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Families of cycles and the Chow scheme

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Abstract

The objects studied in this thesis are *families of cycles* on schemes. A space — the *Chow variety* — parameterizing effective equidimensional cycles was constructed by Chow and van der Waerden in the first half of the twentieth century. Even though cycles are simple objects, the Chow variety is a rather intractable object. In particular, a good functorial description of this space is missing. Consequently, descriptions of the corresponding families and the infinitesimal structure are incomplete. Moreover, the Chow variety is not intrinsic but has the unpleasant property that it depends on a given projective embedding. A main objective of this thesis is to construct a closely related space which has a good functorial description. This is partly accomplished in the last paper.

The first three papers are concerned with families of *zero-cycles*. In the first paper, a functor parameterizing zero-cycles is defined and it is shown that this functor is represented by a scheme — *the scheme of divided powers*. This scheme is closely related to the symmetric product. In fact, the scheme of divided powers and the symmetric product coincide in many situations.

In the second paper, several aspects of the scheme of divided powers are discussed. In particular, a universal family is constructed. A different description of the families as *multi-morphisms* is also given. Finally, the set of k-points of the scheme of divided powers is described. Somewhat surprisingly, cycles with certain rational coefficients are included in this description in positive characteristic.

The third paper explains the relation between the Hilbert scheme, the Chow scheme, the symmetric product and the scheme of divided powers. It is shown that the last three schemes coincide as topological spaces and that all four schemes are isomorphic outside the degeneracy locus.

The last paper gives a definition of families of cycles of arbitrary dimension and a corresponding Chow functor. In characteristic zero, this functor agrees with the functors of Barlet, Guerra, Kollár and Suslin-Voevodsky when these are defined. There is also a monomorphism from Angéniol's functor to the Chow functor which is an isomorphism in many instances. It is also confirmed that the morphism from the Hilbert functor to the Chow functor is an isomorphism over the locus parameterizing normal subschemes and a local immersion over the locus parameterizing reduced subschemes — at least in characteristic zero.

Contents

Included papers

Paper I: **Families of zero-cycles and divided powers: I. Representability**, submitted, arXiv:0803.0618v1.

Paper II: **Families of zero-cycles and divided powers: II. The universal family**.

Paper III: **Hilbert and Chow schemes of points, symmetric products and divided powers**.

Paper IV: **Families of cycles**.

Related papers

A minimal set of generators for the ring of multisymmetric functions, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 6, 1741–1769, arXiv:0710.0470.

Submersions and effective descent of étale morphisms, To appear in *Bull. Sci. Math. France*, arXiv:0710.2488v3.

Existence of quotients by finite groups and coarse moduli spaces, Preprint, Aug 2007, arXiv:0708.3333v1.

Representability of Hilbert schemes and Hilbert stacks of points, Preprint, Feb 2008, arXiv:0802.3807v1.

(with Roy Skjelnes) **The space of generically étale families**, Preprint, Mar 2007, arXiv:math.AG/0703329.

Introduction

Families of cycles

Let X be a scheme. The cycles on X is the free abelian group $C(X)$ generated by the set of reduced and irreducible closed subschemes of X. A cycle on X is thus a formal sum $\sum_i m_i[Z_i]$ where the m_i's are integers and the Z_i's are closed subvarieties of X. A cycle is effective if all the m_i's are positive. We let $C_r(X)$ be the subgroup of cycles which are equidimensional of dimension r. Given a projective embedding $X \hookrightarrow \mathbb{P}^n$, every r-dimensional cycle comes with a *degree*. Geometrically the degree can be interpreted as the number of points, counted with multiplicity, after intersecting the cycle with r general hyperplanes.

Let X be a quasi-projective variety with a given embedding $X \hookrightarrow \mathbb{P}^n$. A fundamental result [CW37] in classical algebraic geometry, due to Chow and van der Waerden, is the existence of a quasi-projective variety $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$ — the *Chow variety* — parameterizing cycles of dimension r and degree d on X. The main goal of this thesis is to obtain a better understanding of this variety.

A family of cycles, parameterized by a variety S, is roughly a collection of cycles $\{\mathcal{Z}_s\}_{s\in S}$ on X which is "continuous". A natural interpretation of continuity is that we require $\{\mathcal{Z}_s\}$ to be induced by a morphism $S \to \mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$. As expected, such a family is then represented by a single cycle $\mathcal{Z}$ on $X \times S$. There are, however, several serious problems with this approach.

- It can be shown that $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$ depends on the chosen projective embedding in positive characteristic. Thus, we do not have a good notion of "continuous" in this case.

- The cycle $\mathcal{Z}$ on $X \times S$ representing the family $\{\mathcal{Z}_s\}$ is not "flat", as the objects usually are in other similar problems. This has several drawbacks, for example, $\mathcal{Z}_s$ is not simply the fiber of $\mathcal{Z}$ over s.

