1.2.5) that $\Gamma_A(M) \twoheadrightarrow \Gamma_A(N)$ is surjective. This also follows from the right-exactness of $\Gamma_A(\cdot)$ as any right-exact functor from modules to rings takes surjections onto surjections, cf. (1.2.8)

(1.2.8) *Presentation* — Let $M = G/R$ be a presentation of the A-module M. Then $\Gamma_A(M) = \Gamma_A(G)/I$ where I is the ideal of $\Gamma_A(G)$ generated by the images in $\Gamma_A(G)$ of $\gamma^d(x)$ for every $x \in R$ and $d \geq 1$ [Rob63, Prop. IV.8, p. 284]. In fact, denoting the inclusion of R in G by i, we can write M as a coequalizer of A-modules

$$\begin{tikzcd} R \oplus G \arrow[r, shift left, "i\oplus \mathrm{id}_G"] \arrow[r, shift right, "0\oplus \mathrm{id}_G"'] & G \arrow[r, "h"] & M \end{tikzcd}$$

which by (1.2.6) gives the exact sequence

$$\begin{tikzcd}\Gamma_A(R\oplus G) \arrow[r, shift left, "\Gamma(i\oplus\mathrm{id}_G)"] \arrow[r, shift right, "\Gamma(0\oplus\mathrm{id}_G)"'] & \Gamma_A(G) \arrow[r, "\Gamma(h)"] & \Gamma_A(M)\end{tikzcd}$$

of A-algebras. Since $\Gamma^0_A(0)=\Gamma^0_A(i)=\mathrm{id}_A$ and $\Gamma^d_A(0)=0$ for $d>0$ it follows that $\Gamma_A(M)$ is the quotient of $\Gamma_A(G)$ by $\bigoplus_{d\geq 1,e\geq 0}\Gamma^{d,e}(R\oplus G)$, i.e., the quotient of $\Gamma_A(G)$ by the ideal generated by $\Gamma(i)\bigl(\bigoplus_{d\geq 1}\Gamma^d(R)\bigr)$.

(1.2.9) *Exactness of* $\Gamma^d_A(\cdot)$ — If $M\twoheadrightarrow N$ is a surjection then $\Gamma^d_A(M)\twoheadrightarrow\Gamma^d_A(N)$ is surjective since $\Gamma_A(M)\twoheadrightarrow\Gamma_A(N)$ is surjective. This does, however, not imply that $\Gamma^d_A(\cdot)$ is right exact. In fact, in general it is not since we have that $\Gamma^d_A(M\oplus M')\neq\Gamma^d_A(M)\oplus\Gamma^d_A(M')$.

(1.2.10) *Presentation of* $\Gamma^d_A(\cdot)$ — If $M=G/R$ is a quotient of A-modules then $\Gamma^d_A(M)=\Gamma^d_A(G)/I$ where I is the A-submodule generated by the elements $\gamma^k(x)\times y$ for $1\leq k\leq d$, $x\in R$ and $y\in\Gamma^{d-k}_A(G)$. This follows immediately from (1.2.8).

(1.2.11) *Filtered direct limits* — The functor $\Gamma^d_A(\cdot)$ commutes with *filtered* direct limits. In fact, if (M_α) is a directed filtered system of A-modules then

$$\begin{aligned}\bigoplus_{d\geq 0}\Gamma^d_A\bigl(\varinjlim_{A\text{-}\mathbf{Mod}} M_\alpha\bigr) &= \varinjlim_{A\text{-}\mathbf{Alg}}\bigoplus_{d\geq 0}\Gamma^d_A(M_\alpha) = \\ &= \varinjlim_{A\text{-}\mathbf{Mod}}\bigoplus_{d\geq 0}\Gamma^d_A(M_\alpha) = \bigoplus_{d\geq 0}\varinjlim_{A\text{-}\mathbf{Mod}}\Gamma^d_A(M_\alpha).\end{aligned}$$

The first equality follows from (1.2.6) and the second from the fact that a filtered direct limit in the category of A-algebras coincides with the corresponding filtered direct limit in the category of A-modules [GV72, Cor. 2.9].

(1.2.12) If M is a free (resp. flat) A-module then $\Gamma^d_A(M)$ is a free (resp. flat) A-module. This follows from (1.2.5) and (1.2.11) as any flat module is a filtered direct limit of free modules [Laz69, Thm. 1.2].

(1.2.13) Γ *and* TS — The homogeneous polynomial law $M\to\mathrm{TS}^d_A(M)$ of degree d given by $x\mapsto x^{\otimes_A d}=x\otimes_A\cdots\otimes_A x$ corresponds by the universal property (1.2.4) to an A-module homomorphism $\varphi:\Gamma^d_A(M)\to\mathrm{TS}^d_A(M)$. This extends to an A-algebra homomorphism $\Gamma_A(M)\to\mathrm{TS}_A(M)$, where the multiplication in $\mathrm{TS}_A(M)$ is the shuffle product (1.1.5), cf. [Rob63, Prop. III.1, p. 254].

When M is a free A-module the homomorphisms $\Gamma^d_A(M)\to\mathrm{TS}^d_A(M)$ and $\Gamma_A(M)\to\mathrm{TS}_A(M)$ are isomorphisms of A-modules respectively A-algebras [Rob63, Prop. IV.5, p. 272]. The functors TS^d_A and Γ^d_A commute with filtered direct limits by (1.1.4) and (1.2.11). Since any flat A-module is the filtered direct limit of free A-modules [Laz69, Thm. 1.2], it thus follows that $\Gamma_A(M)\to\mathrm{TS}_A(M)$ is an isomorphism of graded A-algebras for any flat A-module M.

Moreover by [Rob63, Prop. III.3, p. 256], there are natural A-module homomorphisms $\mathrm{TS}^d_A(M) \hookrightarrow \mathrm{T}^d_A(M) \twoheadrightarrow \mathrm{S}^d_A(M) \to \Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$ such that going around one turn in the diagram

$$\begin{tikzcd}
 & \mathrm{S}^d_A(M) \arrow[dl] & \\
\Gamma^d_A(M) \arrow[rr] & & \mathrm{TS}^d_A(M) \arrow[ul]
\end{tikzcd}$$

is multiplication by $d!$. Here $\mathrm{S}^d_A(M)$ denotes the degree d part of the symmetric algebra. Thus if $d!$ is invertible then $\Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$ is an isomorphism. In particular, this is the case when A is purely of characteristic zero, i.e., contains the field of rationals.

(1.2.14) *Universal multiplication of laws* — Let $d, e \in \mathbb{N}$. There is a canonical homomorphism

$$\rho_{d,e} : \Gamma^{d+e}_A(M) \to \Gamma^d_A(M) \otimes_A \Gamma^e_A(M)$$

given by the homogeneous polynomial law $x \mapsto \gamma^d(x) \otimes \gamma^e(x)$ of degree $d+e$ and the universal property (1.2.4). In particular

$$\rho_{d,e}\big(\gamma^{\nu}(x)\big) = \sum_{\substack{\nu'+\nu''=\nu \\ |\nu'|=d,\ |\nu''|=e}} \gamma^{\nu'}(x) \otimes \gamma^{\nu''}(x). \tag{1.2.14.1}$$

We can factor $\rho_{d,e}$ as $\pi_{d,e} \circ \Gamma^{d+e}(p)$ where $p : M \to M \oplus M$ is the diagonal map $x \mapsto x \oplus x$ and $\pi_{d,e}$ is the projection on the factor of bidegree (d, e) of $\Gamma^{d+e}(M \oplus M)$, cf. Equation (1.2.6.1).

If $F_1 : M \to N_1$ and $F_2 : M \to N_2$ are polynomial laws homogeneous of degrees d and e respectively we can form the polynomial law $F_1 \otimes F_2 : M \to N_1 \otimes_A N_2$ given by $(F_1 \otimes F_2)(x) = F_1(x) \otimes F_2(x)$. The law $F_1 \otimes F_2$ is homogeneous of degree $d+e$. If $f_1 : \Gamma^d(M) \to N_1$, $f_2 : \Gamma^e(M) \to N_2$ and $f_{1,2} : \Gamma^{d+e}(M) \to N_1 \otimes_A N_2$ are the corresponding homomorphisms then $f_{1,2} = (f_1 \otimes f_2) \circ \rho_{d,e}$.

1.3. **Multiplicative structure.** Let M, N be A-modules and d a positive integer. There is a unique homomorphism

$$\mu : \Gamma^d_A(M) \otimes_A \Gamma^d_A(N) \to \Gamma^d(M \otimes_A N)$$

sending $\mu(\gamma^d(x) \otimes \gamma^d(y))$ to $\gamma^d(x \otimes y)$ [Rob80]. When B is an A-algebra, the composition of μ and the multiplication homomorphism $B \otimes_A B \to B$ induces a multiplication on $\Gamma^d_A(B)$ which we will denote by juxtaposition. The multiplication is such that $\gamma^d(x)\gamma^d(y) = \gamma^d(xy)$ and this makes γ^d into a multiplicative polynomial law homogeneous of degree d. The unit in $\Gamma^d_A(B)$ is $\gamma^d(1)$.

If B is an A-algebra and M is a B-module, then μ together with the module structure $B \otimes_A M \to M$ induces a $\Gamma^d_A(B)$-module structure on $\Gamma^d_A(M)$.

(1.3.1) *Universal property* — Let B and C be A-algebras. Then the map

$$\operatorname{Hom}_{A\text{-}\mathbf{Alg}}\big(\Gamma^d_A(B), C\big) \to \operatorname{Pol}^d_{\mathrm{mult}}(B, C)$$

given by $f \to f \circ \gamma^d$ is a bijection [Rob80]. Also see [Fer98, Prop. 2.5.1].

(1.3.2) Γ *and* TS — The homogeneous polynomial law $M \to \mathrm{TS}^d_A(M)$ of degree d given by $x \mapsto x^{\otimes_A d} = x \otimes_A \cdots \otimes_A x$ is multiplicative. The homomorphism $\varphi : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ in (1.2.13) is thus an A-algebra homomorphism. It is an isomorphism when B is a flat over A or when A is of pure characteristic zero (1.2.13). The morphism $\operatorname{Spec}\big(\mathrm{TS}^d_A(B)\big) \to \operatorname{Spec}\big(\Gamma^d_A(B)\big)$ is a universal homeomorphism with trivial residue field extensions, see Corollary (4.2.5). Further results about this morphism is found in [**III**].

(1.3.3) *Filtered direct limits* — The functor $B \mapsto \Gamma^d_A(B)$ commutes with filtered direct limits. This follows from (1.2.11) and the fact that a filtered direct limit in the category of A-algebras coincides with the corresponding filtered direct limit in the category of A-modules [GV72, Cor. 2.9].

(1.3.4) The isomorphism of A-modules given by equation (1.2.6.1) gives an isomorphism of A-algebras

$$\begin{aligned}\Gamma^d_A(B \times C) &= \prod_{a+b=d} \big(\Gamma^a_A(B) \otimes_A \Gamma^b_A(C)\big)\\ \gamma^d\big((x,y)\big) &= \big(\gamma^a(x) \otimes \gamma^b(y)\big)_{a+b=d}.\end{aligned}$$

(1.3.5) *Universal multiplication of laws* — If M is an A-algebra in (1.2.14), then the polynomial law defining the homomorphism $\rho_{d,e}$ is multiplicative. The homomorphism $\rho_{d,e}$ is thus an A-algebra homomorphism. For a geometrical interpretation of $\rho_{d,e}$ as "addition of cycles" see section §4.1.

Formula (1.3.6) (Multiplication formula [Fer98, Form. 2.4.2]). *Let $(x_\alpha)_{\alpha\in\mathcal{I}}$ be a set of elements in B and let $\mu, \nu \in \mathbb{N}^{(\mathcal{I})}$ with $d = |\mu| = |\nu|$. Then we have the following identity in $\Gamma^d_A(B)$*

$$\gamma^\mu(x)\gamma^\nu(x) = \sum_{\xi\in N_{\mu,\nu}} \gamma^\xi\big(x_{(1)}x_{(2)}\big) = \sum_{\xi\in N_{\mu,\nu}} \mathop{\times}_{(\alpha,\beta)\in\mathcal{I}\times\mathcal{I}} \gamma^{\xi_{\alpha,\beta}}(x_\alpha x_\beta)$$

where $N_{\mu,\nu}$ is the set of multi-indices $\xi \in \mathbb{N}^{(\mathcal{I}\times\mathcal{I})}$ such that $\sum_{\beta\in\mathcal{I}} \xi_{\alpha,\beta} = \mu_\alpha$ for every $\alpha \in \mathcal{I}$ and $\sum_{\alpha\in\mathcal{I}} \xi_{\alpha,\beta} = \nu_\beta$ for every $\beta \in \mathcal{I}$.

Proposition (1.3.7). *If B is an A-algebra of finite type (resp. of finite presentation, resp. finite over A, resp. integral over A) then $\Gamma^d_A(B)$ is an A-algebra of finite type (resp. of finite presentation, resp. finite, resp. integral).*

Proof. If B is an A-algebra of finite type then B is a quotient of a polynomial ring $A[x_1, x_2, \dots, x_n]$. The induced homomorphism $\Gamma^d(A[x_1, x_2, \dots, x_n]) \to \Gamma^d_A(B)$ is surjective, and thus it is enough to show that $\Gamma^d(A[x_1, x_2, \dots, x_n])$ is an A-algebra of finite type. As Γ^d commutes with base change it is further enough to show that $\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[x_1,x_2,\dots,x_n]) = \mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}[x_1,x_2,\dots,x_n])$ is a $\mathbb{Z}$-algebra of finite type. This is well-known, cf. [Bou64, Ch. V, §1, No. 9, Thm. 2].

If B is an A-algebra of finite presentation then there is a noetherian ring A_0 and a A_0-algebra of finite type B_0 such that $B = B_0 \otimes_{A_0} A$. The first part of the proposition shows that $\Gamma^d_{A_0}(B_0)$ is an A_0-algebra of finite type and thus also of finite presentation as A_0 is noetherian. As Γ^d commutes with base change this shows that $\Gamma^d_A(B)$ is an A-algebra of finite presentation.

If B is a finite A-algebra then $\Gamma^d_A(B)$ is a finite A-algebra by (1.2.5). If B is an integral A-algebra then B is a filtered direct limit of finite A-algebras. As Γ^d commutes with filtered direct limits this shows that $\Gamma^d_A(B)$ is an integral A-algebra. □

1.4. **The scheme $\Gamma^d(X/S)$ for X/S affine.** Let S be any scheme and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. As the construction of $\Gamma^d_A(B)$ commutes with localization with respect to multiplicatively closed subsets of A we may define a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\Gamma^d_{\mathcal{O}_S}(\mathcal{A})$. This extends the definition of the covariant functor Γ^d to the category of quasi-coherent algebras on S. If $f : X \to S$ is an affine morphism we let $\Gamma^d(X/S) = \mathrm{Spec}\bigl(\Gamma^d_{\mathcal{O}_S}(f_*\mathcal{O}_X)\bigr)$. This defines a covariant functor

$$\Gamma^d : \mathbf{Aff}_{/S} \to \mathbf{Aff}_{/S}, \qquad X/S \mapsto \Gamma^d(X/S)$$

where $\mathbf{Aff}_{/S}$ is the category of schemes affine over S. When it is not likely to cause confusion, we will sometimes abbreviate $\Gamma^d(X/S)$ with $\Gamma^d(X)$.

A polynomial law in this setting is a natural transformation of functors from quasi-coherent $\mathcal{O}_S$-algebras to sheaves of sets on S. We obtain an isomorphism $\mathrm{Hom}_S\bigl(S',\Gamma^d(X/S)\bigr) \to \mathrm{Pol}^d_{\mathrm{mult},\mathcal{O}_S}(\mathcal{O}_X,\mathcal{O}_{S'})$ for any affine S-scheme S'. Also observe that

$$\begin{aligned}\mathrm{Hom}_S\bigl(S',\Gamma^d(X/S)\bigr) &\cong \mathrm{Hom}_{S'}\bigl(S',\Gamma^d(X/S)\times_S S'\bigr)\\ &\cong \mathrm{Hom}_{S'}\bigl(S',\Gamma^d(X'/S')\bigr).\end{aligned}$$

More generally, if S is an algebraic space and $X \to S$ is affine we define $\Gamma^d(X/S)$ by étale descent.

Defining $\Gamma^d(X/S)$ for any S-scheme X is non-trivial. In the following sections we will give a functorial description of $\Gamma^d(X/S)$ and then show that this functor is represented by a scheme or algebraic space $\Gamma^d(X/S)$.

A very useful fact that will repeatedly be used in the sequel is the following rephrasing of paragraph (1.3.4):

Proposition (1.4.1). *Let S be an algebraic space and let $X_1,X_2,\dots,X_n$ be algebraic spaces affine over S. Then*

$$\Gamma^d\left(\coprod_{i=1}^n X_i\right) = \coprod_{\substack{d_i\in\mathbb{N}\\ \sum_i d_i=d}} \Gamma^{d_1}(X_1)\times_S \Gamma^{d_2}(X_2)\times_S\cdots\times_S \Gamma^{d_n}(X_n).$$

Similarly, the following Proposition is a translation of paragraph (1.2.9):

Proposition (1.4.2). *If Y is an algebraic space affine over S and $X \hookrightarrow Y$ a closed subspace, then $\Gamma^d(X/S)$ is a closed subspace of $\Gamma^d(Y/S)$.*

2. SUPPORT AND IMAGE OF A FAMILY OF ZERO-CYCLES

Let X/S be a scheme or an algebraic space, affine over S. In this section we will show that a "family of zero-cycles" α on X parameterized by S, that is, a morphism $\alpha : S \to \Gamma^d(X/S)$, has a unique minimal closed subspace $Z = \mathrm{Image}(\alpha) \hookrightarrow X$, the *image of* α, such that α factors through the closed subspace $\Gamma^d(Z/S) \hookrightarrow \Gamma^d(X/S)$. The reduction Z_{red} will be denoted the *support of* α and written as $\mathrm{Supp}(\alpha)$.

For general X/S a family of zero-cycles α, parameterized by a S-scheme T, should be thought of as one of the following

(i) A morphism $T \to \Gamma^d(X/S)$.
(ii) An "object" living over $\mathrm{Image}(\alpha) \hookrightarrow X \times_S T$.
(iii) A "multi-section" $T \to X \times_S T$ with image $\mathrm{Image}(\alpha)$.

Note that in contrast to ordinary sections and families of closed subschemes, a family of zero-cycles is *not* uniquely determined by its image. If α is a family over a reduced scheme T, then $\mathrm{Supp}(\alpha) = \mathrm{Image}(\alpha)$ is reduced, cf. Proposition (2.1.4). In this case, the "object" in (ii) can be interpreted as a cycle in the ordinary sense.

We will show the following results about the image and the support:

(i) The image is integral over S. (§2.1)
(ii) The image commutes with essentially smooth base change $S' \to S$ and projective limits. In particular it commutes with étale base change and henselization. (§2.2)
(iii) The support commutes with any base change. (§2.3)
(iv) The support has *universally topologically finite fibers*, i.e., each fiber over S consists of a finite number of points and the separable degrees of the corresponding field extensions are finite. (§2.4)
(v) The support is universally open over S. (§2.5)

Many of the results require rather technical but standard demonstrations. In particular we will often need to reduce from the integral to the finite case by the standard limit techniques of [$\mathrm{EGA}_{\mathrm{IV}}$, §8]. The fact that the support is universally open over S will not be needed in the following sections but this result, as well as the fact that the support has universally topologically finite fibers, shows that topologically the support behaves as if it was of finite presentation over S.

2.1. **Kernel of a multiplicative law.** We will first define the *kernel* of a multiplicative polynomial law $F : B \to C$ of A-algebras. If F is of degree 1, i.e., a ring homomorphism, then the kernel is the usual kernel. In general, the kernel of F is the largest ideal I such that F factors through $B \twoheadrightarrow B/I$. We will focus our attention on the case when $C = A$. Then $B/\ker(F)$ is integral over A as shown in

Proposition (2.1.6) and there is a canonical filtration of $\ker(F)$ which degenerates in characteristic zero.

Definition (2.1.1). Let B and C be A-algebras. Given a multiplicative law $F : B \to C$ we define its *kernel* $\ker(F)$ as the largest ideal I such that F factors as $B \twoheadrightarrow B/I \to C$. This is a well-defined ideal since if F factors through $B \twoheadrightarrow B/I$ and $B \twoheadrightarrow B/J$ then F factors through $B/(I+J)$.

Note that F factors through $B \twoheadrightarrow B/I$ if and only if $F_{A'}(b' + IB') = F_{A'}(b')$ for any A-algebra A' and $b' \in B' = B \otimes_A A'$. Also note that the kernel $\ker(F_{A'})$ contains $\ker(F)B'$ but this inclusion is often strict.

