COHERENT TANNAKA DUALITY AND ALGEBRAICITY OF HOM-STACKS

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Abstract. We establish Tannaka duality for noetherian algebraic stacks with affine stabilizer groups. Our main application is the existence of $\underline{\mathrm{Hom}}$-stacks in great generality.

1. Introduction

Classically, Tannaka duality reconstructs a group from its category of finite-dimensional representations [Tan39]. Various incarnations of Tannaka duality have been studied for decades. The focus of this article is a recent formulation for algebraic stacks [Lur04] which we now recall.

Let X be a noetherian algebraic stack. We denote its abelian category of coherent sheaves by $\mathsf{Coh}(X)$. If $f\colon T \to X$ is a morphism of noetherian algebraic stacks, then there is an induced pullback functor

$$f^*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(T).$$

It is well-known that f^* has the following three properties:

(i) f^* sends $\mathcal{O}_X$ to $\mathcal{O}_T$,
(ii) f^* preserves the tensor product of coherent sheaves, and
(iii) f^* is a right exact functor of abelian categories.

Hence, there is a functor

$$\mathrm{Hom}(T, X) \to \mathrm{Hom}_{r\otimes,\simeq}\big(\mathsf{Coh}(X), \mathsf{Coh}(T)\big),$$
$$(f\colon T \to X) \mapsto \big(f^*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(T)\big),$$

where the right hand side denotes the category with objects the functors $F\colon \mathsf{Coh}(X) \to \mathsf{Coh}(T)$ satisfying conditions (i)–(iii) above and morphisms given by natural isomorphisms of functors.

If X has affine diagonal (e.g., X is the quotient of a variety by an affine algebraic group), then the functor above is known [Lur04] to be fully faithful with image consisting of *tame* functors. Even though tameness of a functor is a difficult condition to verify, Lurie was able to establish some striking applications to algebraization problems.

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Various stacks of singular curves [AK16, §4.1] and log stacks can fail to have affine, quasi-affine, or even separated diagonals. In particular, for applications in moduli theory, the results of [Lur04] are insufficient. The main result of this article is the following theorem, which besides removing Lurie's hypothesis of affine diagonal, obviates tameness.

Theorem 1.1. *Let X be a noetherian algebraic stack with affine stabilizers. If T is an algebraic stack that is locally the spectrum of a G-ring (e.g., locally excellent), then the functor:*

$$\mathrm{Hom}(T, X) \to \mathrm{Hom}_{r\otimes,\simeq}\big(\mathsf{Coh}(X), \mathsf{Coh}(T)\big)$$

is an equivalence.

That X has *affine stabilizers* means that $\mathrm{Aut}(x)$ is affine for every field k and point $x\colon \operatorname{Spec} k \to X$; equivalently, the diagonal of X has affine fibers. Examples of algebraic stacks that are locally the spectrum of a G-ring include those that are locally of finite type over a field, over $\mathbb{Z}$, over a complete local noetherian ring, or over an excellent ring (see Remark 7.2). We also wish to emphasize that we do not assume that the diagonal of X is separated in Theorem 1.1. The restriction to stacks with affine stabilizers is a necessary condition for the equivalence in Theorem 1.1 (see Theorem 10.1).

Theorem 1.1 is a consequence of Theorem 8.4, which also gives various refinements in the non-noetherian situation and when X has quasi-affine or quasi-finite diagonal.

Main applications. In work with J. Alper [AHR15], Theorem 1.1 is applied to resolve Alper's conjecture on the local quotient structure of algebraic stacks [Alp10]. A more immediate application of Theorem 1.1 is the following algebraicity result for Hom-stacks, generalizing all previously known results and answering [AOV11, Question 1.4].

Theorem 1.2. *Let $Z \to S$ and $X \to S$ be morphisms of algebraic stacks such that $Z \to S$ is proper and flat of finite presentation, and $X \to S$ is locally of finite presentation, quasi-separated, and has affine stabilizers. Then*

(i) *the stack $\underline{\mathrm{Hom}}_S(Z, X)\colon T \mapsto \mathrm{Hom}_S(Z \times_S T, X)$, is algebraic;*
(ii) *the morphism $\underline{\mathrm{Hom}}_S(Z, X) \to S$ is locally of finite presentation, quasi-separated, and has affine stabilizers; and*
(iii) *if $X \to S$ has affine (resp. quasi-affine, resp. separated) diagonal, then so has $\underline{\mathrm{Hom}}_S(Z, X) \to S$.*

Theorem 1.2 has already seen applications to log geometry [Wis16], an area which provides a continual source of stacks that are neither Deligne–Mumford nor have separated diagonals. In general, the condition that X has affine stabilizers is necessary (see Theorem 10.4). That the Hom-stacks above are quasi-separated is non-trivial, and is established in Appendix B. The main result in Appendix B is a substantial generalization of the strongest boundedness result in the existing literature [Ols06b, Prop. 5.10].

There are analogous algebraicity results for Weil restrictions (that is, restrictions of scalars).

Theorem 1.3. *Let $f\colon Z \to S$ and $g\colon X \to Z$ be morphisms of algebraic stacks such that f is proper and flat of finite presentation and $f \circ g$ is locally of finite presentation, quasi-separated and has affine stabilizers. Then*

(i) *the stack $f_*X = \mathbf{R}_{Z/S}(X)\colon T \mapsto \mathrm{Hom}_Z(Z \times_S T, X)$ is algebraic;*
(ii) *the morphism $\mathbf{R}_{Z/S}(X) \to S$ is locally of finite presentation, quasi-separated and has affine stabilizers; and*
(iii) *if g has affine (resp. quasi-affine, resp. separated) diagonal, then so has $f_*X \to S$.*

When Z has finite diagonal and X has quasi-finite and separated diagonal, Theorems 1.2 and 1.3 were proved in [HR14, Thms. 3 & 4]. In Corollary 9.2, we also excise the finite presentation assumptions on $X \to S$ in Theorems 1.2 and 1.3, generalizing the results of [HR15b, Thm. 2.3 & Cor. 2.4] for stacks with quasi-finite diagonal.

Application to descent. If X has quasi-affine diagonal, then it is well-known that it is a stack for the fpqc topology [LMB, Cor. 10.7]. In general, it is only known that algebraic stacks satisfy effective descent for fppf coverings. Nonetheless, using that QCoh is a stack for the fpqc-topology and Tannaka duality, we are able to establish the following result.

Corollary 1.4. *Let X be a quasi-separated algebraic stack with affine stabilizers. Let $\pi\colon T' \to T$ be an fpqc covering such that T is locally the spectrum of a G-ring and T' is locally noetherian. Then X satisfies effective descent for π.*

Application to completions. Another application concerns completions.

Corollary 1.5. *Let A be a noetherian ring and let $I \subseteq A$ be an ideal. Assume that A is complete with respect to the I-adic topology. Let X be a noetherian algebraic stack and consider the natural morphism*

$$X(A) \to \varprojlim X(A/I^n)$$

of groupoids. This morphism is an equivalence if either

(i) *X has affine stabilizers and A is a G-ring (e.g., excellent); or*
(ii) *X has quasi-affine diagonal; or*
(iii) *X has quasi-finite diagonal.*

Using methods from derived algebraic geometry, Corollary 1.5(ii) was recently proved for non-noetherian complete rings A [BH17, Bha16]. That X has affine stabilizers in Corollary 1.5 is necessary (see Theorem 10.5).

On the proof of Tannaka duality. We will discuss the proof of Theorem 8.4, the refinement of Theorem 1.1. The reason for this is that it is much more convenient from a technical standpoint to consider the problem in the setting of quasi-coherent sheaves on potentially non-noetherian algebraic stacks.

So let T and X be algebraic stacks and let $\mathsf{QCoh}(T)$ and $\mathsf{QCoh}(X)$ denote their respective abelian categories of quasi-coherent sheaves. We will assume that X is quasi-compact and quasi-separated. Our principal concern is the properties of the functor

$$\begin{aligned}\omega_X(T)\colon \operatorname{Hom}(T,X) &\to \operatorname{Hom}_{c\otimes}(\mathsf{QCoh}(X),\mathsf{QCoh}(T)),\\ (f\colon T\to X) &\mapsto (f^*\colon \mathsf{QCoh}(X)\to\mathsf{QCoh}(T)),\end{aligned}$$

where the right hand side denotes the additive functors $F\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ satisfying

(i) $F(\mathcal{O}_X) = \mathcal{O}_T$,
(ii) F preserves the tensor product, and
(iii) F is right exact and preserves (small) direct sums.

We call such F *cocontinuous tensor functors*.

An algebraic stack X has the *resolution property* if every quasi-coherent sheaf is a quotient of a direct sum of vector bundles. In Theorem 4.11 we establish the equivalence of $\omega_X(T)$ when X has affine diagonal and the resolution property. This result has appeared in various forms in the work of others (cf. Schäppi [Sch17, Thm. 1.3.2], Savin [Sav06] and Brandenburg [Bra14, Cor. 5.7.12]) and forms an essential stepping stone in the proof of our main theorem (Theorem 8.4).

In general, there are stacks—even schemes—that do not have the resolution property. Indeed, if X has the resolution property, then X has at least affine diagonal [Tot04, Prop. 1.3]. Our proof uses the following three ideas to overcome this problem:

(i) If $U \subseteq X$ is a quasi-compact open immersion and $\mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ is a tensor functor, then there is an induced tensor functor $\mathsf{QCoh}(U) \to \mathsf{QCoh}(V)$ where $V \subseteq T$ is the "inverse image of U". The proof of this is based on ideas of Brandenburg and Chirvasitu [BC14]. (Section 5)
(ii) If X is an infinitesimal neighborhood of a stack with the resolution property, then $\omega_X(T)$ is an equivalence for all T. (Section 6)
(iii) There is a constructible stratification of X into stacks with affine diagonal and the resolution property (Proposition 8.2). We deduce the main theorem by induction on the number of strata using formal gluings [MB96, HR16]. This step uses special cases of Corollaries 1.4 and 1.5. (Sections 7 and 8)

In the third step, we assume that our functors preserve sheaves of finite type.

Open questions. Concerning (ii), it should be noted that we do not know the answers to the following two questions.

Question 1.6. *If X_0 has the resolution property and $X_0 \hookrightarrow X$ is a nilpotent closed immersion, then does X have the resolution property?*

The question has an affirmative answer if X_0 is cohomologically affine, e.g., $X_0 = B_kG$ where G is a linearly reductive group scheme over k. The question is open if $X_0 = B_kG$ where G is not linearly reductive, even if $X = B_{k[\epsilon]}G_\epsilon$ where G_ϵ is a deformation of G over the dual numbers [Con10].

Question 1.7. *If $X_0 \hookrightarrow X$ is a nilpotent closed immersion and $\omega_{X_0}(T)$ is an equivalence, is then $\omega_X(T)$ an equivalence?*

Step (ii) answers neither of these questions but uses a special case of the first question (Lemma 6.2) and the conclusion (Main Lemma 6.1) is a special case of the second question.

The following technical question also arose in this research.

Question 1.8. *Let X be an algebraic stack with quasi-compact and quasi-separated diagonal and affine stabilizers. Let k be a field. Is every morphism $\operatorname{Spec} k \to X$ affine?*

If X étale-locally has quasi-affine diagonal, then Question 1.8 has an affirmative answer (Lemma 4.7). This makes finding counterexamples extraordinarily difficult and thus very interesting. This question arose because if $\operatorname{Spec} k \to X$ is non-affine, then $\omega_X(\operatorname{Spec} k)$ is not fully faithful (Theorem 10.2). This explains our restriction to natural isomorphisms in Theorem 1.1. Note that every morphism $\operatorname{Spec} k \to X$ as in Question 1.8 is at least quasi-affine [Ryd11a, Thm. B.2]. We do not know the answer to the question even if X has separated diagonal and is of finite type over a field.

On the applications. Let T be a noetherian algebraic stack that is locally the spectrum of a G-ring, and let Z be a closed substack defined by a coherent ideal $J \subseteq \mathcal{O}_T$. Let $Z^{[n]}$ be the closed substack defined by J^{n+1}. Assume that the natural functor $\mathsf{Coh}(T) \to \varprojlim_n \mathsf{Coh}(Z^{[n]})$ is an equivalence of categories. Then an immediate consequence of Tannaka duality (Theorem 1.1) is that

$$\operatorname{Hom}(T, X) \to \varprojlim \operatorname{Hom}(Z^{[n]}, X)$$

is an equivalence of categories for every noetherian algebraic stack X with affine stabilizers. This applies in particular if A is excellent and I-adically complete and $T = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/I$; this gives Corollary 1.5. More generally, it also applies if T is proper over $\operatorname{Spec} A$ and $Z = T \times_{\operatorname{Spec} A} \operatorname{Spec} A/I$ (Grothendieck's existence theorem). This latter case is fed into Artin's criterion to prove Theorem 1.2 (the remaining hypotheses have largely been verified elsewhere).

There are also non-proper stacks T satisfying $\mathsf{Coh}(T) \to \varprojlim_n \mathsf{Coh}(Z^{[n]})$, such as global quotient stacks with proper good moduli spaces (see [GZB15, AB05] for some special cases). This featured in the resolution of Alper's conjecture [AHR15].

Such statements, and their derived versions, were also recently considered by Halpern-Leistner–Preygel [HP14]. There, they considered variants of our Theorem 1.2. For their algebraicity results, their assumption was similar to assuming that $\mathsf{Coh}(T) \to \varprojlim_n \mathsf{Coh}(Z^{[n]})$ was an equivalence (though they also considered other derived versions), and that $X \to S$ was locally of finite presentation with affine diagonal.

Relation to other work. As mentioned in the beginning of the Introduction, Lurie identifies the image of $\omega_X(T)$ with the *tame* functors when X is quasi-compact with affine diagonal [Lur04]. Tameness means that faithful flatness of objects is preserved. This is a very strong assumption that makes it possible to directly pull back a smooth presentation of X to a smooth covering of T and deduce the result by descent. Note that every tensor functor preserves coherent flat objects—these are vector bundles and hence dualizable—but this does not imply that flatness of quasi-coherent objects are preserved. Lurie's methods also work for non-noetherian T.

Brandenburg and Chirvasitu have shown that $\omega_X(T)$ is an equivalence for every quasi-compact and quasi-separated scheme X [BC14], also for non-noetherian T. The key idea of their proof is the tensor localization that we have adapted in Section 5. Using this technique, we give a slightly simplified proof of their theorem in Theorem 5.10.

When X has quasi-affine diagonal, derived variants of Theorem 1.1 have recently been considered by various authors [FI13, Bha16, BH17]. Specifically, they were concerned with symmetric monoidal ∞-functors $G\colon D(X) \to D(T)$ between stable ∞-categories of quasi-coherent sheaves. These functors are assumed to preserve derived tensor products, connective complexes (i.e., are right t-exact) and pseudo-coherent complexes. Hence, such a functor induces a right-exact tensor functor $H^0(G)\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ preserving sheaves of finite type. When T is locally noetherian, our main result (Theorem 8.4) thus implies that of [BH17, Thm. 1.4]. When X has finite stabilizers, the right t-exactness is sometimes automatic [FI13, Bha16, BZ10].

Conversely, given a tensor functor $F\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$, it is not obvious how to derive it to a symmetric monoidal ∞-functor $\mathsf{L}F\colon D(X) \to D(T)$ without additional assumptions. In particular, without additional assumptions, our results cannot be deduced from the derived variants. If, however, we assume the following conditions:

(i) there are enough flat quasi-coherent sheaves on X,
(ii) F takes exact sequences of flat quasi-coherent sheaves to exact sequences,
(iii) $D(X)$ and $D(T)$ are compactly generated, and
(iv) X and T have affine diagonal or are noetherian and affine-pointed,

then $\mathsf{L}F$ exists. Indeed, the first two conditions permit us to derive F in the usual way to a symmetric monoidal ∞-functor $D(\mathsf{QCoh}(X)) \to D(\mathsf{QCoh}(T))$. The last two conditions, combined with [HNR18, Thm. 1.2], establish the equivalences $D(X) \simeq D(\mathsf{QCoh}(X))$ and $D(T) \simeq D(\mathsf{QCoh}(T))$. Condition (i) is known to hold when X has the resolution property or is a scheme with affine diagonal. Condition (ii) is part of the tameness assumption in [Lur04]. Condition (iii) is known to hold if X has quasi-finite and separated diagonal or étale-locally has the resolution property in characteristic 0 [HR17].

We do not address the Tannaka *recognition* problem, i.e., which symmetric monoidal categories arise as the category of quasi-coherent sheaves on an algebraic stack. For gerbes, this has been done in characteristic zero by Deligne [Del90, Thm. 7.1]. For stacks with the resolution property, this has been done by Schäppi [Sch14, Thm. 1.4], [Sch15, Thms. 1.2.2, 5.3.10]. Similar results from the derived perspective have been considered by Wallbridge [Wal12] and Iwanari [Iwa18].

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2. Symmetric monoidal categories

A *symmetric monoidal category* is the data of a category $\mathbf{C}$, a tensor product $\otimes_{\mathbf{C}}\colon \mathbf{C}\times\mathbf{C}\to\mathbf{C}$, and a unit $\mathcal{O}_{\mathbf{C}}$ that together satisfy various naturality, commutativity, and associativity properties [ML98, VII.7]. A symmetric monoidal category $\mathbf{C}$ is *closed* if for any $M\in\mathbf{C}$ the functor $-\otimes_{\mathbf{C}} M\colon \mathbf{C}\to\mathbf{C}$ admits a right adjoint, which we denote as $\mathcal{H}om_{\mathbf{C}}(M,-)$.

Example 2.1. Let A be a commutative ring; then the category of A-modules, $\mathsf{Mod}(A)$, together with its tensor product $\otimes_A$, is a symmetric monoidal category with unit A. In fact, $\mathsf{Mod}(A)$ is even closed: the right adjoint to $-\otimes_A M$ is the A-module $\mathrm{Hom}_A(M,-)$. If A is noetherian, then the subcategory of finite A-modules, $\mathsf{Coh}(A)$, is also a closed symmetric monoidal category.

A functor $F\colon\mathbf{C}\to\mathbf{D}$ between symmetric monoidal categories is *lax symmetric monoidal* if for each M and M' of $\mathbf{C}$ there are natural maps $F(M)\otimes_{\mathbf{D}} F(M')\to F(M\otimes_{\mathbf{C}} M')$ and $\mathcal{O}_{\mathbf{D}}\to F(\mathcal{O}_{\mathbf{C}})$ that are compatible with the symmetric monoidal structure. If these maps are both isomorphisms, then F is *symmetric monoidal*. Note that if $F\colon\mathbf{C}\to\mathbf{D}$ is a symmetric monoidal functor, then a right adjoint $G\colon\mathbf{D}\to\mathbf{C}$ to F is always lax symmetric monoidal.

Example 2.2. Let $\phi\colon A\to B$ be a ring homomorphism. The functor $-\otimes_A B\colon \mathsf{Mod}(A)\to\mathsf{Mod}(B)$ is symmetric monoidal. It admits a right adjoint, $\mathsf{Mod}(B)\to\mathsf{Mod}(A)$, which is given by the forgetful functor. This forgetful functor is lax monoidal, but not monoidal.