- It is desirable to have a notion of families of cycles also over non-reduced *schemes*. In particular, it is important to have an infinitesimal theory to be able to study deformations of cycles. The classical construction of the Chow variety comes without any infinitesimal structure, that is, it is a variety and not a scheme. It is therefore not at all clear what a family parameterized by a scheme is.

Methods

The first problem is due to a deficiency in the classical construction. After choosing a sufficiently ample projective embedding, the classical construction gives the “correct” Chow variety, at least for zero-cycles. Also, families of cycles can be defined without referring to the Chow variety so this is not a serious problem.

The second problem is mainly technical. It also indicates that representing $\{\mathscr{Z}_s\}$ as a cycle on $X \times S$, although appealing, is not an ideal approach.

The third problem is paramount. A possible solution, closely related to the construction of the Chow variety, is to represent families of cycles as certain families of divisors on a grassmannian parameterizing linear subspaces of complementary dimension. This seems to be somewhat cumbersome and has not been systematically studied. The method introduced by Barlet [Bar75] is of a dual nature. Instead of intersecting with subspaces, he studies *projections* onto affine spaces of the same dimension as the cycle. A family is then an object, in his case a cycle $\mathscr{Z}$ on $X \times S$, which induces a family of zero-cycles over every projection.

When the parameter scheme S is not reduced, then in general there is not such a simple object as a cycle on $X \times S$ which induces a family of zero-cycles over each projection. The method advocated in this thesis is to not require the existence of such an object a priori. Instead, a family α is defined to be a collection of zero-cycles, indexed by all projections, satisfying natural compatibility conditions. Under appropriate conditions on S and the family α, one can then find simpler geometric objects inducing this family. For example, if S is reduced then α is represented by a cycle $\mathscr{Z}$ on $X \times S$ as above. If the cycles of the family α are without multiplicities or are divisors, then there is a subscheme Z of $X \times S$ inducing α. If S is of characteristic zero, then α is represented by its relative fundamental class.

Angéniol [Ang80], working exclusively in characteristic zero, starts from the opposite end and defines a family as a class, the relative fundamental class, and imposes conditions on this class ensuring that it induces a zero-cycle over each projection. It is reasonable to believe that Angéniol’s definition of a family agrees with the definition outlined above but this is not at all clear.

Angéniol’s approach, applying duality and residue theory, requires deeper theory and is arguably more complicated. On the other hand, he is able to give a deformation theory for families of cycles and to show representability. My method, although also technical and sometimes cumbersome, has the advantage of giving a definition in great generality without assuming projectivity, smoothness, characteristic zero etc. It is also more geometric and easier to relate with other definitions, such as those of Kollár [Kol96] and Suslin-Voevodsky [SV00].

Overview of the thesis and results

In Paper I, a functor $\underline{\Gamma}^d_{X/S}$ parameterizing families of zero-cycles on X/S is defined and shown to be represented by an algebraic space $\Gamma^d(X/S)$. This space — the space of divided powers — is closely related to the divided powers algebra and can be viewed as a functorially well-behaved version of the symmetric product $\mathrm{Sym}^d(X/S)$. The algebraicity of $\Gamma^d(X/S)$, for an arbitrary algebraic space X/S, is obtained via an explicit étale covering.

With similar methods, the existence of geometric quotients [Ryd07b] and the algebraicity of Hilbert stacks [Ryd08] can be shown.

In Paper II, several aspects of the space of divided powers are discussed. A universal family of zero-cycles is constructed and a description of the k-points of $\Gamma^d(X/S)$ is given. Also, a different description of the functor $\underline{\Gamma}^d_{X/S}$ in terms of multi-morphisms is given.

In Paper III, the relation between the Hilbert scheme of points, the symmetric product, the space of divided powers and the Chow variety of zero-cycles is studied. It is shown that all four of these schemes coincide over the locus parameterizing non-degenerate families and it is shown that the last three schemes coincide as topological spaces. The dependence of the Chow variety on a given projective embedding is explained using weighted projective spaces. This is related to the fact that the ring of multisymmetric functions is not generated by elementary multisymmetric functions in positive characteristic [Ryd07a]. The morphism from the Hilbert scheme of points to the Chow variety, is essentially a blow-up [Hai98, ES04, RS07, Ran08] and has been used to study the Hilbert scheme of points [Fog68, Göt94].

In Paper IV, families of cycles of arbitrary dimension are defined. This definition generalizes previous definitions by Barlet [Bar75], Guerra [Gue96], Kollár [Kol96] and Suslin-Voevodsky [SV00]. Conjecturally, this definition also coincides with Angéniol's definition [Ang80] in characteristic zero. Indeed, this is so in many cases. The Chow functor, parameterizing families of proper and equidimensional cycles, is representable in similar situations.