Notation (2.1.2). We will in the following denote homogeneous laws by upper-case Latin letters and the corresponding homomorphisms by lower-case letters. For example, if $F : B \to C$ is a homogeneous multiplicative polynomial law of degree d we let $f : \Gamma^d_A(B) \to C$ be the corresponding homomorphism. If A' is an A-algebra we denote by $F' : B' \to C'$ the multiplicative law given by $F'_R = F_R$ for every A'-algebra R. The corresponding homomorphism $f' : \Gamma^d_{A'}(B') \to C'$ is then the base change of f along $A \to A'$.

Lemma (2.1.3). *Let A be a ring and let B and C be A-algebras. Given a multiplicative law $F : B \to C$ homogeneous of degree d, or equivalently given a morphism $f : \Gamma^d_A(B) \to C$, define the following subsets of B*

$$\begin{aligned} L_1 &= \bigl\{ b \in B \ : \ f\bigl(\gamma^k(b) \times y\bigr) = 0, \ \forall k, y \bigr\} \\ L_2 &= \bigl\{ b \in B \ : \ f\bigl(\gamma^k(bx) \times y\bigr) = 0, \ \forall k, x, y \bigr\} \\ L_3 &= \bigl\{ b \in B \ : \ f'\bigl(\gamma^k(bx') \times y'\bigr) = 0, \ \forall k, A', x', y' \bigr\} \end{aligned}$$

where $1 \le k \le d$, $x \in B$, $y \in \Gamma^{d-k}_A(B)$, $x' \in B'$, $y' \in \Gamma^{d-k}_{A'}(B')$ and $A \to A'$ is a ring homomorphism. Then $\ker(F) = L_1 = L_2 = L_3$. *In particular, these sets are ideals.*

Proof. Clearly $L_3 \subseteq L_2 \subseteq L_1$. Let $b \in L_1$ and let $x \in B$. The multiplication formula (1.3.6) shows that for any $y \in \Gamma^{d-k}_A(B)$

$$\gamma^k(bx) \times y = \bigl(\gamma^k(b) \times y\bigr)\bigl(\gamma^k(x) \times \gamma^{d-k}(1)\bigr) + \sum_{i=1}^{k} \gamma^i(b) \times y_i$$

for some $y_i \in \Gamma^{d-i}_A(B)$. Thus $b \in L_2$ and hence $L_1 = L_2$. From Equations (1.2.1.3) and (1.2.1.4) it follows that $L_2 = L_3$ and that this set is an ideal.

If I is an ideal in B then $\Gamma^d_A(B/I) = \Gamma^d_A(B)/J$ where J is the ideal generated by $\gamma^k(b) \times y$ where $b \in I$, $1 \le k \le d$ and $y \in \Gamma^{d-k}_A(B)$, cf. (1.2.10). Thus $\ker(F)$ is contained in L_2. On the other hand, if b is contained in L_3 then for any A-algebra A' and $b', x' \in B' = B \otimes_A A'$ we have that

$$F_{A'}(b' + bx') = \sum_{k=0}^{d} f'\bigl(\gamma^k(bx') \times \gamma^{d-k}(b')\bigr) = f'\bigl(\gamma^d(b')\bigr) = F_{A'}(b')$$

and thus $b \in \ker(F)$. □

Proposition (2.1.4) ([Zip88, Lem. 7.6]). *Let A be a ring and B, C be A-algebras together with a multiplicative law $F : B \to C$ homogeneous of degree d. If C is reduced then $B/\ker(F)$ is reduced.*

Proof. Let $f : \Gamma^d_A(B) \to C$ be the homomorphism corresponding to F. Let $b \in B$ such that $b^n \in \ker(F)$ for some $n \in \mathbb{N}$. Then by Lemma (2.1.3) we have that $f\big(\gamma^k(b^n x) \times y\big) = 0$ for every $1 \leq k \leq d$, $x \in B$ and $y \in \Gamma^{d-k}_A(B)$. An easy calculation using the multiplication formula (1.3.6) shows that the element $\big(\gamma^k(b) \times y\big)^{\lceil dn/k \rceil}$ is in the kernel of f for every $1 \leq k \leq d$ and $y \in \Gamma^{d-k}_A(B)$. As C is reduced this implies that $\gamma^k(b) \times y$ is in the kernel of f and thus $b \in \ker(F)$. □

Definition (2.1.5). Let $F : B \to A$ be a multiplicative law homogeneous of degree d. For any $b \in B$ we define its characteristic polynomial as

$$\chi_{F,b}(t) = F_{A[t]}(b - t) = \sum_{k=0}^{d} (-1)^k f\big(\gamma^{d-k}(b) \times \gamma^k(1)\big) t^k \in A[t].$$

We let

$$I_{\mathrm{CH}}(F) = \big(\chi_{F,b}(b)\big)_{b \in B} \subseteq B$$

be the *Cayley-Hamilton ideal* of F. Here $\chi_{F,b}(b)$ is the evaluation of $\chi_{F,b}(t)$ at $b \in B$, i.e., the image of $\chi_{F,b}(t)$ along $A[t] \to B[t] \to B[t]/(t - b) = B$.

Proposition (2.1.6) ([Ber65, Satz 4]). *Let $F : B \to A$ be a multiplicative law. Then $I_{\mathrm{CH}}(F) \subseteq \ker(F) \subseteq \sqrt{I_{\mathrm{CH}}(F)}$. In particular it follows that $B/\ker(F)$ is integral over A.*

Proof. Let $P \twoheadrightarrow B$ be a surjection from a flat A-algebra P and let $F' : P \to A$ be the multiplicative law given as the composition of F with $P \twoheadrightarrow B$. As the images of $I_{\mathrm{CH}}(F')$ and $\ker(F')$ in B are $I_{\mathrm{CH}}(F)$ and $\ker(F)$ respectively, we can, replacing B with P and F with F', assume that B is flat over A. Then $\Gamma^d_A(B) = \mathrm{TS}^d_A(B)$.

We will first show the inclusion $I_{\mathrm{CH}}(F) \subseteq \ker(F)$. By definition this is equivalent with the following: For every base change $A \to A'$, every $b \in B$ and every $b', x' \in B' = B \otimes_A A'$, the identity $F_{A'}\big(\chi_{F,b}(b)x' + b'\big) = F_{A'}(b')$ holds.

For any ring R we let $\mathrm{Diag}_d(R) = R^d$ denote the diagonal $d \times d$-matrices with coefficients in R. Let $\Psi : B \to \mathrm{Diag}_d\big(\mathrm{T}^d_A(B)\big)$ be the ring homomorphism such that $\Psi(b) = \mathrm{diag}\,(b_1, b_2, \dots, b_d)$ where $b_k = 1^{\otimes k-1} \otimes b \otimes 1^{\otimes d-k} \in \mathrm{T}^d_A(B)$. The determinant gives a multiplicative law

$$\det : \mathrm{Diag}_d\big(\mathrm{T}^d_A(B)\big) \to \mathrm{T}^d_A(B)$$

which is homogeneous of degree d. Let $E = \mathrm{TS}^d_A(A[t]) = A[e_1, e_2, \dots, e_d]$ be the polynomial ring over A in d variables. Here e_k denotes the elementary symmetric function $t^{\otimes k} \times 1^{\otimes d-k}$. Let $b \in B$ be any element. We have a homomorphism $\rho_b : E \hookrightarrow \mathrm{TS}^d_A(B)$ induced by the morphism $A[t] \to B$ mapping t on b. More explicitly $\rho_b(e_k) = b^{\otimes k} \times 1^{\otimes d-k}$.

Let $A \to A'$ be any ring homomorphism and let $B' = B \otimes_A A'$, $E' = E \otimes_A A'$. We have a commutative diagram

$$\begin{tikzcd}
B' \arrow[r, "\gamma^d"] & \mathrm{TS}^d_{A'}(B') \arrow[r, "f'"] & A' \\
B' \otimes_{A'} E' \arrow[u, twoheadrightarrow, "{(\mathrm{id},f'\circ\rho'_b)}"'] \arrow[r, "\gamma^d"] \arrow[d, "\Psi"] \arrow[ur, phantom, "\circ"] & \mathrm{TS}^d_{A'}(B') \otimes_{A'} E' \arrow[u, twoheadrightarrow, "{(\mathrm{id},f'\circ\rho'_b)}"'] \arrow[r, twoheadrightarrow, "{(\mathrm{id},\rho'_b)}"] \arrow[d, hook] \arrow[ur, phantom, "\circ"] & \mathrm{TS}^d_{A'}(B') \arrow[u, "f'"] \arrow[dd, hook] \\
\mathrm{Diag}_d\big(\mathrm{T}^d_{A'}(B') \otimes_{A'} E'\big) \arrow[r, "\det"] \arrow[d, twoheadrightarrow, "{\mathrm{Diag}(\mathrm{id},\rho'_b)}"] \arrow[ur, phantom, "\circ"] & \mathrm{T}^d_{A'}(B') \otimes_{A'} E' \arrow[dr, twoheadrightarrow, "{(\mathrm{id},\rho'_b)}"] \arrow[ur, phantom, "\circ"] & \\
\mathrm{Diag}_d\big(\mathrm{T}^d_{A'}(B')\big) \arrow[rr, "\det"] \arrow[ur, phantom, "\circ"] & & \mathrm{T}^d_{A'}(B').
\end{tikzcd}$$

Let $\chi(t) = \sum_{k=0}^{d} (-1)^k e_{d-k} t^k \in E[t]$ where we let $e_0 = 1$. Let

$$\chi_b(t) = \rho_b \circ \chi(t) = \sum_{k=0}^{d} (-1)^k \gamma^{d-k}(b) \times \gamma^k(1) t^k \in \mathrm{TS}^d_A(B)[t].$$

Then $f(\chi_b(t)) = \chi_{F,b}(t) \in B[t]$. Let $b', x' \in B'$ be any elements. We begin with the elements $\chi_{F,b}(b)x' + b'$ and b' in the upper-left corner B' of the diagram and want to show that their images by $F_{A'} = f' \circ \gamma^d$ in the upper-right corner A' coincide. As $\chi_{F,b}(b)x' + b'$ lifts to $\chi(b)x' + b' \in B' \otimes_{A'} E'$ it is enough to show that images of $b', \chi(b)x' + b' \in B' \otimes_{A'} E'$ in the lower-left corner $\mathrm{Diag}_d\big(\mathrm{T}^d_{A'}(B')\big)$ are equal.

For any ring R and diagonal matrix $D \in \mathrm{Diag}_d(R)$ let $P_D(t) \in R[t]$ be the characteristic polynomial of D. Then by Cayley-Hamilton's theorem $P_D(D) = 0$ in $\mathrm{Diag}_d(R)$. Note that the determinant and the characteristic polynomial commute with arbitrary base change $R \to R'$. Now, the image of $\chi(b)$ by $\mathrm{Diag}(\mathrm{id}, \rho_b) \circ \Psi$ is easily seen to be $\chi_b(\Psi(b)) = P_{\Psi(b)}\big(\Psi(b)\big) = 0$. Thus the images of $\chi(b)x' + b'$ and b' in the lower-left corner are equal. This concludes the proof of the inclusion $I_{\mathrm{CH}}(F) \subseteq \ker(F)$.

If $b \in \ker(F)$ then by Lemma (2.1.3) $f\big(\gamma^k(b) \times \gamma^{d-k}(1)\big) = 0$ for every $k = 1, 2, \ldots, d$. Thus $\chi_{F,b}(t) = t^d$ and hence $b^d \in I_{\mathrm{CH}}(F)$ which shows the second inclusion. Finally $B/I_{\mathrm{CH}}(F)$ is clearly integral over A and thus also $B/\ker(F)$. □

Remark **(2.1.7).** Ziplies defines the *radical* of a not necessarily homogeneous polynomial law in [Zip88, Def. 6.7]. When the polynomial law is homogeneous the radical coincides with the kernel as defined in (2.1.5). Ziplies further proves in [Zip88, Lem. 7.4] that if $I_{\mathrm{CH}}(F)$ is zero in B then $\ker(F)$ is contained in the Jacobson radical of B. Proposition (2.1.6) shows more generally that under this assumption $\ker(F)$ is contained in the nilradical of B. Note that both inclusions $I_{\mathrm{CH}}(F) \subseteq \ker(F) \subseteq \sqrt{I_{\mathrm{CH}}(F)}$ can be strict[1].

In [Zip86, 3.4] Ziplies also shows that $I_{\mathrm{CH}}(F)$ is contained in the ideal

$$\begin{aligned} I_F^{(1)} &= \bigl\{ b \in B \;:\; f\bigl(bx \times \gamma^{d-1}(1)\bigr) = 0,\ \forall x \in B \bigr\} \\ &= \bigl\{ b \in B \;:\; f(b \times y) = 0,\ \forall y \in \Gamma_A^{d-1}(B) \bigr\}. \end{aligned}$$

As this ideal by Lemma (2.1.3) clearly contains $\ker(F)$, the first inclusion of Proposition (2.1.6) is a generalization of this result.

2.2. **Kernel and base change.**

Definition (2.2.1). Let A be a ring and let B and C be A-algebras. Given a multiplicative law $F : B \to C$ homogeneous of degree d, or equivalently given a morphism $f : \Gamma_A^d(B) \to C$, we let

$$I_F^{(k)} = \bigl\{ b \in B \;:\; f\bigl(\gamma^i(b) \times y\bigr) = 0,\ \forall 1 \leq i \leq k,\ y \in \Gamma_A^{d-i}(B) \bigr\}.$$

for $k = 0, 1, 2, \dots, d$.

Proposition (2.2.2). *Let B and C be A-algebras and let $F : B \to C$ be a multiplicative law homogeneous of degree d. Then the sets $I_F^{(k)}$ are ideals of B and we have a filtration*

$$B = I_F^{(0)} \supseteq I_F^{(1)} \supseteq \cdots \supseteq I_F^{(d)} = \ker(F).$$

If A' is an A-algebra and $B' = B \otimes_A A'$ then $I_{F_{A'}}^{(k)} \supseteq I_F^{(k)} B'$. In particular $\ker(F_{A'}) \supseteq \ker(F)B'$.

Proof. That $I_F^{(k)}$ are ideals follows exactly as in the proof of Lemma (2.1.3). That $I_F^{(d)} = \ker(F)$ is Lemma (2.1.3) and the other assertions are trivial. □

The main application for the filtration $I_F^{(0)} \supseteq I_F^{(1)} \supseteq \cdots \supseteq I_F^{(d)}$ is that the elements in $I_F^{(k-1)}$ behave "quasi-linear" modulo $I_F^{(k)}$ with respect to γ^k in a certain sense. This will be utilized in Lemma (2.2.10).

Lemma (2.2.3). *Let $n \in \mathbb{N}$ and p be a prime. Then $p \mid \binom{n}{k}$ for every $1 \leq k \leq n-1$ if and only if $n = p^s$.*

Proof. Assume that $p \mid \binom{n}{k}$ for $1 \leq k \leq n-1$. It easily follows that $a^n = a$ in $\mathbb{F}_p$ for every $a \in \mathbb{F}_p$. Thus $x^p - x$ divides $x^n - x$ in $\mathbb{F}_p[x]$ which shows that $p \mid n$. We obtain that $a^{n/p} = a$ for every $a \in \mathbb{F}_p$ and by induction on s that $n = p^s$. The converse is easy. □

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[1] There is a misprint in [Zip88, Lem. 7.4]. "equals" should be replaced with "is contained in". Also A should be a B-algebra as well as an R-algebra in his notation.

Proposition (2.2.4). *Let A be either a $\mathbb{Z}_{(p)}$-algebra with p a prime or a $\mathbb{Q}$-algebra in which case we let $p = 1$. Let $k \geq 1$ be an integer. Then $I_F^{(k)} = I_F^{(k-1)}$ if $k \neq p^s$. In particular, if A is a $\mathbb{Q}$-algebra then* $\ker(F) = I_F^{(1)}$.

Proof. Let $A' = A[t]$ and $b'_1, b'_2 \in I_{F'}^{(k-1)}$. Then for any $y' \in \Gamma_{A'}^{d-k}(B')$

$$f'\big(\gamma^k(b'_1 + b'_2) \times y'\big) = f'\big(\gamma^k(b'_1) \times y'\big) + f'\big(\gamma^k(b'_2) \times y'\big).$$

In particular for any $b \in I_F^{(k-1)}$ and $y \in \Gamma_A^{d-k}(B)$

$$(1+t)^k f'\big(\gamma^k(b) \times y\big) = f'\big(\gamma^k((1+t)b) \times y\big) = (1+t^k) f\big(\gamma^k(b) \times y\big)$$

which shows that $\binom{k}{i}$ annihilates $f\big(\gamma^k(b) \times y\big)$ for any $1 \leq i \leq k-1$. If $k \neq p^s$ then $f\big(\gamma^k(b) \times y\big) = 0$ by Lemma (2.2.3) and thus $b \in I_F^{(k)}$. □

Lemma (2.2.5). *Let A be a ring and $B = \varinjlim B_\lambda$ be a filtered direct limit of A-algebras with induced homomorphisms $\varphi_\lambda : B_\lambda \to B$. Let $f : \Gamma_A^d(B) \to C$ and denote by f_λ the composition of $\Gamma_A^d(\varphi_\lambda) : \Gamma_A^d(B_\lambda) \to \Gamma_A^d(B)$ and f. Then $I_F^{(k)} = \varinjlim I_{F_\lambda}^{(k)}$ for every $k = 0, 1, \dots, d$. In particular* $\ker(F) = \varinjlim \ker(F_\lambda)$.

Proof. As f_λ factors as $\Gamma_A^d(B_\lambda) \to \Gamma_A^d(B) \to C$ it follows that $\varphi_\lambda^{-1}\big(I_F^{(k)}\big) \subseteq I_{F_\lambda}^{(k)}$. Thus $I_F^{(k)} \subseteq \varinjlim I_{F_\lambda}^{(k)}$. Conversely, for any $b \in B \setminus I_F^{(k)}$ there is an $i \leq k$ and $y \in \Gamma_A^{d-i}(B)$ such that $f\big(\gamma^i(b) \times y\big) \neq 0$. If we let α be such that ${\varphi_\alpha}^{-1}(b) \neq \emptyset$ and $\Gamma^{d-i}(\varphi_\alpha)^{-1}(y) \neq \emptyset$ then for any $\lambda \geq \alpha$ and $b_\lambda \in B_\lambda$ such that $\varphi_\lambda(b_\lambda) = b$ we have that $b_\lambda \notin I_{F_\lambda}^{(k)}$. Thus $\varinjlim I_{F_\lambda}^{(k)} \subseteq I_F^{(k)}$. □

Proposition (2.2.6). *Let A be a ring and S a multiplicative closed subset. Let $F : B \to A$ be a multiplicative homogeneous law of degree d and denote by $S^{-1}F : S^{-1}B \to S^{-1}A$ the map corresponding to the A-algebra $S^{-1}A$. Then $S^{-1}I_F^{(k)} = I_{S^{-1}F}^{(k)}$. In particular $S^{-1}\ker(F) = \ker(S^{-1}F)$, i.e., the kernel commutes with localization.*

Proof. By Proposition (2.1.6) the quotient $B/\ker(F)$ is integral over A. Replacing B by $B/\ker(F)$ we can thus assume that B is integral over A. As B is the filtered direct limit of its finite sub-A-algebras and both the kernel of a multiplicative law, Lemma (2.2.5), and tensor products commute with filtered direct limits we can assume that B is a finite A-algebra. Then $\Gamma_A^i(B)$ is a finite A-algebra for all $i = 0, 1, \dots, d$ by Proposition (1.3.7).

Let $x/s \in I_{S^{-1}F}^{(k)}$, i.e., by definition $x/s \in S^{-1}B$ such that $S^{-1}f\big(\gamma^i(x/s) \times y\big) = 0$ for all $1 \leq i \leq k$ and $y \in \Gamma_A^{d-i}(B)$. For any $y \in \Gamma_A^{d-i}(B)$ there is then a $t \in S$ such that $tf\big(\gamma^i(x) \times y\big) = 0$ in A. As $\Gamma_A^{d-i}(B)$ is a finite A-algebra we can find a common t that works for all $i \leq k$ and y. Then $f\big(\gamma^i(tx) \times y\big) = t^i f\big(\gamma^i(x) \times y\big) = 0$ for all $i \leq k$ and y. As $x/s = tx/st$, this shows that $I_{S^{-1}F}^{(k)} = S^{-1}I_F^{(k)}$. □

Proposition (2.2.7). *Let A be a ring and B an A-algebra. Let $A' = \varinjlim A'_\lambda$ be a filtered direct limit of A-algebras with induced homomorphisms $\varphi_\lambda : A'_\lambda \to A'$. Let $F : B \to A$ be a multiplicative polynomial law of degree d. Then $I^{(k)}_{F_{A'}} = \varinjlim I^{(k)}_{F_{A'_\lambda}}$ for every $k = 0, 1, \dots, d$. In particular* $\ker(F_{A'}) = \varinjlim \ker(F_{A'_\lambda})$.