If $\mathbf{C}$ is a symmetric monoidal category, then a *commutative* $\mathbf{C}$*-algebra* consists of an object A of $\mathbf{C}$ together with a multiplication $m\colon A\otimes_{\mathbf{C}} A\to A$ and a unit $e_A\colon\mathcal{O}_{\mathbf{C}}\to A$ with the expected properties [ML98, VII.3]. Let $\mathsf{CAlg}(\mathbf{C})$ denote the category of commutative $\mathbf{C}$-algebras. The category $\mathsf{CAlg}(\mathbf{C})$ is naturally endowed with a symmetric monoidal structure that makes the forgetful functor $\mathsf{CAlg}(\mathbf{C})\to\mathbf{C}$ symmetric monoidal.

Example 2.3. If A is a ring, then $\mathsf{CAlg}(\mathsf{Mod}(A))$ is the category of commutative A-algebras.

The following observation will be used frequently: if $F\colon\mathbf{C}\to\mathbf{D}$ is a lax symmetric monoidal functor and A is a commutative $\mathbf{C}$-algebra, then $F(A)$ is a commutative $\mathbf{D}$-algebra.

3. Abelian tensor categories

An *abelian tensor category* is a symmetric monoidal category that is abelian and the tensor product is right exact and preserves finite direct sums in each variable (i.e., preserves all finite colimits in each variable).

Recall that an abelian category is *Grothendieck* if it is closed under small direct sums, filtered colimits are exact, and it has a generator [Stacks, Tag

079A]. Also, recall that a functor $F\colon \mathbf{C} \to \mathbf{D}$ between two Grothendieck abelian categories is *cocontinuous* if it is right-exact and preserves small direct sums, equivalently, it preserves all small colimits.

A *Grothendieck abelian tensor category* is an abelian tensor category such that the underlying abelian category is Grothendieck abelian and the tensor product is cocontinuous in each variable. By the Special Adjoint Functor Theorem [KS06, Prop. 8.3.27(iii)], if $\mathbf{C}$ is a Grothendieck abelian tensor category, then it is also closed.

Example 3.1. Let A be a ring. Then $\mathsf{Mod}(A)$ is a Grothendieck abelian tensor category. If A is noetherian, then $\mathsf{Coh}(A)$ is an abelian tensor category but not Grothendieck abelian—it is not closed under small direct sums.

A *tensor functor* $F\colon \mathbf{C} \to \mathbf{D}$ is an additive symmetric monoidal functor between abelian tensor categories. Let $\mathbf{GTC}$ be the 2-category of Grothendieck abelian tensor categories and cocontinuous tensor functors. By the Special Adjoint Functor Theorem, if $F\colon \mathbf{C} \to \mathbf{D}$ is a cocontinuous tensor functor, then F admits a (lax symmetric monoidal) right adjoint.

Example 3.2. Let T be a ringed site. The category of $\mathcal{O}_T$-modules $\mathsf{Mod}(T)$ is a Grothendieck abelian tensor category with unit $\mathcal{O}_T$ and the internal Hom is the functor $\mathcal{H}om_{\mathcal{O}_T}(M,-)$ [KS06, §§18.1-2].

Example 3.3. Let X be an algebraic stack. The category of quasi-coherent sheaves $\mathsf{QCoh}(X)$ is a Grothendieck abelian tensor category with unit $\mathcal{O}_X$ [Stacks, Tag 0781]. The internal Hom is $\mathrm{QC}(\mathcal{H}om_{\mathcal{O}_X}(M,-))$, where QC denotes the quasi-coherator (the right adjoint to the inclusion of the category of quasi-coherent sheaves in the category of lisse-étale $\mathcal{O}_X$-modules). If X is an algebraic stack, then $\mathsf{CAlg}(\mathsf{QCoh}(X))$ is the symmetric monoidal category of quasi-coherent $\mathcal{O}_X$-algebras.

If $f\colon X \to Y$ is a morphism of algebraic stacks, then the resulting functor $f^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is a cocontinuous tensor functor. If f is flat, then f^* is exact. We always denote the right adjoint of f^* by $f_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$. If f is quasi-compact and quasi-separated, then f_* coincides with the pushforward of lisse-étale $\mathcal{O}_X$-modules [Ols07, Lem. 6.5(i)]. In particular, if f is quasi-compact and quasi-separated, then $f_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ preserves directed colimits (work smooth-locally on Y and then apply [Stacks, Tag 0738]) and is lax symmetric monoidal.

Definition 3.4. Given abelian tensor categories $\mathbf{C}$ and $\mathbf{D}$, we let $\mathrm{Hom}_{c\otimes}(\mathbf{C},\mathbf{D})$ (resp. $\mathrm{Hom}_{r\otimes}(\mathbf{C},\mathbf{D})$) denote the category of cocontinuous (resp. right exact) tensor functors and natural transformations. The transformations are required to be natural with respect to both homomorphisms and the symmetric monoidal structure. We let $\mathrm{Hom}_{c\otimes,\simeq}(\mathbf{C},\mathbf{D})$ (resp. $\mathrm{Hom}_{r\otimes,\simeq}(\mathbf{C},\mathbf{D})$) denote the groupoid of cocontinuous (resp. right exact) tensor functors and natural isomorphisms.

We conclude this section with some useful facts for the paper. We first consider modules over algebras, which are addressed, for example, in Brandenburg's thesis [Bra14, §5.3] in even greater generality.

3.1. **Modules over an algebra in tensor categories.** Let $\mathbf{C}$ be a Grothendieck abelian tensor category and let A be a commutative $\mathbf{C}$-algebra. Define

$\mathsf{Mod}_{\mathbf{C}}(A)$ to be the category of A-modules. Objects are pairs (M, a), where $M \in \mathbf{C}$ and $a\colon A \otimes_{\mathbf{C}} M \to M$ is an action of A on M. Morphisms $\phi\colon (M, a) \to (M', a')$ in $\mathsf{Mod}_{\mathbf{C}}(A)$ are those $\mathbf{C}$-morphisms $\phi\colon M \to M'$ that preserve the respective actions. We identify A with $(A, m) \in \mathsf{Mod}_{\mathbf{C}}(A)$ where $m\colon A \otimes_{\mathbf{C}} A \to A$ is the multiplication. It is straightforward to show that $\mathsf{Mod}_{\mathbf{C}}(A)$ is a Grothendieck abelian tensor category, with tensor product $\otimes_A$ and unit A, and the natural forgetful functor $\mathsf{Mod}_{\mathbf{C}}(A) \to \mathbf{C}$ preserves all limits and colimits [KS06, §4.3].

If $s\colon A \to B$ is a $\mathbf{C}$-algebra homomorphism, then there is a natural cocontinuous tensor functor

$$s^*\colon \mathsf{Mod}_{\mathbf{C}}(A) \to \mathsf{Mod}_{\mathbf{C}}(B), \quad (M, a) \mapsto (B \otimes_A M, B \otimes_A a).$$

Suppose $f^*\colon \mathbf{C} \to \mathbf{D}$ is a cocontinuous tensor functor with right adjoint $f_*\colon \mathbf{D} \to \mathbf{C}$. If A is a commutative $\mathbf{C}$-algebra, then there is a natural induced cocontinuous tensor functor

$$f^*_A\colon \mathsf{Mod}_{\mathbf{C}}(A) \to \mathsf{Mod}_{\mathbf{D}}(f^*A), \quad (M, a) \mapsto (f^*M, f^*a).$$

Noting that $\epsilon\colon f^*f_*\mathcal{O}_{\mathbf{D}} \to \mathcal{O}_{\mathbf{D}}$ is a $\mathbf{D}$-algebra homomorphism, there is a natural induced cocontinuous tensor functor

$$\bar{f}^*\colon \mathsf{Mod}_{\mathbf{C}}(f_*\mathcal{O}_{\mathbf{D}}) \xrightarrow{f^*_{f_*\mathcal{O}_{\mathbf{D}}}} \mathsf{Mod}_{\mathbf{D}}(f^*f_*\mathcal{O}_{\mathbf{D}}) \xrightarrow{\epsilon^*} \mathsf{Mod}_{\mathbf{D}}(\mathcal{O}_{\mathbf{D}}) = \mathbf{D}.$$

Moreover, if we let $\eta\colon \mathcal{O}_{\mathbf{C}} \to f_*f^*\mathcal{O}_{\mathbf{C}} = f_*\mathcal{O}_{\mathbf{D}}$ denote the unit, then $f^* = \bar{f}^*\eta^*$. We have the following striking characterization of module categories by Brandenburg.

Proposition 3.5 ([Bra14, Prop. 5.3.1])**.** *Let $\mathbf{C}$ be a Grothendieck abelian tensor category and let A be a commutative algebra in $\mathbf{C}$. Then for every Grothendieck abelian tensor category $\mathbf{D}$, there is an equivalence of categories*

$$\begin{aligned}\mathrm{Hom}_{c\otimes}(\mathsf{Mod}_{\mathbf{C}}(A), \mathbf{D}) \simeq \{(F, h)\colon\ & F \in \mathrm{Hom}_{c\otimes}(\mathbf{C}, \mathbf{D}),\\ & h \in \mathrm{Hom}_{\mathsf{CAlg}(\mathbf{D})}(F(A), \mathcal{O}_{\mathbf{D}})\},\end{aligned}$$

where a morphism $(F, h) \to (F', h')$ is a natural transformation $\alpha\colon F \to F'$ such that $h = h' \circ \alpha(A)$.

The following corollary is immediate (see [Bra14, Cor. 5.3.7]).

Corollary 3.6. *Let $p\colon Y' \to Y$ be an affine morphism of algebraic stacks. Let X be an algebraic stack and let $g^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ be a cocontinuous tensor functor. If X' is the affine X-scheme $\mathcal{S}\mathrm{pec}_X(g^*p_*\mathcal{O}_{Y'})$ with structure morphism $p'\colon X' \to X$, then there is a 2-cocartesian diagram in* **GTC***:*

$$\begin{array}{ccc} \mathsf{QCoh}(X') & \xleftarrow{g'^*} & \mathsf{QCoh}(Y') \\ {\scriptstyle p'^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ \mathsf{QCoh}(X) & \xleftarrow{g^*} & \mathsf{QCoh}(Y). \end{array}$$

Moreover, the natural transformation $g^*p_* \Rightarrow p'_*g'^*$ is an isomorphism.

Note that if g^* comes from a morphism $g\colon X \to Y$, then $X' \cong X \times_Y Y'$.

3.2. **Inverse limits of abelian tensor categories.** We will now briefly discuss some inverse limits that will be crucial when we apply Tannaka duality to establish the algebraicity of Hom-stacks in Theorem 1.2. The following notation will be useful.

Notation 3.7. Let $i\colon Z \to X$ be a closed immersion of algebraic stacks defined by a quasi-coherent ideal I. For each integer $n \geq 0$, we let $i^{[n]}\colon Z^{[n]} \to X$ denote the closed immersion defined by the quasi-coherent ideal I^{n+1}, which we call the *nth infinitesimal neighborhood of Z.*

Let X be a noetherian algebraic stack and let $i\colon Z \to X$ be a closed immersion. Let $\mathsf{Coh}(X, Z)$ denote the category $\varprojlim_n \mathsf{Coh}(Z^{[n]})$. The arguments of [Stacks, Tag 087X] easily extend to establish the following:

(i) $\mathsf{Coh}(X, Z)$ is an abelian tensor category with:
  (a) unit: $\{\mathcal{O}_{Z^{[n]}}\}$,
  (b) tensor product: $\{M_n\}_{n\geq 0} \otimes \{N_n\}_{n\geq 0} = \{M_n \otimes_{\mathcal{O}_{Z^{[n]}}} N_n\}_{n\geq 0}$,
  (c) addition: $\{f_n\colon M_n \to N_n\}_{n\geq 0} + \{g_n\colon M_n \to N_n\}_{n\geq 0} = \{f_n + g_n\}_{n\geq 0}$, and
  (d) cokernels: $\operatorname{coker}(\{f_n\colon M_n \to N_n\}_{n\geq 0}) = \{\operatorname{coker} f_n\}_{n\geq 0}$.

(ii) if U is a noetherian algebraic stack and $p\colon U \to X$ is a flat morphism, then the restriction $\mathsf{Coh}(X, Z) \to \mathsf{Coh}(U, U \times_X Z)$ is an exact tensor functor; and

(iii) exactness in $\mathsf{Coh}(X, Z)$ may be checked on a flat, noetherian covering of X.

Computing $\ker(\{f_n\colon M_n \to N_n\}_{n\geq 0})$ is more involved without additional flatness assumptions. The problem is that in general the system of kernels $\{\ker f_n\}_{n\geq 0}$ is not an adic system; that is, the morphism $(\ker f_{n+1}) \otimes_{\mathcal{O}_{Z^{[n+1]}}} \mathcal{O}_{Z^{[n]}} \to \ker(f_n)$ need not be an isomorphism. As shown in the proof of [Stacks, Tag 087X], $\ker\{f_n\}_{n\geq 0}$ ends up being the stable value of the $\ker f_n$ (in the sense of the Artin–Rees Lemma).

The abelian category $\mathsf{Coh}(X, Z)$ is also the limit of $\mathsf{Coh}(Z^{[n]})$ as an abelian tensor category. This is made precise by the following lemma.

Lemma 3.8. $\mathsf{Coh}(X, Z)$ *is the limit of the inverse system of categories* $\{\mathsf{Coh}(Z^{[n]})\}_{n\geq 0}$ *in the* 2*-category of abelian tensor categories with right exact tensor functors and natural isomorphisms of tensor functors.*

Proof. It remains to verify that for every abelian tensor category $\mathbf{C}$, a functor $F\colon \mathbf{C} \to \mathsf{Coh}(X, Z)$ is a right exact tensor functor if and only if the induced functors $q_n \circ F\colon \mathbf{C} \to \mathsf{Coh}(X, Z) \to \mathsf{Coh}(X, Z^{[n]})$ are right exact tensor functors. This follows from the description of the abelian tensor structure of $\mathsf{Coh}(X, Z)$ in (i)(a)–(i)(d) above. □

4. Tensorial algebraic stacks

Let T and X be algebraic stacks. There is an induced functor

$$\omega_X(T)\colon \operatorname{Hom}(T, X) \to \operatorname{Hom}_{c\otimes}\big(\mathsf{QCoh}(X), \mathsf{QCoh}(T)\big)$$

that takes a morphism f to f^*. We also let $\operatorname{Hom}^{\mathrm{ft}}_{c\otimes}\big(\mathsf{QCoh}(X), \mathsf{QCoh}(T)\big)$ denote the full subcategory of functors that preserve sheaves of finite type. Similarly, we let $\operatorname{Hom}_{c\otimes,\simeq}\big(\mathsf{QCoh}(X), \mathsf{QCoh}(T)\big)$ denote the subcategory of natural isomorphisms of functors. Clearly, $\omega_X(T)$ factors through all of these subcategories and we let $\omega_{X,\simeq}(T)$, $\omega_X^{\mathrm{ft}}(T)$ and $\omega_{X,\simeq}^{\mathrm{ft}}(T)$ denote the respective factorizations. Note that when X and T are locally noetherian, the natural functor:

$$\mathrm{Hom}_{r\otimes}\big(\mathsf{Coh}(X), \mathsf{Coh}(T)\big) \to \mathrm{Hom}_{c\otimes}^{\mathrm{ft}}\big(\mathsf{QCoh}(X), \mathsf{QCoh}(T)\big)$$

is an equivalence of categories. Thus, Theorem 1.1 says that $\omega_{X,\simeq}^{\mathrm{ft}}(T)$ is an equivalence.

Since $\mathsf{QCoh}(-)$ is a stack in the fpqc topology, the target categories of the functors ω_X, $\omega_{X,\simeq}$, ω_X^{ft} and $\omega_{X,\simeq}^{\mathrm{ft}}$ are stacks in the fpqc topology when varying T—for an elaborate proof of this, see [LT12, Thm. 1.1]. The source categories $\mathrm{Hom}(T, X)$ are groupoids and, when varying T, form a stack for the fppf topology in general and for the fpqc topology when X has quasi-affine diagonal [LMB, Cor. 10.7].

Definition 4.1. Let T and X be algebraic stacks. We say that a tensor functor $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ is *algebraic* if it arises from a morphism of algebraic stacks $f\colon T \to X$. If $f, g\colon T \to X$ are morphisms, then a natural transformation $\tau\colon f^* \Rightarrow g^*$ of tensor functors is *realizable* if it is induced by a 2-morphism $f \Rightarrow g$. We say that X is *tensorial* if $\omega_X(T)$ is an equivalence for every algebraic stack T, or equivalently, for every affine scheme T [Bra14, Def. 3.4.4].

We begin with a descent lemma.

Lemma 4.2. *Let X be an algebraic stack. Let $p\colon T' \to T$ be a morphism of algebraic stacks that is covering for the fpqc topology. Let $T'' = T' \times_T T'$ and $T''' = T' \times_T T' \times_T T'$. Assume that p is a morphism of effective descent for X (e.g., p is flat and locally of finite presentation).*

- (i) *Let $f_1, f_2\colon T \to X$ be morphisms and let $\tau, \tau'\colon f_1 \Rightarrow f_2$ be 2-morphisms. If $p^*\tau = p^*\tau'\colon f_1 \circ p \Rightarrow f_2 \circ p$ then $\tau = \tau'$.*
- (ii) *Let $f_1, f_2\colon T \to X$ be morphisms and let $\gamma\colon f_1^* \Rightarrow f_2^*$ be a natural transformation. If $p^*\gamma\colon p^*f_1^* \Rightarrow p^*f_2^*$ is realizable and $\omega_X(T'')$ is faithful, then γ is realizable.*
- (iii) *Let $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor. If p^*f^* is algebraic, $\omega_{X,\simeq}(T'')$ is fully faithful and $\omega_X(T''')$ is faithful, then f^* is algebraic.*

Proof. It is sufficient to observe that $\mathrm{Hom}(-, X)$ is a stack in groupoids for the covering p and $\mathrm{Hom}_{c\otimes}(\mathsf{QCoh}(X), \mathsf{QCoh}(-))$ is an fpqc stack in categories, so the result boils down to a straightforward and general result for a 1-morphism of such stacks. □

Lemma 4.3. *Let $\mathbf{C} \subset \mathsf{AlgSt}$ be a full 2-subcategory of algebraic stacks, such that if $p\colon T' \to T$ is representable and smooth with $T \in \mathbf{C}$, then $T' \in \mathbf{C}$. For example, $\mathbf{C}$ could be the 2-category of locally noetherian algebraic stacks or the 2-category of algebraic stacks that are locally the spectra of G-rings. Let $\omega \in \{\omega_X, \omega_{X,\simeq}, \omega_X^{\mathrm{ft}}, \omega_{X,\simeq}^{\mathrm{ft}}\}$. If $\omega(T)$ is faithful (resp. fully faithful, resp. an equivalence) for every affine scheme T in $\mathbf{C}$, then $\omega(T)$ is faithful (resp. fully faithful, resp. an equivalence) for every algebraic stack T in $\mathbf{C}$.*

Proof. Note that ω is faithful if and only if ω_X is faithful and that if ω is fully faithful, then so is $\omega_{X,\simeq}$. The conclusion of the lemma holds for disjoint unions of affine schemes in $\mathbf{C}$ since $\mathsf{QCoh}(\coprod_i T_i) = \prod_i \mathsf{QCoh}(T_i)$. We then deduce the conclusion for every stack T in $\mathbf{C}$ with affine diagonal. This follows from Lemma 4.2 applied to a presentation $T' \to T$ where T' is a disjoint union of affine schemes since T'' and T''' then also are disjoint unions of affine schemes. We may then similarly deduce the conclusion for every stack T in $\mathbf{C}$ with affine double diagonal since T'' and T''' then have affine diagonals. Finally we deduce the conclusion for every stack T in $\mathbf{C}$ since the triple diagonal of T is an isomorphism. □

We next recall two basic lemmas on tensorial stacks. The first is the combination of [Bra14, Cor. 5.3.4 & 5.6.4].