There are natural morphisms from the Hilbert scheme [FGA], the Hilbert stack [Art74, App.], the space of Cohen-Macaulay curves [Høn05], the stack of branchvarieties [AK06] and the Kontsevich space of stable maps [Kon95] into the Chow functor. It is shown that all these morphisms are isomorphisms over the subset parameterizing normal subschemes, at least in characteristic zero. It is also shown that the Hilbert-Chow morphism is a local immersion over the subset parameterizing reduced subschemes.

The interdependence between the papers is as follows. Paper II presupposes Paper I and Papers III and IV depend on both Paper I and II. The fourth paper is to a large extent work in progress.

Lawson homology

There is a natural equivalence relation on $C_r(X)$ called *rational equivalence* and the quotient by this relation is the *Chow group* $A_r(X)$. If X is a smooth and projective scheme of dimension n, then $A^\bullet(X) = \bigoplus_{i=0}^n A_{n-i}(X)$ is a graded ring — the *Chow ring* — under the intersection product. The Chow ring is a central object of study in algebraic geometry and an alternative to usual cohomology theories. In fact, for any Weil cohomology $H^\bullet$ on X, such as Betti cohomology, l-adic cohomology or algebraic singular cohomology, there is a ring homomorphism $A^\bullet(X) \to H^{2\bullet}(X)$.

The cycle map $C_r(X) \to A_r(X) \to H_{2r}(X)$ factors through *algebraic equivalence.* Two cycles $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are algebraically equivalent if there exists an effective cycle $\mathcal{W}$ such that $\mathcal{Z}_1 + \mathcal{W}$ and $\mathcal{Z}_2 + \mathcal{W}$ corresponds to two points in the same connected component of the Chow variety $\mathrm{ChowVar}_r(X)$. The quotient of $C_{n-1}(X)$ by algebraic equivalence is the Néron-Severi group of X.

One of the more spectacular applications of Chow varieties is Lawson (co)homology. The Lawson homology groups of X are defined as $L_rH_k(X) = \pi_{k-2r}(\mathrm{Chow}_r(X)^+)$ where

+ is a topological group completion [Law89, Fri89]. In particular, it immediately follows that $L_r H_{2r}(X)$ is the group of r-dimensional cycles up to algebraic equivalence. Dold-Thom's theorem is that the singular homology group $H_k(X, \mathbb{Z})$ is naturally isomorphic to $\pi_k \left(\left(\coprod_d \mathrm{Sym}^d(X) \right)^+ \right) = \pi_k \left(\mathrm{Chow}_0(X)^+ \right)$. Thus $L_0 H_k(X) = H_k(X, \mathbb{Z})$ and Lawson homology interpolates between topological homology groups and algebraic groups.

When studying Lawson homology, operations such as proper push-forward, flat pull-back and proper intersections are used extensively. These are commonly only defined as algebraic maps between Chow varieties, i.e., as continuous maps induced by algebraic correspondences. This is equivalent with giving morphisms on the *semi-normalizations*. For topological purposes, it is enough to define these operations as algebraic maps. Nevertheless, it is expected that these operations exist as morphisms. In Paper IV, it is shown that the push-forward and the pull-back are defined as morphisms under certain assumptions.

An example

To illustrate the difference between the Chow scheme and other parameter spaces we study curves of degree two in $\mathbb{P}^3$. Recall that such a curve, if reduced, is either a conic contained in a plane ($g = 0$) or two skew lines ($g = -1$). The main distinction between the parameter spaces is the various descriptions of curves with multiplicities, that is, double lines in our example. Note that all these parameter schemes are isomorphic over the locus parameterizing smooth curves. The schemes are illustrated with figures where ovals indicate closed subsets and the numbers are the dimensions of the corresponding closed subsets.

The Chow scheme. The Chow scheme parameterizes one-dimensional cycles $\mathscr{Z}$ of degree two on $\mathbb{P}^3$. It is connected and has two irreducible components. One of these parameterizes conics contained in a plane and lines of multiplicity two. The other component parameterizes pairs of skew lines, singular conics and lines of multiplicity two. In characteristic zero, the Chow scheme is *non-reduced* over the locus parameterizing lines of multiplicity two [Ang80, Rem. 6.4.3].

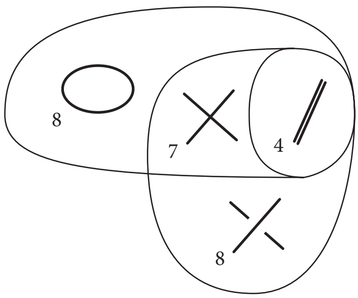

Figure 1: The Chow scheme of degree two curves on $\mathbb{P}^3$.