Proof. As in the proof of Proposition (2.2.6) we can assume that B is finite over A and hence that $\Gamma^i_A(B)$ is a finite A-module. Choose generators $y_{i1}, y_{i2}, \dots, y_{in_i}$ of $\Gamma^{d-i}_A(B)$ as an A-module for $i = 1, 2, \dots, d$. Let $B' = B \otimes_A A'$ and $B'_\lambda = B \otimes_A A'_\lambda$. Let $b' \in I^{(k)}_{F_{A'}}$. Then there exists an α and $b'_\alpha \in B'_\alpha$ such that b' is the image of b'_α by $B'_\alpha \to B$. As the image of $f_{A'_\alpha}(\gamma^i(b'_\alpha) \times y_{ij})$ in A' is $f_{A'}(\gamma^i(b') \times y_{ij})$ and hence zero for $i = 1, 2, \dots, k$, there is a $\beta \geq \alpha$ such that $b'_\alpha \in I^{(k)}_{F_{A'_\lambda}}$ for all $\lambda \geq \beta$. Thus $b' \in \varinjlim_\lambda I^{(k)}_{F_{A'_\lambda}}$ and $I^{(k)}_{F_{A'}} \subseteq \varinjlim_\lambda I^{(k)}_{F_{A'_\lambda}}$. The reverse inclusion is obvious. □

We will now show that the kernel, always commutes with smooth base change and that it commutes with flat base change in characteristic zero.

Proposition (2.2.8). *Let A be a ring and let $F : B \to A$ a multiplicative homogeneous law of degree d. Let A' be a flat A-algebra and denote by F' the multiplicative law corresponding to A'. Then $I^{(1)}_F B' = I^{(1)}_{F'}$. In particular, if A is a $\mathbb{Q}$-algebra then the kernel commutes with flat base change.*

Proof. We reduce to B a finite A-algebra as in the proof of Proposition (2.2.6). For any $y \in \Gamma^{d-1}_A(B)$ let φ_y be the A-module homomorphism $B \to \Gamma^d_A(B)$ given by $b \mapsto b \times y$. Then $I^{(1)}_f = \bigcap_{y \in \Gamma^{d-1}_A(B)} \ker(f \circ \varphi_y)$. As $\Gamma^{d-1}_A(B)$ is a finitely generated A-module and φ_y is linear in y, this intersection coincides with an intersection over a finite number of y's. As both finite intersections and kernels commute with flat base change the first statement of the proposition follows. The last statement follows from Proposition (2.2.4). □

Recall that a monic polynomial $g \in A[t]$ is *separable* if $(g, g') = A[t]$, where g' is the formal derivative of g. Further recall that $A \hookrightarrow A[t]/g$ is *étale* if and only if g is separable. We will need the following basic lemma to which we, for a lack of suitable reference, include a proof.

Lemma (2.2.9). *Let $A \hookrightarrow A' = A[t]/g$ be an étale homomorphism, i.e., such that g is a separable polynomial. If A is a local ring of residue characteristic $p > 0$ then for any prime power $q = p^s$, $s \in \mathbb{N}$, the elements $1, t^q, t^{2q}, \dots, t^{(n-1)q}$ form an A-module basis of A' where $n = \deg(g)$.*

Proof. Let $k = A/\mathfrak{m}_A$. By Nakayama's lemma it is enough to show that a basis of $A'/\mathfrak{m}_A A' = k[t]/\overline{g}$ over k is given by $1, t^q, t^{2q}, \dots, t^{(n-1)q}$. Replacing A, A' and g with k, $A'/\mathfrak{m}_A A'$ and $\overline{g}$ respectively, we can thus assume that $A = k$ is a field of characteristic p.

Let $g = g_1 g_2 \dots g_m$ be a factorization of g into irreducible polynomials. We have that $A' = k[t]/g = k'_1 \times k'_2 \times \dots \times k'_m$ where $k \hookrightarrow k'_i = k[t]/g_i$ are separable field extensions. The subring generated by t^q is the image of $k[t^q]/g^q = \prod k[t^q]/g_i^q$ in $\prod_i k_i'$. To show that t^q generates $k[t]/g$ it is thus enough to show that its image in k_i' generates k_i' for every i. Thus, we can assume that g is irreducible such that $A' = k[t]/g = k'$ is a field.

The field extension $k \hookrightarrow k(t^q) \hookrightarrow k(t) = k'$ is separable which shows that so is $k(t^q) \hookrightarrow k(t)$. Thus $k(t^q) = k(t)$ and t^q generates k'. □

Lemma (2.2.10). *Let $F : B \to A$ be a multiplicative polynomial law of degree d. Let $A' = A[t]/g$ where either $g = 0$ or g is separable. Then $I_F^{(k)}$ and $\ker(F)$ commute with the base change $A \hookrightarrow A'$.*

Proof. If $g = 0$ we let $n = \infty$ and otherwise we let $n = \deg(g)$. A basis of A' as an A-module is then given by $1, t, t^2, \dots, t^{n-1}$. By Proposition (2.2.6) we can assume that A is a local ring. Let p be the exponential characteristic of the residue field $A/\mathfrak{m}_A$, i.e., p equals the characteristic if it is positive and 1 if the characteristic is zero.

We will proceed by induction on k to show that $I_F^{(k)} B' = I_{F'}^{(k)}$. As $I_F^{(0)} = B$ and $I_{F'}^{(0)} = B'$ the case $k = 0$ is obvious. Proposition (2.2.4) shows that $I_F^{(k)} = I_F^{(k-1)}$ if $k \neq p^s$ and we can thus assume that $k = p^s$.

Let $x' \in I_{F'}^{(p^s)} \subseteq I_{F'}^{(p^s-1)}$. By induction $x' \in I_F^{(p^s-1)} B'$ and we can thus write uniquely $x' = \sum_{i=0}^{n-1} x_i t^i$ where $x_i \in I_F^{(p^s-1)}$ are almost all zero. Let $y \in \Gamma_A^{d-p^s}(B)$. Then

$$f'\bigl(\gamma^{p^s}(x') \times y\bigr) = \sum_{i=0}^{n-1} t^{p^s i} f\bigl(\gamma^{p^s}(x_i) \times y\bigr).$$

If $g = 0$ then $1, t^{p^s}, t^{2p^s}, \dots$ are linearly independent in $A' = A[t]$. If g is separable then $1, t^{p^s}, t^{2p^s}, t^{(n-1)p^s}$ are linearly independent by Lemma (2.2.9). This shows that $f\bigl(\gamma^{p^s}(x_i) \times y\bigr) = 0$ for every y and thus $x_i \in I_F^{(p^s)}$ as $x_i \in I_F^{(p^s-1)}$. Hence $x' \in I_F^{(p^s)} B'$ which shows that $I_F^{(p^s)} B' = I_{F'}^{(p^s)}$. □

2.3. **Image and base change.**

As the kernel of a multiplicative law commutes with localization by Proposition (2.2.6) it is possible to define the kernel for a multiplicative law for schemes:

Definition (2.3.1). Let S be a scheme, $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras and $F : \mathcal{A} \to \mathcal{O}_S$ a multiplicative polynomial law, cf. §1.4. We let $\ker(F) \subseteq \mathcal{A}$ be the quasi-coherent ideal sheaf given by $\ker(F)|_U = \ker\bigl(F|_U\bigr)$ for any affine open subset $U \subseteq S$. If $f : X \to S$ is an affine morphism of schemes and $\alpha : S \to \Gamma^d(X/S)$ is a morphism then we let the *image* of α, denoted $\mathrm{Image}(\alpha)$, be the closed subscheme of X corresponding to the ideal sheaf $\ker(F_\alpha)$ where $F_\alpha : f_*\mathcal{O}_X \to \mathcal{O}_S$ is the polynomial law corresponding to α.

We say that a morphism $S' \to S$ is *essentially smooth* if every local ring of S' is a local ring of a scheme which is smooth over S. The results of the previous section are summarized in the following proposition.

Theorem (2.3.2). *Let $f : X \to S$ be an affine morphism of schemes and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. If $S' \to S$ is an* essentially smooth *morphism then* $\mathrm{Image}(\alpha) \times_S S' = \mathrm{Image}(\alpha \times_S S')$*, i.e., the image commutes with essentially smooth base change.*

Proof. As $\mathrm{Image}(\alpha)$ commutes with localization we can assume that $S = \mathrm{Spec}(A)$ is local and that $S' \to S$ is smooth. Further it is enough that for any $x \in S'$ there is an affine neighborhood $S'' \subseteq S'$ such that the image commutes with the base change $S'' \to S$. By [$\mathrm{EGA_{IV}}$, Cor. 17.11.4] we can choose S'' such that $S'' \to S$ is the composition of an étale morphism followed by a morphism $\mathbb{A}^n_S = \mathrm{Spec}(A[t_1, t_2, \dots, t_n]) \to S = \mathrm{Spec}(A)$. We can thus assume that either $S' \to S$ is étale or $S' = \mathbb{A}^1_S$.

If $S' \to S$ is étale and $S = \mathrm{Spec}(A)$ is local, then for any $s' \in S'$ we have that $\mathcal{O}_{S',s'} = A[t]/g$ where $g \in A[t]$ is a separable polynomial [$\mathrm{EGA_{IV}}$, Thm. 18.4.6 (ii)] and it is thus enough to consider base changes $S' \to S$ of the form $A \to A[t]/g$. The result now follows from Lemma (2.2.10). □

Corollary (2.3.3). *Let $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(A')$ such that A' is a direct limit of essentially smooth A-algebras. Let $f : X \to S$ be an affine morphism and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. Then $\mathrm{Image}(\alpha') = \mathrm{Image}(\alpha) \times_S S'$. In particular this holds if S' is the henselization or the strict henselization of a local ring of S.*

Proof. Follows from Proposition (2.2.7) and Theorem (2.3.2). □

Remark **(2.3.4).** If S and S' are locally noetherian and $S' \to S$ is a flat morphism with geometrically regular fibers, then S' is a filtered direct limit of smooth morphisms by Popescu's theorem [Swa98, Spi99]. Thus the image of a family $\alpha : S \to \Gamma^d(X/S)$ commutes with the base change $S' \to S$ under this hypothesis. In particular we can apply this with $S' = \mathrm{Spec}(\widehat{\mathcal{O}_{S,s}})$ for $s \in S$ if S is an excellent scheme [$\mathrm{EGA_{IV}}$, Def. 7.8.2].

Definition (2.3.5). Let $f : X \to S$ be an affine morphism of *algebraic spaces* and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. We let $\mathrm{Image}(\alpha)$ be the closed subspace of X such that for any *scheme* S' and étale morphism $S' \to S$ we have that $\mathrm{Image}(\alpha) \times_S S' = \mathrm{Image}(\alpha \times_S S')$. As étale morphisms descend closed subspaces and the image commutes with étale base change, this is a unique and well-defined closed subspace. When S is a scheme, this definition of $\mathrm{Image}(\alpha)$ and the one in Definition (2.3.1) agree. We let $\mathrm{Supp}(\alpha) = \mathrm{Image}(\alpha)_{\mathrm{red}}$ and call this subscheme the *support* of α.

Theorem (2.3.6). *Let S and X be algebraic spaces such that X is affine over S. Let $\alpha : S \to \Gamma^d(X/S)$ be a morphism and let $S' \to S$ be any morphism. Then* $\bigl(\mathrm{Supp}(\alpha) \times_S S'\bigr)_{\mathrm{red}} = \mathrm{Supp}(\alpha \times_S S')$*, i.e., the support commutes with arbitrary base change.*

Proof. We can assume that $S = \operatorname{Spec}(A)$ and $S' = \operatorname{Spec}(A')$ are affine. Let P be a, possibly infinite-dimensional, polynomial algebra over A such that there is a surjection $P \to A'$. Then as $\operatorname{Spec}(P)$ is a limit of smooth S-schemes we can by Theorem (2.3.2) replace A with P and assume that $A \to A'$ is surjective.

Let $X = \operatorname{Spec}(B)$, let $f : \Gamma^d_A(B) \to A$ correspond to α and let $F : B \to A$ be the corresponding multiplicative law. Pick an element $b' \in \ker(F_{A'}) \subseteq B \otimes_A A'$ and choose a lifting $b \in B$ of b'. Then by Lemma (2.1.3), the elements $f(\gamma^{d-k}(b) \times \gamma^k(1))$, $k = 0, 1, 2, \dots, d-1$ lie in the kernel of $A \to A'$. In particular, the image of $\chi_{F,b}(b)$ in B is b'^d. Thus $\ker(F_{A'}) \subseteq \sqrt{I_{\mathrm{CH}}(F)}(B \otimes_A A')$. As $\sqrt{I_{\mathrm{CH}}(F)} = \sqrt{\ker F}$ by Proposition (2.1.6) the theorem follows. □

Examples (2.3.7). We give two examples. The first shows that $\ker(F)$ does not commute with arbitrary base change even in characteristic zero. The second shows that $\ker(F)$ does not commute with flat base change in positive characteristic.

(i) Let $A = k[x]$ and $B = k[x, y]/(x^2 - y^2)$. Then B is a free A-module of rank 2. The norm $N : B \to A$ is a multiplicative law of degree 2. It can further be seen that $\ker(N) = 0$. Let $A' = k[x]/x$. Then $B' = B \otimes_A A' = k[y]/y^2$ is not reduced and by Proposition (2.1.4) the kernel of N' cannot be trivial. In fact, we have that $\ker(N') = (y)$.

(ii) Let k be a field of characteristic p and $A = B = k$. We let $F : B \to A$ be the polynomial law given by $x \mapsto x^p$, i.e., the Frobenius. Clearly $\ker(F) = 0$. Let $A' = A[t]/t^p$ which is a flat A-algebra. Then $\ker(F') = (t)$ as $(b'' + tx'')^p = b''^p + t^p x''^p = b''^p$ for any $A' \to A''$ and $b'', x'' \in B'' = A''$.

It is further easily seen that $\ker(F)$ does not commute with any base change such that A' is not reduced. In fact, if $t \in A'$ is such that $t^p = 0$ then $t \in \ker(F')$.

2.4. **Various properties of the image and support.** A morphism $\alpha : S \to \Gamma^d(X/S)$ is, as we will see later on, a "family of zero-cycles of degree d on X parameterized by S". The subscheme $\operatorname{Supp}(\alpha) \hookrightarrow X$ is the support of this family of cycles. In particular it should, topologically at least, have finite fibers over S.

Proposition (2.4.1). *Let S be a* connected *algebraic space and X a space affine over S. Let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. If $X = \coprod_{i=1}^n X_i$, then there are uniquely defined integers $d_1, d_2, \dots, d_n \in \mathbb{N}$ such that $d = d_1 + d_2 + \dots + d_n$ and such that α factors through the closed subspace $\Gamma^{d_1}(X_1) \times_S \Gamma^{d_2}(X_2) \times_S \dots \times_S \Gamma^{d_n}(X_n) \hookrightarrow \Gamma^d(X/S)$. The support* $\operatorname{Supp}(\alpha)$ *is contained in the union of the X_i's with $d_i > 0$. In particular* $\operatorname{Supp}(\alpha)$ *has at most d connected components.*

Proof. By Proposition (1.4.1) there is a decomposition

$$\Gamma^d(X/S) = \coprod_{\substack{d_i \in \mathbb{N} \\ \sum_i d_i = d}} \Gamma^{d_1}(X_1) \times_S \Gamma^{d_2}(X_2) \times_S \dots \times_S \Gamma^{d_n}(X_n).$$

As S is connected α factors uniquely through one of the spaces in this decomposition. It is further clear that $X_i \cap \operatorname{Supp}(\alpha) \neq \emptyset$ if and only if $d_i > 0$. The last observation follows after replacing X with $\operatorname{Image}(\alpha)$ as then n is at most d in any decomposition. $\square$

Definition (2.4.2). Let S and X be as in Proposition (2.4.1). The *multiplicity* of α on X_i is the integer d_i.

Proposition (2.4.3). *Let $S = \operatorname{Spec}(k)$ where k is a field and let X/S be an affine scheme. Let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. Then $\operatorname{Image}(\alpha) = \operatorname{Supp}(\alpha) = \coprod_{i=1}^n \operatorname{Spec}(k_i)$ is a disjoint union of a at most d points such that the separable degree of each k_i/k is finite.*

Proof. Propositions (2.1.4) and (2.1.6) shows that $\operatorname{Image}(\alpha)$ is reduced and affine of dimension zero, hence totally disconnected. By Proposition (2.4.1) it is thus a disjoint union of at most d reduced points. As the support commutes with arbitrary base change by Theorem (2.3.6), it follows after considering the base change $k \hookrightarrow \overline{k}$ that the separable degree of k_i/k is finite. $\square$

Corollary (2.4.4). *Let X, Y and S be algebraic spaces with affine morphisms $f : X \to Y$ and $g : Y \to S$. Let $\alpha : S \to \Gamma^d(X/S)$ be a morphism and denote by $f_*\alpha$ the composition of α and the morphism $\Gamma^d(f) : \Gamma^d(X/S) \to \Gamma^d(Y/S)$. Then $\operatorname{Supp}(f_*\alpha) = f\bigl(\operatorname{Supp}(\alpha)\bigr)$.*

Proof. As the support and the set-theoretic image commute with any base change, we can assume that $S = \operatorname{Spec}(k)$ where k is a field. Then

$$\operatorname{Image}(\alpha) = \coprod_{i=1}^{n} \operatorname{Spec}(k_i) = \{x_1, x_2, \dots, x_n\}$$

by Proposition (2.4.3). Further, by Proposition (1.4.1) there are positive integers $d_1, d_2, \dots, d_n$ such that α factors through $\prod_{i=1}^n \Gamma^{d_i}\bigl(\operatorname{Spec}(k_i)\bigr) \hookrightarrow \Gamma^d(X/S)$. Let

$$f\bigl(\operatorname{Image}(\alpha)\bigr) = \coprod_{j=1}^{m} \operatorname{Spec}(k'_j) = \{y_1, y_2, \dots, y_m\}$$

where $m \leq n$. It is then immediately seen that $f_*\alpha$ factors through the closed subspace $\prod_{j=1}^m \Gamma^{e_j}\bigl(\operatorname{Spec}(k'_j)\bigr) \hookrightarrow \Gamma^d(Y/S)$ where $e_j = \sum_{f(x_i)=y_j} d_i$. As d_i is positive so is e_j and thus $y_j \in \operatorname{Supp}(f_*\alpha)$. This shows that $\operatorname{Supp}(f_*\alpha) = f\bigl(\operatorname{Supp}(\alpha)\bigr)$. $\square$

Proposition (2.4.5). *Let X be an algebraic space affine over S and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. Then every irreducible component of $\operatorname{Supp}(\alpha)$ maps onto an irreducible component of S.*

Proof. As the support commutes with any base change it is enough to consider the case where $S = \operatorname{Spec}(A)$ is irreducible, reduced and affine. Let $\operatorname{Image}(\alpha) = \operatorname{Spec}(B)$ and $F : B \to A$ be the multiplicative polynomial law corresponding to α. We have a commutative diagram

$$\begin{tikzcd}
\Gamma^d_A(B) \arrow[r, two heads] \arrow[d, "f"'] & \Gamma^d_A(B/I) \arrow[r] \arrow[d, phantom, "\circ"] & \Gamma^d_{K(A)}\big(B \otimes_A K(A)\big) \arrow[d, "f"'] \\
A \arrow[rr, hook] & & K(A)
\end{tikzcd}$$

where $I = \ker\big(B \to B \otimes_A K(A)\big)$. This shows that $I \subseteq \ker(F) = 0$. As $V(I)$ is the union of the irreducible components of $\operatorname{Supp}(\alpha)$ which dominate S this shows that every component surjects onto S. □

In the following theorem we restate the main properties of the image and support of a family of cycles:

Theorem (2.4.6). *Let X be an algebraic space affine over S and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. Then*

(i) *If S is reduced then* $\operatorname{Image}(\alpha)$ *is reduced.*
(ii) $\operatorname{Image}(\alpha) \to S$ *is integral.*
(iii) *If S is connected then* $\operatorname{Supp}(\alpha)$ *has at most d connected components.*
(iv) *If $S = \operatorname{Spec}(k)$ where k is a field then* $\operatorname{Image}(\alpha) = \coprod_{i=1}^{n} \operatorname{Spec}(k_i)$ *is a disjoint union of a finite number of points, at most d, such that the separable degree of each k_i/k is finite.*
(v) *Each geometric fiber of* $\operatorname{Supp}(\alpha) \to S$ *has at most d points. In particular,* $\operatorname{Supp}(\alpha) \to S$ *has universally topologically finite fibers, cf. Definition* (A.2.1).
(vi) *If S is a semi-local scheme, i.e., the spectrum of a semi-local ring, then* $\operatorname{Supp}(\alpha)$ *is semi-local.*
(vii) *Every irreducible component of* $\operatorname{Supp}(\alpha)$ *maps onto an irreducible component of S.*

Proof. Properties (i) and (ii) follows from Propositions (2.1.4) and (2.1.6) respectively. Properties (iii) and (iv) are Propositions (2.4.1) and (2.4.3) respectively. Property (v) follows from (iv) as the support commutes with any base change and property (vi) follows immediately from (ii) and (v). Property (vii) is Proposition (2.4.5). □

The following examples show that the support is not always finite.