Lemma 4.4. *Let $q\colon X' \to X$ be a quasi-affine morphism of algebraic stacks. If T is an algebraic stack and $\omega_X(T)$ is faithful, fully faithful or an equivalence; then so is $\omega_{X'}(T)$. In particular, if X is tensorial, then so is X'.*

Proof. Since q is the composition of a quasi-compact open immersion followed by an affine morphism, it suffices to treat these two cases separately. When q is affine the result is an easy consequence of Proposition 3.5. If q is a quasi-compact open immersion, then the counit $q^*q_* \to \mathrm{id}_{\mathsf{QCoh}(X')}$ is an isomorphism; the result now follows from [BC14, Prop. 2.3.6]. □

The second lemma is well-known (e.g., it is a very special case of [BC14, Thm. 3.4.2]).

Lemma 4.5. *Every quasi-affine scheme is tensorial.*

Proof. By Lemma 4.4, it is sufficient to prove that $X = \operatorname{Spec} \mathbb{Z}$ is tensorial, which is well-known. We refer the interested reader to [BC14, Cor. 2.2.4] or [Bra14, Cor. 5.2.3]. □

Lacking an answer to Question 1.8 in general, we are forced to make the following definition to treat natural transformations that are not isomorphisms.

Definition 4.6. An algebraic stack X is *affine-pointed* if every morphism $\operatorname{Spec} k \to X$, where k is a field, is affine.

Note that if X is affine-pointed, then it has affine stabilizers. The following lemma shows that many algebraic stacks with affine stabilizers that are encountered in practice are affine-pointed.

Lemma 4.7. *Let X be an algebraic stack.*

(i) *If X has quasi-affine diagonal, then X is affine-pointed.*
(ii) *Let $g\colon V \to X$ be a quasi-finite and faithfully flat morphism of finite presentation (not necessarily representable). If V is affine-pointed, then X is affine-pointed.*

Proof. Throughout, we fix a field k and a morphism $x\colon \operatorname{Spec} k \to X$.

For (i), since k is a field, every extension in $\mathsf{QCoh}(\operatorname{Spec} k)$ is split; thus x_* is cohomologically affine [Alp13, Def. 3.1]. Since X has quasi-affine diagonal, this property is preserved after pulling back x along a smooth morphism

$p\colon U \to X$, where U is an affine scheme [Alp13, Prop. 3.10(vii)]. By Serre's Criterion [EGA, II.5.2.2], the morphism $\operatorname{Spec} k \times_X U \to U$ is affine; and this case follows.

For (ii), the pullback of g along x gives a quasi-finite and faithfully flat morphism $g_0\colon V_0 \to \operatorname{Spec} k$. Since V_0 is discrete with finite stabilizers, there exists a finite surjective morphism $W_0 \to V_0$ where W_0 is a finite disjoint union of spectra of fields. By assumption $W_0 \to V_0 \to V$ is affine; hence so is $V_0 \to V$ (by Chevalley's Theorem [Ryd15, Thm. 8.1] applied smooth-locally on V). By descent, $\operatorname{Spec} k \to X$ is affine and the result follows. □

The following lemma highlights the benefits of affine-pointed stacks.

Lemma 4.8. *Let f_1, $f_2\colon T \to X$ be morphisms of algebraic stacks and let $\gamma\colon f_1^* \Rightarrow f_2^*$ be a natural transformation of cocontinuous tensor functors. If X is affine-pointed, then the induced maps of topological spaces $|f_1|$, $|f_2|\colon |T| \to |X|$ coincide.*

Proof. It suffices to prove that if $T = \operatorname{Spec} k$, where k is a field, then γ is realizable. Since X is affine-pointed, the morphisms f_1 and f_2 are affine. Also, the natural transformation γ induces, by adjunction, a morphism of quasi-coherent $\mathcal{O}_X$-algebras $\gamma^\vee(\mathcal{O}_T)\colon (f_2)_*\mathcal{O}_T \to (f_1)_*\mathcal{O}_T$. In particular, $\gamma^\vee(\mathcal{O}_T)$ induces a morphism $v\colon T \to T$ over X. We are now free to replace X by T, f_2 by id_T, and f_1 by v. Since T is affine, the result now follows from Lemma 4.5. □

We can now prove the following proposition (generalizing Lurie's corresponding result for an algebraic stack with affine diagonal).

Proposition 4.9. *Let X be an algebraic stack.*

(i) *If T is an algebraic stack and X has quasi-affine diagonal, then the functor $\omega_X(T)$ is fully faithful.*
(ii) *Let T be a quasi-affine scheme and let f_1, $f_2\colon T \to X$ be quasi-affine morphisms.*
    (a) *If α, $\beta\colon f_1 \Rightarrow f_2$ are 2-morphisms and $\alpha^* = \beta^*$ as natural transformations $f_1^* \Rightarrow f_2^*$, then $\alpha = \beta$.*
    (b) *Let $\gamma\colon f_1^* \Rightarrow f_2^*$ be a natural transformation of cocontinuous tensor functors. If γ is an isomorphism or X is affine-pointed, then γ is realizable.*

Proof. For (i), we may assume that T is an affine scheme (Lemma 4.3). Then every morphism $T \to X$ is quasi-affine and the result follows by (ii) and Lemma 4.7(i).

For (ii), there are quasi-compact open immersions $i_k\colon T \hookrightarrow V_k$ over X, where $V_k := \mathcal{S}\mathrm{pec}_X((f_k)_*\mathcal{O}_T)$ and $k = 1, 2$. Let $v_k\colon V_k \to X$ be the induced 1-morphism.

We first treat (ii)(a). The hypotheses imply that $\alpha_* = \beta_*$ as natural isomorphisms of functors from $(f_2)_*$ to $(f_1)_*$. In particular, α_* and β_* induce the same 1-morphism from V_1 to V_2 over X. Since i_1 and i_2 are open immersions, they are monomorphisms; hence $\alpha = \beta$.

We now treat (ii)(b). The natural transformation $\gamma\colon f_1^* \Rightarrow f_2^*$ uniquely induces a natural transformation of lax symmetric monoidal functors $\gamma^\vee\colon (f_2)_* \Rightarrow$

$(f_1)_*$. In particular, there is an induced morphism of quasi-coherent $\mathcal{O}_X$-algebras $\gamma^\vee(\mathcal{O}_T)\colon (f_2)_*\mathcal{O}_T \to (f_1)_*\mathcal{O}_T$; hence a morphism of algebraic stacks $g\colon V_1 \to V_2$ over X. Note that $\gamma^\vee$ uniquely induces a natural transformation of lax symmetric monoidal functors $(i_2)_* \Rightarrow g_*(i_1)_*$, and by adjunction we have a uniquely induced natural transformation of tensor functors $\gamma'\colon (g \circ i_1)^* \Rightarrow i_2^*$.

Replacing X by V_2, f_1 by $g \circ i_1$, f_2 by i_2, and γ by γ', we may assume that f_2 is a quasi-compact open immersion such that $\mathcal{O}_X \to (f_2)_*\mathcal{O}_T$ is an isomorphism.

If γ is an isomorphism, then f_1 is also a quasi-compact open immersion. Let Z_1 and Z_2 denote closed substacks of X whose complements are $f_1(T)$ and $f_2(T)$, respectively. Then $f_2^*\mathcal{O}_{Z_2} \cong 0$; indeed, by definition we have $Z_2 \cap f_2(T) = \emptyset$. In particular, the isomorphism $\gamma(\mathcal{O}_Z)$ implies that $f_1^*\mathcal{O}_{Z_2} \cong f_2^*\mathcal{O}_{Z_2} \cong 0$; hence, $f_1(T) \subseteq f_2(T)$. Arguing similarly, we obtain the reverse inclusion and we see that $f_1(T) = f_2(T)$. Since f_1 and f_2 are open immersions, we obtain the result when γ is assumed to be an isomorphism.

Otherwise, Lemma 4.8 implies that f_1 factors through $f_2(T) \subseteq X$. We may now replace X by T and γ with $(f_2)_*(\gamma)$ [BC14, Prop. 2.3.6]. Then X is quasi-affine and the result follows from Lemma 4.5. □

From Proposition 4.9(ii)(b), we obtain an analogue of Lemma 4.8 for natural isomorphisms of functors when X has affine stabilizers (as opposed to affine-pointed).

Corollary 4.10. *Let f_1, $f_2\colon T \to X$ be morphisms of algebraic stacks and let $\gamma\colon f_1^* \simeq f_2^*$ be a natural isomorphism of cocontinuous tensor functors. If X has affine stabilizers and quasi-compact diagonal, then the induced maps of topological spaces $|f_1|$, $|f_2|\colon |T| \to |X|$ coincide.*

Proof. It suffices to prove the result when $T = \operatorname{Spec} k$, where k is a field. Since X has affine stabilizers and quasi-compact diagonal the morphisms f_1 and f_2 are quasi-affine [Ryd11a, Thm. B.2]. The result now follows from Proposition 4.9(ii)(b). □

The following result, in a slightly different context, was proved by Schäppi [Sch17, Thm. 1.3.2]. Using the Totaro–Gross theorem, we can simplify Schäppi's arguments in the algebro-geometric setting.

Theorem 4.11. *Let X be a quasi-compact and quasi-separated algebraic stack with affine stabilizers. If X has the resolution property, then it is tensorial.*

Proof. By Totaro–Gross [Gro17], there is a quasi-affine morphism $g\colon X \to \mathrm{BGL}_{N,\mathbb{Z}}$. By Lemma 4.4, it is enough to prove that $X = \mathrm{BGL}_{N,\mathbb{Z}}$ is tensorial.

We must prove that $\omega_X(T)$ is an equivalence for every algebraic stack T. Since X is quasi-compact with affine diagonal, the functor $\omega_X(T)$ is fully faithful for every T (Proposition 4.9). Thus, it remains to prove that for every algebraic stack T, every cocontinuous tensor functor $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ is algebraic. To this end, we note the following. Let Y be an algebraic stack. Then

(i) the dualizable objects in $\mathsf{QCoh}(Y)$ are the vector bundles [Bra14, Prop. 4.7.5]; and
(ii) every tensor functor $g^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(T)$ preserves dualizable objects and exact sequences of dualizable objects [Bra14, Def. 4.7.1 & Lem. 4.7.10].

Now let $p\colon \operatorname{Spec}\mathbb{Z} \to \mathrm{BGL}_{N,\mathbb{Z}}$ be the universal GL_N-bundle and let $\mathcal{A} = p_*\mathbb{Z}$ be the regular representation. There is an exact sequence

$$0 \to \mathcal{O}_{\mathrm{BGL}_{N,\mathbb{Z}}} \to \mathcal{A} \to \mathcal{Q} \to 0$$

of flat quasi-coherent sheaves. Write $\mathcal{A}$ as the directed colimit of its subsheaves $\mathcal{A}_\lambda$ of finite type containing the unit and let $\mathcal{Q}_\lambda = \mathcal{A}_\lambda/\mathcal{O}_{\mathrm{BGL}_{N,\mathbb{Z}}} \subseteq \mathcal{Q}$. Then $\mathcal{A}_\lambda$ and $\mathcal{Q}_\lambda$ are vector bundles.

Thus, let $f^*\colon \mathsf{QCoh}(\mathrm{BGL}_{N,\mathbb{Z}}) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor. Then by (i) and (ii) above there are exact sequences of vector bundles:

$$0 \to \mathcal{O}_T \to f^*\mathcal{A}_\lambda \to f^*\mathcal{Q}_\lambda \to 0.$$

Since f^* is cocontinuous, we also obtain an exact sequence

$$0 \to \mathcal{O}_T \to f^*\mathcal{A} \to f^*\mathcal{Q} \to 0$$

of flat quasi-coherent sheaves. In particular, $f^*\mathcal{A}$ is a faithfully flat algebra.

Let $V = \mathcal{S}\mathrm{pec}_T(f^*\mathcal{A})$; then $r\colon V \to T$ is faithfully flat. By Corollary 3.6, we have a cocartesian diagram

$$\begin{array}{ccc} \mathsf{QCoh}(V) & \xleftarrow{f'^*} & \mathsf{QCoh}(\operatorname{Spec}\mathbb{Z}) \\ {\scriptstyle r^*}\big\uparrow & & \big\uparrow{\scriptstyle p^*} \\ \mathsf{QCoh}(T) & \xleftarrow{f^*} & \mathsf{QCoh}(X). \end{array}$$

Since $\operatorname{Spec}\mathbb{Z}$ is tensorial (Lemma 4.5), the functor f'^* is algebraic. Thus, $f'^*p^* \simeq r^*f^*$ is algebraic. Descent along $r\colon V \to T$ (Lemma 4.2(iii)) implies that f^* is algebraic. □

5. Tensor localizations

The goal of this section is to prove the following theorem.

Theorem 5.1. *Let X be a quasi-compact and quasi-separated algebraic stack. Let $i\colon Z \to X$ be a finitely presented closed immersion defined by an ideal sheaf I. Let $j\colon U \to X$ be the open complement of Z. Let T be an algebraic stack and let $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor. Let $i_T\colon Z_T \to T$ be the closed immersion defined by the ideal $I_T := \operatorname{Im}(f^*I \to \mathcal{O}_T)$. Let $j_T\colon U_T \to T$ denote the complement of Z_T.*

(i) *There exists an essentially unique cocontinuous tensor functor*

$$f_U^*\colon \mathsf{QCoh}(U) \to \mathsf{QCoh}(U_T),$$

such that there is an isomorphism of tensor functors $j_T^*f^* \simeq f_U^*j^*$.

(ii) *For each integer $n \geq 0$,*

$$f^*_{Z^{[n]}} := (i_T^{[n]})^* f^* (i^{[n]})_*\colon \mathsf{QCoh}(Z^{[n]}) \to \mathsf{QCoh}(Z_T^{[n]})$$

is a cocontinuous tensor functor and there is a canonical isomorphism of tensor functors $(i_T^{[n]})^* f^* \simeq f^*_{Z^{[n]}} (i^{[n]})^*$. *Moreover,* $f^*(i^{[n]})_* \simeq (i_T^{[n]})_* f^*_{Z^{[n]}}$.

In addition, if f^* *preserves sheaves of finite type, then the same is true of* f^*_U *and* $f^*_{Z^{[n]}}$ *for all* $n \geq 0$.

Theorem 5.1 features in a key way in the proof of our main theorem (Theorem 8.4), which we prove via stratifications and formal gluings. From this context, we hope that the long and technical statement of Theorem 5.1 should appear to be quite natural. While Theorem 5.1(ii) follows easily from the results of §3.1, Theorem 5.1(i) is more subtle. It turns out, however, that it is a consequence of a more general result about Grothendieck abelian tensor categories (Theorem 5.8), which is what we will spend most of this section proving.

Let $\mathbf{C}$ be a Grothendieck abelian category. A *Serre subcategory* is a full non-empty subcategory $\mathbf{K} \subseteq \mathbf{C}$ closed under taking subquotients and extensions. Serre subcategories are abelian and the inclusion functor is exact. A Serre subcategory is *localizing* if it is also closed under small direct sums in $\mathbf{C}$, equivalently, it is closed under small colimits in $\mathbf{C}$.

If $\mathbf{K} \subseteq \mathbf{C}$ is a Serre subcategory, then there is a quotient $\mathbf{Q}$ of $\mathbf{C}$ by $\mathbf{K}$ and an exact functor $q^* \colon \mathbf{C} \to \mathbf{Q}$, which is universal for exact functors out of $\mathbf{C}$ that vanish on $\mathbf{K}$ [Gab62, Ch. III]. Note that $\mathbf{K}$ is localizing if and only if the quotient $q^* \colon \mathbf{C} \to \mathbf{Q}$ is a *localization*, that is, q^* admits a right adjoint $q_* \colon \mathbf{Q} \to \mathbf{C}$; it follows that $\mathbf{Q}$ is Grothendieck abelian, q^* is cocontinuous, q_* is fully faithful and $q^* q_* \simeq \mathrm{id}_{\mathbf{Q}}$. This statement follows by combining the Gabriel–Popescu Theorem (e.g., [BD68, Thm. 6.25]) with [BD68, Prop. 6.21].

Let $\mathbf{C}$ be a Grothendieck abelian tensor category and let $\mathbf{K} \subseteq \mathbf{C}$ be a Serre subcategory. We say that $\mathbf{K}$ is a *tensor ideal* if $\mathbf{K}$ is closed under tensor products with objects in $\mathbf{C}$. If $\mathbf{K}$ is also localizing, then we say that $\mathbf{K}$ is a *localizing tensor ideal*.

If $f^* \colon \mathbf{C} \to \mathbf{D}$ is an exact cocontinuous tensor functor between Grothendieck abelian tensor categories, then $\ker(f^*)$ is a localizing tensor ideal. Conversely, if $\mathbf{K} \subseteq \mathbf{C}$ is a localizing tensor ideal, then the quotient $\mathbf{Q} = \mathbf{C}/\mathbf{K}$ is a Grothendieck abelian tensor category, the localization $q^* \colon \mathbf{C} \to \mathbf{Q}$ is an exact cocontinuous tensor functor and $\ker(q^*) = \mathbf{K}$; in this situation, we will refer to q^* as a *tensor localization*.

Example 5.2. Let $f \colon X \to Y$ be a morphism of algebraic stacks. If f is flat, then f^* is exact. If f is a quasi-compact flat monomorphism (e.g., a quasi-compact open immersion), then $\mathsf{QCoh}(X)$ is the quotient of $\mathsf{QCoh}(Y)$ by $\ker(f^*)$. This follows from the fact that the counit $f^* f_* \to \mathrm{id}$ is an isomorphism so that f_* is a section of f^* [Gab62, Prop. III.2.5].

Definition 5.3. Let $\mathbf{C}$ be a Grothendieck abelian tensor category. For $M \in \mathbf{C}$ let $\varphi_M \colon \mathcal{O}_{\mathbf{C}} \to \mathcal{H}om_{\mathbf{C}}(M, M)$ denote the adjoint to the canonical isomorphism $\mathcal{O}_{\mathbf{C}} \otimes_{\mathbf{C}} M \to M$. Let the annihilator $\mathrm{Ann}_{\mathbf{C}}(M)$ of M be the kernel of φ_M, which we consider as an ideal of $\mathcal{O}_{\mathbf{C}}$.

Example 5.4. Let X be an algebraic stack and let $\mathcal{F} \in \mathsf{QCoh}(X)$. Then $\mathrm{Ann}_{\mathsf{QCoh}(X)}(\mathcal{F}) = \mathrm{QC}\big(\mathrm{Ann}_{\mathsf{Mod}(X)}(\mathcal{F})\big)$. In particular, if $\mathcal{F}$ is of finite type, then $\mathrm{Ann}_{\mathsf{QCoh}(X)}(\mathcal{F}) = \mathrm{Ann}_{\mathsf{Mod}(X)}(\mathcal{F})$.