The Hilbert scheme. The Hilbert scheme parameterizes one-dimensional subschemes of $\mathbb{P}^3$ of degree two. The Hilbert scheme has an infinite number of connected components indexed by the (arithmetic) genus g of the curve and is non-empty for any integer $g \leq 0$. In fact, there are even subschemes of arbitrary negative genus which are Cohen-Macaulay, that is, without components of dimension zero [Har04]. When $g = 0$, the Hilbert scheme is smooth and irreducible and parameterizes conics. When $g = -1$, the Hilbert scheme consists of several irreducible components. The main component, with generic member a pair of skew lines, is generically smooth.

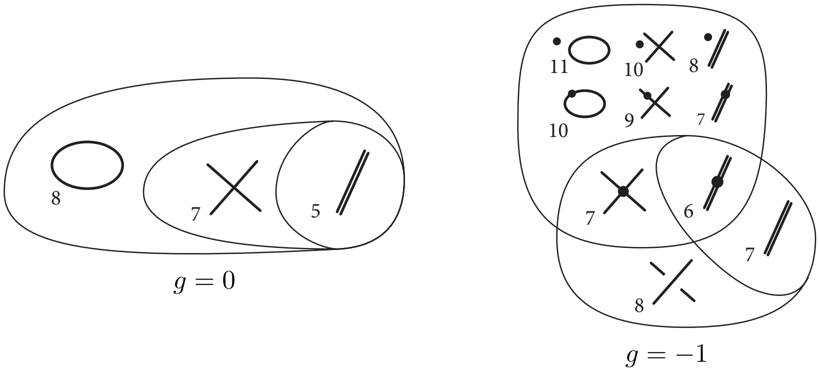

Figure 2: Part of the Hilbert scheme of degree two curves on $\mathbb{P}^3$.

The stack of Branch-varieties. The stack of Branch varieties parameterizes reduced curves C together with a finite morphism to $\mathbb{P}^3$. It has an infinite number of connected components indexed by the genus g of the curve C. It is non-empty for every integer $g \geq -1$. For positive g, it parameterizes reduced, possibly singular, genus g curves C equipped with a ramified degree two covering $C \to \mathbb{P}^1$ of a line in $\mathbb{P}^3$. In particular, C is hyperelliptic.

When $g = 0$ then C is either a smooth rational curve or two secant lines. The map $C \to \mathbb{P}^3$ either embedds C as a conic in a plane or is a ramified cover of degree two over a line in $\mathbb{P}^3$.

When $g = -1$ then C is a pair of skew lines which either sits inside $\mathbb{P}^3$, maps onto two secant lines of $\mathbb{P}^3$ or maps onto a single line of $\mathbb{P}^3$. This component is the stack quotient of a product of grassmannians $[\mathrm{Gr}(2,4)^2/\mathfrak{S}_2]$ and hence smooth.

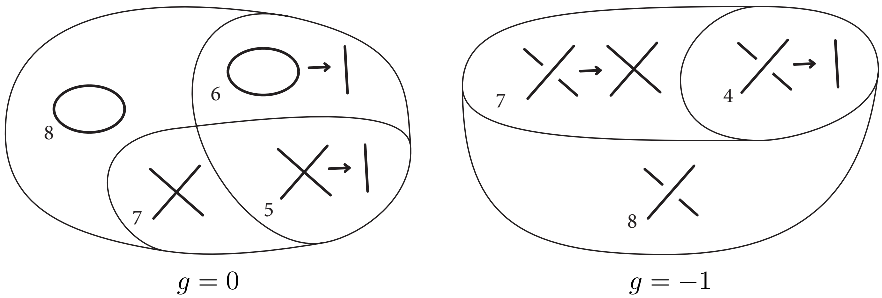

Figure 3: Part of the stack of Branch-varieties of degree two curves mapping to $\mathbb{P}^3$.

Open questions

- An explicit description of the deformation theory of multiplicity-free cycles is yet missing. This should be a far more amenable problem than a description of the deformation theory of general cycles.
- Representability of the Chow functor is not shown except in the cases when the functor is shown to coincide with other descriptions. To show the representability using Artin's criteria, knowledge of the deformation theory is crucial. In the projective case, a projective embedding is conjecturally given by the classical Chow construction.
- Many of the operations on families of cycles, such as push-forward, pull-back and intersections, have only been defined for families satisfying certain properties. It should be possible to define these operations in general.
- Given a sheaf $\mathcal{F}$ on X, there is an induced sheaf on $\mathrm{Chow}_{0,d}(X)$. Are there similar Chow sheaves in higher dimension?
- Even though there is a good functorial description of the Chow-scheme, there are some features similar to that of a coarse functor to a stack. In characteristic zero $\mathrm{Chow}_{0,d}(X) = \mathrm{Sym}^d(X)$ is the coarse moduli space of the symmetric stack. Is there a similar stack in higher dimension? This stack would probably be without automorphisms over the locus parameterizing multiplicity-free cycles.

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