Example (2.4.7). Let $k = \mathbb{F}_p(t_1, t_2, \dots)$ and $K = \mathbb{F}_p(t_1^{1/p}, t_2^{1/p}, \dots)$. We have a polynomial law $F : K \to k$ given by $a \mapsto a^p$. The support of the corresponding family $\alpha : \operatorname{Spec}(k) \to \Gamma^d(\operatorname{Spec}(K))$ is $\operatorname{Spec}(K)$ and $k \hookrightarrow K$ is not finite.

The following example shows that even if $X \to S$ is of finite presentation then the image of a family $\alpha : S \to \Gamma^d(X/S)$ need not be of finite presentation.

Example (2.4.8). Let $X = S = \operatorname{Spec}(A)$ where $A = k[t_1, t_2, \dots]/(t_1^p, t_2^p, \dots)$ and k is a field of characteristic p. Let α correspond to the multiplicative polynomial law

$F : A \to A$, $x \mapsto x^p$. Then, as in Examples (2.3.7) the kernel of F is $(t_1, t_2, \dots)$ which is not finitely generated. Hence $\mathrm{Image}(\alpha) = \mathrm{Spec}(k) \hookrightarrow X$ is not finitely presented over S.

2.5. **Topological properties of the support.**

Definition (2.5.1) ([$\mathrm{EGA}_{\mathrm{I}}$, Def. 3.9.2])**.** We say that a morphism of algebraic spaces $f : X \to Y$ is *generizing* if for any $x \in X$ and generization $y' \in Y$ of $y = f(x)$ there exists a generization x' of x such that $f(x') = y'$. Equivalently, if X and Y are schemes, the image of $\mathrm{Spec}(\mathcal{O}_{X,x})$ by f is $\mathrm{Spec}(\mathcal{O}_{Y,y})$. We say that f is *component-wise dominating* if every irreducible component of X dominates an irreducible component of Y. We say that f is *universally generizing* (resp. *universally component-wise dominating*) if $f' : X' \to Y'$ is generizing (resp. dominating) for any morphism $g : Y' \to Y$ where $X' = X \times_Y Y'$.

Remark **(2.5.2).** A morphism $f : X \to Y$ is generizing (resp. universally generizing) if and only if f_{red} is generizing (resp. universally generizing). If $g : Y' \to Y$ is a generizing surjective morphism, we have that f is generizing if f' is generizing. If $g : Y' \to Y$ is a universally generizing surjective morphism, then f is generizing (resp. universally generizing) if and only if f' is generizing (resp. universally generizing). Any flat morphism $Y' \to Y$ of algebraic spaces is universally generizing.

Lemma (2.5.3). *Let $f : X \to Y$ be a morphism of algebraic spaces. Then f is universally generizing if and only if it is universally component-wise dominating.*

Proof. A generizing morphism is component-wise dominating so the condition is necessary. For sufficiency, assume that f is universally component-wise dominating. Let $x \in X$, $y = f(x)$ and choose a generization $y' \in Y$. Let $Y' = \overline{\{y'\}}$ with the reduced structure and consider the base change $Y' \hookrightarrow Y$. As f' is component-wise dominating, there is a generization x' of x above y'. ☐

Proposition (2.5.4). *Let $f : X \to S$ be an affine morphism of algebraic spaces. Let $\alpha : S \to \Gamma^d(X/S)$ be a family with support $Z = \mathrm{Supp}(\alpha) \hookrightarrow X$. Then $f|_Z$ is universally generizing.*

Proof. Follows immediately from Lemma (2.5.3) as the support of a family of cycles is universally component-wise dominating by Theorems (2.4.6) (vii) and (2.3.6). ☐

Remark **(2.5.5).** If $Z \to S$ is of finite presentation, e.g., if S is locally noetherian and $X \to S$ is locally of finite type, then it immediately follows that $f|_Z$ is *universally open* from [$\mathrm{EGA}_{\mathrm{I}}$, Prop. 7.3.10]. We will show that $f|_Z$ is universally open without any hypothesis on f. The following lemma settles the case when $X \to S$ is locally of finite type.

Lemma (2.5.6). *Let S and X be* affine *schemes and $f : X \to S$ a morphism of* finite type*. Let $\alpha : S \to \Gamma^d(X/S)$ be a family of cycles and $Z = \mathrm{Supp}(\alpha)$ its* *support. There is then a bijective closed immersion $Z \hookrightarrow Z'$ such that Z' is of finite presentation over S.*

Proof. Let $S = \operatorname{Spec}(A)$, $Z = \operatorname{Spec}(B)$ and let $F : B \to A$ be the multiplicative law corresponding to α restricted to its image. Let $C = A[t_1, t_2, \dots, t_n] \to B$ be a surjection. The multiplicative law F induces a multiplicative law $G : C \twoheadrightarrow B \to A$. Note that $B = C/\ker(G)$. Corresponding to G is a homomorphism $g : \Gamma^d_A(C) \twoheadrightarrow \Gamma^d_A(B) \to A$. As $\Gamma^d_A(C)$ is a finitely presented A-algebra, cf. Proposition (1.3.7), this homomorphism descends to a homomorphism $g_0 : \Gamma^d_{A_0}(C_0) \to A_0$ with A_0 noetherian such that $C = C_0 \otimes_{A_0} A$ and $g = g_0 \otimes_{A_0} \mathrm{id}_A$. As A_0 is noetherian $C_0/\ker(G_0)$ is a finite A_0-algebra of finite presentation.

Let $Z_0 = \operatorname{Spec}\bigl(C_0/\ker(G_0)\bigr)$ and $Z' = Z_0 \times_{\operatorname{Spec}(A_0)} \operatorname{Spec}(A)$. As the support commutes with base change by Theorem (2.3.6) we have that $Z \hookrightarrow Z'$ is a bijective closed immersion. □

Proposition (2.5.7). *Let S and X be algebraic spaces and $f : X \to S$ an affine morphism. Let Z be the support of a family $\alpha : S \to \Gamma^d(X/S)$. Then the restriction of f to Z is* universally open*.*

Proof. The statement is étale-local so we can assume that $S = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(B)$. Further as the support commutes with any base change, cf. Theorem (2.3.6), it is enough to show that $f|_Z : Z \to S$ is open.

We can write B as a filtered direct limit of finite A-subalgebras $B_\lambda \hookrightarrow B$. Let $Z_\lambda = \operatorname{Spec}(B_\lambda)$. As $B_\lambda \hookrightarrow B$ is integral and injective it follows that $Z \to Z_\lambda$ is closed and dominating and thus surjective. Let $\alpha : S \to \Gamma^d(Z/S)$ be a family with support Z and let $\alpha_\lambda : S \to \Gamma^d(Z_\lambda/S)$ be the family given by push-forward along $\varphi_\lambda : Z \to Z_\lambda$.

By Corollary (2.4.4) we have that $\operatorname{Supp}(\alpha_\lambda) = \varphi_\lambda(Z_{\mathrm{red}}) = (Z_\lambda)_{\mathrm{red}}$. Further by Lemma (2.5.6) there is a scheme Z'_λ of finite presentation over S such that $\operatorname{Supp}(\alpha_\lambda)$ and Z_λ are homeomorphic to Z'_λ. As $\operatorname{Supp}(\alpha_\lambda) \to S$ is generizing by Proposition (2.5.4) so is $Z'_\lambda \to S$. As $Z'_\lambda \to S$ is also of finite presentation it is *open* by [$\mathrm{EGA}_{\mathrm{I}}$, Prop. 7.3.10] and hence so is $Z_\lambda \to S$.

To show that $f|_Z : Z \to S$ is open it is enough to show that the image of any quasi-compact open subset of Z is open. Let $U \subseteq Z$ be a quasi-compact open subset. Then according to [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 8.2.11] there is a λ and $U_\lambda \subseteq Z_\lambda$ such that $U = {\varphi_\lambda}^{-1}(U_\lambda)$. As φ_λ is surjective and $Z_\lambda \to S$ is open this shows that $f|_Z(U)$ is open. □

3. Definition and representability of $\underline{\Gamma}^d_{X/S}$

We will define a functor $\underline{\Gamma}^d_{X/S}$ and show that when X/S is affine it is represented by $\Gamma^d(X/S)$. It is then easy to prove that $\underline{\Gamma}^d_{X/S}$ is represented by a scheme for any AF-scheme X/S. To prove representability in general, i.e., when X/S is any separated algebraic space, is more difficult. For *any* morphism $f : X \to Y$ there is a natural transformation $f_* : \underline{\Gamma}^d_{X/S} \to \underline{\Gamma}^d_{Y/S}$ which is "push-forward of cycles". If

f is étale, then f_* is étale over a certain open subset of $\Gamma^d(X/S)$. We will use this result to show representability of $\underline{\Gamma}^d_{X/S}$ giving an explicit étale covering.

3.1. **The functor $\underline{\Gamma}^d_{X/S}$.**

Recall that a morphism of algebraic spaces $f : X \to S$ is said to be *integral* if it is affine and the corresponding homomorphism $\mathcal{O}_S \to f_*\mathcal{O}_X$ is integral. Equivalently, for any affine scheme $T = \operatorname{Spec}(A)$ and morphism $T \to S$ the space $X \times_S T = \operatorname{Spec}(B)$ is affine and $A \to B$ is integral. Further recall, Proposition (1.4.2), that if X/S is affine and Z is a closed subspace of X, then $\Gamma^d(Z/S)$ is a closed subspace of $\Gamma^d(X/S)$.

Definition (3.1.1). Let S be an algebraic space and X/S an algebraic space *separated* over S. A family of zero-cycles of degree d consists of a closed subscheme $Z \hookrightarrow X$ such that $Z \hookrightarrow X \to S$ is *integral* together with a morphism $\alpha : S \to \Gamma^d(Z/S)$. Two families (Z_1, α_1) and (Z_2, α_2) are equivalent if there is a closed subscheme Z of both Z_1 and Z_2 and a morphism $\alpha : S \to \Gamma^d(Z/S)$ such that α_i is the composition of α and the morphism $\Gamma^d(Z/S) \hookrightarrow \Gamma^d(Z_i/S)$ for $i = 1, 2$.

If $g : S' \to S$ is a morphism of spaces and (Z, α) a family of cycles on X/S, we let $g^*(Z, \alpha) = \big(g^*(Z), g^*\alpha\big)$ be the pull-back along g. The image and support of a family of cycles (Z, α) is the image and support of α, cf. Definitions cf. (2.3.1) and (2.3.5).

Remark **(3.1.2).** It is clear that the pull-backs of equivalent families are equivalent and that the image and support of equivalent families coincide. If (Z, α) is a family then the family $(\operatorname{Image}(\alpha), \alpha')$ is a minimal representative in the same equivalence class. Here α' is the restriction of α to its image, i.e., the morphism $S \to \Gamma^d(\operatorname{Image}(\alpha)/S)$ which composed with $\Gamma^d(\operatorname{Image}(\alpha)/S) \hookrightarrow \Gamma^d(Z/S)$ is α.

The pull-back $g^*\alpha$ of a minimal representative α will not in general be a minimal representative. However note that by Theorem (2.3.6) we have a canonical bijective closed immersion $\operatorname{Image}(g^*\alpha) \hookrightarrow g^*\operatorname{Image}(\alpha)$.

Definition (3.1.3). We let $\underline{\Gamma}^d_{X/S}$ be the contravariant functor from S-schemes to sets defined as follows. For any S-scheme T we let $\underline{\Gamma}^d_{X/S}(T)$ be the set of equivalence classes of families of zero-cycles (Z, α) of degree d of $X \times_S T/T$. For any morphism $g : T' \to T$ of S-schemes, the map $\underline{\Gamma}^d_{X/S}(g)$ is the pull-back of families of cycles as defined above.

In the sequel we will suppress the space of definition Z and write $\alpha \in \underline{\Gamma}^d_{X/S}(T)$. We will not make explicit use of Z. Instead, we will use the subspace $\operatorname{Image}(\alpha) \hookrightarrow X \times_S T$ which is independent on the choice of Z by Remark (3.1.2).

Proposition (3.1.4). *If X is* affine *over S then the functor $\underline{\Gamma}^d_{X/S}$ is represented by the algebraic space $\Gamma^d(X/S)$, defined in §1.4, which is affine over S.*

Proof. There is a natural transformation from $\underline{\Gamma}^d_{X/S}$ to $\operatorname{Hom}_S(-, \Gamma^d(X/S))$ given by composing a family $\alpha : T \to \Gamma^d(Z/T)$ with $\Gamma^d(Z/T) \hookrightarrow \Gamma^d(X \times_S T/T) =$

$\Gamma^d(X/S) \times_S T \to \Gamma^d(X/S)$. If $\alpha : T \to \Gamma^d(X/S)$ is any morphism then $\alpha \times_S \mathrm{id}_T$ factors through $\Gamma^d(Z/T) \hookrightarrow \Gamma^d(X \times_S T/T)$ where $Z \hookrightarrow X \times_S T$ is the image of $\alpha \times_S \mathrm{id}_T$. As Z is integral over S by Theorem (2.4.6) (ii), we have that the morphism α corresponds to a unique equivalence class of families. It is thus clear that $\Gamma^d(X/S)$ represents $\underline{\Gamma}^d_{X/S}$. □

Remark **(3.1.5).** For an affine morphism of algebraic spaces $X \to S$, we have that $\Gamma^1(X/S) = X$ and that the T-points of $\Gamma^1(X/S)$ parameterizes sections of $X \times_S T \to T$. Thus, for any separated algebraic space X/S it follows that $\underline{\Gamma}^1_{X/S}$ parameterizes sections of $X \to S$ and that $\underline{\Gamma}^1_{X/S}$ is represented by X.

Proposition (3.1.6). *The functor $\underline{\Gamma}^d_{X/S}$ is a sheaf in the étale topology.*

Proof. Let T be an S-scheme and $f : T' \to T$ an étale surjective morphism. Let $T'' = T' \times_T T'$ with projections π_1 and π_2. Given an element $\alpha' \in \underline{\Gamma}^d_{X/S}(T')$ such that $\pi_1^*\alpha' = \pi_2^*\alpha'$ we have to show that there is a unique $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ such that $f^*\alpha = \alpha'$. Let $Z' \hookrightarrow X \times_S T'$ be the image of α'. As the image commutes with étale base change, cf. Theorem (2.3.2), the image of α'' is $Z'' = \pi_1^{-1}(Z') = \pi_2^{-1}(Z')$. As closed immersions satisfy effective descent with respect to étale morphisms [SGA$_1$, Exp. VIII, Cor. 1.9], there is a closed subspace $Z \hookrightarrow X \times_S T$ such that $Z' = Z \times_T T'$. Moreover Z is affine over T. Any $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ such that $f^*\alpha = \alpha'$ is then in the subset $\underline{\Gamma}^d_{Z/T}(T) \subseteq \underline{\Gamma}^d_{X/S}(T)$. It is thus enough to show that $\underline{\Gamma}^d_{Z/S}$ is a sheaf in the étale topology. But $\underline{\Gamma}^d_{Z/S}$ is represented by the space $\Gamma^d(Z/S)$ which is affine over S. As the étale topology is sub-canonical, it follows that $\underline{\Gamma}^d_{Z/S}$ is a sheaf. □

Proposition (3.1.7). *Let X/S and Y/S be separated algebraic spaces. If $f : X \to Y$ is an immersion (resp. a closed immersion, resp. an open immersion) then $\underline{\Gamma}^d_{X/S}$ is a locally closed subfunctor (resp. a closed subfunctor, resp. an open subfunctor) of $\underline{\Gamma}^d_{Y/S}$.*

Proof. Let T be an S-scheme and let $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ be a family with $Z = \mathrm{Image}(\alpha) \hookrightarrow X_T$. Then $Z \hookrightarrow X_T$ is a closed subscheme such that $Z \to T$ is integral and hence universally closed. As $Y \to S$ is separated it thus follows that $Z \hookrightarrow X_T \hookrightarrow Y_T$ is a closed subscheme. It follows that $\underline{\Gamma}^d_{X/S}$ is a subfunctor of $\underline{\Gamma}^d_{Y/S}$.

Let $\alpha : T \to \underline{\Gamma}^d_{Y/S}$ be a family of cycles. We have to show that if f is a closed (resp. open) immersion then there is a closed (resp. open) subscheme $U \hookrightarrow T$ such that if $g : T' \to T$ and $\alpha' = g^*\alpha \in \underline{\Gamma}^d_{X/S}(T')$ then g factors through U. Let $X_T = X \times_S T$, $Y_T = Y \times_S T$, $Z = \mathrm{Image}(\alpha) \subset Y_T$ and $W = Z \cap X_T = Z \times_{Y_T} X_T \hookrightarrow X_T$.

If f is an open immersion we let V be the closed subset $Y_T \setminus X_T$ and U be the complement of the image of $V \cap Z = Z \setminus W$ by $Z \to T$. Thus U is the open subset of T such that $t \in U$ if and only if the fiber Z_t does not meet V or equivalently is contained in W. As the support commutes with arbitrary base change, see Theorem (2.3.6), it is easily seen that $Z \times_T T'$ factors through $X_{T'}$ if and only if

$T' \to T$ factors through U. Hence $T \times_{\underline{\Gamma}^d_{Y/S}} \underline{\Gamma}^d_{X/S} = T|_U$ which shows that $\underline{\Gamma}^d_{X/S}$ is an open subfunctor.

If f is a closed immersion we consider the cartesian diagram

$$\begin{tikzcd}
T \times_{\underline{\Gamma}^d_{Z/T}} \underline{\Gamma}^d_{W/T} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{W/T} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{X_T/T} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{X/S} \arrow[d, hook] \\
T \arrow[r] & \underline{\Gamma}^d_{Z/T} \arrow[r, hook] & \underline{\Gamma}^d_{Y_T/T} \arrow[r] & \underline{\Gamma}^d_{Y/S}.
\end{tikzcd}$$

As W and Z are affine over S, the functors $\underline{\Gamma}^d_{W/T}$ and $\underline{\Gamma}^d_{Z/T}$ are represented by $\Gamma^d(W/T)$ and $\Gamma^d(Z/T)$ respectively. As $\Gamma^d(W/T) \hookrightarrow \Gamma^d(Z/T)$ is a closed immersion by Proposition (1.4.2) it follows that $\underline{\Gamma}^d_{X/S}$ is a closed subfunctor of $\underline{\Gamma}^d_{Y/S}$. □

Proposition (3.1.8). *Let S be an algebraic space and let $X_1, X_2, \dots, X_n$ be algebraic spaces separated over S. Then*

$$\underline{\Gamma}^d_{\coprod_{i=1}^n X_i} = \coprod_{\substack{d_i \in \mathbb{N} \\ \sum_i d_i = d}} \underline{\Gamma}^{d_1}_{X_1} \times_S \underline{\Gamma}^{d_2}_{X_2} \times_S \cdots \times_S \underline{\Gamma}^{d_n}_{X_n}.$$

Proof. Follows from Proposition (1.4.1). □

Corollary (3.1.9). *Let X/S be a separated algebraic space. Let k be an algebraically closed field and $s : \operatorname{Spec}(k) \to S$ a geometric point of S. There is a one-to-one correspondence between k-points of $\underline{\Gamma}^d_{X/S}$ and effective zero-cycles of degree d on X_s. In this correspondence, a zero-cycle $\sum_{i=1}^n d_i[x_i]$ on X_s corresponds to the family (Z, α) where $Z = \{x_1, x_2, \dots, x_n\} \subseteq X_s$ and α is the morphism*

$$\alpha \,:\, \operatorname{Spec}(k) \cong \Gamma^{d_1}(x_1/k) \times_k \Gamma^{d_2}(x_2/k) \times_k \cdots \times_k \Gamma^{d_n}(x_n/k) \hookrightarrow \Gamma^d(Z/k)$$

Proof. Let $\alpha \in \underline{\Gamma}^d_{X/S}(k)$ be a k-point. By Theorem (2.4.6) (iv) we have that $Z = \operatorname{Image}(\alpha) \hookrightarrow X_s$ is a finite disjoint union of points $x_1, x_2, \dots, x_n$, all with residue field k as k is algebraically closed. According to Proposition (3.1.8), there are positive integers $d_1, d_2, \dots, d_n$ such that $d = d_1 + d_2 + \cdots + d_n$ and such that $\alpha : k \to \Gamma^d(Z/k)$ factors through the open and closed subscheme $\Gamma^{d_1}(x_1/k) \times_k \Gamma^{d_2}(x_2/k) \times_k \cdots \times_k \Gamma^{d_n}(x_n/k)$. As $k(x_i) = k$, we have that $\Gamma^{d_i}(x_i/k) \cong k$. The point α corresponds to $\sum_{i=1}^n d_i[x_i]$. □

Proposition (3.1.10). *Let X/S be a separated algebraic space. Let $\{U_\beta\}$ be an open covering of X such that any set of d points in X above the same point in S lies in one of the U_β's. Then $\coprod_\beta \underline{\Gamma}^d_{U_\beta/S} \to \underline{\Gamma}^d_{X/S}$ is an open covering. If X/S is an* AF*-scheme then such a covering with the U_β's* affine *exists.*

Proof. Let k be a field and $\alpha \in \underline{\Gamma}^d_{X/S}(k)$. Then by Theorem (2.4.6) (iv) there is a β such that $\alpha \in \underline{\Gamma}^d_{U_\beta/S}(k) \subseteq \underline{\Gamma}^d_{X/S}(k)$. Thus $\coprod_\beta \underline{\Gamma}^d_{U_\beta/S} \to \underline{\Gamma}^d_{X/S}$ is an open covering by Proposition (3.1.7). □

Theorem (3.1.11). *Let S be a scheme and X/S an* AF*-scheme. The functor $\underline{\Gamma}^d_{X/S}$ is then represented by an* AF*-scheme $\Gamma^d(X/S)$.*

Proof. As $\underline{\Gamma}^d_{X/S}$ is a sheaf in the Zariski topology, we can assume that S is affine. Let $\{U_\beta\}$ be an open covering of X by affines such that any set of d points in X above the same point in S lies in one of the U_β's. As $\underline{\Gamma}^d_{U_\beta/S}$ is represented by an affine scheme, Proposition (3.1.10) shows that $\underline{\Gamma}^d_{X/S}$ is represented by a scheme $\Gamma^d(X/S)$.