Recall that an object $c \in \mathbf{C}$ is *finitely generated* if the natural map:

$$\varinjlim_{\lambda} \mathrm{Hom}_{\mathbf{C}}(c, d_\lambda) \to \mathrm{Hom}_{\mathbf{C}}(c, \varinjlim_{\lambda} d_\lambda)$$

is bijective for every direct system $\{d_\lambda\}_\lambda$ in $\mathbf{C}$ with monomorphic bonding maps. A category $\mathbf{C}$ is *locally finitely generated* if it is cocomplete (all small colimits exist) and has a set $\mathcal{A}$ of finitely generated objects such that every object c of $\mathbf{C}$ is a directed colimit of objects from $\mathcal{A}$.

Example 5.5. Let X be a quasi-compact and quasi-separated algebraic stack. The finitely generated objects in $\mathsf{QCoh}(X)$ are the quasi-coherent sheaves of finite type. Thus $\mathsf{QCoh}(X)$ is locally finitely generated [Ryd16].

We also require the following definition.

Definition 5.6. Let $q^*\colon \mathbf{C} \to \mathbf{Q}$ be a tensor localization. Then it is *supported* if $q^*(\mathcal{O}_{\mathbf{C}}/\operatorname{Ann}(K)) \cong 0$ for every finitely generated object K of $\mathbf{C}$ such that $q^*(K) \cong 0$.

The notion of a supported tensor localization is very natural.

Example 5.7. If $f\colon X \to Y$ is a flat monomorphism of quasi-compact and quasi-separated algebraic stacks, then the tensor localization $f^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ of Example 5.2 is supported. Indeed, if M is a quasi-coherent $\mathcal{O}_Y$-module of finite type in the kernel of f^*, then $f^* \operatorname{Ann}_{\mathsf{QCoh}(Y)}(M) = \operatorname{Ann}_{\mathsf{QCoh}(X)}(f^*M) = \mathcal{O}_X$.

We now have our key result, which also generalizes [BC14, Lem. 3.3.6].

Theorem 5.8. *Let $\mathbf{C}$ be a locally finitely generated Grothendieck abelian tensor category and let $q^*\colon \mathbf{C} \to \mathbf{Q}$ be a supported tensor localization. Let $\mathbf{D}$ be a Grothendieck abelian tensor category. If $f^*\colon \mathbf{C} \to \mathbf{D}$ is a cocontinuous tensor functor such that $f^*(K) \cong 0$ for every finitely generated object K of $\mathbf{C}$ such that $q^*(K) \cong 0$, then f^* factors essentially uniquely through a cocontinuous tensor functor $g^*\colon \mathbf{Q} \to \mathbf{D}$. If f^* preserves finitely generated objects, then so does g^*.*

Note that Theorem 5.8 is trivial if f^* is exact. The challenge is to use the symmetric monoidal structure to deduce this also when f^* is merely right-exact. The proof we give is a straightforward generalization of [BC14, Lem. 3.3.6]. First, we will see how Theorem 5.8 implies Theorem 5.1.

Proof of Theorem 5.1. For (ii), note that $(i^{[n]})_*$ identifies $\mathsf{QCoh}(Z^{[n]})$ with the category of modules over the algebra $A_n = \mathcal{O}_X/I^{n+1}$. The algebra f^*A_n is $\mathcal{O}_T/I_T^{n+1}$ and $(f_{Z^{[n]}})^* = (f_{A_n})^*$ in the terminology of §3.1.

For (i), recall that $\mathsf{QCoh}(X)$ is locally finitely generated (Example 5.5) and that $j^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(U)$ is a supported localization (Example 5.7). If $K \in \mathsf{QCoh}(X)$ is finitely generated and $j^*K = 0$, then $I^mK = 0$ for sufficiently large m. Thus, the natural map $I^m \otimes_{\mathcal{O}_X} K \to \mathcal{O}_X \otimes_{\mathcal{O}_X} K \cong K$ is zero. Applying $j_T^* f^*$, the map becomes the identity since $j_T^* f^*(I^m) \to$

$j_T^* f^*(\mathcal{O}_X) = \mathcal{O}_{U_T}$ is an isomorphism. It follows that $j_T^* f^* K = 0$. We may thus apply Theorem 5.8 and deduce that $j_T^* f^*$ factors via j^* and a tensor functor $f_U^* \colon \mathsf{QCoh}(U) \to \mathsf{QCoh}(U_T)$. □

To prove Theorem 5.8 we require the following lemma.

Lemma 5.9 ([BC14, Lem. 3.3.2]). *Let $f^* \colon \mathbf{C} \to \mathbf{D}$ be a cocontinuous tensor functor. If $I \subseteq \mathcal{O}_{\mathbf{C}}$ is an $\mathcal{O}_{\mathbf{C}}$-ideal such that $f^*(\mathcal{O}_{\mathbf{C}}/I) \cong 0$, then $f^*(I) \to f^*(\mathcal{O}_{\mathbf{C}})$ is an isomorphism.*

Proof. Since f^* is right-exact and $f^*(\mathcal{O}_{\mathbf{C}}/I) = 0$, it follows that $f^*(I) \to f^*(\mathcal{O}_{\mathbf{C}}) = \mathcal{O}_{\mathbf{D}}$ is surjective. Let $J = f^*(I)$ and let $\varphi \colon J \to \mathcal{O}_{\mathbf{D}}$ denote the surjection. The multiplication $I \otimes_{\mathbf{C}} I \to I$ factors through $I \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{C}}$ and $\mathcal{O}_{\mathbf{C}} \otimes_{\mathbf{C}} I$ and gives rise to the commutative diagram

$$\begin{array}{ccc} J \otimes_{\mathbf{D}} J & \xrightarrow{\mathrm{id}_J \otimes \varphi} & J \otimes_{\mathbf{D}} \mathcal{O}_{\mathbf{D}} \\ {\scriptstyle \varphi \otimes \mathrm{id}_J} \downarrow & & \downarrow \cong \\ \mathcal{O}_{\mathbf{D}} \otimes_{\mathbf{D}} J & \xrightarrow{\cong} & J \end{array}$$

Let η_F denote the unit of the adjunction between $- \otimes_{\mathbf{D}} F$ and $\mathcal{H}om_{\mathbf{D}}(F, -)$. Then we obtain the commutative diagram

$$\begin{array}{ccccc} J & \xrightarrow{\eta_J(J)} & \mathcal{H}om_{\mathbf{D}}(J, J \otimes J) & \xrightarrow{\mathcal{H}om(-, \mathrm{id}_J \otimes \varphi)} & \mathcal{H}om_{\mathbf{D}}(J, J \otimes \mathcal{O}_{\mathbf{D}}) \\ {\scriptstyle \varphi} \downarrow & & \downarrow {\scriptstyle \mathcal{H}om(-, \varphi \otimes \mathrm{id}_J)} & & \downarrow \cong \\ \mathcal{O}_{\mathbf{D}} & \xrightarrow{\eta_J(\mathcal{O}_{\mathbf{D}})} & \mathcal{H}om_{\mathbf{D}}(J, \mathcal{O}_{\mathbf{D}} \otimes J) & \xrightarrow{\cong} & \mathcal{H}om_{\mathbf{D}}(J, J). \end{array}$$

But the top row also factors as

$$J \xrightarrow{\eta_{\mathcal{O}_{\mathbf{D}}}(J)} \mathcal{H}om_{\mathbf{D}}(\mathcal{O}_{\mathbf{D}}, J \otimes \mathcal{O}_{\mathbf{D}}) \xrightarrow{\mathcal{H}om(\varphi, -)} \mathcal{H}om_{\mathbf{D}}(J, J \otimes \mathcal{O}_{\mathbf{D}})$$

which is injective since $\eta_{\mathcal{O}_{\mathbf{D}}}$ is an isomorphism and φ is surjective. It follows that $J \to \mathcal{H}om_{\mathbf{D}}(J, J)$ is injective, hence so is $\varphi \colon J \to \mathcal{O}_{\mathbf{D}}$. □

Proof of Theorem 5.8. If $K \in \mathbf{C}$, since $\mathbf{C}$ is locally finitely generated, it may be written as a directed colimit $K = \varinjlim_\lambda K_\lambda$, where $K_\lambda \subseteq K$ and K_λ is finitely generated. If $K \in \ker(q^*)$, then $q^* K_\lambda \subseteq q^* K \cong 0$. In particular, $\mathbf{K} := \ker(q^*) \subseteq \ker(f^*)$.

Let $0 \to K \to M \to N \to Q \to 0$ be an exact sequence in $\mathbf{C}$ with $K, Q \in \mathbf{K}$. We have to prove that $f^*(M \to N)$ is an isomorphism in $\mathbf{D}$. Let N_0 be the image of M in N. By right-exactness, we have an exact sequence $f^*(K) \to f^*(M) \to f^*(N_0) \to 0$. Since $f^*(K) = 0$, we have that $f^*(M) = f^*(N_0)$. We may thus replace M with N_0 and assume that $K = 0$ and $M \to N$ is injective.

Write N as the directed colimit of finitely generated subobjects $N_\lambda^\circ \subseteq N$. Let $N_\lambda = M + N_\lambda^\circ \subseteq N$ and $I_\lambda = \mathrm{Ann}(N_\lambda/M)$. By definition, we have that $I_\lambda \otimes N_\lambda/M \to N_\lambda/M$ is zero; hence $I_\lambda \otimes N_\lambda \to N_\lambda$ factors through M.

Note that $N_\lambda/M = N_\lambda^\circ/(N_\lambda^\circ \cap M)$ is a quotient of a finitely generated object and a subobject of Q, so $\mathcal{O}_{\mathbf{C}}/I_\lambda \in \mathbf{K}$ since q^* is supported. We conclude that $f^*(I_\lambda) \to f^*(\mathcal{O}_{\mathbf{C}})$ is an isomorphism using Lemma 5.9. Now consider the commutative diagrams:

$$\begin{array}{ccc} I_\lambda \otimes M & \longrightarrow & M \\ \downarrow & \nearrow & \downarrow \\ I_\lambda \otimes N_\lambda & \longrightarrow & N_\lambda \end{array} \quad \text{and} \quad \begin{array}{ccc} f^*(M) & \xrightarrow{\cong} & f^*(M) \\ \downarrow & \nearrow & \downarrow \\ f^*(N_\lambda) & \xrightarrow{\cong} & f^*(N_\lambda), \end{array}$$

where the right diagram is obtained by applying f^* to the left diagram. It follows that $f^*(M) \to f^*(N_\lambda)$ is an isomorphism. Since f^* is cocontinuous, it follows that $f^*(M) \to f^*(N) = \varinjlim f^*(N_\lambda)$ is an isomorphism.

This proves that $f^* = g^*q^*$ where $g^* = f^*q_*$. It is readily verified that g^* is cocontinuous (it preserves small direct sums and is right-exact). If $M \in \mathbf{Q}$ is a finitely generated object, then we may find a finitely generated object $N \in \mathbf{C}$ such that $M = q^*N$. Indeed, by assumption q_*M is a filtered colimit of finitely generated objects. It follows that there is a finitely generated subobject $N \subseteq q_*M$ such that $q^*N \to M$ is an isomorphism. If f^* preserves finitely generated objects, then $g^*M = f^*N$ is finitely generated. □

To show how powerful tensor localization is, we can quickly prove that tensoriality is local for the Zariski topology—even for stacks.

Theorem 5.10. *Let X be a quasi-compact and quasi-separated algebraic stack. Let $X = \bigcup_{k=1}^n X_k$ be an open covering by quasi-compact open substacks. If every X_k is tensorial, then so is X.*

Proof. Let $j_k \colon X_k \to X$ denote the open immersion and let I_k be an ideal of finite type defining a closed substack complementary to X_k [Ryd16, Prop. 8.2].

Let T be an algebraic stack. First we will show that $\omega_X(T)$ is fully faithful. Thus, let $f, g \colon T \to X$ be two morphisms and suppose that we are given a natural transformation of cocontinuous tensor functors $\gamma \colon f^* \Rightarrow g^*$. Then $f^*(\mathcal{O}_X/I_k) \twoheadrightarrow g^*(\mathcal{O}_X/I_k)$ so there is an inclusion $f^{-1}(X_k) \subseteq g^{-1}(X_k)$ for every k. Let $T_k = f^{-1}(X_k)$, let $j_{k,T} \colon T_k \to T$ denote the corresponding open immersion and let $f_k, g_k \colon T_k \to X_k$ denote the restrictions of f and g. Since $(f_k)^* = j_{k,T}^* f^*(j_k)_*$ and $(g_k)^* = j_{k,T}^* g^*(j_k)_*$, we obtain a natural transformation $\gamma_k \colon f_k^* \Rightarrow g_k^*$, hence a unique 2-isomorphism $f_k \Rightarrow g_k$. Since $T = \bigcup_{k=1}^N T_k$, it follows by fppf-descent, that $\omega_X(T)$ is faithful (Lemma 4.2(i)). As this holds for all T, we also have that $\omega_X(T_k \cap T_{k'})$ is faithful and it follows by fppf-descent that $\omega_X(T)$ is full (Lemma 4.2(ii)).

For essential surjectivity, let $f^* \colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor. The surjection $\mathcal{O}_T \twoheadrightarrow f^*(\mathcal{O}_X/I_k)$ defines a closed subscheme and we let $j_{k,T} \colon T_k \to T$ denote its open complement. By Theorem 5.1(i), $j_{k,T}^* f^*$ factors via j_k^* and a tensor functor $f_k^* \colon \mathsf{QCoh}(X_k) \to \mathsf{QCoh}(T_k)$. The latter is algebraic by assumption; hence, so is $j_{k,T}^* f^* = f_k^* j_k^*$.

Finally, since $\mathcal{O}_X/I_1 \otimes \cdots \otimes \mathcal{O}_X/I_n = 0$, it follows that $f^*(\mathcal{O}_X/I_1) \otimes \cdots \otimes f^*(\mathcal{O}_X/I_n) = 0$ so $T = \bigcup_{k=1}^n T_k$ is an open covering. We conclude that f^* is algebraic by fppf descent (Lemma 4.2(iii)). □

Combining Theorem 5.10 with Lemma 4.5 we obtain a short proof of the main result of [BC14].

Corollary 5.11 (Brandenburg–Chirvasitu)**.** *Every quasi-compact and quasi-separated scheme is tensorial.*

6. The Main Lemma

The main result of this section is the following technical lemma, which proves that the tensorial property extends over nilpotent thickenings of quasi-compact algebraic stacks with affine stabilizers having the resolution property.

Lemma 6.1 (Main Lemma)**.** *Let $i\colon X_0 \to X$ be a closed immersion of algebraic stacks defined by a quasi-coherent ideal I such that $I^n = 0$ for some integer $n > 0$. Suppose that X_0 is quasi-compact and quasi-separated with affine stabilizers. If X_0 has the resolution property, then X is tensorial.*

We have another lemma that will be crucial for proving Lemma 6.1.

Lemma 6.2. *Consider a 2-cocartesian diagram of algebraic stacks:*

$$\begin{array}{ccc} U_0 & \overset{i}{\hookrightarrow} & U \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ X_0 & \overset{j}{\hookrightarrow} & X, \end{array}$$

such that the following conditions are satisfied.

(i) *i is a nilpotent closed immersion;*
(ii) *U_0 is an affine scheme; and*
(iii) *X_0 is quasi-compact and quasi-separated with affine stabilizers.*

If X_0 has the resolution property, then so has X.

Proof. Note that X_0 has affine diagonal by the Totaro–Gross theorem; hence p_0 is affine. By [Hal17, Prop. A.2], the square is a geometric pushout. In particular, j is a nilpotent closed immersion, p is affine, and the natural map $\mathcal{O}_X \to p_*\mathcal{O}_U \times_{p_*i_*\mathcal{O}_{U_0}} j_*\mathcal{O}_{X_0}$ is an isomorphism. By the Totaro–Gross Theorem [Gro17, Cor. 5.9], there exists a vector bundle V_0 on X_0 such that the total space of the frame bundle of V_0 is quasi-affine. Let $E_0 = p_0^*V_0$; then, since U_0 is affine, there exists a vector bundle E on U equipped with an isomorphism $\alpha\colon i^*E \to E_0$. Let V be the quasi-coherent $\mathcal{O}_X$-module $p_*E \times_\alpha j_*V_0$. By [Fer03, Thm. 2.2(iv)], V is a vector bundle on X and there is an isomorphism $j^*V \cong V_0$. By [Gro17, Prop. 5.7], it follows that X has the resolution property. □

Proof of Lemma 6.1. We prove the result by induction on $n > 0$. The case $n = 1$ is Theorem 4.11. So we let $n > 1$ be an integer and we will assume that if $W_0 \hookrightarrow W$ is any closed immersion of algebraic stacks defined by an ideal J such that $J^{n-1} = 0$ and W_0 has the resolution property, then W is tensorial. We now fix a closed immersion of algebraic stacks $i\colon X_0 \to X$ defined by an ideal I such that $I^n = 0$ and X_0 has the resolution property. It remains to prove that X is tensorial.

We observe that the Totaro–Gross Theorem [Gro17, Cor. 5.9] implies that X_0 has affine diagonal; thus, X has affine diagonal. We have seen that $\omega_X(T)$ is fully faithful (Proposition 4.9) so it remains to prove that $\omega_X(T)$

is essentially surjective. By descent, it suffices to prove that if T is an affine scheme and $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ is a cocontinuous tensor functor, then there exists an étale and surjective morphism $c\colon T' \to T$ such that c^*f^* is algebraic (Lemma 4.2(iii)).

By Corollary 3.6, there is a 2-cocartesian diagram in **GTC**

$$\begin{array}{ccc} \mathsf{QCoh}(T_0) & \xleftarrow{f_0^*} & \mathsf{QCoh}(X_0) \\ {\scriptstyle k^*}\uparrow & & \uparrow{\scriptstyle i^*} \\ \mathsf{QCoh}(T) & \xleftarrow{f^*} & \mathsf{QCoh}(X), \end{array}$$

where $k\colon T_0 \to T$ is the closed immersion defined by the image K of f^*I in $\mathcal{O}_T$. In particular, $K^n = 0$. Since X_0 has the resolution property, f_0^* is given by a morphism of algebraic stacks $f_0\colon T_0 \to X_0$ (Theorem 4.11).

Let $p\colon U \to X$ be a smooth and surjective morphism, where U is an affine scheme; then, p is affine. The pullback of p along the morphism $i \circ f_0\colon T_0 \to X$ results in a smooth and affine surjective morphism of schemes $q_0\colon V_0 \to T_0$. By [EGA, IV.17.16.3(ii)], there exists an affine étale and surjective morphism $c_0\colon T_0' \to T_0$ such that the pullback $q_0'\colon V_0' \to T_0'$ of q_0 to T_0' admits a section. By [EGA, IV.18.1.2], there exists a unique affine étale morphism $c\colon T' \to T$ lifting $c_0\colon T_0' \to T_0$. After replacing T with T' and f^* with c^*f^*, we may thus assume that q_0 admits a section (Lemma 4.2(iii)).

Let $X' = \mathcal{S}\mathrm{pec}_X(f_*\mathcal{O}_T)$. Let $I' = I(f_*\mathcal{O}_T)$ be the $\mathcal{O}_{X'}$-ideal generated by I and let $X_0' = V(I')$. Then X' is a quasi-compact stack with affine diagonal, $X_0' \to X'$ is a closed immersion defined by an ideal whose nth power vanishes and X_0' has the resolution property. Let $f'^* = \bar{f}^*\colon \mathsf{QCoh}(X') \to \mathsf{QCoh}(T)$ be the resulting tensor functor.

Since f'^* is right-exact, it follows that $K = \mathrm{im}(f'^*I' \to \mathcal{O}_T)$. Also, $I' \subseteq f'_*K \subseteq \mathcal{O}_{X'}$. Thus $V(f'_*K) \subseteq X_0'$, so has the resolution property. Note that $f'^*I' \to f'^*f'_*K \to K$ is surjective. Since f'_* is lax symmetric monoidal, for each integer $l \geq 1$ the morphism $(f'_*K)^{\otimes l} \to \mathcal{O}_{X'}$ factors through $f'_*(K^{\otimes l}) \to \mathcal{O}_{X'}$. In particular, $(f'_*(K^l))^2 \subseteq f'_*(K^{l+1})$ and $(f'_*K)^n = 0$. We may thus replace X by X', X_0 by $V(f'_*K)$, f^* by f'^*, I by f_*K and assume henceforth that

(i) $\mathcal{O}_X \to f_*\mathcal{O}_T$ is an isomorphism,
(ii) $I = f_*K$ for some $\mathcal{O}_T$-ideal K with $K^n = 0$,
(iii) $f_*(K^l)^2 \subseteq f_*(K^{l+1})$ for each integer $l \geq 1$,
(iv) $(f_*K)^l \subseteq f_*(K^l)$ for $l \geq 1$, and
(v) $q_0\colon V_0 \to T_0$ admits a section.

For each integer $l \geq 0$ let $I_l = f_*(K^{l+1})$, which is a quasi-coherent sheaf of ideals on X. Let $i_l\colon X_l \to X$ be the closed immersion defined by I_l and let $k_l\colon T_l \to T$ be the closed immersion defined by K^{l+1}. Since $f^*f_*(K^{l+1}) \to f^*\mathcal{O}_X = \mathcal{O}_T$ factors through K^{l+1}, it follows that $k_l^*f^*(i_l)_*(\mathcal{O}_{X_l}) = \mathcal{O}_{T_l}$. Hence, $f_l^* = k_l^*f^*(i_l)_*\colon \mathsf{QCoh}(X_l) \to \mathsf{QCoh}(T_l)$ is a tensor functor and $k_l^*f^* \simeq f_l^*(i_l)^*$ (Theorem 5.1(ii)).

By condition (iv), we see that $i_l\colon X_0 \to X_l$ is a closed immersion of algebraic stacks defined by an ideal whose $(l+1)$th power is zero. In particular, if $l < n-1$, then X_l is tensorial by the inductive hypothesis. Thus, the tensor functor f_l^* is given by an affine morphism $f_l\colon T_l \to X_l$.

We will now prove by induction on $l \geq 0$ that X_l has the resolution property. Since $X_{n-1} = X$, the result will then follow from Theorem 4.11. Note that (iii) implies that the closed immersion $X_l \to X_{l+1}$ is a square zero extension of X_l by I_l/I_{l+1}. Let $m = n-2$.

Claim 1. If $M \in \mathsf{QCoh}(T_m)$, then the natural map $f_*(k_m)_*M \to p_*p^*f_*(k_m)_*M$ is split injective.

Proof of Claim 1. Form the cartesian diagram of algebraic stacks:

$$\begin{array}{ccccccc} V_0 & \longrightarrow & V_m & \xrightarrow{g_m} & U_m & \xrightarrow{u_m} & U \\ {\scriptstyle q_0}\downarrow & & {\scriptstyle q_m}\downarrow & & {\scriptstyle p_m}\downarrow & & \downarrow{\scriptstyle p} \\ T_0 & \longrightarrow & T_m & \xrightarrow{f_m} & X_m & \xrightarrow{i_m} & X. \end{array}$$

Now observe that $f_*(k_m)_*M \cong (i_m)_*(f_m)_*M$. Since f_m^* is given by a morphism $f_m\colon T_m \to X_m$, there are natural isomorphisms:

$$\begin{aligned} p_*p^*f_*(k_m)_*M &\cong p_*p^*(i_m)_*(f_m)_*M \cong p_*(u_m)_*p_m^*(f_m)_*M \\ &\cong p_*(u_m)_*(g_m)_*q_m^*M \cong (i_m)_*(f_m)_*(q_m)_*q_m^*M. \end{aligned}$$

Hence, it remains to prove that the natural map $M \to (q_m)_*q_m^*M$ is split injective. But q_m is affine, so $(q_m)_*q_m^*M \cong (q_m)_*\mathcal{O}_{V_m} \otimes_{\mathcal{O}_{T_m}} M$. Thus, we are reduced to proving that $\mathcal{O}_{T_m} \to (q_m)_*\mathcal{O}_{V_m}$ is split injective. By (v), q_0 admits a section. Since q_m is smooth and T_m is affine, the section that q_0 admits lifts to a section of q_m. This implies that the morphism $\mathcal{O}_{T_m} \to (q_m)_*\mathcal{O}_{V_m}$ is split injective. $\triangle$

Claim 2. If $0 \leq l < n-1$, then the natural maps $I_l/I_{l+1} \to p_*p^*(I_l/I_{l+1})$ are split injective.

Proof of Claim 2. If $N \in \mathsf{QCoh}(T_m)$, then $f_*(k_m)_*N = (i_m)_*(f_m)_*N$. Since f_m is an affine morphism, it follows that $f_*(k_m)_*\colon \mathsf{QCoh}(T_m) \to \mathsf{QCoh}(X)$ is exact. If P is one of the modules K^{l+1}, K^{l+2}, or K^{l+1}/K^{l+2}, then $K^{m+1}P = 0$, so the natural map $P \to (k_m)_*k_m^*P$ is an isomorphism. In particular, $f_*P \cong f_*(k_m)_*k_m^*P$. Hence, $I_l/I_{l+1} \cong f_*(k_m)_*k_m^*(K^{l+1}/K^{l+2})$ and the claim now follows from Claim 1. $\triangle$

So we let $l \geq 0$ be an integer, which we assume to be $< n-1$. We will assume that X_l has the resolution property and we will now prove that X_{l+1} has the resolution property. Retaining the notation of Claim 1, there is a 2-commutative diagram of algebraic stacks:

$$\begin{array}{ccc} U_l & \longrightarrow & U_{l+1} \\ \Big\downarrow{\scriptstyle p_l} & & \downarrow \\ & \nearrow & \tilde{X}_{l+1} \\ & & \downarrow \\ X_l & \longrightarrow & X_{l+1}, \end{array}$$

where both the inner and outer squares are 2-cartesian and the inner square is 2-cocartesian. Let $Q_l = I_l/I_{l+1}$. The morphism $X_l \to \tilde{X}_{l+1}$ is a square zero extension of X_l by $(p_l)_*p_l^*Q_l \cong p_*p^*Q_l$ and the morphism $\tilde{X}_{l+1} \to X_{l+1}$ is the morphism of X_l-extensions given by the natural map $Q_l \to (p_l)_*p_l^*Q_l$. By Claim 2, the morphism $Q_l \to (p_l)_*p_l^*Q_l$ is split injective and so there is an induced splitting $X_{l+1} \to \tilde{X}_{l+1}$ which is affine. By [Gro17, Prop. 4.3(i)], it remains to prove that $\tilde{X}_{l+1}$ has the resolution property, which is just Lemma 6.2. □

7. Formal gluings

Let T be an algebraic stack, let $i\colon Z \hookrightarrow T$ be a finitely presented closed immersion and let $j\colon U \to T$ denote its complement. A *flat Mayer–Vietoris square* is a cartesian square of algebraic stacks

$$\begin{array}{ccc} U' & \xrightarrow{j'} & T' \\ {\scriptstyle \pi_U}\downarrow & \square & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{j} & T \end{array}$$

such that π is flat and the induced morphism $\pi_Z\colon T' \times_T Z \to Z$ is an isomorphism [MB96, HR16].

Given an affine noetherian scheme $T = \operatorname{Spec} A$ and a closed subscheme $Z = V(I)$ we obtain a flat Mayer–Vietoris square as follows: let $U = T \smallsetminus Z$, let $T' = \operatorname{Spec} \widehat{A}$, where $\widehat{A}$ is the I-adic completion of A, and let $U' = U \times_T T'$. We call such squares *formal gluings*. While we will state our results more generally, the flat Mayer–Vietoris squares of relevance to this article will always be formal gluings.

If $F\colon \mathsf{AlgSt}^{\mathrm{op}} \to \mathbf{Cat}$ is a pseudo-functor, then there is a natural functor:

$$\Phi_F\colon F(T) \to F(T') \times_{F(U')} F(U).$$

Here AlgSt denotes the 2-category of algebraic stacks. For the purposes of this paper, it is enough to consider pseudo-functors defined on affine schemes, that is, fibered categories over affine schemes.

In this article, we will only consider two examples of pseudo-functors F. Let X be an algebraic stack.

(i) We may view X as a pseudo-functor via the 2-Yoneda lemma: $T \mapsto \operatorname{Hom}(T, X) \simeq X(T)$. Note that the flat Mayer–Vietoris square is a pushout (i.e., cocartesian) if and only if Φ_X is an equivalence.
(ii) There is also the pseudo-functor $X_\otimes(-) = \operatorname{Hom}_\otimes(\mathsf{QCoh}(X), \mathsf{QCoh}(-))$. Note that $\Phi_{X_\otimes}$ is an equivalence if and only if quasi-coherent sheaves can be glued along the flat Mayer–Vietoris square.

The following theorem follows from the main results of [HR16] (and almost from [MB96]).

Theorem 7.1. *Consider a flat Mayer–Vietoris square as above. Then*

(i) $\Phi_{X_\otimes}$ *is an equivalence of categories;*
(ii) Φ_X *is fully faithful; and is an equivalence if:*
  (a) Δ_X *is quasi-affine; or*
  (b) X *is Deligne–Mumford; or*

(c) *T is locally the spectrum of a G-ring (cf. Remark 7.2).*

Proof. By [HR16, Thm. B(1)] (or one of [MB96, 0.3] and [FR70, App.] when π is affine), there is an equivalence

$$\mathsf{QCoh}(T) \to \mathsf{QCoh}(T') \times_{\mathsf{QCoh}(U')} \mathsf{QCoh}(U).$$

Thus, we have (i). Claims (ii) and (ii)(a) are [HR16, Thm. B(3)] and claims (ii)(b) and (ii)(c) are [HR16, Thm. E and Thm. A] respectively. Under some additional assumptions: π is affine, Δ_X is quasi-compact and separated, and in (ii)(c) T' is locally noetherian; claims (ii)(a)–(ii)(c) also follow from [MB96, 6.2 and 6.5.1]. □

Remark 7.2. Recall that a noetherian ring A is *excellent* [Mat89, p. 260], [Mat80, Ch. 13] or [EGA, IV.7.8.2], if

(i) A is a G-ring, that is, $A_\mathfrak{p} \to \widehat{A_\mathfrak{p}}$ has geometrically regular fibers for every prime ideal $\mathfrak{p} \subset A$;
(ii) the regular locus $\operatorname{Reg} B \subseteq \operatorname{Spec} B$ is open for every finitely generated A-algebra B; and
(iii) A is universally catenary.

If (i) and (ii) hold, then we say that A is *quasi-excellent*. All G-ring assumptions in this paper originate from [MB96, HR16] via Theorem 7.1. The assumptions are used to guarantee that the formal fibers are geometrically regular so that Néron–Popescu desingularization applies. Note that whereas being a G-ring and being quasi-excellent are local for the smooth topology [Mat89, 32.2], excellency does not descend even for finite étale coverings [EGA, IV.18.7.7].

Corollary 7.3. *Let X be an algebraic stack. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $T = \operatorname{Spec} A$, $Z = V(I)$ and $U = T \smallsetminus Z$. Let $i\colon Z \to T$ and $j\colon U \to T$ be the resulting immersions.*

(i) *Let f_1, $f_2\colon T \to X$ be morphisms of algebraic stacks.*
  (a) *Assume that $\ker(\mathcal{O}_T \to j_*\mathcal{O}_U) \cap \bigcap_{n=0}^{\infty} I^n = 0$. Let α, $\beta\colon f_1 \Rightarrow f_2$ be 2-morphisms. If $\alpha_U = \beta_U$ and $\alpha_{Z^{[n]}} = \beta_{Z^{[n]}}$ for all n, then $\alpha = \beta$.*
  (b) *Assume that T is noetherian and that $\omega_X(T)$ is faithful for all noetherian T. Let $t\colon f_1^* \Rightarrow f_2^*$ be a natural transformation of cocontinuous tensor functors. If $j^*(t)$ and $(i^{[n]})^*(t)$ are realizable for all n, then t is realizable.*

(ii) *Assume either (a) T is the spectrum of a G-ring, or (b) T is noetherian and X has quasi-affine or unramified diagonal. Further, assume that $\omega_{X,\simeq}(T)$ is fully faithful for all noetherian T. Let $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor that* preserves sheaves of finite type. *If j^*f^* and $(i^{[n]})^*f^*$ are algebraic for all n, then f^* is algebraic.*

The assumption in (i)(a) says that the filtration $\{\emptyset \hookrightarrow Z \hookrightarrow T\}$ is *separating* (Definition A.1). This is automatic if T is noetherian (Lemma A.2).

Proof of Corollary 7.3. First, we show (ii). By assumption, the induced functor $(i^{[n]})^*f^*$ comes from a morphism $f^{[n]}\colon Z^{[n]} \to X$. Pick an étale cover $q\colon \tilde{Z} \to Z$ such that $f^{[0]} \circ q\colon \tilde{Z} \to X$ has a lift $g\colon \tilde{Z} \to W$, where

$p\colon W \to X$ is a smooth covering and W is affine. Descent (Lemma 4.2(iii)) implies that we are free to replace T with an étale cover, so we may assume that f also has a lift $g\colon Z \to W$ [EGA, IV.18.1.1].

Since p is smooth, we may choose compatible lifts $g^{[n]}\colon Z^{[n]} \to W$ of $f^{[n]}$ for all n. But W is affine, so there is an induced morphism $\hat{g}\colon \hat{T} \to W$, where $\hat{T} = \operatorname{Spec} \hat{A}$ and $\hat{A}$ denotes the completion of A at the ideal I. Let $\hat{f} = p \circ \hat{g}$. Then $(i^{[n]})^* \hat{f}^* = (f^{[n]})^* = (i^{[n]})^* f^*$ for all n. Since $\mathsf{Coh}(\hat{T}) = \varprojlim_n \mathsf{Coh}(Z_n)$ (Lemma 3.8), it follows that $\hat{f}^* \simeq \pi^* f^*$ where $\pi\colon \hat{T} \to T$ is the completion morphism. Indeed, this last equivalence may be verified after restricting both sides to quasi-coherent $\mathcal{O}_X$-modules of finite type (Example 5.5) and both sides send quasi-coherent $\mathcal{O}_X$-modules of finite type to $\mathsf{Coh}(\hat{T})$.

Let $\hat{\jmath}\colon \hat{U} \to \hat{T}$ be the pullback of j along π; then we obtain a flat Mayer–Vietoris square:

$$\begin{array}{ccc} \hat{U} & \xrightarrow{\hat{\jmath}} & \hat{T} \\ {\scriptstyle \pi_U}\downarrow & \square & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{j} & T. \end{array}$$

Since U and $\hat{U}$ are noetherian, $\omega_{X,\simeq}(U)$ and $\omega_{X,\simeq}(\hat{U})$ are fully faithful. Thus, there is an essentially unique morphism of algebraic stacks $h\colon U \to X$ such that $h^* \simeq j^* f^*$. But there are isomorphisms:

$$\hat{\jmath}^* \hat{f}^* \simeq \hat{\jmath}^* \pi^* f^* \simeq \pi_U^* j^* f^* \simeq \pi_U^* h^*,$$

so $\hat{f} \circ \hat{\jmath} \simeq h \circ \pi_U$. That f^* is algebraic now follows from Theorem 7.1.

For (i)(b), we proceed similarly. Consider the representable morphism $E \to T$ given by the equalizer of f_1 and f_2. Then 2-isomorphisms between f_1 and f_2 correspond to T-sections of E. By assumption, we have compatible sections $\tau_U \in E(U)$ and $\tau^{[n]} \in E(Z^{[n]})$ for all n. Choose an étale presentation $E' \to E$ by an affine scheme E'. We may replace T with an étale cover (Lemma 4.2(ii)) and thus assume that $\tau^{[0]}$ lifts to E'. In particular, there are compatible lifts of all the $\tau^{[n]}$ to E'. Since E' is affine, we get an induced morphism $\hat{T} \to E'$; thus, a morphism $\hat{T} \to E$. Equivalently, we get a 2-isomorphism between $f_1 \circ \pi$ and $f_2 \circ \pi$. The induced 2-isomorphism between $\pi^* f_1^*$ and $\pi^* f_2^*$ equals $\pi^* t$ since it coincides on the truncations. We may now apply Theorem 7.1 to deduce that t is realized by a 2-morphism $\tau\colon f_1 \Rightarrow f_2$.

For (i)(a), we consider the representable morphism $r\colon R \to T$ given by the equalizer of α and β. It suffices to prove that r is an isomorphism. Note that r is always a monomorphism and locally of finite presentation. By assumption, there are compatible sections of r over U and $Z^{[n]}$ for all n, thus r_U and $r_{Z^{[n]}}$ are isomorphisms for all n. By Proposition A.3, r is an isomorphism. □

Remark 7.4. We do not know if the condition that f^* preserves sheaves of finite type in (ii) is necessary. We do know that for any sheaf F of finite type, the restrictions of $f^* F$ to U and $Z^{[n]}$ are coherent but this does not imply that $f^* F$ is coherent. For example, if $A = k[[x]]$, and $I = (x)$, then the A-module $k((x))/k[[x]]$ is not finitely generated but becomes 0 after tensoring with $A/(x^n)$ or A_x.

8. Tannaka duality

In this section, we prove our general Tannaka duality result (Theorem 8.4) and as a consequence also establish Theorem 1.1. To accomplish this, we consider the following refinement of [HR15a, Def. 2.4].

Definition 8.1. Let X be a quasi-compact algebraic stack. A *finitely presented filtration of* X is a sequence of finitely presented closed immersions $\emptyset = X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots \hookrightarrow X_r \hookrightarrow X$ such that $|X_r| = |X|$. The *strata* of the filtration are the locally closed finitely presented substacks $Y_k := X_k \smallsetminus X_{k-1}$.

Stacks that have affine stabilizers can be stratified into stacks with the resolution property.