If $\alpha_1, \alpha_2, \dots, \alpha_m$ are points of $\Gamma^d(X/S)$ above the same point of S, then the union of their supports consists of at most dm points and there is thus an affine subset $U \subseteq X$ such that $\alpha_1, \alpha_2, \dots, \alpha_m \in \Gamma^d(U/S)$. This shows that $\Gamma^d(X/S)/S$ is an AF-scheme. $\square$

3.2. **Effective pro-representability of $\underline{\Gamma}^d_{X/S}$.** Let A be a henselian local ring and $T = \operatorname{Spec}(A)$ together with a morphism $T \to S$. The image of a family of cycles $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ over T is then a semi-local scheme Z, integral over T by Theorem (2.4.6) (ii), (vi). Furthermore, Proposition (A.2.7) implies that Z is a finite disjoint union of local henselian schemes.

Let $z_1, z_2, \dots, z_n$ be the closed points of $Z \hookrightarrow X_T$ and $\{x_1, x_2, \dots, x_m\}$ their images in X where the x_i's are chosen to be distinct. As z_i lies over the closed point of T, all x_i lies over a common point $s \in S$. Let ${}^{\mathrm{h}}X_{x_i} = \operatorname{Spec}({}^{\mathrm{h}}\mathcal{O}_{X,x_i})$, ${}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m} = \coprod_{i=1}^m {}^{\mathrm{h}}X_{x_i}$ and ${}^{\mathrm{h}}S_s = \operatorname{Spec}({}^{\mathrm{h}}\mathcal{O}_{S,s})$ be the henselizations of X and S at the x_i's and s. As $\mathcal{O}_{Z,z_i}$ is henselian it follows that $Z \hookrightarrow X_T \to X$ factors uniquely through ${}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m} \to X$. Thus $Z \hookrightarrow X_T$ factors uniquely through ${}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m} \times_{{}^{\mathrm{h}}S_s} T \to X_T$ and α corresponds to a unique element of $\underline{\Gamma}^d_{{}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s}(T)$. As ${}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}$ is affine, we have a unique morphism $T \to \Gamma^d({}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s)$.

Further, by Proposition (1.4.1)

$$\Gamma^d\left({}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s\right) = \coprod_{\substack{d_i \in \mathbb{N} \\ \sum_i d_i = d}} \prod_{i=1}^m \Gamma^{d_i}\left({}^{\mathrm{h}}X_{x_i}/{}^{\mathrm{h}}S_s\right).$$

and as T is connected $T \to \Gamma^d({}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s)$ factors through one of these components.

To conclude, there are uniquely determined points $x_1, x_2, \dots, x_m \in X$, unique positive integers d_i and a unique morphism

$$\varphi : T \to \prod_{i=1}^m \Gamma^{d_i}\left({}^{\mathrm{h}}X_{x_i}/{}^{\mathrm{h}}S_s\right) \hookrightarrow \Gamma^d\left({}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s\right)$$

such that α is equivalent to $\varphi \times_{{}^{\mathrm{h}}S_s} \mathrm{id}_T$. This implies the following:

Proposition (3.2.1). *Let X/S be a separated algebraic space and assume that $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space $\Gamma^d(X/S)$. Let $\beta \in \Gamma^d(X/S)$ be a point with residue field k and s its image in S. The point β corresponds uniquely to points*

$x_1, x_2, \dots, x_m \in X$, positive integers $d_1, d_2, \dots, d_m$ with sum d and morphisms $\varphi_i : k \to \Gamma^d\big(k(x_i)/k(s)\big)$. The local henselian ring (resp. strictly local ring) at β is the local henselian ring (resp. strictly local ring) of $\prod_{i=1}^m \Gamma^{d_i}({}^{\mathrm{h}}X_{x_i}/{}^{\mathrm{h}}S_s)$ at the point corresponding to the morphisms φ_i.

(3.2.2) If X/S is *locally of finite type* and A is a complete local *noetherian* ring, then the support of any family of cycles α on X parameterized by $T = \operatorname{Spec}(A)$ is finite over T. Thus $\operatorname{Image}(\alpha)$ is a disjoint union of a finite number of complete local rings. Let $s \in S$ and $x_i \in X$ be defined as above and let $\widehat{X}_{x_i} = \operatorname{Spec}(\widehat{\mathcal{O}}_{X,x_i})$, $\widehat{S}_s = \operatorname{Spec}(\widehat{\mathcal{O}}_{S,s})$ and $\widehat{X}_{x_1,x_2,\dots,x_m} = \coprod_{i=1}^m \widehat{X}_{x_i}$ be the completions of X and S at the corresponding points. Repeating the reasoning above we conclude that there is a unique morphism

$$\varphi : T \to \prod_{i=1}^m \Gamma^{d_i}(\widehat{X}_{x_i}/\widehat{S}_s) \hookrightarrow \Gamma^d(\widehat{X}_{x_1,x_2,\dots,x_m}/\widehat{S}_s)$$

such that α is equivalent to $\varphi \times_{\widehat{S}_s} \mathrm{id}_T$. Thus we obtain:

Proposition (3.2.3). *Let S be locally noetherian and X an algebraic space separated and locally of finite type over S and assume that $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space $\Gamma^d(X/S)$. Let $\beta \in \Gamma^d(X/S)$ be a point with residue field k and s its image in S. The point β corresponds uniquely to points $x_1, x_2, \dots, x_m \in X$, positive integers $d_1, d_2, \dots, d_m$ with sum d and morphisms $\varphi_i : k \to \Gamma^d\big(k(x_i)/k(s)\big)$. The formal local ring at β is the formal local ring of $\prod_{i=1}^m \Gamma^{d_i}(\widehat{X}_{x_i}/\widehat{S}_s)$ at the point corresponding to the morphisms φ_i.*

Corollary (3.2.4). *Let S be locally noetherian and X an algebraic space separated and locally of finite type over S. The functor $\underline{\Gamma}^d_{X/S}$ is* effectively pro-representable *by which we mean the following: Let k be any field and $\beta_0 \in \underline{\Gamma}^d_{X/S}(k)$. There is then a complete local noetherian ring $\widehat{A}$ and an object $\widehat{\beta} \in \underline{\Gamma}^d_{X/S}(\operatorname{Spec}(A))$ such that for any local artinian scheme T and family $\alpha \in \underline{\Gamma}^d_{X/S}(T)$, coinciding with β_0 at the closed point of T, there is a unique morphism $f : T \to \operatorname{Spec}(\widehat{A})$ such that $\alpha = f^*\widehat{\beta}$.*

Remark **(3.2.5).** Assume that $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space $\Gamma^d(X/S)$. Questions about properties of $\Gamma^d(X/S)$ which only depend on the strictly local rings, such as being flat or reduced, can be reduced to the case where X is affine using Proposition (3.2.1). As some properties cannot be read from the strictly local rings we will need the stronger result of Proposition (3.4.2) which shows that any point in $\Gamma^d(X/S)$ has an étale neighborhood which is an open subset of $\Gamma^d(U/S)$ for some affine scheme U.

3.3. **Push-forward of families of cycles.**

Definition (3.3.1). Let $f : X \to Y$ be a morphism of algebraic spaces separated over S. If $(Z,\alpha) \in \underline{\Gamma}^d_{X/S}(T)$ is a family of cycles over T we let $f_*(Z,\alpha) = (f_T(Z), f_*\alpha)$ where $f_T(Z)$ is the schematic image of Z along $X \times_S T \to Y \times_S T$ and $f_*\alpha$ is the composition of $\alpha : T \to \Gamma^d(Z/T)$ and $\Gamma^d(Z/T) \to \Gamma^d(f_T(Z)/T)$. This induces a natural transformation of functors $f_* : \underline{\Gamma}^d_{X/S} \to \underline{\Gamma}^d_{Y/S}$ denoted the push-forward.

Remark **(3.3.2).** If $g : Y \to Z$ is another morphism of S-spaces then clearly $g_* \circ f_* = (g \circ f)_*$. If X and Y are affine over S, the push-forward $f_* : \underline{\Gamma}^d_{X/S} \to \underline{\Gamma}^d_{Y/S}$ coincides with the morphism $\Gamma^d(X/S) \to \Gamma^d(Y/S)$ given by the covariance of the functor Γ^d.

Definition (3.3.1) only makes sense after we have checked that $f_T(Z)$ is integral over T. If Y/S is locally of finite type then $f_T(Z)$ is quasi-finite and proper and hence finite, cf. Proposition (A.2.3). More generally, as $Z \to T$ is integral with topological finite fibers by Theorem (2.4.6) (v), it follows from Theorem (A.2.2) that $f_T(Z)$ is integral without any hypothesis on Y/S except the separatedness.

Definition (3.3.3). Let X/S and Y/S be separated algebraic spaces and let $f : X \to Y$ be any morphism of S-spaces. We say that $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ is *regular* (resp. *quasi-regular*) with respect to f if $f_T|_{\mathrm{Image}(\alpha)}$ is a closed immersion (resp. universally injective) or equivalently if $f_T|_{\mathrm{Image}(\alpha)} : \mathrm{Image}(\alpha) \to f_T\big(\mathrm{Image}(\alpha)\big)$ is an isomorphism (resp. a universal bijection). We let $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f}(T)$ (resp. $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f}(T)$) be the elements which are regular (resp. quasi-regular) with respect to f.

Definition (3.3.4). Let $\mathcal{F}$ and $\mathcal{G}$ be contravariant functors from S-schemes to sets. We say that a morphism of functors $f : \mathcal{F} \to \mathcal{G}$ is *topologically surjective* if for any field k and element $y \in \mathcal{G}\big(\mathrm{Spec}(k)\big)$ there is a field extension $g : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ and an element $x \in \mathcal{F}\big(\mathrm{Spec}(k')\big)$ such that $f(x) = g^*y$ in $\mathcal{G}\big(\mathrm{Spec}(k')\big)$. If $\mathcal{F}$ and $\mathcal{G}$ are represented by algebraic spaces, we have that f is topologically surjective if and only if the corresponding morphism of spaces is surjective.

Definition (3.3.5). A morphism $f : X \to Y$ is *unramified* if it is formally unramified and locally of finite type.

In [EGA$_{\mathrm{IV}}$] unramified morphisms are locally of finite presentation but the above definition is more useful and also commonly used.

Proposition (3.3.6). *Let X/S and Y/S be separated algebraic spaces and let $f : X \to Y$ be a morphism of S-spaces. Let $\alpha \in \underline{\Gamma}^d_{X/S}(T)$. If f is unramified then α is quasi-regular if and only if α is regular.*

Proof. If α is quasi-regular and f unramified then $\mathrm{Image}(\alpha) \hookrightarrow X \times_S T \to Y \times_S T$ is unramified and universally injective. By [EGA$_{\mathrm{IV}}$, Prop. 17.2.6] this implies that $f_T|_{\mathrm{Image}(\alpha)} : \mathrm{Image}(\alpha) \to Y \times_S T$ is a monomorphism. As $\mathrm{Image}(\alpha) \to T$ is universally closed and $Y_T \to T$ is separated it follows that $f_T|_{\mathrm{Image}(\alpha)}$ is a proper monomorphism and hence a closed immersion [EGA$_{\mathrm{IV}}$, Cor. 18.12.6]. □

Proposition (3.3.7). *Let X/S and Y/S be separated algebraic spaces and let $f : X \to Y$ be a morphism of S-spaces. Let T be an S-scheme and $f_T : X \times_S T \to Y \times_S T$ the base change of f along $T \to S$. Let $\alpha \in \underline{\Gamma}^d_{X/S}(T)$. Then*

(i) $\mathrm{Image}(f_*\alpha) \hookrightarrow f_T\big(\mathrm{Image}(\alpha)\big)$.
(ii) $\mathrm{Supp}(f_*\alpha) = f_T\big(\mathrm{Supp}(\alpha)\big)$.
(iii) $\mathrm{Supp}(\alpha) \to f_T\big(\mathrm{Supp}(\alpha)\big) = \mathrm{Supp}(f_*\alpha)$ *is a bijection if α is quasi-regular with respect to f_*.*
(iv) $\mathrm{Image}(\alpha) \cong f_T\big(\mathrm{Image}(\alpha)\big) = \mathrm{Image}(f_*\alpha)$ *if α is regular with respect to f_*.*

Proof. (i) follows immediately by the definition of f_* and (ii) follows from Corollary (2.4.4). (iii) follows from the definition of a quasi-regular family and (ii). (iv) follows by the definition of regular as $\mathrm{Image}(\alpha) \cong f_T\big(\mathrm{Image}(\alpha)\big)$ easily implies that $f_T\big(\mathrm{Image}(\alpha)\big) = \mathrm{Image}(f_*\alpha)$. □

Examples (3.3.8). We give two examples on bad behavior of the image with respect to push-forward. In the first example f is étale, α not (quasi-)regular and $\mathrm{Image}(f_*\alpha) \hookrightarrow f_T\big(\mathrm{Image}(\alpha)\big)$ is not an isomorphism. In the second example f is universally injective and α quasi-regular but not regular.

(i) Let $S = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $X = Y \amalg Y = \mathrm{Spec}(B \times B)$ where $A = k[\epsilon]/\epsilon^2$ and $B = k[\epsilon,\delta]/(\epsilon^2,\delta^2,\epsilon\delta)$. We let $f : X \to Y$ be the étale map given by the identity on the two components. Finally we let $\alpha \in \underline{\Gamma}^2_{X/S}(S)$ be the family of cycles corresponding to the multiplicative polynomial law $F : B \times B \to B/(\delta - \epsilon) \times B/(\delta + \epsilon) \cong A \times A \to A \otimes_A A \cong A$ which is homogeneous of degree 2. The support of α corresponds to $\ker(F) = \big((\delta-\epsilon),(\delta+\epsilon)\big) \subset B \times B$. It is easily seen that $f\big(\mathrm{Image}(\alpha)\big) = V(0)$. On the other hand an easy calculation shows that $\mathrm{Image}(f_*\alpha) = V(\delta)$.
(ii) Let k be a field of characteristic different from 2. Let $S = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(C)$ where $A = k[\epsilon]/\epsilon^2$, $B = k[\epsilon,\delta]/(\epsilon,\delta)^2$ and $C = k[\epsilon,\delta,\tau]/\big(\epsilon^2,\epsilon\delta,\epsilon\tau,\delta^2,\tau^2,\delta\tau-\epsilon\big)$. Let $f : X \to Y$ be the natural morphism. An easy calculation shows that $\Gamma^2_A(C)$ is generated by $\gamma^2(\delta)$, $\gamma^2(\tau)$, $\delta \times 1$, $\tau \times 1$ and $\delta \times \tau$. After finding explicit relations for these generators in $\Gamma^2_A(C)$, it can also be shown that $\gamma^2(\delta), \gamma^2(\tau), \delta \times 1, \tau \times 1 \mapsto 0$ and $\delta \times \tau \mapsto -2\epsilon$ defines a family $\alpha : S \to \Gamma^2(X/S)$. It is easy to check that $\mathrm{Image}(\alpha) = X$, $f(\mathrm{Image}(\alpha)) = Y$ but $\mathrm{Image}(f_*\alpha) = V(\delta)$.

Proposition (3.3.9). *Let $f : X \to Y$ be a morphism between algebraic spaces separated over S. Then:*

(i) $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f}$ *and* $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f}$ *are subfunctors of* $\underline{\Gamma}^d_{X/S}$.
(ii) *If $f : X \to Y$ is unramified then* $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f} = \underline{\Gamma}^d_{X/S,\mathrm{qreg}/f}$ *is an open subfunctor of* $\underline{\Gamma}^d_{X/S}$.

(iii) *If f is an immersion then* $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f} = \underline{\Gamma}^d_{X/S,\mathrm{qreg}/f} = \underline{\Gamma}^d_{X/S}$.
(iv) *If f is surjective then* $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \to \underline{\Gamma}^d_{Y/S}$ *is topologically surjective.*

Proof. (i) As the support commutes with arbitrary base change it follows that the requirement for $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ to be quasi-regular is stable under arbitrary base change. Thus the pull-back $\underline{\Gamma}^d_{X/S}(T) \to \underline{\Gamma}^d_{X/S}(T')$ induced by $T' \to T$ restricts to $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f}$. If $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ is regular then by definition $\mathrm{Image}(\alpha) \cong f_T\big(\mathrm{Image}(\alpha)\big) = \mathrm{Image}(f_*\alpha)$. If $g : T' \to T$ is any morphism then clearly $\mathrm{Image}(g^*\alpha) \cong \mathrm{Image}(g^* f_*\alpha) = \mathrm{Image}(f_* g^*\alpha)$ and thus $g^*\alpha \in \underline{\Gamma}^d_{X/S,\mathrm{reg}/f}(T')$.

(ii) Proposition (3.3.6) shows that $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f} = \underline{\Gamma}^d_{X/S,\mathrm{reg}/f}$. To show that $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \subseteq \underline{\Gamma}^d_{X/S}$ is open we let $\alpha : T \to \underline{\Gamma}^d_{X/S}$ be a morphism. This factors through $T \to \Gamma^d(Z/T)$ where $Z = \mathrm{Image}(\alpha) \hookrightarrow X_T$ and $X_T = X \times_S T$. As f is unramified $(f_T)|_Z : Z \hookrightarrow X_T \to Y_T$ is unramified. In particular $(f_T)|_Z : Z \to f_T(Z)$ is finite and unramified. By Nakayama's lemma, the rank of the fibers of a finite morphism is upper semicontinuous. Thus, the subset W of $f_T(Z)$ over which the geometric fibers of $(f_T)|_Z$ contain more than one point is closed. Let $U = T \setminus g_T(W)$, where $g : Y \to S$ is the structure morphism. Then $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f} \times_{\underline{\Gamma}^d_{X/S}} T = U$ which shows that $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f} \subseteq \underline{\Gamma}^d_{X/S}$ is an open subfunctor.

(iii) Obvious from the definitions.

(iv) Let $\beta \in \underline{\Gamma}^d_{Y/S}(k)$ where $k = \overline{k}$ is an algebraically closed field. Then by Theorem (2.4.6) (iv) the image $W := \mathrm{Image}(\beta) \hookrightarrow Y_k$ is a finite disjoint union of reduced points, each with residue field k. As f is surjective we can then find a field extension $k \hookrightarrow k'$ and a closed subspace $Z \hookrightarrow X_{k'}$ such that $f_{k'}(Z) = W_{k'}$ and $f_{k'}|_Z : Z \to W_{k'}$ is an isomorphism. This gives an element $\alpha \in \underline{\Gamma}^d_{X/S}(k')$ such that $f_*\alpha = \beta$. □

Proposition (3.3.10). *Let*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

be a cartesian square of algebraic spaces separated over S. Let

$$\begin{aligned}
\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} &= \underline{\Gamma}^d_{X'/S} \times_{\underline{\Gamma}^d_{Y'/S}} \underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g} \\
&= \Big\{ \alpha \in \underline{\Gamma}^d_{X'/S} \;:\; f'_*\alpha \text{ is regular with respect to } g \Big\}.
\end{aligned}$$

Then

(i) *If g is unramified or f is an immersion then*

$$\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} \subseteq \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g'}.$$

(ii) *If g is étale or f is an immersion then we have a cartesian diagram*

$$\begin{tikzcd}
\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} \arrow[r, "g'_*"] \arrow[d, "f'_*"] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{X/S} \arrow[d, "f_*"] \\
\underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g} \arrow[r, "g_*"] & \underline{\Gamma}^d_{Y/S}.
\end{tikzcd}$$

(iii) *For arbitrary g the results of* (i) *and* (ii) *are true over reduced S-schemes, i.e., for any reduced S-scheme T we have that*

$$\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g}(T) \subseteq \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g'}(T)$$

and the diagram in (ii) *is cartesian in the subcategory of functors from reduced S-schemes.*

Proof. (i) Let $\alpha' \in \underline{\Gamma}^d_{X'/S}(T)$. If f is an immersion then $\mathrm{Image}(f'_*\alpha') = \mathrm{Image}(\alpha')$ and $\mathrm{Image}(f_*g'_*\alpha') = \mathrm{Image}(g'_*\alpha')$. It is thus obvious that α' is regular if and only if $f'_*\alpha'$ is regular, i.e., $\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} = \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g'}$.