Proposition 8.2. *Let X be an algebraic stack. The following are equivalent:*

(i) *X is quasi-compact and quasi-separated with affine stabilizers;*
(ii) *X has a finitely presented filtration (X_k) with strata of the form $Y_k = [U_k/\mathrm{GL}_{N_k}]$ where U_k is quasi-affine.*
(iii) *X has a finitely presented filtration (X_k) with strata Y_k that are quasi-compact with affine diagonal and the resolution property.*

Proof. That (i) $\Longrightarrow$ (ii) is [HR15a, Prop. 2.6(i)]. That (ii) $\Longleftrightarrow$ (iii) is the Totaro–Gross theorem [Gro17]. That (iii) $\Longrightarrow$ (i) is straightforward. □

When in addition X is noetherian or, more generally, X has finitely presented inertia, this result is due to Kresch [Kre99, Prop. 3.5.9] and Drinfeld–Gaitsgory [DG13, Prop. 2.3.4]. They construct stratifications by quotient stacks of the form $[V_k/\mathrm{GL}_{N_k}]$, where each V_k is quasi-projective and the action is linear. This implies that the strata have the resolution property. When X has finitely presented inertia the situation is simpler since X can be stratified into gerbes [Ryd16, Cor. 8.4], something which is not possible in general.

Remark 8.3. In [DG13, Def. 1.1.7], Drinfeld and Gaitsgory introduces the notion of a QCA stack. These are (derived) algebraic stacks that are quasi-compact and quasi-separated with affine stabilizers and finitely presented inertia. The condition on the inertia is presumably only used for [DG13, Prop. 2.3.4] and could be excised using Proposition 8.2.

We now state and prove the main result of the paper.

Theorem 8.4. *Let T and X be algebraic stacks and consider the functor*

$$\omega_X(T)\colon \operatorname{Hom}(T,X) \to \operatorname{Hom}_{c\otimes}(\mathsf{QCoh}(X),\mathsf{QCoh}(T))$$

and its variants $\omega_X^{\mathrm{ft}}(T)$, $\omega_{X,\simeq}(T)$ and $\omega_{X,\simeq}^{\mathrm{ft}}(T)$ (see §4). Assume that X is quasi-compact and quasi-separated with affine stabilizers and that T is locally noetherian. Then

(i) *$\omega_{X,\simeq}(T)$ is fully faithful;*
(ii) *$\omega_X(T)$ is fully faithful if X is affine-pointed;*
(iii) *$\omega_{X,\simeq}^{\mathrm{ft}}(T)$ is an equivalence of groupoids if either*
  (a) *T is locally the spectrum of a G-ring; (Remark 7.2)*
  (b) *X has quasi-affine diagonal; or*

(c) *X is Deligne–Mumford.*

In particular, $\omega^{\mathrm{ft}}_{X,\simeq}(T)$ is an equivalence if T is locally excellent and X has affine stabilizers, and $\omega^{\mathrm{ft}}_{X}(T)$ is an equivalence if T is locally noetherian and X either has quasi-affine diagonal or is Deligne–Mumford.

When X has quasi-affine diagonal, we have already seen that $\omega_X(T)$ is fully faithful without any noetherian assumptions on T (Proposition 4.9(i)). In some situations, we can also prove faithfulness for non-noetherian T, see Proposition 8.5 below.

Proof of Theorem 8.4. We will first prove that $\omega_X(T)$ is faithful, then prove that $\omega_{X,\simeq}(T)$ and $\omega_X(T)$ are fully faithful (the latter under the assumption that X is affine-pointed) and finally prove that $\omega^{\mathrm{ft}}_{X,\simeq}(T)$ is an equivalence under the assumptions in (iii). By Lemma 4.3, it is enough to prove these results when $T = \operatorname{Spec} A$ is affine.

Stratification on X. Choose a filtration (X_k) as in Proposition 8.2. We will prove the theorem by induction on the number of strata r. If $r = 0$, then $X = \emptyset$ and there is nothing to prove. If $r \geq 1$, then $U := X \smallsetminus X_1$ has a filtration of length $r-1$; thus by induction the theorem holds over U.

Let $I \subseteq \mathcal{O}_X$ be the ideal defining $Z := X_1$. Let $i^{[n]}\colon Z^{[n]} \hookrightarrow X$ be the closed substack defined by I^{n+1} and let $j\colon U \to X$ be its complement. The filtration was chosen such that Z has the resolution property. Thus $\omega_{Z^{[n]}}(T)$ is an equivalence of categories for every $n \geq 0$ by the Main Lemma 6.1. In particular, the theorem holds over $Z^{[n]}$ for every $n \geq 0$.

Setup. For faithfulness, pick two maps $f_1, f_2\colon T \to X$ and two 2-isomorphisms $\tau_1, \tau_2 : f_1 \Rightarrow f_2$ and assume that $\omega_X(T)(\tau_1) = \omega_X(T)(\tau_2)$. We need to prove that $\tau_1 = \tau_2$.

For fullness of $\omega_{X,\simeq}(T)$ (resp. $\omega_X(T)$), pick two maps $f_1, f_2\colon T \to X$ and a natural isomorphism (resp. transformation) $\gamma\colon f_1^* \Rightarrow f_2^*$ of cocontinuous tensor functors. We need to prove that γ is realizable.

For essential surjectivity, pick a cocontinuous tensor functor $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ preserving sheaves of finite type. We need to prove that f^* is algebraic.

Pulled-back stratification on T. For faithfulness and fullness, let $I_T = \operatorname{Im}(f_2^* I \to f_2^*\mathcal{O}_X = \mathcal{O}_T)$, which is a finitely generated ideal because f_2 is a morphism. For essential surjectivity, let $I_T = \operatorname{Im}(f^* I \to f^*\mathcal{O}_X = \mathcal{O}_T)$, which is a finitely generated ideal since f^* is assumed to preserve finite type objects. Let $i_T^{[n]} : Z_T^{[n]} \hookrightarrow T$ be the finitely presented closed immersion defined by I_T^{n+1} and let $j_T\colon U_T \hookrightarrow T$ be its complement, a quasi-compact open immersion.

Result holds on strata. For faithfulness and fullness, we have that $U_T = f_1^{-1}(U) = f_2^{-1}(U)$; for faithfulness this is obvious and for fullness of $\omega_{X,\simeq}(T)$ and $\omega_X(T)$ this follows from Corollary 4.10 and Lemma 4.8 (when X affine-pointed), respectively. We also have that $Z_T^{[n]} = f_2^{-1}(Z^{[n]}) \hookrightarrow f_1^{-1}(Z^{[n]})$. Thus, by the inductive assumption and the case $r = 1$, after restricting to either U_T or $Z_T^{[n]}$ we have that $\tau_1 = \tau_2$ (for faithfulness), and that γ is realizable (for fullness).

For essential surjectivity, Theorem 5.1 produces for every $n \geq 0$ essentially unique cocontinuous tensor functors $f_U^*\colon \mathsf{QCoh}(U) \to \mathsf{QCoh}(U_T)$ and $f_{Z^{[n]}}^*\colon \mathsf{QCoh}(Z^{[n]}) \to \mathsf{QCoh}(Z_T^{[n]})$ such that $j_T^* f^* \simeq f_U^* j^*$ and $(i_T^{[n]})^* f^* \simeq (f_{Z^{[n]}})^*(i^{[n]})^*$. By the inductive assumption, f_U^* is algebraic and the case $r = 1$ implies that $f_{Z^{[n]}}^*$ is algebraic for each $n \geq 0$. In particular, $j_T^* f^*$ and $(i_T^{[n]})^* f^*$ is algebraic for each $n \geq 0$.

Formal gluing. The result now follows from Corollary 7.3 which uses the noetherian assumption on T. □

We also have some partial results in the non-noetherian situation.

Proposition 8.5. *Let $f_1, f_2\colon T \to X$ be morphisms of algebraic stacks. Assume that X is quasi-compact and quasi-separated with affine stabilizers. Further assume that either*

(i) *T has no embedded associated points; or*
(ii) *f_2 factors as $T \to S \to X$ where S is locally noetherian and $\pi\colon T \to S$ is flat.*

Then $\omega_X(T)\colon \mathrm{Hom}(f_1, f_2) \to \mathrm{Hom}(f_1^*, f_2^*)$ is injective. In particular, if T has no embedded associated points, then $\omega_X(T)$ is faithful.

Proof. The proof is identical with the proof of faithfulness in Theorem 8.4. We only have to argue that Corollary 7.3(i)(a) applies. That is, we have to show that the filtration $\{\emptyset \hookrightarrow Z_T \hookrightarrow T\}$ is separating.

If T has no embedded associated points, then the stratification $\emptyset \subset Z_T \subset T$ is separating by Lemma A.2. If f_2 factors as in (ii), then the stratification $\{\emptyset \hookrightarrow Z_T \hookrightarrow T\}$ is the pull-back along π of a stratification $\{\emptyset \hookrightarrow Z_S \hookrightarrow S\}$, hence separating by Lemma A.2. □

We conclude with the proof of Theorem 1.1.

Proof of Theorem 1.1. First, observe that if Y is a noetherian algebraic stack, then $\mathsf{QCoh}(Y)$ may be identified as the ind-category of $\mathsf{Coh}(Y)$ [Lur04, 3.9-10]. Essentially by definition, this induces an equivalence of categories:

$$\mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \mathsf{Coh}(T)) \to \mathrm{Hom}^{\mathrm{ft}}_{c\otimes,\simeq}(\mathsf{QCoh}(X), \mathsf{QCoh}(T)).$$

It is thus enough to prove that $\omega^{\mathrm{ft}}_{X,\simeq}(T)$ is an equivalence of groupoids, which follows from Theorem 8.4. □

9. Applications

In this section, we address the applications outlined in the introduction.

Proof of Corollary 1.4. Let $T' \to T$ be an fpqc covering where T is an algebraic stack, locally the spectrum of a G-ring, and T' is a locally noetherian algebraic stack. Since X is an fppf-stack, we may assume that T and T' are affine and that $T' \to T$ is faithfully flat. Let $T'' = T' \times_T T'$. Since X has affine stabilizers, the functor $\omega_{X,\simeq}(T)$ is an equivalence, the functor $\omega_{X,\simeq}(T')$ is fully faithful and the functor $\omega_X(T'')$ is faithful for morphisms $T'' \to T' \to X$ (Theorem 8.4 and Proposition 8.5). Since $\mathrm{Hom}_{c\otimes,\simeq}(\mathsf{QCoh}(X), \mathsf{QCoh}(-))$ is an fpqc stack, it follows that $T' \to T$ is a morphism of effective descent for X. □

Proof of Corollary 1.5. It is readily verified that we can assume that X is quasi-compact. As A is noetherian, $\mathsf{Coh}(A) = \varprojlim_n \mathsf{Coh}(A/I^n)$. Thus,

$$\begin{aligned} X(A) &\cong \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \mathsf{Coh}(A)) \\ &\cong \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \varprojlim \mathsf{Coh}(A/I^n)) \\ &\cong \varprojlim \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \mathsf{Coh}(A/I^n)) \\ &\cong \varprojlim X(A/I^n). \end{aligned}$$

□

Proof of Theorem 1.2. First, we prove (i). We begin with the following standard reductions: we can assume that S is affine; $X \to S$ is quasi-compact, so is of finite presentation; and S is of finite type over $\operatorname{Spec} \mathbb{Z}$.

Since S is now assumed to be excellent, we can prove the algebraicity of $\underline{\mathrm{Hom}}_S(Z, X)$ using a variant of Artin's criterion for algebraicity due to the first author [Hal17, Thm. A]. Hence, it is sufficient to prove that $\underline{\mathrm{Hom}}_S(Z, X)$ is

[1] a stack for the étale topology;
[2] limit preserving, equivalently, locally of finite presentation;
[3] homogeneous, that is, satisfies a strong version of the Schlessinger–Rim criteria;
[4] effective, that is, formal deformations can be algebraized;
[5] the automorphisms, deformations, and obstruction functors are coherent.

The main result of this article provides a method to prove [4] in maximum generality, which we address first. Thus, let $T = \operatorname{Spec} B \to S$, where $(B, \mathfrak{m})$ is a complete local noetherian ring. Let $T_n = \operatorname{Spec}(B/\mathfrak{m}^{n+1})$. Since $Z \to S$ is proper, for every noetherian algebraic stack W with affine stabilizers there are equivalences

$$\begin{aligned} \mathrm{Hom}(Z \times_S T, W) &\cong \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(W), \mathsf{Coh}(Z \times_S T)) && \text{(Theorem 1.1)} \\ &\cong \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(W), \varprojlim \mathsf{Coh}(Z \times_S T_n)) && \text{[Ols05, Thm. 1.4]} \\ &\cong \varprojlim \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(W), \mathsf{Coh}(Z \times_S T_n)) && \text{(Lemma 3.8)} \\ &\cong \varprojlim \mathrm{Hom}(Z \times_S T_n, W) && \text{(Theorem 1.1).} \end{aligned}$$

Since X and S have affine stabilizers, it follows that

$$\mathrm{Hom}_S(Z \times_S T, X) \cong \varprojlim \mathrm{Hom}_S(Z \times_S T_n, X);$$

that is, the stack $\underline{\mathrm{Hom}}_S(Z, X)$ is effective and so satisfies [4].

The remainder of Artin's conditions are routine, so we will just sketch the arguments and provide pointers to the literature where they are addressed in more detail. Condition [1] is just étale descent and [2] is standard—see, for example, [LMB, Prop. 4.18]. For conditions [3] and [5], it will be convenient to view $\underline{\mathrm{Hom}}_S(Z, X)$ as a substack of another moduli problem. This lets us avoid having to directly discuss the deformation theory of non-representable morphisms of algebraic stacks.

If $W \to S$ is a morphism of algebraic stacks, let $\underline{\mathrm{Rep}}_{W/S}$ denote the S-groupoid that assigns to each S-scheme T the category of *representable* morphisms of algebraic stacks $V \to W \times_S T$ such that the composition $V \to W \times_S T \to T$ is proper, flat and of finite presentation. There is a morphism of S-groupoids: $\Gamma\colon \underline{\mathrm{Hom}}_S(Z,X) \to \underline{\mathrm{Rep}}_{Z\times_S X/S}$, which is given by sending a T-morphism $f\colon Z\times_S T \to X\times_S T$ to its graph $\Gamma(f)\colon Z\times_S T \to (Z\times_S X)\times_S T$. It is readily seen that Γ is formally étale since $Z \to S$ is flat. Hence, it is sufficient to verify conditions [3] and [5] for $\underline{\mathrm{Rep}}_{Z\times_S X/S}$ [Hal17, Lemmas 1.5(9), 6.3 & 6.11]. That $\underline{\mathrm{Rep}}_{Z\times_S X/S}$ is homogeneous follows immediately from [Hal17, Lem. 9.3]. A description of the automorphism, deformation and obstruction functors of $\underline{\mathrm{Rep}}_{Z\times_S X/S}$ in terms of the cotangent complex are given on [Hal17, p. 173], which mostly follows from the results of [Ols06a]. That these functors are coherent is [Hal14, Thm. C]. This completes the proof of (i).

We now address (ii) and (iii), that is, the separation properties of the algebraic stack $\underline{\mathrm{Hom}}_S(Z,X)$ relative to S. Let T be an affine scheme. Let Z_T and X_T denote $Z\times_S T$ and $X\times_S T$, respectively. Suppose we are given two T-morphisms $f_1, f_2\colon Z_T \to X_T$ and consider $Q := \underline{\mathrm{Isom}}_{Z_T}(f_1,f_2) = X\times_{X\times_S X} Z_T$. Then $Q \to Z_T$ is representable and of finite presentation. If $\pi\colon Z_T \to T$ denotes the structure morphism, then π_*Q is an algebraic space which is locally of finite presentation, being the pull-back of the diagonal of $\underline{\mathrm{Hom}}_S(Z,X)$ along the morphism $T \to \underline{\mathrm{Hom}}_S(Z,X)\times_S \underline{\mathrm{Hom}}_S(Z,X)$ corresponding to (f_1,f_2).

Let P be one of the properties: affine, quasi-affine, separated, quasi-separated. Assume that Δ_X has P; then $Q \to Z_T$ has P. We claim that the induced morphism $\pi_*Q \to T$ has P. For the properties affine and quasi-affine, this is [HR15b, Thm. 2.3 (i),(ii)]. For quasi-separated (resp. separated), this is [HR15b, Thm. 2.3 (ii),(iv)] applied to the quasi-affine morphism (resp. closed immersion) $Q \to Q\times_Z Q$ and the Weil restriction $\pi_*Q \to \pi_*Q \times_T \pi_*Q = \pi_*(Q\times_Z Q)$. In particular, we have proved that $\underline{\mathrm{Hom}}_S(Z,X)$ is algebraic and locally of finite presentation with quasi-separated diagonal over S.

Now by Theorem B.1, $\Delta_{\underline{\mathrm{Hom}}_S(Z,X)/S} = \underline{\mathrm{Hom}}_S(Z,\Delta_{X/S})$ is of finite presentation, so $\underline{\mathrm{Hom}}_S(Z,X)$ is also quasi-separated. It remains to prove that it has affine stabilizers. To see this, we may assume that T is the spectrum of an algebraically closed field. In this situation, either π_*Q is empty or $f_1 \simeq f_2$; it suffices to treat the latter case. In the latter case, $T \to X\times_S X$ factors through the diagonal $\Delta_{X/S}\colon X \to X\times_S X$, so it is sufficient to prove that $\underline{\mathrm{Hom}}_S(Z, I_{X/S})$, where $I_{X/S}\colon X\times_{X\times_S X} X \to X$ is the inertia stack, has affine fibers. But $I_{X/S}$ defines a group over X with affine fibers, and the result follows from Theorem B.1. □

Lemma 9.1. *Let $f\colon Z \to S$ be a proper and flat morphism of finite presentation between algebraic stacks. For any morphism $X \to Z$ of algebraic stacks, the forgetful morphism $f_*X \to \underline{\mathrm{Hom}}_S(Z,X)$ is an open immersion.*

Proof. It is sufficient to prove that if T is an affine S-scheme and $h\colon Z\times_S T \to X\times_S T$ is a T-morphism, then the locus of points where $f_T\circ h\colon Z\times_S T \to Z\times_S T$ is an isomorphism is open on T.

First, consider the diagonal of $f_T\circ h$. This morphism is proper and representable and the locus on T where this map is a closed immersion is open [Ryd11b, Lem. 1.8 (iii)]. We may thus assume that $f_T\circ h$ is representable. Repeating the argument on $f_T\circ h$, we may assume that $f_T\circ h$ is a closed immersion. That the locus in T where $f_T \circ h$ is an isomorphism is open now follows easily by studying the étale locus of $f_T \circ h$, cf. [Ols06b, Lem. 5.2]. The result follows. □

Proof of Theorem 1.3. That $f_*X \to S$ is algebraic, locally of finite presentation, with quasi-compact and quasi-separated diagonal and affine stabilizers follows from Theorem 1.2 and Lemma 9.1. The additional separation properties of f_*X follows from [HR15b, Thm. 2.3 (i), (ii) & (iv)] applied to the diagonal and double diagonal of $X \to Z$. □

As claimed in the introduction, we now extend [HR15b, Thm. 2.3 & Cor. 2.4]. The statement of the following corollary uses the notion of a morphism of algebraic stacks that is *locally of approximation type* [HR15b, §1]. A trivial example of a morphism locally of approximation type is a quasi-separated morphism that is locally of finite presentation. It is hoped that every quasi-separated morphism of algebraic stacks is locally of approximation type, but this is currently unknown. It is known, however, that morphisms of algebraic stacks that have quasi-finite and locally separated diagonal are locally of approximation type [Ryd15]. In particular, all quasi-separated morphisms of algebraic stacks that are relatively Deligne–Mumford are locally of approximation type.

Corollary 9.2. *Let $f\colon Z \to S$ be a proper and flat morphism of finite presentation between algebraic stacks.*

(i) *Let $h\colon X \to S$ be a morphism of algebraic stacks with affine stabilizers that is locally of approximation type. Then $\underline{\mathrm{Hom}}_S(Z, X)$ is algebraic and locally of approximation type with affine stabilizers. If h is locally of finite presentation, then so is $\underline{\mathrm{Hom}}_S(Z, X) \to S$. If the diagonal of h is affine (resp. quasi-affine, resp. separated), then so is the diagonal of $\underline{\mathrm{Hom}}_S(Z, X) \to S$.*

(ii) *Let $g\colon X \to Z$ be a morphism of algebraic stacks such that $f \circ g : X \to S$ has affine stabilizers and is locally of approximation type. Then the S-stack f_*X is algebraic and locally of approximation type with affine stabilizers. If g is locally of finite presentation, then so is $f_*X \to S$. If the diagonal of g is affine (resp. quasi-affine, resp. separated), then so is the diagonal of $f_*X \to S$.*

Proof. For (i), we may immediately reduce to the situation where S is an affine scheme. Since f is quasi-compact, we may further assume that h is quasi-compact. By [HR15b, Lem. 1.1], there is an fppf covering $\{S_i \to S\}$ such that each S_i is affine and $X \times_S S_i \to S_i$ factors as $X \times_S S_i \to X_i^0 \to S_i$, where $X_i^0 \to S_i$ is of finite presentation and $X \times_S S_i \to X_i^0$ is affine. Combining the results of [HR15a, Thm. 2.8] with [Ryd15, Thms. D & 7.10], we can arrange so that each $X_i^0 \to S$ has affine stabilizers (or has one of the other desired separation properties).

Thus, we may now replace S by S_i and may assume that $X \to S$ factors as $X \xrightarrow{q} X_0 \to S$, where q is affine and $X_0 \to S$ is of finite presentation with the appropriate separation condition. By Theorem 1.2, the stack $\underline{\mathrm{Hom}}_S(Z, X_0)$ is algebraic and locally of finite presentation with the appropriate separation condition. By [HR15b, Thm. 2.3(i)], the morphism

$\underline{\mathrm{Hom}}_S(Z, X) \to \underline{\mathrm{Hom}}_S(Z, X_0)$ is representable by affine morphisms; the result follows.

For (ii) we argue exactly as in the proof of Theorem 1.3. □

10. Counterexamples

In this section we give four counter-examples (Theorems 10.1, 10.2, 10.4, and 10.5):

- in Theorems 1.1 and 8.4, and Proposition 8.5 it is necessary that X has affine stabilizer groups;
- in Theorem 8.4(ii), it is necessary that X is affine-pointed;
- in Theorem 1.2, it is necessary that X has affine stabilizer groups; and
- in Corollary 1.5, it is necessary that X has affine stabilizer groups.

For this section, the following definition will be important. Let k be a field and let G be an algebraic group scheme over k; we say that G is *anti-affine* if $\Gamma(G, \mathcal{O}_G) = k$ [Bri09]. Abelian varieties are always anti-affine, but there are many other anti-affine group schemes [Bri09, §2]. Anti-affine group schemes are always smooth, connected, and commutative [DG70, III.3.8.3]. In general, there is always a largest anti-affine k-subgroup scheme G_{ant} contained in the center of G such that the resulting quotient G/G_{ant} is affine. In fact, $G_{\mathrm{ant}} = \ker(G \to \operatorname{Spec}\Gamma(G, \mathcal{O}_G))$; in particular, if G is not affine, then G_{ant} is non-trivial [DG70, III.3.8.2].

Theorem 10.1. *Let X be a quasi-separated algebraic stack. If k is an algebraically closed field and $x\colon \operatorname{Spec} k \to X$ is a point with non-affine stabilizer, then $\operatorname{Aut}(x) \to \operatorname{Aut}_{\otimes}(x^*)$ is not injective. In particular, $\omega_X(\operatorname{Spec} k)$ is not faithful and X is not tensorial.*

Proof. By assumption, the stabilizer group scheme G_x of x is not affine. Let $H = (G_x)_{\mathrm{ant}}$ be the largest anti-affine subgroup of G_x; then H is a non-trivial anti-affine group scheme over k and the quotient group scheme G_x/H is affine [DG70, §III.3.8]. The induced morphism $B_k H \to B_k G_x \to X$ is thus quasi-affine by [Ryd11a, Thm. B.2].

By [Bri09, Lem. 1.1], the morphism $p\colon \operatorname{Spec} k \to B_k H$ induces an equivalence of Grothendieck abelian tensor categories $p^*\colon \mathsf{QCoh}(B_k H) \to \mathsf{QCoh}(\operatorname{Spec} k)$. Since $\operatorname{Aut}(p) = H(k) \neq \{\mathrm{id}_p\} = \operatorname{Aut}_{\otimes}(p^*)$, the functor $\omega_{B_k H}(\operatorname{Spec} k)$ is not faithful. Hence $\omega_X(\operatorname{Spec} k)$ is not faithful by Lemma 4.4. □

We also have the following theorem.

Theorem 10.2. *Let X be a quasi-compact and quasi-separated algebraic stack with affine stabilizers. If k is a field and $x_0\colon \operatorname{Spec} k \to X$ is a non-affine morphism, then there exists a field extension K/k and a point $y\colon \operatorname{Spec} K \to X$ such that $\operatorname{Isom}(y, x) \to \operatorname{Hom}_{\otimes}(y^*, x^*)$ is not surjective, where x denotes the K-point corresponding to x_0. In particular, $\omega_X(\operatorname{Spec} K)$ is not full.*

Proof. To simplify notation, we let $x = x_0$. Since X has quasi-compact diagonal, x is quasi-affine [Ryd11a, Thm. B.2]. By Lemma 4.4, we may replace X by $\mathcal{S}\mathrm{pec}_X(x_* k)$ and consequently assume that x is a quasi-compact open immersion and $\mathcal{O}_X \to x_* k$ is an isomorphism. In particular, x is a section to a morphism $f\colon X \to \operatorname{Spec} k$. Since x is not affine, it follows that there exists a closed point y disjoint from the image of x. In particular, there is a field extension K/k and a k-morphism $y\colon \operatorname{Spec} K \to X$ whose image is a closed point disjoint from x.

We now base change the entire situation by $\operatorname{Spec} K \to \operatorname{Spec} k$. This results in two morphisms $x_K, y_K\colon \operatorname{Spec} K \to X \otimes_k K$, where x_K is a quasi-compact open immersion such that $\mathcal{O}_{X\otimes_k K} \cong (x_K)_*K$ and y_K has image a closed point disjoint from the image of x_K. We replace X, k, x, and y by $X \otimes_k K$, K, x_K, and y_K respectively.

Let $\mathcal{G}_y \subseteq X$ be the residual gerbe associated to y, which is a closed immersion. We define a natural transformation $\gamma^\vee\colon x_* \Rightarrow y_*$ at k to be the composition $x_*k \cong \mathcal{O}_X \twoheadrightarrow \mathcal{O}_{\mathcal{G}_y} \to y_*k$ and extend to all of $\mathsf{QCoh}(\operatorname{Spec} k)$ by taking colimits. By adjunction, there is an induced natural transformation $\gamma\colon y^* \to x^*$. A simple calculation shows that γ is a natural transformation of cocontinuous tensor functors. Since its adjoint $\gamma^\vee$ is not an isomorphism, γ is not an isomorphism; thus γ is not realizable. The result follows. □

The following lemma is a variant of [Bha16, Ex. 4.12], which B. Bhatt communicated to the authors.

Lemma 10.3. *Let k be an algebraically closed field and let G/k be an anti-affine group scheme of finite type. Let Z/k be a regular scheme with a closed subscheme C that is a nodal curve over k. Then there is a compatible system of G-torsors $E_n \to C^{[n]}$ such that there does not exist a G-torsor $E \to Z$ that restricts to the E_n's.*

Proof. Recall that G is smooth, connected and commutative [DG70, §III.3.8]. Furthermore, by Chevalley's theorem, there is an extension $0 \to H \to G \to A \to 0$, where A is an abelian variety (of positive dimension) and H is affine. Let $x_A \in A(k)$ be an element of infinite order and let $x \in G(k)$ be any lift of x_A.

Let $\widetilde{C}$ be the normalization of C. Let $F_0 \to C$ be the G-torsor obtained by gluing the trivial G-torsor on $\widetilde{C}$ along the node by translation by x. Note that the induced A-torsor $F_0/H \to C$ is not torsion as it is obtained by gluing along the non-torsion element x_A.

We may now lift $F_0 \to C$ to G-torsors $F_n \to C^{[n]}$. Indeed, the obstruction to lifting F_{n-1} to F_n lies in $\operatorname{Ext}^1_{\mathcal{O}_C}(\mathsf{L}g_0^* L^\bullet_{BG/k}, I^n/I^{n+1})$, where $g_0\colon C \to BG$ is the morphism corresponding to $F_0 \to C$ and I is the ideal defining C in Z. Since G is smooth, the cotangent complex $L^\bullet_{BG/k}$ is concentrated in degree 1 and since C is a curve, it has cohomological dimension 1. It follows that the obstruction group is zero.

Now given a G-torsor $F \to Z$, there is an induced A-torsor $F/H \to Z$. Since Z is regular, the torsor $F/H \to Z$ is torsion in $H^1(Z, A)$ [Ray70, XIII 2.4 & 2.6]. Thus, $F/H \to Z$ cannot restrict to $F_0/H \to C$ and the result follows. □

We now have the following theorem, which is a counterexample to [Aok06a, Thm. 1.1] and [Aok06b, Case I].

Theorem 10.4. *Let $X \to S$ be a quasi-separated morphism of algebraic stacks. If k is an algebraically closed field and $x\colon \operatorname{Spec} k \to X$ is a point with* *non-affine stabilizer, then there exists a morphism* $\mathbb{A}^1_k \to S$ *and a proper and flat family of curves* $Z \to \mathbb{A}^1_k$*, where* Z *is regular, such that* $\underline{\mathrm{Hom}}_{\mathbb{A}^1_k}(Z, X \times_S \mathbb{A}^1_k)$ *is not algebraic.*

Proof. Let Q be the stabilizer group scheme of x and let G be the largest anti-affine subgroup scheme of Q; thus, G is a non-trivial anti-affine group scheme over k and the quotient group scheme Q/G is affine [DG70, §III.3.8].

Let Z be a proper family of curves over $T = \mathbb{A}^1_k = \operatorname{Spec} k[t]$ with regular total space and a nodal curve C as the fiber over the origin; for example, take $Z = \operatorname{Proj}_T(k[t][x,y,z]/(y^2z - x^2z - x^3 - tz^3))$ over T. Let $T_n = V(t^{n+1})$, $\hat{T} = \operatorname{Spec} \hat{\mathcal{O}}_{T,0}$, $Z_n = Z \times_T T_n$, and $\hat{Z} = Z \times_T \hat{T}$. We now apply Lemma 10.3 to C in $\hat{Z}$ and G. Since $Z_n = C^{[n]}$, this produces an element in

$$\varprojlim_n \underline{\mathrm{Hom}}_T(Z, BG_T)(T_n) = \varprojlim_n \mathrm{Hom}(Z_n, BG)$$

that does not lift to

$$\underline{\mathrm{Hom}}_T(Z, BG_T)(\hat{T}) = \mathrm{Hom}(\hat{Z}, BG).$$

This shows that $\underline{\mathrm{Hom}}_T(Z, BG_T)$ is not algebraic.

By [Ryd11a, Thm. B.2], the morphism x factors as $\operatorname{Spec} k \to BQ \to \mathcal{Q} \to X$, where $\mathcal{Q}$ is the residual gerbe, $\mathcal{Q} \to X$ is quasi-affine and $BQ \to \mathcal{Q}$ is affine. Since Q/G is affine, it follows that the induced morphism $BG \to BQ \to X$ is quasi-affine. By [HR15b, Thm. 2.3(ii)], the induced morphism $\underline{\mathrm{Hom}}_T(Z, BG_T) \to \underline{\mathrm{Hom}}_T(Z, X \times_S T)$ is quasi-affine. In particular, if $\underline{\mathrm{Hom}}_T(Z, X \times_S T)$ is algebraic, then $\underline{\mathrm{Hom}}_T(Z, BG_T)$ is algebraic, which is a contradiction. The result follows. □

The following theorem extends [Bha16, Ex. 4.12].

Theorem 10.5. *Let* X *be an algebraic stack with quasi-compact diagonal. If* X *does not have affine stabilizers, then there exists a noetherian two-dimensional regular ring* A*, complete with respect to an ideal* I*, such that* $X(A) \to \varprojlim X(A/I^n)$ *is not an equivalence of categories.*

Proof. Let $x \in |X|$ be a point with non-affine stabilizer group. Arguing as in the proof of Theorem 10.4, there exists an algebraically closed field k, an anti-affine group scheme G/k of finite type and a quasi-affine morphism $BG \to X$. An easy calculation shows that it is enough to prove the theorem for $X = BG$.

Let $A_0 = k[x,y]$ and let A be the completion of A_0 along the ideal $I = (y^2 - x^3 - x^2)$. Then $Z = \operatorname{Spec} A$ and $C = \operatorname{Spec} A/I$ satisfies the conditions of Lemma 10.3 and we obtain an element in $\varprojlim_n X(A/I^n)$ that does not lift to $X(A)$. □

Appendix A. Monomorphisms and stratifications

In this appendix, we introduce some notions and results needed for the faithfulness part of Theorem 8.4 when T is not noetherian. This is essential for the proof of Corollary 1.4.

We recall Definition 8.1: let X be a quasi-compact algebraic stack. A *finitely presented filtration of* X is a sequence of finitely presented closed immersions $\emptyset = X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_r \hookrightarrow X$ such that $|X_r| = |X|$.

The *strata* of the filtration are the locally closed finitely presented substacks $Y_k := X_k \smallsetminus X_{k-1}$.

As in Notation 3.7, the nth *infinitesimal neighborhood of* X_k is the finitely presented closed immersion $X_k^{[n]} \hookrightarrow X$ which is given by the ideal I_k^{n+1} where $X_k \hookrightarrow X$ is given by I_k. The nth *infinitesimal neighborhood of the stratum* Y_k is the locally closed finitely presented substack $Y_k^{[n]} := X_k^{[n]} \smallsetminus X_{k-1}$.

Definition A.1. A finitely presented filtration (X_k) of X is *separating* if the family $\{j_k^n \colon Y_k^{[n]} \to X\}_{k,n}$ is separating [EGA, IV.11.9.1]; that is, if the intersection $\bigcap_{k,n} \ker\big(\mathcal{O}_X \to (j_k^n)_* \mathcal{O}_{Y_k^{[n]}}\big)$ is zero as a lisse-étale sheaf.

Lemma A.2. *Every finitely presented filtration (X_k) on X is separating if either*

(i) *X is noetherian; or*

(ii) *X has no embedded (weakly) associated point.*

If X is noetherian with a filtration (X_k) and $X' \to X$ is flat, then $(X_k \times_X X')$ is a separating filtration on X'.

Proof. As the question is smooth-local, we can assume that X and X' are affine schemes. If X is noetherian, then by primary decomposition there exists a separating family $\coprod_{i=1}^m \operatorname{Spec} A_i \to X$ where the A_i are artinian. As every $\operatorname{Spec} A_i$ factors through some $Y_k^{[n]}$, it follows that (X_k) is separating. In general, $\{\operatorname{Spec} \mathcal{O}_{X,x} \to X\}_{x \in \operatorname{Ass}(X)}$ is separating [Laz64, 1.2, 1.5, 1.6]. If x is a non-embedded associated point, then $\operatorname{Spec} \mathcal{O}_{X,x}$ is a one-point scheme and factors through some $Y_k^{[n]}$ and the first claim follows.

For the last claim, we note that a finite number of the infinitesimal neighborhoods of the strata suffices in the noetherian case and that flat morphisms preserve kernels and finite intersections. □

Proposition A.3. *Let X be an algebraic stack with a finitely presented filtration (X_k). Let $f \colon Z \to X$ be a morphism locally of finite type. If $f|_{Y_k^{[n]}}$ is an isomorphism for every k and n, then f is a surjective closed immersion. If in addition (X_k) is separating, then f is an isomorphism.*

Proof. First note that f is a surjective and quasi-compact monomorphism.

We will prove that f is a closed immersion by induction on the number of strata r. If $r = 0$, then $X = \emptyset$ and there is nothing to prove. If $r = 1$, then $X = X_1^{[n]} = Y_1^{[n]}$ for sufficiently large n and the result follows. If $r \geq 2$, then let $U = X \smallsetminus X_1$. By the induction hypothesis, $f|_U$ is a surjective closed immersion. We may also assume that $X_1 \neq \emptyset$ and $|X_1| \neq |X|$ since these cases are trivial.

Verifying that f is a closed immersion is local on X, so we may assume that X, and hence Z, are schemes [LMB, Thm. A.1]. Then f is a closed immersion if and only if f is proper [EGA, IV.18.12.6]. By the valuative criterion for properness, we may thus assume that $X = \operatorname{Spec} V$ is the spectrum of a henselian valuation ring. Note that $f|_U$ is now an isomorphism since X is reduced.

By [EGA, IV.18.12.3], there is a decomposition $Z = Z_1 \amalg Z_2$, where $Z_1 \to X$ is a closed immersion and $Z_2 \cap f^{-1}(\mathfrak{m}) = \emptyset$, where $\mathfrak{m}$ is the maximal ideal of V. It remains to prove that $Z_2 = \emptyset$.

Since X_1 is local, hence connected, and $f|_{X_1}$ is an isomorphism, it follows that $Z_2 \cap f^{-1}(X_1) = \emptyset$. Similarly, since U is integral, hence connected, and $f|_U$ is an isomorphism, it follows that either $Z_2 = \emptyset$ or $Z_2 = U$. In the former case we are done. We will now show that the latter case is impossible.

Since V is a valuation ring and $X_1 \subseteq \operatorname{Spec} V$ is a finitely presented closed immersion, there is an $a \in V$ such that $X_1 = \operatorname{Spec} V/(a)$ and $U = D(a)$. Write $Z_1 = \operatorname{Spec} V/J$ for some ideal J. Since the closed immersion $Z_1 \hookrightarrow X$ is an isomorphism over $X_1^{[n]}$ for all $n \geq 0$, we have that $J \subseteq (a^n)$ for all $n \geq 0$. The condition that $Z_1 \cap D(a) = \emptyset$ is equivalent to $a^n \in J$ for all $n \gg 0$. That is, $J = (a^n)$ for all $n \gg 0$. This implies that $(a^{n+1}) = (a^n)$ is an equality for all $n \gg 0$, which is absurd since a is neither zero nor a unit. Hence, $Z_2 = \emptyset$ and f is a surjective closed immersion.

For general X, if (X_k) is separating, then the schematic image of f contains all the $Y_k^{[n]}$, and hence equals X by definition. This proves the last claim. □

The following example illustrates that a closed immersion $f\colon Z \to X$ as in Proposition A.3 need not be an isomorphism even if f is of finite presentation.