Assume instead that f is arbitrary but g is unramified. Let $Z' = \mathrm{Image}(\alpha')$ and $W' = f'_T(Z')$. If $\alpha' \in \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g}(T)$, i.e., if $f'_*\alpha'$ is regular with respect to g, we have that $\mathrm{Image}(f'_*\alpha') \hookrightarrow W' \hookrightarrow Y'_T \to Y_T$ is a closed immersion. But $\mathrm{Image}(f'_*\alpha') \hookrightarrow W'$ is universally bijective and thus $W' \to Y_T$ is universally injective and unramified. By [EGA$_{\mathrm{IV}}$, Prop. 17.2.6] this implies that $W' \to Y_T$ is a monomorphism and hence a closed immersion. Thus $Z' \hookrightarrow W' \times_{Y'_T} X'_T = W' \times_{Y_T} X_T \hookrightarrow X_T$ is a closed immersion which shows that α' is regular with respect to g'.

(ii) The commutativity of the diagrams is obvious. This gives us a canonical morphism

$$\Lambda : \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} \to \underline{\Gamma}^d_{X/S} \times_{\underline{\Gamma}^d_{Y/S}} \underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g}.$$

We construct an inverse Λ^{-1} of this morphism as follows: Let T be an S-scheme, $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ and $\beta' \in \underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g}(T)$ such that $\beta = g_*\beta' = f_*\alpha \in \underline{\Gamma}^d_{Y/S}(T)$. As β' is regular with respect to g we have that $\mathrm{Image}(\beta') \hookrightarrow Y'_T$ is isomorphic to $\mathrm{Image}(\beta) \hookrightarrow Y_T$. Let $Z = \mathrm{Image}(\alpha) \hookrightarrow X_T$. If f is an immersion then α is regular with respect to f and $Z \hookrightarrow X_T$ is isomorphic to $\mathrm{Image}(\beta)$ and we let $Z' = \mathrm{Image}(\beta') \times_{\mathrm{Image}(\beta)} \mathrm{Image}(\alpha) \cong Z$.

For arbitrary f but étale g, let $W = f_T(Z)$. Then $\mathrm{Image}(\beta) \hookrightarrow W$ is a bijective closed immersion. By the regularity of β', we have that $\mathrm{Image}(\beta')$ is a section of $g_T^{-1}\bigl(\mathrm{Image}(\beta)\bigr) \to \mathrm{Image}(\beta)$. As g is unramified it thus follows that $\mathrm{Image}(\beta')$ is open and closed in $g_T^{-1}\bigl(\mathrm{Image}(\beta)\bigr) \hookrightarrow g_T^{-1}W$. Let W' be the corresponding open and closed subscheme of $g_T^{-1}W$. As g is étale $W' \cong W$ and we let $Z' = W' \times_W Z$.

In both cases we have obtained a canonical closed subscheme $Z' \hookrightarrow X'_T$ such that $Z' \cong Z$. This gives a unique lifting of the family $\alpha \in \underline{\Gamma}^d_Z(T)$ to a family $\alpha' \in \underline{\Gamma}^d_{Z'}(T) \subseteq \underline{\Gamma}^d_{X'/S}(T)$. By the construction of Z' and the regularity of β', it is clear that $f'_*\alpha' = \beta'$. We let $\Lambda^{-1}(T)(\alpha, \beta') = \alpha'$ and it is obvious that Λ is a morphism since the construction is functorial. By construction $\Lambda \circ \Lambda^{-1}$ is the identity and as $\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} \subseteq \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g'}$ it follows that $\Lambda^{-1} \circ \Lambda$ is the identity as well.

(iii) Over reduced schemes, all images involved are reduced by Theorem (2.4.6) (i) and the support of the push-forward coincides with the image. The arguments of (i) and (ii) then simplify and go through without any hypotheses on f and g. $\square$

Corollary (3.3.11). *Let $f : X \to Y$ and $g : Y' \to Y$ be morphism of algebraic spaces, separated over S. Assume that for every involved space Z, the functor $\underline{\Gamma}^d_{Z/S}$ is represented by a space which we denote by $\Gamma^d(Z/S)$.*

(i) *If g is unramified, then $\underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g}$ is represented by an open subspace $U = \mathrm{reg}(g)$ of $\Gamma^d(Y'/S)$.*

(ii) *If g is étale, then we have a cartesian diagram*

$$\begin{tikzcd}
\Gamma^d(X'/S)|_{{f'_*}^{-1}(U)} \arrow[r, "g'_*"] \arrow[d, "f'_*"] \arrow[dr, phantom, "\square"] & \Gamma^d(X/S) \arrow[d, "f_*"] \\
\Gamma^d(Y'/S)|_U \arrow[r, "g_*"] & \Gamma^d(Y/S).
\end{tikzcd}$$

(iii) *If g is unramified, the canonical morphism*

$$\Lambda : \Gamma^d(X'/S)|_{{f'_*}^{-1}(U)} \to \Gamma^d(Y'/S)|_U \times_{\Gamma^d(Y/S)} \Gamma^d(X/S)$$

is a universal homeomorphism such that Λ_{red} is an isomorphism.

Proof. Follows immediately from Propositions (3.3.9) and (3.3.10). $\square$

Corollary (3.3.12). *Let $f_i : X_i \to Y$, $i = 1, 2$ be morphism of algebraic spaces, separated over S. Let $\pi_i : X_1 \times_Y X_2 \to X_i$ be the projections. Assume that for every involved space Z, the functor $\underline{\Gamma}^d_{Z/S}$ is represented by a space which we denote by $\Gamma^d(Z/S)$. Assume that f_1 and f_2 are both étale and let $U_i = \mathrm{reg}(f_i)$ and $U_{12} = \mathrm{reg}(f_1 \circ \pi_1) = \mathrm{reg}(f_2 \circ \pi_2)$. Then*

(i) *$U_{12} = ((\pi_1)_*)^{-1}(U_1) \cap ((\pi_2)_*)^{-1}(U_2)$.*

(ii) *The diagram*

$$\begin{tikzcd}
\Gamma^d(X_1 \times_Y X_2/S)|_{U_{12}} \arrow[r, "(\pi_2)_*"] \arrow[d, "(\pi_1)_*"] \arrow[dr, phantom, "\square"] & \Gamma^d(X_2/S)|_{U_2} \arrow[d, "(f_2)_*"] \\
\Gamma^d(X_1/S)|_{U_1} \arrow[r, "(f_1)_*"] & \Gamma^d(Y/S)
\end{tikzcd}$$

is cartesian.

Proof. It follows from (i) of Proposition (3.3.10) that

$$((\pi_1)_*)^{-1}(U_1) \cap ((\pi_2)_*)^{-1}(U_2) \subseteq U_{12}$$

and the reverse inclusion is obvious. That the diagram is cartesian now follows from Corollary (3.3.11). $\square$

Remark **(3.3.13).** The diagrams in Proposition (3.3.10) and Corollary (3.3.11) are not always cartesian if g is unramified but not étale. In fact, by Examples (3.3.8) there is a morphism $f : X \to Y$ and a family $\alpha \in \underline{\Gamma}^d_{X/S}(S)$ such that $\mathrm{Image}(\alpha) = X$, $f(\mathrm{Image}(\alpha)) = Y$ and such that $\mathrm{Image}(f_*\alpha) \hookrightarrow Y$ is not an isomorphism. If we let $Y' = \mathrm{Image}(f_*\alpha)$ and $\beta' = f_*\alpha \in \underline{\Gamma}^d_{Y'/S}(S)$, then we cannot lift (α, β') to a family $\alpha' \in \underline{\Gamma}^d_{X'/S}(S)$. On the other hand, it is easily seen that Corollary (3.3.12) remains valid if we replace étale with unramified.

Remark **(3.3.14).** Let X, Y, U, f and g as in Corollary (3.3.11) and let U' be the open subscheme of $\Gamma^d(X'/S)$ which represents $\underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g'}$. Then ${f'_*}^{-1}(U) \subseteq U'$ by Proposition (3.3.10) (i), i.e., the points of $\Gamma^d(X'/S)|_{{f'_*}^{-1}(U)}$ are regular with respect to g'. On the other hand, a point which is regular with respect to g' need not be regular with respect to g, i.e., the inclusion ${f'_*}^{-1}(U) \subseteq U'$ is strict in general.

Proposition (3.3.15). *If $f : X/S \to Y/S$ is an étale (resp. étale and surjective) morphism of algebraic spaces separated over S, then the push-forward $f_* : \underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \to \underline{\Gamma}^d_{Y/S}$ is representable and étale (resp. étale and surjective).*

Proof. If f is surjective then $f_* : \underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \to \underline{\Gamma}^d_{Y/S}$ is topologically surjective by Proposition (3.3.9) (iv).

I) *Reduction to $X \to S$ quasi-compact.* Let $\{U_\beta\}$ be an open cover of X such that U_β is quasi-compact and any set of d points in X over the same point in S lies in some U_β. Then $\{\underline{\Gamma}^d_{U_\beta,\mathrm{reg}/f|_{U_\beta}} \to \underline{\Gamma}^d_{X,\mathrm{reg}/f}\}$ is an open cover by Proposition (3.1.10). Replacing X with U_β we can thus assume that X is quasi-compact.

II) *Reduction to X, Y and S affine and Y integral over S.* Let T be an affine scheme and $T \to \underline{\Gamma}^d_{Y/S}$ a morphism. Then it factors as $T \to \Gamma^d(W/T)$ where $W \hookrightarrow Y_T = Y \times_S T$ is a closed subspace such that $W \to T$ is integral. Let $Z = f_T^{-1}(W)$. Note that f is separated and quasi-compact as $X \to S$ is separated and quasi-compact. Hence f is quasi-affine as well as $Z \to W \to T$ which is the composition of two quasi-affine morphisms. Thus $\underline{\Gamma}_{Z/T}$ and $\underline{\Gamma}_{W/T}$ are both representable by Theorem (3.1.11). As $W \hookrightarrow Y_T$ is a closed immersion it follows from Proposition (3.3.10) (ii) that we have a cartesian diagram

$$\begin{tikzcd}
\Gamma^d(Z/T)|_{\mathrm{reg}(f_T|_Z)} \arrow[r, hook] \arrow[d, "(f_T|_Z)_*"] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{X_T/T,\mathrm{reg}/f_T} \arrow[r] \arrow[d, "(f_T)_*"] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \arrow[d, "f_*"] \\
\Gamma^d(W/T) \arrow[r, hook] & \underline{\Gamma}^d_{Y_T/T} \arrow[r] & \underline{\Gamma}^d_{Y/S}.
\end{tikzcd}$$

This shows that f_* is representable. To show that $f_* : \underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \to \underline{\Gamma}^d_{Y/S}$ is étale it is thus enough to show that $\Gamma^d(Z/T) \to \Gamma^d(W/T)$ is étale over the open subset $\mathrm{reg}\,(f_T|_Z)$. Further, as $\Gamma^d(Z/T)$ is covered by open affine subsets of the form $\Gamma^d(U/T)$ where $U \subseteq Z$ is an affine open subset by Proposition (3.1.10), we can assume that Z/T is affine. Replacing X, Y and S with Z, W and T we can then assume that X and S are affine and Y is integral over S.

III) *Reduction to X and Y quasi-finite and finitely presented over S.* Let $S = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $X = \operatorname{Spec}(C)$. We can write B as a filtered direct limit of finite and finitely presented A-algebras B_λ. As $B \to C$ is of finite presentation, we can find an μ and a B_μ-algebra C_μ such that $C = C_\mu \otimes_{B_\mu} B$. Let $C_\lambda = C_\mu \otimes_{B_\mu} B_\lambda$, $X_\lambda = \operatorname{Spec}(C_\lambda)$ and $Y_\lambda = \operatorname{Spec}(B_\lambda)$ for every $\lambda \geq \mu$. As Γ^d commutes with filtered direct limits, cf. paragraph (1.3.3), we have that $\Gamma^d_A(B) = \varinjlim_\lambda \Gamma^d_A(B_\lambda)$ and $\Gamma^d_A(C) = \varinjlim_\lambda \Gamma^d_A(C_\lambda)$.

Let $U = \operatorname{reg}(f) \subseteq \Gamma^d(X/S)$ and let $u \in U$ be a point with residue field k and let $\alpha \in \underline{\Gamma}_{X/S}(k)$ be the corresponding family of cycles with image $Z \hookrightarrow X_k$. Let $\beta = f_*\alpha$ and $W = \operatorname{Image}(\beta)$. As α is regular $Z \to W$ is an isomorphism. Now as W consists of a finite number of points each with a residue field of finite separable degree over k, it is easily seen that there is a $\lambda \geq \mu$ such that $(Y \times_S k)|_W \to Y_\lambda \times_S k$ is universally injective. Thus the push-forward of α along $\psi_\lambda : X \to X_\lambda$ is quasi-regular with respect to f_λ and thus regular as f_λ is étale. Corollary (3.3.11) gives the cartesian diagram

$$\begin{tikzcd}
\Gamma^d(X/S)|_{\psi_\lambda^{-1}(V)} \arrow[r, "f_*"] \arrow[d, "\psi_{\lambda *}"] \arrow[dr, phantom, "\square"] & \Gamma^d(Y/S) \arrow[d] \\
\Gamma^d(X_\lambda)|_V \arrow[r, "(f_\lambda)_*"] & \Gamma^d(Y_\lambda/S)
\end{tikzcd}$$

where $V = \operatorname{reg}(f_\lambda)$ and $u \in \psi_\lambda^{-1}(V)$ as $(\psi_\lambda)_*\alpha$ is regular.

Replacing X and Y with X_λ and Y_λ we can thus assume that X and Y are of finite presentation over S. Further as f is quasi-finite and of finite presentation and $Y \to S$ is finite and of finite presentation it follows that $X \to S$ is quasi-finite and of finite presentation. Proposition (1.3.7) then shows that $\Gamma^d(X/S)$ and $\Gamma^d(Y/S)$ are of finite presentation over S. Thus $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ is also of finite presentation.

IV) *Reduction to S strictly local.* Let $\alpha \in \Gamma^d(X/S)$ and let $\beta = f_*(\alpha)$ and $s \in S$ be its images. Let $S' \to S$ be a flat morphism such that s is in its image. Then, as f_* is of finite presentation, f_* is étale at a point $\alpha \in \Gamma^d(X/S)$ if the morphism $\Gamma^d(X'/S') \to \Gamma^d(Y'/S')$ is étale at a point $\alpha' \in \Gamma^d(X'/S')$ above α [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 17.7.1]. We take S' as the strict henselization of $\mathcal{O}_{S,s}$. As $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f}$ is an open subfunctor of $\underline{\Gamma}^d_{X/S}$ we have that $\operatorname{reg}(f) \times_S S' = \operatorname{reg}(f')$. We can thus replace X, Y and S with X', Y' and S' and assume that S is strictly local.

V) *Conclusion* We have now reduced the proposition to the following situation: S is strictly local, $X \to S$ is quasi-finite and finitely presented and $Y \to S$ is finite and finitely presented. The support of $\alpha \in \Gamma^d(X/S)$ consists of a finite number of points $x_1, x_2, \dots, x_m \in X$ lying above the closed point $s \in S$. As $X \to S$ is quasi-finite and S is henselian it follows that $X = \left(\coprod_{i=1}^m X_i\right) \amalg X'$ where X_i are strictly local schemes, finite over S, such that $x_i \in X_i$. Then $\alpha \in \Gamma^d\bigl(\coprod_{i=1}^m X_i\bigr) \hookrightarrow \Gamma^d(X/S)$ and we can thus assume that $X = \coprod_{i=1}^m X_i$ is finite over S.

As S is strictly local and $Y \to S$ is finite it follows that $Y = \coprod_{j=1}^n Y_j$ is a finite disjoint union of strictly local schemes. For every $i = 1, 2, \dots, m$ there is a $j(i)$ such that $f(X_i) \hookrightarrow Y_{j(i)}$ and $f|_{X_i} : X_i \to Y_{j(i)}$ is an isomorphism as f is étale. We have further by Proposition (1.4.1) that

$$\Gamma^d(X/S) = \coprod_{\sum_i d_i = d} \prod_{i=1}^m \Gamma^{d_i}(X_i), \quad \Gamma^d(Y/S) = \coprod_{\sum_j e_j = d} \prod_{j=1}^n \Gamma^{e_j}(Y_j).$$

It is obvious that the regular subset $U \subseteq \Gamma^d(X/S)$ is given by the connected components with $d_1, d_2, \dots, d_m$ such that for every $j = 1, 2, \dots, n$ there is at most one i with $d_i > 0$ such that $j(i) = j$. As

$$\prod_{i=1}^m \Gamma^{d_i}(X_i) \to \prod_{i=1}^m \Gamma^{d_i}\left(Y_{j(i)}\right)$$

is an isomorphism this completes the demonstration. □

Corollary (3.3.16). *Let X/S be a separated algebraic space and $\{f_\alpha : U_\alpha \to X\}_\alpha$ an étale separated cover. Assume that for every involved space Z, the functor $\underline{\Gamma}^d_{Z/S}$ is represented by a space which we denote by $\Gamma^d(Z/S)$. Then*

$$\coprod_{\alpha,\beta} \Gamma^d(U_\alpha \times_X U_\beta/S)|_{\mathrm{reg}} \rightrightarrows \coprod_{\alpha} \Gamma^d(U_\alpha/S)|_{\mathrm{reg}} \longrightarrow \Gamma^d(X/S) \tag{3.3.16.1}$$

is an étale equivalence relation. Here reg *denotes the regular locus with respect to the push-forward to X.*

Proof. This follows from Corollary (3.3.12) and Proposition (3.3.15). □

3.4. Representability of $\underline{\Gamma}^d_{X/S}$ by an algebraic space.

In this subsection, it will be shown that for *any* algebraic space X separated over S, the functor $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space, separated over S.

Theorem (3.4.1). *Let S be an algebraic space and X/S a separated algebraic space. Then the functor $\underline{\Gamma}^d_{X/S}$ is represented by a separated algebraic space $\Gamma^d(X/S)$.*

Proof. Let $f : X' \to X$ be an étale cover such that X' is a disjoint union of affine schemes. Then X' is an AF-scheme and $\underline{\Gamma}^d_{X'/S}$ is represented by the scheme $\Gamma^d(X'/S)$, cf. Theorem (3.1.11). By Propositions (3.1.6) and (3.3.15), the functor $\underline{\Gamma}^d_{X/S}$ is a sheaf in the étale topology and the push-forward $f_* : \Gamma^d(X'/S)|_{\mathrm{reg}(f)} \to \underline{\Gamma}^d_{X/S}$ is an étale presentation.

To show that $\underline{\Gamma}^d_{X/S}$ is a separated algebraic space, it is thus sufficient to show that the diagonal is represented by closed immersions. Let T be an S-scheme and $\alpha, \beta \in \underline{\Gamma}^d_{X/S}(T)$. Let $Z_\alpha, Z_\beta \hookrightarrow X \times_S T$ be the images of α and β. Let $Z_0 = Z_\alpha \cap Z_\beta = Z_\alpha \times_{X_T} Z_\beta$. We then let $T_0 = \alpha^{-1}(\Gamma^d(Z_0/S)) \cap \beta^{-1}(\Gamma^d(Z_0/S))$ where we have considered α and β as morphisms $T \to \Gamma^d(Z_\alpha/T)$ and $T \to \Gamma^d(Z_\beta/T)$ respectively. Then $T_0 \hookrightarrow T$ is a closed subscheme and

$$\begin{aligned}(\alpha,\beta)^*\Delta_{\underline{\Gamma}^d_{X/S}/S} &= \underline{\Gamma}^d_{X/S} \times_{\underline{\Gamma}^d_{X/S}\times_S\underline{\Gamma}^d_{X/S}} T\\ &= \underline{\Gamma}^d(Z_0/T) \times_{\underline{\Gamma}^d(Z_0/T)\times_S\underline{\Gamma}^d(Z_0/T)} T_0\\ &= (\alpha|_{T_0},\beta|_{T_0})^*\Delta_{\Gamma^d(Z_0/T)/T}\end{aligned}$$

which is a closed subscheme of T_0 as $\Gamma^d(Z_0/T) \to T$ is affine. □

Proposition (3.4.2). *Let X/S be a separated algebraic space. Let $s \in S$ and let $\alpha \in \Gamma^d(X/S)$ be a point over $s \in S$. There is then a finite number of points $x_1, x_2, \dots, x_n \in X$ with $n \leq d$ such that the following condition holds:*

(*) *Choose an étale neighborhood $S' \to S$ of s and étale neighborhoods $\{U_i \to X\}$ of $\{x_i\}$ such that the U_i's are algebraic S'-spaces. There is then an open subset V of $\Gamma^d\bigl(\coprod_{i=1}^n U_i/S'\bigr)$ such that $V \to \Gamma^d(X/S)$ is an étale neighborhood of α.*

Furthermore, if we choose the U_i's such that there is a point above x_i with trivial residue field extension, then there is a point in V above α with trivial residue field extension.