Example A.4. Let $A = k[x, z_1, z_2, \dots]/(xz_1, \{z_k - xz_{k+1}\}_{k\geq 1}, \{z_i z_j\}_{i,j\geq 1})$ and $B = A/(z_1)$. Then $A/(x^n) = k[x]/(x^n) = B/(x^n)$ and $A_x = k[x]_x = B_x$ but the surjection $A \to B$ is not an isomorphism.

Appendix B. A relative boundedness result for Hom stacks

Here we prove the following relative boundedness result for Hom stacks.

Theorem B.1. *Let $f\colon Z \to S$ be a proper, flat and finitely presented morphism of algebraic stacks. Let X and Y be algebraic stacks that are locally of finite presentation and quasi-separated over S and have affine stabilizers over S. Let $g\colon X \to Y$ be a finitely presented S-morphism. If g has affine fibers, then*

$$\underline{\operatorname{Hom}}_S(Z, g)\colon \underline{\operatorname{Hom}}_S(Z, X) \to \underline{\operatorname{Hom}}_S(Z, Y)$$

is of finite presentation. If in addition $g\colon X \to Y$ is a group, then $\underline{\operatorname{Hom}}_S(Z, g)$ is a group with affine fibers.

Theorem B.1 is used in Theorems 1.2 and 1.3 to establish the quasi-compactness of the diagonal of Hom-stacks and Weil restrictions. Prior to Theorem B.1, the strongest boundedness result is due to Olsson [Ols06b, Prop. 5.10]. There it is assumed that g is finite and f is representable.

Without using Theorem B.1, the proof of Theorem 1.2 give the algebraicity of the Hom-stacks and that they have quasi-separated diagonals. In the setting of Theorem B.1, we may conclude that $\underline{\operatorname{Hom}}_S(Z, g)$ is a quasi-separated morphism of algebraic stacks that are locally of finite presentation over S. It remains to prove that the morphism $\underline{\operatorname{Hom}}_S(Z, g)$ is quasi-compact.

Preliminary reductions. If W and T are algebraic stacks over S, let $W_T = W \times_S T$; similarly for morphisms between stacks over S. We will use this notation throughout this appendix.

As the question is local on S, we may assume that S is an affine scheme. We may also assume that X and Y are of finite presentation over S since it is enough to prove the theorem after replacing Y with an open quasi-compact substack and X with its inverse. By standard approximation results, we may then assume that S is of finite type over $\operatorname{Spec} \mathbb{Z}$. For the remainder of this article, all stacks will be of finite presentation over S and hence excellent with finite normalization.

By noetherian induction on S, to prove that $\underline{\mathrm{Hom}}_S(Z, g)$ is quasi-compact, we may assume that S is integral and replace S with a suitable dense open subscheme. Moreover, we may also replace $Z \to S$ with the pull-back along a dominant map $S' \to S$. Recall that there exists a field extension $K'/K(S)$ such that $(Z_{K'})_{\mathrm{red}}$ (resp. $(Z_{K'})_{\mathrm{norm}}$) is geometrically reduced (resp. geometrically normal) over K'. After replacing S with a dense open subset of the normalization in K', we may thus assume that

(i) $Z_{\mathrm{red}} \to S$ is flat with geometrically reduced fibers; and
(ii) $Z_{\mathrm{norm}} \to S$ is flat with geometrically normal fibers;

since these properties are constructible [EGA, IV.9.7.7 (iii) and 9.9.4 (iii)].

We now prove three reduction results. Throughout, we will assume the following:

- Z is proper and flat over S,
- X and Y are finitely presented algebraic stacks over S with affine stabilizers, and
- $g\colon X \to Y$ is a representable morphism over S.

Our first reduction result is similar to [Ols06b, Lem. 5.11].

Lemma B.2. *If $\underline{\mathrm{Hom}}_{S'}(Z', g_{S'})$ is quasi-compact for every scheme S', morphism $S' \to S$ and nil-immersion $Z' \to Z_{S'}$ such that $Z' \to S'$ is proper and flat with geometrically reduced fibers, then $\underline{\mathrm{Hom}}_S(Z, g)$ is quasi-compact.*

Proof. Assume that the condition holds. To prove that $\underline{\mathrm{Hom}}_S(Z, g)$ is quasi-compact, we may assume that S is integral. We may also assume that $Z_{\mathrm{red}} \to S$ has geometrically reduced fibers. Pick a sequence of square-zero nil-immersions $Z_{\mathrm{red}} = Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n = Z$. After replacing S with a dense open subset, we may assume that all the $Z_i \to S$ are flat. Thus, it suffices to show that if $j\colon Z_0 \to Z$ is a square-zero closed immersion where Z_0 is flat over S and $\underline{\mathrm{Hom}}_S(Z_0, g)$ is quasi-compact, then $\underline{\mathrm{Hom}}_S(Z, g)$ is quasi-compact. Now argue as in [Ols06b, Lem. 5.11], but this time using the deformation theory of [Ols06a, Thm. 1.5] and the Semicontinuity Theorem of [Hal14, Thm. A]. □

Before we proceed, we make the following observation: fix an S-scheme T and an S-morphism $y\colon Z_T \to Y$. This corresponds to a map $T \to \underline{\mathrm{Hom}}_S(Z, Y)$. The pullback of $\underline{\mathrm{Hom}}_S(Z, g)$ along this map is isomorphic to the Weil restriction $\mathbf{R}_{Z_T/T}(X \times_{g,Y,y} Z_T)$, which we will denote as $H_{Z/S,g}(y)$. Note that our hypotheses guarantee that $H_{Z/S,g}(y)$ is locally of finite type and quasi-separated over T.

The second reduction is for a (partial) normalization.

Lemma B.3. *If* $\underline{\mathrm{Hom}}_{S'}(Z', g_{S'})$ *is quasi-compact for every scheme* S'*, morphism* $S' \to S$ *and finite morphism* $Z' \to Z_{S'}$ *such that* $Z' \to S'$ *is proper and flat with geometrically normal fibers, then* $\underline{\mathrm{Hom}}_S(Z, g)$ *is quasi-compact.*

Proof. By Lemma B.2, we may assume that $Z \to S$ is flat with geometrically reduced fibers. We will use induction on the maximal fiber dimension d of $Z \to S$. After modifying S, we may assume that $W := Z_{\mathrm{norm}} \to S$ is flat with geometrically normal fibers. Let $Z_0 \hookrightarrow Z$ and $W_0 \hookrightarrow W$ be the closed substacks given by the conductor ideal of $W \to Z$.

After replacing S with a dense open subset, we may assume that $Z_0 \to S$ and $W_0 \to S$ are flat and that $W \to Z$ is an isomorphism over an open subset $U \subseteq Z$ that is dense in every fiber. In particular, since $Z_0 \cap U = \emptyset$, the dimensions of the fibers of $Z_0 \to S$ are strictly smaller than d. Thus, by induction we may assume that $\underline{\mathrm{Hom}}_S(Z_0, g)$ is quasi-compact. But

$$\begin{array}{ccc} W_0 & \overset{i}{\hookrightarrow} & W \\ {\scriptstyle h_0}\downarrow & \square & \downarrow{\scriptstyle h} \\ Z_0 & \overset{j}{\hookrightarrow} & Z \end{array}$$

is a bicartesian square and remains so after arbitrary base change over S since $W_0 \to S$ is flat. Indeed, that it is cartesian is [Hal17, Lem. A.3(i)]. That it is cocartesian and the commutes with arbitrary base change over S follows from the arguments of [Hal17, Lem. A.4, A.8] and the existence of pinchings of algebraic spaces [Kol11, Thm. 38].

It remains to prove that $H_{Z/S,g}(y) \to T$ is quasi-compact, where T is an integral scheme of finite type over S and $y\colon Z_T \to Y$ is a morphism. The bicartesian square above implies that

$$H_{Z/S,g}(y) \simeq H_{Z_0/S,g}(yj) \times_{H_{W_0/S,g}(yhi)} H_{W/S,g}(yh).$$

The result follows, since $H_{Z_0/S,g}(yj)$ and $H_{W/S,g}(yh)$ are quasi-compact and $H_{W_0/S,g}(yhi)$ is quasi-separated. $\square$

We have the following variant of h-descent [Ryd10, Thm. 7.4].

Lemma B.4. *Let* S *be an algebraic stack, let* T *be an algebraic* S*-stack and let* $g\colon T' \to T$ *be a universally subtrusive (e.g., proper and surjective) morphism of finite presentation such that* g *is flat over an open substack* $U \subseteq T$*. If* T *is weakly normal in* U *(e.g.,* T *normal and* U *open dense), then for every representable morphism* $X \to S$*, the following sequence of sets is exact:*

$$X(T) \longrightarrow X(T') \rightrightarrows X(T' \times_T T')$$

where $X(T) = \mathrm{Hom}_S(T, X)$ *etc.*

Proof. It is enough to prove that given a morphism $f\colon T' \to X$ such that $f \circ \pi_1 = f \circ \pi_2 \colon T' \times_T T' \to X$, there exists a unique morphism $h\colon T \to X$ such that $f = h \circ g$. By fppf-descent over U, there is a unique $h|_U\colon U \to X$ such that $f|_{g^{-1}(U)} = h|_U \circ g|_{g^{-1}(U)}$. Consider the morphism $\tilde{g}\colon \widetilde{T'} = T' \amalg U \to T$. The morphism $\tilde{f} = (f, h|_U)\colon \widetilde{T'} \to X$ satisfies $\tilde{f} \circ \tilde{\pi}_1 = \tilde{f} \circ \tilde{\pi}_2$ where $\tilde{\pi}_i$ denotes the projections of $\widetilde{T'} \times_X \widetilde{T'} \to \widetilde{T'}$. By assumption, $\tilde{g}$ is universally subtrusive and weakly normal. Thus, by h-descent [Ryd10, Thm. 7.4], we have an exact sequence

$$X(T) \longrightarrow X(\widetilde{T}') \rightrightarrows X\big((\widetilde{T}' \times_T \widetilde{T}')_{\mathrm{red}}\big).$$

Indeed, by smooth descent we can assume that S, T and $\widetilde{T}'$ are schemes so that [Ryd10, Thm. 7.4] applies. We conclude that $\tilde{f}$ comes from a unique morphism $h\colon T \to X$. □

We now have our last general reduction result.

Proposition B.5. *Let $w\colon W \to Z$ be a proper surjective morphism over S. Assume that $Z \to S$ has geometrically normal fibers and $W \to S$ is flat. If $\underline{\mathrm{Hom}}_S(W, g)$ is quasi-compact, then so is $\underline{\mathrm{Hom}}_S(Z, g)$.*

Proof. We may assume that S is an integral scheme. After replacing S with an open subscheme, we may also assume that $W \to Z$ is flat over an open subset $U \subseteq Z$ that is dense in every fiber over S and $W \times_Z W$ is flat over S. It remains to prove that $H_{Z/S,g}(y) \to T$ is quasi-compact, where T is an integral scheme of finite type over S and $y\colon Z_T \to Y$ is a morphism. By assumption, $H_{W/S,g}(yw) \to T$ is quasi-compact. Now consider the sequence:

$$H_{Z/S,g}(y) \longrightarrow H_{W/S,g}(yw) \rightrightarrows H_{W\times_Z W/S,g}(yv),$$

where $v\colon W \times_Z W \to Z$ is the natural map. There is a canonical morphism $\varphi\colon H_{Z/S,g}(y) \to E$, where E denotes the equalizer of the parallel arrows. Since $H_{W/S,g}(yw)$ is quasi-compact (and $H_{W\times_V W/S,g}(yv)$ is quasi-separated), the equalizer E is quasi-compact. It is thus enough to show that φ is quasi-compact. Thus, pick a scheme T' and a morphism $T' \to E$ and let us show that $H_{Z/S,g}(y) \times_E T'$ is quasi-compact.

By noetherian induction on T', we may assume that T' is normal. The morphism $T' \to E$ gives an element of $\mathrm{Hom}_Y(W_{T'}, X)$ such that the two images in $\mathrm{Hom}_Y(W_{T'} \times_{Z_{T'}} W_{T'}, X)$ coincide. Noting that $Z_{T'}$ is normal, Lemma B.4 applies to $W_{T'} \to Z_{T'}$ and gives a unique element in $\mathrm{Hom}_Y(Z_{T'}, X) = \mathrm{Hom}_T(T', H_{Z/S,g}(y))$. Thus, the morphism $\varphi_{T'}\colon H_{Z/S,g}(y) \times_E T' \to T'$ has a section. Repeating the argument with $T' = \operatorname{Spec} \kappa(t')$ for every point $t' \in T'$, we see that $\varphi_{T'}$ is injective, so the section is surjective. It follows that $H_{Z/S,g}(y) \times_E T'$ is quasi-compact. □

Proof of the main result.

Proof of Theorem B.1. As usual, we may assume that S is an affine integral scheme. By Lemma B.3, we may in addition assume that $Z \to S$ has geometrically normal fibers. Let $W \to Z$ be a proper surjective morphism with W a projective S-scheme [Ols05]. By replacing S with a dense open, we may assume that $W \to S$ is flat. By Proposition B.5, we may replace Z with W and assume that Z is a (projective) scheme. Repeating the first reduction, we may still assume that $Z \to S$ has geometrically normal fibers.

As before, it remains to prove that $H_{Z/S,g}(y) \to T$ is quasi-compact, where T is an integral S-scheme of finite type and $y\colon Z_T \to Y$ is an S-morphism. Hence, it suffices to prove the following claim.

Claim: Let S be integral. If $Z \to S$ is projective with geometrically normal fibers and $q\colon Q \to Z$ is representable with affine fibers, then $\mathbf{R}_{Z/S}(Q) \to S$ is quasi-compact.

Proof of Claim: Let $\overline{Q} = \operatorname{Spec}_Z(q_*\mathcal{O}_Q)$ and let $Q \to \overline{Q} \to Z$ be the induced factorization. Since $Q \to Z$ has affine fibers, $Q \to \overline{Q}$ is an isomorphism over an open dense subset $U \subseteq Z$. After replacing S with a dense open subscheme, we may assume that U is dense in every fiber over S. Since $\mathbf{R}_{Z/S}(\overline{Q}) \to S$ is affine [HR15b, Thm. 2.3(i)], it is enough to prove that $\mathbf{R}_{Z/S}(Q) \to \mathbf{R}_{Z/S}(\overline{Q})$ is quasi-compact. We may thus replace Q, Z, U and S with $Q \times_{\overline{Q}} (Z \times_S \mathbf{R}_{Z/S}(\overline{Q}))$, $Z \times_S \mathbf{R}_{Z/S}(\overline{Q})$, $U \times_S \mathbf{R}_{Z/S}(\overline{Q})$ and $\mathbf{R}_{Z/S}(\overline{Q})$. We may thus assume that $Q \to Z$ is an isomorphism over U.

Since Q is an algebraic space, there exists a finite surjective morphism $\tilde{Q} \to Q$ such that $\tilde{Q}$ is a scheme. In particular, there is a finite field extension $L/K(U)$ such that the normalization of Q in L is a scheme. Take a splitting field L'/L and let Z' be the normalization of Z in L'. Then $Q' := (Q \times_Z Z')_{\mathrm{norm}} = Q_{\mathrm{norm}/L'}$ is a scheme. By replacing S with a normalization in an extension of $K(S)$ and shrinking, we may assume that $Z' \to S$ and $Q' \to S$ are flat with geometrically normal fibers. By Proposition B.5, it is enough to prove that $\mathbf{R}_{Z'/S}(Q \times_Z Z')$ is quasi-compact.

There is a natural morphism $\mathbf{R}_{Z'/S}(Q') \to \mathbf{R}_{Z'/S}(Q \times_Z Z')$, which we claim is surjective. To see this, we may assume that S is the spectrum of an algebraically closed field. Then Z' and Q' are normal and any section $Z' \to Q \times_Z Z'$ lifts uniquely to a section $Z' \to Q'$. Indeed, $Z' \times_{Q \times_Z Z'} Q' \to Z'$ is finite and an isomorphism over U, hence has a canonical section. We can thus replace Q and Z with Q' and Z' and assume that Q is a scheme.

Since Q is a scheme, it is locally separated; hence, there is a U-admissible blow-up $Z' \to Z$ such that the strict transform $Q' \to Z'$ of $Q \to Z$ is étale [RG71, Thm. 5.7.11]. After shrinking S, we may assume that $Z' \to S$ is flat. Then since $U \subseteq Z'$ remains dense after arbitrary pull-back over S, we have that $\mathbf{R}_{Z'/S}(Q \times_Z Z') = \mathbf{R}_{Z'/S}(Q')$. Replacing $Q \to Z$ with $Q' \to Z'$ (Proposition B.5), we may thus assume that $Q \to Z$ in addition is étale.

Finally, we note that the étale morphism $Q \to Z$ corresponds to a constructible sheaf on $Z_{\text{Ét}}$ and that $\mathbf{R}_{Z/S}(Q)$ is nothing but the étale sheaf $f_{\text{Ét},*}Q$. By a special case of the proper base change theorem [SGA4$_3$, XIV.1.1], $f_{\text{Ét},*}Q$ is constructible, so $\mathbf{R}_{Z/S}(X) \to S$ is of finite presentation.

For the second part of the theorem on groups: let T be the the spectrum of an algebraically closed field and let $y\colon Z_T \to Y$ be a morphism. By the first part $H_{Z/S,g}(y) \simeq \mathbf{R}_{Z_T/T}(Q)$ is then a group scheme G of finite type over T, where $Q = X \times_Y Z_T$. Let $K = G_{\mathrm{ant}}$ be the largest anti-affine subgroup of G; it is normal, connected and smooth and the quotient G/K is affine [DG70, §III.3.8].

The universal family $G \times_T Z_T \to Q$ is a group homomorphism and induces a group homomorphism $K \times_T Z_T \to Q$. It is enough to show that this factors through the unit section of $Q \to Z_T$, because this forces $K = 0$. Note that for every stack $W \to T$, the pull-back $K \times_T W \to W$ is an anti-affine group in the sense that the push-forward of $\mathcal{O}_{K \times_T W}$ is $\mathcal{O}_W$ (flat base change).

We will now use the results on finitely presented filtrations in Appendix A. Since $Q \to Z_T$ has affine fibers, there is a finitely presented filtration $(Z_{T,i})$ of Z_T with strata $V_i = Z_{T,i} \smallsetminus Z_{T,i-1}$ such that $Q \times_{Z_T} V_i \to V_i$ are affine for every

i. By Chevalley's theorem, $Q \times_{Z_T} V_i^{[n]} \to V_i^{[n]}$ is affine for every i and $n \geq 0$. Since $K \times_T V_i^{[n]} \to V_i^{[n]}$ is anti-affine, it follows that $K \times_T V_i^{[n]} \to Q \times_{Z_T} V_i^{[n]}$ factors through the unit section $V_i^{[n]} \to Q \times_{Z_T} V_i^{[n]}$ for every i and n.

Let E be the equalizer of $K \times_T Z_T \to Q$ and the constant map $K \times_T Z_T \to Q$ to the unit. The above discussion shows that the monomorphism $E \to K \times_T Z_T$ is an isomorphism over every infinitesimal neighborhood $V_i^{[n]}$ of every stratum V_i, hence an isomorphism (Proposition A.3, using that the filtration is separating since Z_T is noetherian). $\square$

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