In particular, $\Gamma^d(X/S)$ has an étale covering of the form $\coprod_i \Gamma^d(X_i/S_i)|_{V_i}$ where S_i and X_i are affine and $S_i \to S$ and $X_i \to X$ étale.

Proof. The point α corresponds to a family $\operatorname{Spec}(k(\alpha)) \to \Gamma^d(X/S)$ where $k(\alpha)$ is the residue field. Let $Z \hookrightarrow X \times_S \operatorname{Spec}(k(\alpha))$ be the image of this family. Then Z is reduced and consists of a finite number of points $z_1, z_2, \dots, z_m$ such that $m \leq d$. Let $W = \{x_1, x_2, \dots, x_n\}$ be the projection of Z on X. Then α lies in the closed subset $\Gamma^d(W/S) \hookrightarrow \Gamma^d(X/S)$.

If $f : U \to X$ is an étale neighborhood of W then it is obvious that there is a lifting of α to $V = \Gamma^d(U/S)|_{\mathrm{reg}(f)}$. Furthermore, if f has trivial residue field extensions over W, then we can choose a lifting with the residue field $k(\alpha)$. That $V \to \Gamma^d(X/S)$ is étale is Proposition (3.3.15). □

4. Further properties of $\Gamma^d(X/S)$

4.1. **Addition of cycles and non-degenerate families.** In paragraphs (1.2.14) and (1.3.5) we defined the universal multiplication of laws $\rho_{d,e} : \Gamma^{d+e}_A(B) \to \Gamma^d_A(B) \otimes_A \Gamma^e_A(B)$. We will give a corresponding morphism $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$ for arbitrary X/S.

Definition-Proposition (4.1.1). *Let X/S be a separated algebraic space and let d, e be positive integers. Then there exists a morphism*

$$+ : \Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$$

which on points is addition of cycles. When X/S is affine, this morphism corresponds to the homomorphism $\rho_{d,e}$. The operation $+$ makes the space $\Gamma(X/S) = \coprod_{d\geq 0} \Gamma^d(X/S)$ into a graded commutative monoid.

Proof. The morphism $+$ is the composition of the open and closed immersion $\Gamma^d(X/S) \times \Gamma^e(X/S) \hookrightarrow \Gamma^{d+e}(X \amalg X/S)$ of Proposition (3.1.8) and the push-forward along $X \amalg X \to X$. It is clear that this is an associative and commutative operation as push-forward is functorial. When X/S is affine, it is clear from (1.2.14) that the addition of cycles corresponds to the homomorphism $\rho_{d,e}$. ☐

Proposition (4.1.2). *Let X/S be a separated algebraic space and T an S-scheme. Let $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ and $\beta \in \underline{\Gamma}^e_{X/S}(T)$.*

- (i) *If T is connected and* $\mathrm{Image}(\alpha) = \coprod_{i=1}^n Z_i$ *then there are integers $d_i \geq 1$ and families of cycles $\alpha_i \in \underline{\Gamma}^{d_i}_{Z_i/S}(T)$ such that $d = d_1 + d_2 + \cdots + d_n$ and $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.*
- (ii) $\mathrm{Supp}(\alpha + \beta) = \mathrm{Supp}(\alpha) \cup \mathrm{Supp}(\beta)$.
- (iii) *Let $f : X \to Y$ be a morphism of separated algebraic spaces. Then $f_*(\alpha + \beta) = f_*\alpha + f_*\beta$.*

Proof. (i) and (ii) follows from Propositions (3.1.8) and (3.3.7) (ii) respectively. (iii) follows easily from the definitions and the functoriality of the push-forward. ☐

Proposition (4.1.3). *The morphism $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$ is étale over the open subset $U \subseteq \Gamma^d(X/S) \times_S \Gamma^e(X/S)$ where $(\alpha, \beta) \in U$ if* $\mathrm{Supp}(\alpha)$ *and* $\mathrm{Supp}(\beta)$ *are disjoint.*

Proof. The morphism $X \amalg X \to X$ is étale. By Propositions (3.1.8) and (3.3.15) we have that $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$ is étale at (α, β) if $\alpha \amalg \beta$ is regular with respect to $X \amalg X \to X$. This is fulfilled if and only if $\mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\beta)$ are disjoint. ☐

Notation (4.1.4). We let $(X/S)^d$ denote the fiber product $X \times_S X \times_S \cdots \times_S X$ of d copies of X over S.

Proposition (4.1.5). *Let X/S be a separated algebraic space. The symmetric group on d letters $\mathfrak{S}_d$ acts on $(X/S)^d$ by permutation of factors. We equip $\Gamma^d(X/S)$ with the trivial $\mathfrak{S}_d$-action. Then:*

- (i) *There is a canonical $\mathfrak{S}_d$-equivariant morphism $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$.*
- (ii) *Ψ_X is integral and universally open. Its fibers are the orbits of $(X/S)^d$ and this also holds after base change.*
- (iii) *Ψ_X is étale outside the diagonals of $(X/S)^d$.*
- (iv) *If $f : X \to Y$ is a morphism of separated algebraic spaces we have a commutative diagram*

$$\begin{tikzcd}
(X/S)^d \arrow[r, "f^d"] \arrow[d, "\Psi_X"] \arrow[dr, phantom, "\circ"] & (Y/S)^d \arrow[d, "\Psi_Y"] \\
\Gamma^d(X/S) \arrow[r, "f_*"] & \Gamma^d(Y/S).
\end{tikzcd}$$

If f is unramified (resp. étale) and $U = \operatorname{reg}(f)$ then the canonical morphism

$$\Lambda : (X/S)^d|_{\Psi_X^{-1}(U)} \to \Gamma^d(X/S)|_U \times_{\Gamma^d(Y/S)} (Y/S)^d$$

is a universal homeomorphism (resp. an isomorphism).

Proof. (i) As $\operatorname{Hom}_S\bigl(T, (X/S)^d\bigr) = \operatorname{Hom}_S(T, X)^d = \underline{\Gamma}^1_{X/S}(T)^d$ by Remark (3.1.5) we obtain by addition of cycles the morphism $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$ and this is clearly an $\mathfrak{S}_d$-equivariant morphism as addition of cycles is commutative.

(iii) Follows immediately from Proposition (4.1.3).

(iv) Follows from the definition of Ψ and Corollary (3.3.11) since

$$\begin{tikzcd}
(X/S)^d \arrow[r, "f^d"] \arrow[d] \arrow[dr, phantom, "\square"] & (Y/S)^d \arrow[d] \\
\Gamma^d(\coprod_{i=1}^d X) \arrow[r] & \Gamma^d(\coprod_{i=1}^d Y)
\end{tikzcd}$$

is cartesian.

(ii) We first show that the fibers of Ψ are the $\mathfrak{S}_d$-orbits and that this holds after any base change. Let $f : \operatorname{Spec}(k) \to \Gamma^d(X/S)$ be a morphism. Then f factors through $\Gamma^d(Z/k) \to \Gamma^d(X/S)$ where $Z \hookrightarrow X \times_S \operatorname{Spec}(k)$ is a closed subspace integral over k.

As Γ^d commutes with base change, we can replace S with $\operatorname{Spec}(k)$. Furthermore, using the unramified part of (iv), we can replace X with Z. We can thus assume that $S = \operatorname{Spec}(k)$ and that $X = Z = \operatorname{Spec}(B)$. Then $(X/k)^d = \operatorname{Spec}\bigl(\mathrm{T}^d_k(B)\bigr)$ and $\Gamma^d(X/k) = \operatorname{Spec}\bigl(\mathrm{TS}^d_k(B)\bigr) = \operatorname{Sym}^d(X/k)$. As the fibers of $(X/k)^d \to \operatorname{Sym}^d(X/k)$ are the $\mathfrak{S}_d$-orbits it follows that the same holds for Ψ.

If $U \hookrightarrow (X/S)^d$ is an open (resp. closed subset) then $\Psi^{-1}\bigl(\Psi(U)\bigr) = \bigcup_{\sigma \in \mathfrak{S}_d} \sigma U$. As this also holds after any base change $T \to \Gamma^d(X/S)$ it follows that Ψ is universally closed and universally open.

We will now show that Ψ_X is affine. As Ψ_X is universally closed it then follows that Ψ_X is integral by [EGA$_{\mathrm{IV}}$, Prop. 18.12.8]. As affineness is local in the étale topology we can assume that S is affine. Let $f : X' \to X$ be an étale covering such that X' is a disjoint union of affine schemes and in particular an AF-scheme. By Proposition (3.3.15) the push-forward morphism $f_* : \Gamma^d(X'/S)|_{\operatorname{reg}(f)} \to \Gamma^d(X/S)$ is an étalecover. Using (iv) and replacing X with X' we can thus assume that X is AF. Proposition (3.1.10) then shows that $\Gamma^d(X/S)$ is covered by open subsets $\Gamma^d(U/S)$ where U is affine. Finally Ψ_U is affine as $(U/S)^d$ is affine. $\square$

Definition (4.1.6). Let X/S be a separated algebraic space, T an S-space and $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ a family of cycles. Let $t \in T$ be a point and let $\overline{k}$ be an algebraic closure of its residue field k. We say that α is *non-degenerated* in a point $t \in T$ if the support of the cycle $\alpha_t \times_k \overline{k}$ consists of d distinct points. Here $\alpha_t \times_k \overline{k}$ denotes the family given by the composition of $\operatorname{Spec}(\overline{k}) \to \operatorname{Spec}(k) \to T$ and α.

The *non-degeneracy locus* is the set of points $t \in T$ such that α is non-degenerate in t.

Definition (4.1.7). We let $\Gamma^d(X/S)_{\mathrm{nondeg}} \subseteq \Gamma^d(X/S)$ denote the subset of non-degenerate families.

Proposition (4.1.8). *The subset* $\Gamma^d(X/S)_{\mathrm{nondeg}} \subseteq \Gamma^d(X/S)$ *is open. The morphism* $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$ *is étale of rank* $d!$ *over* $\Gamma^d(X/S)_{\mathrm{nondeg}}$ *and the addition morphism* $+ : \Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$ *is étale of rank* $((d,e))$ *over* $\Gamma^{d+e}(X/S)_{\mathrm{nondeg}}$.

Proof. Let U be the complement of the diagonals of $(X/S)^d$, which is an open subset. Then $\Gamma^d(X/S)_{\mathrm{nondeg}} = \Psi_X(U)$ which is an open subset as Ψ_X is open. The last two statements follow from Proposition (4.1.3). □

4.2. **The** $\mathrm{Sym}^d \to \Gamma^d$ **morphism.**

Definition (4.2.1) ([Kol97, Ryd07])**.** If G is a group and $f : X \to Y$ a G-equivariant morphism, then we say that f is fixed-point reflecting, or fpr, at $x \in X$ if the stabilizer of x coincides with the stabilizer of $f(x)$. The subset of X where G is fixed-point reflecting is G-stable and denoted $\mathrm{fpr}(f)$.

Remark **(4.2.2).** Let X/S be a separated algebraic space. There is then a uniform geometric and categorical quotient $\mathrm{Sym}^d(X/S) := (X/S)^d/\mathfrak{S}_d$, cf. [Ryd07]. Furthermore we have that $q : (X/S)^d \to \mathrm{Sym}^d(X/S)$ is integral, universally open and a topological quotient, i.e., it satisfies (ii) of Proposition (4.1.5). Moreover (iii) and the étale part of (iv) also holds for q instead of Ψ if we replace $\mathrm{reg}(f)$ with $\mathrm{fpr}(f)$, cf. [Ryd07].

As $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$ is $\mathfrak{S}_d$-equivariant and $\mathrm{Sym}^d(X/S)$ is a *categorical* quotient, we obtain a canonical morphism $\mathrm{SG}_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ such that $\Psi_X = \mathrm{SG}_X \circ q$.

Lemma (4.2.3). *Let* $f : X \to Y$ *be a morphism of algebraic spaces and let* $\alpha \in \Gamma^d(X/S)$ *be a point. Then* α *is quasi-regular with respect to* f *if and only if* f^d *is fixed-point reflecting at* $\Psi_X^{-1}(\alpha)$ *with respect to the action of* $\mathfrak{S}_d$.

Proof. Let k be the algebraic closure of the residue field $k(\alpha)$. The supports of α and $f_*\alpha$ are finite disjoint unions of points. Thus $\alpha : \mathrm{Spec}(k) \to \Gamma^d(X/S)$ and $f_*\alpha : \mathrm{Spec}(k) \to \Gamma^d(Y/S)$ factors as

$$\mathrm{Spec}(k) \to \prod_{i=1}^{n} \Gamma^{d_i}(x_i/k) \to \Gamma^d(X/S)$$

and

$$\mathrm{Spec}(k) \to \prod_{j=1}^{m} \Gamma^{e_i}(y_j/k) \to \Gamma^d(Y/S)$$

where x_i and y_j are points of $X \times_S \operatorname{Spec}(k)$ and $Y \times_S \operatorname{Spec}(k)$ respectively and $k(x_i) = k(y_j) = k$. Every point of $(X/S)^d$ (resp. $(Y/S)^d$) above α (resp. $f_*\alpha$) is thus such that, after a permutation, the first d_1 (resp. e_1) projections agree, the next d_2 (resp. e_2) projections agree, etc, but no other two projections are equal. Thus the stabilizers of the points of $\Psi_X^{-1}(\alpha)$ (resp. $\Psi_Y^{-1}(f_*\alpha)$) are $\mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2} \times \dots \times \mathfrak{S}_{d_n}$ (resp. $\mathfrak{S}_{e_1} \times \mathfrak{S}_{e_2} \times \dots \times \mathfrak{S}_{e_m}$). Equality holds if and only if f is quasi-regular. $\square$

Proposition (4.2.4). *Let $f : X \to Y$ be an étale morphism of algebraic spaces. Then $\Psi_X^{-1}\bigl(\operatorname{reg}(f)\bigr) = \operatorname{fpr}(f^d)$, and we have a cartesian diagram*

$$\begin{tikzcd}
(X/S)^d|_{\operatorname{fpr}(f^d)} \arrow[r, "q"] \arrow[d, "f^d"] \arrow[dr, phantom, "\square"] & \operatorname{Sym}^d(X/S)|_{\operatorname{fpr}(f^d)} \arrow[r, "\operatorname{SG}_X"] \arrow[d, "f^d/\mathfrak{S}_d"] \arrow[dr, phantom, "\square"] & \Gamma^d(X/S)|_{\operatorname{reg}(f)} \arrow[d, "f_*"] \\
(Y/S)^d \arrow[r, "q"] & \operatorname{Sym}^d(Y/S) \arrow[r, "\operatorname{SG}_Y"] & \Gamma^d(Y/S)
\end{tikzcd}$$

In particular $f^d/\mathfrak{S}_d$ is étale over the open subset $q\bigl(\operatorname{fpr}(f^d)\bigr) = \operatorname{SG}_X^{-1}\bigl(\operatorname{reg}(f)\bigr)$.

Proof. As f is unramified $\operatorname{reg}(f) = \operatorname{qreg}(f)$ by Proposition (3.3.6), the first statement follows from Lemma (4.2.3). The outer square is cartesian by Proposition (4.1.5) (iv) and as q is a uniform quotient the formation of the quotient commutes with étale base change which shows that the right square is cartesian. It follows that the left square is cartesian too. $\square$

Corollary (4.2.5). *Let X/S be a separated algebraic space. The canonical morphism $\operatorname{SG}_X : \operatorname{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a universal homeomorphism with trivial residue field extensions. If S has pure characteristic zero or X/S is flat, then SG_X is an isomorphism.*

Proof. Using Proposition (4.2.4) and the covering in Proposition (3.4.2) we can assume that $X = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(A)$ are affine. Then $(X/S)^d = \operatorname{Spec}\bigl(\mathrm{T}_A^d(B)\bigr)$, $\Gamma^d(X/S) = \operatorname{Spec}\bigl(\Gamma_A^d(B)\bigr)$ and $\operatorname{Sym}^d(X/S) = \operatorname{Spec}\bigl(\mathrm{TS}_A^d(B)\bigr)$ are all affine. As $\Gamma_A^d(B) \to \mathrm{TS}_A^d(B) \hookrightarrow \mathrm{T}_A^d(B)$ is integral by Proposition (4.1.5) (ii), we have that $\operatorname{SG}_X : \operatorname{Spec}\bigl(\mathrm{TS}_A^d(B)\bigr) \to \operatorname{Spec}\bigl(\Gamma_A^d(B)\bigr)$ is integral.

The geometric fibers of both Ψ_X and $q : (X/S)^d \to \operatorname{Sym}^d(X/S)$ are the geometric orbits of $(X/S)^d$. Thus SG_X is universally bijective and hence a universal homeomorphism. That SG_X is an isomorphism when S is purely of characteristic zero or X/S is flat follows from paragraph (1.3.2) as X and S are affine.

Let $a \in \operatorname{Sym}^d(X/S)$ be any point, $b = \operatorname{SG}_X(a) \in \Gamma^d(X/S)$ and s its image in S. We have a commutative diagram

$$\begin{tikzcd}
\operatorname{Sym}^d\bigl(X_s/k(s)\bigr) \arrow[r, "\operatorname{SG}_{X_s}", "\cong"'] \arrow[d] \arrow[dr, phantom, "\circ"] & \Gamma^d\bigl(X_s/k(s)\bigr) \arrow[d, "\cong"] \\
\operatorname{Sym}^d(X/S) \times_S k(s) \arrow[r, "\operatorname{SG}_X \times_S \operatorname{id}_{k(s)}"] & \Gamma^d(X/S) \times_S k(s)
\end{tikzcd}$$

which gives a commutative diagram of residue fields

$$\begin{tikzcd}
k(\overline{a}) & k(\overline{b}) \arrow[l, hook', "\cong"'] \\
k(a) \arrow[u, hook] \arrow[ur, phantom, "\circ"] & k(b) \arrow[l, hook'] \arrow[u, hook, "\cong"'] .
\end{tikzcd}$$

and thus $k(a) = k(b)$. ☐

Proposition (4.2.6). *Let X/S be a separated algebraic space. The canonical morphism* $\mathrm{SG}_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ *is an isomorphism over* $\Gamma^d(X/S)_{\mathrm{nondeg}}$.

Proof. Let U be the complement of the diagonals in $(X/S)^d$. Then $\Psi_X(U) = \Gamma^d(X/S)_{\mathrm{nondeg}}$ and $\mathfrak{S}_d$ acts freely on U. By Proposition (4.1.8) the morphism Ψ_X is étale of rank $d!$ over $\Gamma^d(X/S)_{\mathrm{nondeg}}$. It is further well-known that $q : (X/S)^d \to \mathrm{Sym}^d(X/S)$ is étale of rank $d!$ over $q(U)$. In fact, $\mathrm{Sym}^d(X/S)|_{q(U)}$ is the quotient sheaf in the étale topology of the étale equivalence relation $\mathfrak{S}_d \times U \rightrightarrows U$. ☐

4.3. Properties of $\Gamma^d(X/S)$ and the push-forward.

Proposition (4.3.1). *Let S be an algebraic space and let X be an algebraic space separated over S. Consider for a morphism the property of being*

(i) *quasi-compact*
(ii) *finite type*
(iii) *finite presentation*
(iv) *locally of finite type*
(v) *locally of finite presentation*
(vi) *flat*
(vii) *finite*
(viii) *integral*
(ix) *affine*
(x) *quasi-affine*

If $X \to S$ has one of these properties then so does $\Gamma^d(X/S) \to S$.

Proof. If $X \to S$ is quasi-compact then $(X/S)^d \to S$ is quasi-compact. As there is a surjective morphism $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$ it follows that $\Gamma^d(X/S)$ is quasi-compact over S. This shows (i). As $\Gamma^d(X/S)$ is separated, (ii) and (iii) follows from (i), (iv) and (v). It is thus enough to show (iv)–(x).

As the question is local over S we can assume that S is affine. If $X \to S$ is affine (resp. quasi-affine) then $\Gamma^d(X/S)$ is affine (resp. quasi-affine) by Propositions (3.1.4) and (3.1.7). If $X \to S$ is finite (resp. integral), then $X \to S$ is affine and $\Gamma^d(X/S) \to S$ is finite (resp. integral) by Proposition (1.3.7).

By Proposition (3.4.2) any point of $\Gamma^d(X/S)$ has an étale neighborhood V such that V is an open subset of $\Gamma^d(U/S)$ where U is an affine scheme and $U \to X$ étale. If $V \to S$ is locally of finite type (resp. locally of finite presentation, resp. flat) for any such neighborhood V then it follows by [$\mathrm{EGA}_{\mathrm{IV}}$, Lem. 17.7.5] and

[EGA$_{\mathrm{IV}}$, Cor. 2.2.11 (iv)] that $\Gamma^d(X/S)$ is locally of finite type (resp. locally of finite presentation, resp. flat) over S. Replacing X with U we can thus assume that X is affine. The proposition now follows from Proposition (1.3.7) and paragraph (1.2.12). □

Corollary (4.3.2). *Let S and X be algebraic spaces. If $f : X \to S$ is flat with geometric reduced fibers then $\Gamma^d(X/S) \to S$ is flat with geometric reduced fibers. In particular, if in addition S is reduced then $\Gamma^d(X/S)$ is reduced.*

Proof. Proposition (4.3.1) shows that $\Gamma^d(X/S) \to S$ is flat. It is thus enough to show that $\Gamma^d(X_k/k)$ reduced for any algebraic closed field k and morphism $\mathrm{Spec}(k) \to S$. As X_k is reduced by hypothesis and hence also $(X_k/k)^d$ it follows that $\mathrm{Sym}^d(X_k/k)$ is reduced and $\Gamma^d(X_k/k) = \mathrm{Sym}^d(X_k/k)$ by Corollary (4.2.5). The last statement follows by [Pic98, Prop. 5.17]. □

Proposition (4.3.3). *Let S and X be algebraic spaces. If $f : X \to S$ is smooth of relative dimension 0 (resp. 1, resp. at most 1) then $\Gamma^d(X/S) \to S$ is smooth of relative dimension 0 (resp. d, resp. at most d).*

Proof. As $\Gamma^d(X/S) \to S$ is flat and locally of finite presentation by Proposition (4.3.1), it is enough to show that its geometric fibers are regular [EGA$_{\mathrm{IV}}$, Thm. 17.5.1]. Thus we can assume that $S = \mathrm{Spec}(k)$ where k is algebraically closed. Let $y \in \Gamma^d(X/k)$. Then by Proposition (3.2.3), the formal local ring $\widehat{\mathcal{O}}_{\Gamma^d(X/k),y}$ is the completion at a point of the scheme $\prod_{i=1}^n \Gamma^{d_i}(\widehat{X}_{x_i}/k)$ where $x_1, x_2, \dots, x_n$ are points of X and $d = d_1 + d_2 + \dots + d_n$. If f has relative dimension 0 at x_i then $\mathcal{O}_{X,x_i} = k$ and if f has relative dimension 1 at x_i then $\widehat{\mathcal{O}}_{X,x_i} = k[[t]]$, cf. [EGA$_{\mathrm{IV}}$, Prop. 17.5.3]. The proposition now easily follows if we can show that $\Gamma^e\big(\mathrm{Spec}(k[t])/\mathrm{Spec}(k)\big)$ is smooth of relative dimension e. But $\Gamma^e_k(k[t]) = \mathrm{TS}^e_k(k[t]) = k[s_1, s_2, \dots, s_e]$ where $s_1, s_2, \dots, s_e$ are the elementary symmetric functions. □

Remark **(4.3.4).** If X/S is smooth of relative dimension ≥ 2 then $\Gamma^d(X/S)$ is not smooth for $d \geq 2$. This can be seen by an easy tangent space calculation. If X/S is smooth of relative dimension 2 then on the other hand $\mathrm{Hilb}^d(X/S)$ is smooth and gives a resolution of $\Gamma^d(X/S)$ [Fog68, Cor. 2.6 and Thm. 2.9]. Moreover $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is a blow-up in this case [Hai98, ES04].

Proposition (4.3.5). *If $f : X \to Y$ has one of the following properties, then so has $f^d/\mathfrak{S}_d : \mathrm{Sym}^d(X/S) \to \mathrm{Sym}^d(Y/S)$:*

(i) *quasi-compact*
(ii) *closed*
(iii) *open*
(iv) *universally closed*
(v) *universally open*
(vi) *open immersion*

(vii) *affine*
(viii) *quasi-affine*
(ix) *integral*

If f has one of the above properties or one of the following

(x) *closed immersion*
(xi) *immersion*

then so has $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$.

Proof. Use that Ψ_X and $q : (X/S)^d \to \operatorname{Sym}^d(X/S)$ are universally closed, universally open, quasi-compact and surjective for (i)-(v). Property (vi) is well-known. For (vii) reduced to Y/S affine using Proposition (3.4.2) and then use that $\Gamma^d(X/S)$ and $\operatorname{Sym}^d(X/S)$ are affine if X/S is affine. The combination of (i), (vi) and (vii) gives (viii). Finally (ix) follows from (vii) and (iv). The last two properties for f_* follow from Proposition (3.1.7). □

Remark **(4.3.6).** If f has one of the properties (x) or (xi), then $f^d/\mathfrak{S}_d$ need not have that property. If f has one of the properties

(i) finite
(ii) locally of finite type
(iii) locally of finite presentation
(iv) unramified
(v) flat
(vi) étale

then neither $f^d/\mathfrak{S}_d$ nor f_* need to have that property.

Corollary (4.3.7). *The addition morphism $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$ is integral and universally open.*

Proof. The morphism $X \amalg X \to X$ is finite and étale and hence integral and universally open. Thus $\Gamma^{d+e}(X \amalg X/S) \to \Gamma^{d+e}(X/S)$ is integral and universally open by Proposition (4.3.5). As the addition morphism is the composition of the open and closed immersion $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \hookrightarrow \Gamma^{d+e}(X \amalg X/S)$ and the push-forward along $X \amalg X \to X$ the corollary follows. □

Appendix A. Appendix

A.1. **The (AF) condition.** The (AF) condition has frequently been used as a natural setting for a wide range of problems. It guarantees the existence of finite quotients [SGA$_1$, Exp. V], push-outs [Fer03] and the Hilbert scheme of points [Ryd08b]. Moreover, under the (AF) condition, étale cohomology can be calculated using Čech cohomology [Art71, Cor. 4.2], [Sch03].

Definition (A.1.1). We say that a scheme X/S is AF if it satisfies the following condition.

(AF) Every finite set of points $x_1, x_2, \dots, x_n \in X$ over the same point $s \in S$ is contained in an open subset $U \subseteq X$ such that $U \to S$ is quasi-affine.

Remark **(A.1.2).** Let X/S and Y/S be AF-schemes. Then $X \times_S Y/S$ is an AF-scheme. If $S' \to S$ is any morphism, then $X \times_S S'/S'$ is an AF-scheme. This is obvious as the class of quasi-affine morphisms is stable under products and base change. It is also clear that the (AF) condition is local on S and that the subset U in the condition can be chosen such that U is an affine scheme. Moreover, if S is quasi-separated, then we can replace the condition that $U \to S$ is quasi-affine with the condition that U is affine.

Proposition (A.1.3). *Let X be an S-scheme. If X has an ample invertible sheaf $\mathcal{O}_X(1)$ relative to S then X/S is an* AF*-scheme. In particular, it is so if X/S is (quasi-)affine or (quasi-)projective.*

Proof. Follows immediately from [EGA$_{\text{II}}$, Cor. 4.5.4] since we can assume that $S = \operatorname{Spec}(A)$ is affine. □

Proposition (A.1.4). *Let X/S be an* AF*-scheme. Then X/S is separated.*

Proof. Let z be a point in the closure of $\Delta_{X/S}(X)$, where $\Delta_{X/S} : X \hookrightarrow X \times_S X$ is the diagonal morphism, and let $x_1, x_2 \in X$ be its two projections. Choose an affine neighborhood U containing x_1 and x_2. Then $\Delta_{U/S} : U \hookrightarrow U \times_S U$ is closed and $\Delta_{U/S}$ is the pull-back of $\Delta_{X/S}$ along the open immersion $U \times_S U \subset X \times_S X$. Taking closure commutes with restricting to open subsets and thus $z \in U \subset X$. This shows that $\Delta_{X/S}(X)$ is closed and hence that X/S is separated. □

The following conjecture was proved by Kleiman [Kle66].

Theorem (A.1.5) (Chevalley's conjecture). *Let X/k be a proper regular algebraic scheme. Then X is projective if and only if X/k is an* AF*-scheme.*

It is however not true that a proper singular scheme always is projective if it is AF. In fact, there are singular, proper but non-projective AF-surfaces [Hor71].

A.2. **A theorem on integral morphisms.**

Definition (A.2.1). We say that a morphism $f : X \to Y$ has *topologically finite fibers* if the underlying topological space of every fiber is a finite set. We say that f has *universally topologically finite fibers* if the base change of f by any morphism $Y' \to Y$ has topologically finite fibers, equivalently the underlying topological space of every fiber is a finite set and each residue field extension has finite separable degree.

The purpose of this section is to prove the following theorem:

Theorem (A.2.2). *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of algebraic spaces. If $g \circ f$ is integral with topologically finite fibers and g is separated then the "schematic" image Y' of f exists and $Y' \to S$ is integral with topologically finite fibers.*

Let us first note that this is easy to proof when g is locally of finite type:

Proposition (A.2.3). *Let X and Y be schemes locally of finite type and separated over the base scheme S. Let $f : X \to Y$ and $g : Y \to S$ be S-morphisms. If $g \circ f$ is finite then the schematic image Y' of f exists and $Y' \to S$ is finite.*

Proof. As $g \circ f$ is separated, f is separated. As $g \circ f$ is quasi-compact and universally closed and g is separated, f is quasi-compact and universally closed. Thus the image Y' exists [EGA$_{\text{I}}$, Prop. 6.10.5] and $X \to Y'$ is surjective. As $g \circ f$ is universally closed and $X \to Y'$ is surjective it follows that $Y' \to S$ is universally closed. Further it is immediately seen that $Y' \to S$ has discrete fibers. Thus $Y' \to S$ is quasi-finite, universally closed and separated. By Deligne's theorem [EGA$_{\text{IV}}$, Cor. 18.12.4] this implies that $Y' \to S$ is finite. □

Remark **(A.2.4).** It is easy to generalize Proposition (A.2.3) to the case where X and Y are *algebraic spaces*. In [Knu71, Thm. 6.15] Deligne's theorem is proven for algebraic spaces under a finite presentation hypothesis. The full version of Deligne's theorem for algebraic spaces is given in [LMB00, Thm. A.2].

Remark **(A.2.5).** Now instead assume as in Theorem (A.2.2) that X and Y are arbitrary schemes and $g \circ f$ is integral with topologically finite fibers. The first part of the proof of Proposition (A.2.3) then shows as before that the schematic image Y' exists and that $Y' \to S$ is separated and universally closed. It is further easily seen that every fiber Y'_s is a discrete finite topological space.

Under the hypothesis that Y/S is an AF-scheme it easily follows that $Y' \to S$ is integral. In fact, then Y'/S is AF and any neighborhood of Y'_s in Y' contains an affine neighborhood of Y'_s. Thus $Y' \to S$ is affine by [EGA$_{\text{IV}}$, Lem. 18.12.7.1] and therefore integral by [EGA$_{\text{IV}}$, Prop. 18.12.8].

In general, note that Y'_s is affine and hence integral over $k(s)$ as a morphism is integral if and only if it is universally closed and affine, cf. [EGA$_{\text{IV}}$, Prop. 18.12.8]. Theorem (A.2.2) thus follows by the following conjecture of Grothendieck (for schemes):

Conjecture (A.2.6) ([EGA$_{\text{IV}}$, Rem. 18.12.9]). *If $X \to S$ is a separated, universally closed morphism of algebraic spaces, such that X_s is integral, then $X \to S$ is integral.*

This conjecture will be proved in [Ryd08a]. In the remainder of this appendix, we will give an independent proof of Theorem (A.2.2) without using Grothendieck's conjecture. We first establish the following preliminary results.

(i) If $X \to Y$ is integral, X is a semi-local scheme and Y is henselian, then X is henselian, cf. Proposition (A.2.7).

(ii) Affineness is descended by (not necessarily quasi-compact) flat morphisms if we a priori know that the morphism in question is quasi-compact and quasi-separated, cf. Proposition (A.2.8).
(iii) A criterion for an algebraic space to be a scheme, cf. Lemma (A.2.12).

Proposition (A.2.7). *If A is semi-local and henselian and B is an integral semi-local A-algebra, then B is henselian. In particular B is a finite direct product of local henselian rings.*

Proof. Follows immediately from [Ray70, Ch. XI, §2, Prop. 2]. $\square$

Proposition (A.2.8). *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes with g faithfully flat. Let $f' : X' \to Y'$ be the base-change of f along g. Then*

(i) *f is a homeomorphism if f is quasi-compact and f' is a homeomorphism.*
(ii) *f is an isomorphism if and only if f is quasi-compact and f' is an isomorphism.*
(iii) *f is affine if and only if f is quasi-compact and quasi-separated and f' is affine.*

Proof. The conditions in (ii) and (iii) are clearly necessary. Assume that f' is a homeomorphism (resp. an isomorphism, resp. affine). Let $Y'' = \coprod_{y\in Y} \operatorname{Spec}(\mathcal{O}_{Y,y})$ and choose for every $y \in Y$ a point $y' \in g^{-1}(y)$. If we let $Y''' = \coprod_{y\in Y} \operatorname{Spec}(\mathcal{O}_{Y',y'})$ then f''' is a homeomorphism (resp. an isomorphism, resp. affine) and we can factor $Y''' \to Y' \to Y$ through the natural faithfully flat and quasi-compact morphism $Y''' \to Y''$. As the statement of the proposition is true when g is quasi-compact by [EGA$_{\text{IV}}$, Prop. 2.6.2 (iv), Prop. 2.7.1 (viii), (xiii)] it follows that f'' is a homeomorphism (resp. an isomorphism, resp. affine). Replacing Y' with Y'' we can thus assume that $Y' = \coprod_{y\in Y} \operatorname{Spec}(\mathcal{O}_{Y,y})$.

(i) In order to show that f is a homeomorphism it is enough to show that f is open since it is clearly bijective. As f is generizing, see [EGA$_{\text{I}}$, Def. 3.9.2], it follows by [EGA$_{\text{I}}$, Thm. 7.3.1] that f is open if and only it is open in the constructible topology [EGA$_{\text{I}}$, 7.2.11]. But as f is quasi-compact and bijective it follows from [EGA$_{\text{I}}$, Prop. 7.2.12 (iv)] that f is a homeomorphism in the constructible topology and in particular open.

(ii) From (i) it follows that f is a homeomorphism and since f' is an isomorphism, we have that f is an isomorphism on the stalks. This shows that f is an isomorphism.

(iii) Taking direct images along quasi-compact and quasi-separated morphisms commutes with flat pull-back by [EGA$_{\text{IV}}$, Lem. 2.3.1]. Thus we have a cartesian diagram:

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec}(f'_*\mathcal{O}_{X'}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[d] \\
X \arrow[r] & \operatorname{Spec}(f_*\mathcal{O}_X) \arrow[r] & Y
\end{tikzcd}$$

Since $f' : X' \to Y'$ is affine we have that $X' \to \mathrm{Spec}(f'_*\mathcal{O}_{X'})$ is an isomorphism and it is enough to show that $X \to \mathrm{Spec}(f_*\mathcal{O}_X)$ is an isomorphism. This follows from (ii). $\square$

Definition (A.2.9). We say that an algebraic space X is *local* if there exist a point $x \in X$ such that every closed subset $Z \subseteq X$ contains x.

Remark **(A.2.10).** If X is a local algebraic space then there is exactly one closed point $x \in X$. If X is a local scheme then X is the spectrum of a local ring and in particular affine.

Lemma (A.2.11). *Let $f : X \to Y$ be a closed surjective morphism of algebraic spaces. Let $y \in Y$ be a closed point such that $f^{-1}(y)$ is discrete and such that for any $x \in f^{-1}(y)$ we can write $X = X'_x \amalg X''$ where X'_x is local and contains x. Then $Y = Y' \amalg Y''$ where Y' is local and contains y. Furthermore $f^{-1}(Y') = \coprod_{x\in f^{-1}(y)} X'_x$.*

Proof. For every $x \in f^{-1}(y)$ let $X'_x \subseteq X$ be a local subspace containing x and choose X'' such that $X = \left(\coprod_{x\in f^{-1}(y)} X'_x\right) \amalg X''$. Let Y' be the subset of Y consisting of every generization of y. As $f(X'')$ is closed and does not contain y, it does not intersect Y'. On the other hand $f(X'_x)$ is contained in Y'. Since f is surjective this shows that $f(X'') = Y \setminus Y'$ and $f(\bigcup X'_x) = Y'$. Thus Y' and $Y'' = Y \setminus Y'$ are both open and closed. $\square$

Lemma (A.2.12). *Let $X = \coprod_{\alpha\in\mathcal{I}} X_\alpha$ and Y be algebraic spaces such that X_α is local with closed point x_α and Y is local with closed point y. Let $f : X \to Y$ be a universally closed schematically dominant morphism such that $f^{-1}(y) = \{x_\alpha : \alpha \in \mathcal{I}\}$. If X_α is a henselian scheme for every $\alpha \in \mathcal{I}$ then Y is affine.*

Proof. There is an étale quasi-compact separated surjective morphism $g : Y' \to Y$ such that Y' is a scheme and such that there is a point $y' \in g^{-1}(y)$ with $k(y') = k(y)$. Let $X' = X \times_Y Y'$ with projections $h : X' \to X$ and $f' : X' \to Y'$. Similarly we let $X'_\alpha = X_\alpha \times_Y Y'$ and we have that $X' = \coprod_{\alpha\in\mathcal{I}} X'_\alpha$. As $k(y') = k(y)$ we have that $f'^{-1}(y') = \{x'_\alpha\}$ such that $x'_\alpha \in X'_\alpha$ and $h(x'_\alpha) = x_\alpha$.

Since X_α is henselian and h is quasi-finite and separated it follows by [EGA$_{\mathrm{IV}}$, Thm. 18.5.11 c)] that $\mathrm{Spec}(\mathcal{O}_{X'_\alpha,x'_\alpha}) \to X_\alpha$ is finite and that $\mathrm{Spec}(\mathcal{O}_{X'_\alpha,x'_\alpha}) \subseteq X'$ is open and closed. Further as X_α is henselian, $k(x'_\alpha) = k(x_\alpha)$ and $X'_\alpha \to X_\alpha$ is étale it follows that $\mathrm{Spec}(\mathcal{O}_{X'_\alpha,x'_\alpha}) \to X_\alpha$ is an isomorphism. By Lemma (A.2.11) we then have a decomposition $Y' = Y'_1 \amalg Y'_2$ where Y'_1 is local and $f'^{-1}(Y'_1) = \coprod_\alpha \mathrm{Spec}(\mathcal{O}_{X'_\alpha,x'_\alpha}) \cong X$. Thus we can, replacing Y' with Y'_1, assume that Y' is a local scheme and $X' \cong X$.

Let $Y'' = Y' \times_Y Y'$, which is a quasi-affine scheme, and $X'' = X \times_Y Y'' = X' \times_X X' \cong X$. Lemma (A.2.11) shows as before that Y'' is local and hence affine. Let $Y' = \mathrm{Spec}(A')$, $Y'' = \mathrm{Spec}(A'')$, $X' = \mathrm{Spec}(B')$ and $X'' = \mathrm{Spec}(B'')$

where $B'' = B'$. As $A' \hookrightarrow A''$ is faithfully flat it follows that A''/A' is a flat A'-algebra. Further $A' \to B'$ is injective since $X \to Y$ is schematically dominant. Thus $A''/A' \hookrightarrow (A''/A') \otimes_{A'} B' = B''/B' = 0$ which shows that $A'' = A'$. This shows that Y is the quotient of the étale equivalence relation $\mathrm{Spec}(A'') \rightrightarrows \mathrm{Spec}(A')$ where the two morphisms are the identity. Thus $Y = \mathrm{Spec}(A')$ is a local *scheme*. $\square$

Proof of Theorem (A.2.2). As $g \circ f$ is separated, f is separated. As $g \circ f$ is quasi-compact and universally closed and g is separated, f is quasi-compact and universally closed. Thus the image Y' exists [$\mathrm{EGA}_{\mathrm{I}}$, Prop. 6.10.5] and [Knu71, Prop. 4.6] and $X \to Y'$ is surjective. As $g \circ f$ is universally closed and $X \to Y'$ is surjective it follows that $Y' \to S$ is universally closed. Further it is obvious that $Y' \to S$ has topologically finite fibers.

Since the question is local over S, we can assume that S is affine. Then X is affine and we will show that $Y' \to S$ is *affine*. It then follows that $Y' \to S$ is integral since $\mathcal{O}_S \to g_* \mathcal{O}_{Y'} \hookrightarrow g_* f_* \mathcal{O}_X$ is integral.

Using Proposition (A.2.8) we are allowed to replace S with the henselization $\mathrm{Spec}({}^{\mathrm{h}}\mathcal{O}_{S,s})$ at an arbitrary point s and thus assume that S is local and henselian. Then by Proposition (A.2.7) X is henselian and a disjoint union of local schemes.

Let $x_1, x_2, \dots, x_n$ be the closed points of X and $X = X_1 \amalg X_2 \amalg \cdots \amalg X_n$ the corresponding partition into local henselian schemes. Then by Lemma (A.2.11) $Y = Y_1 \amalg Y_2 \amalg \cdots \amalg Y_m$ where Y_k is a local space with closed point $y_k \in f(x_j)$ for some j depending on k. Further Lemma (A.2.12) shows that Y_k is a local *scheme* and hence affine. $\square$

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Department of Mathematics, KTH, 100 44 Stockholm, Sweden
E-mail address: `dary@math.kth.se`