Local structure of stacks

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Outline

Deligne–Mumford stacks

Equivalent conditions for a stack $\mathscr{X}$ to be **Deligne–Mumford**:

- There is an étale atlas $p\colon U \to \mathscr{X}$.
- $\mathscr{X} = [R \rightrightarrows U]$ (étale groupoid).
- $\mathscr{X}$ algebraic w/ finite (étale) stabilizer groups.

Orbifold description

If $\mathscr{X}$ has a coarse moduli space X, then $\forall x \in |\mathscr{X}| = |X|$ exists:

- U affine
- $G_x = \mathrm{stab}(x)$ acting on U
- $\exists u \in U$ fix-point
- f étale, $f(u) = x$
- $\mathrm{stab}(u) \to \mathrm{stab}(x)$ isomorphism

$$\begin{array}{ccc} [U/G_x] & \xrightarrow{f} & \mathscr{X} \\ \downarrow & \square & \downarrow \\ U/G_x & \longrightarrow & X \end{array}$$

Tame Artin stacks

Abramovich, Olsson and Vistoli 2008 give a similar description for tame Artin stacks with finite stabilizers (positive characteristic).

Artin stacks

Many moduli problems ($\mathcal{M}_{g,n}$, $\mathcal{A}_g$, …) are Deligne–Mumford (or at least have finite stabilizers) stacks. Not all though: moduli of vector bundles, moduli of singular curves, … are **Artin stacks.**

Example

- $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$,
- $\mathrm{Aut}(\mathbb{P}^1, 0) = \mathbb{G}_a \rtimes \mathbb{G}_m$,
- $\mathrm{Aut}(\mathbb{P}^1, 0, \infty) = \mathbb{G}_m$,
- Aut(nodal cubic curve) $= \mathbb{G}_m$,
- Aut(cuspidal cubic curve) $= \mathbb{G}_a$.

GIT and good moduli spaces

Example (GIT, affine)

G reductive group (e.g., GL_n or SL_n) acting on affine scheme $U = \operatorname{Spec} A$. Then $[U/G]$ is an Artin stack. GIT quotient is $U/\!\!/G = \operatorname{Spec} A^G$.

$$\pi \colon [U/G] \to U/\!\!/G \quad \text{good/adequate moduli space}$$

Example (GIT, projective)

G reductive acting linearly on projective $(X, \mathcal{L})$. Then $X^{ss} \to X^{ss}/\!\!/G := \mathrm{Proj}\big(\bigoplus H^0(X, (\mathcal{L}^n)^G)\big)$.

$$\pi \colon [X^{ss}/G] \to X^{ss}/\!\!/G \quad \text{good/adequate moduli space}$$

- π **good moduli space** if G **linearly reductive** (e.g., $G = (\mathbb{G}_m)^r$ or char. zero).

GIT quotients as orbit spaces

Closed points of $U /\!\!/ G$ correspond to closed orbits of U. If $\overline{Gx} \cap \overline{Gy} \neq \emptyset$, then $\pi(x) = \pi(y)$. There is a unique closed orbit in every fiber of $\pi \colon U \to U /\!\!/ G$.

Example

1. $\mathscr{X} = [\mathbb{A}^1/\mathbb{G}_m]$ weights 1,
2. $\mathscr{X} = [\mathbb{A}^2/\mathbb{G}_m^2]$ weights (1,0) and (0,1),
3. $\mathscr{X} = [\mathbb{A}^2/\mathbb{G}_m]$ weights 1 and -1,
4. $\mathscr{X} = [C/\mathbb{G}_m]$ where C nodal cubic curve.

[pictures of $\mathscr{X} \to X = \mathscr{X}_{\text{gms}}$]

Local structure theorem

Theorem (Alper–Hall–R 2015)

$\mathscr{X}$ algebraic stack of finite type over $k = \overline{k}$ and $x \in \mathscr{X}(k)$ with

1. ***linearly reductive stabilizer*** G_x
2. *affine stabilizers G_y for all $y \in |\mathscr{X}|$*

Then there exists:

- *U affine*
- $G_x = \mathrm{stab}(x)$ *acting on U*
- $\exists u \in U$ *fix-point*
- *f étale, $f(u) = x$*
- $\mathrm{stab}(u) \to \mathrm{stab}(x)$ *isomorphism*

$$\begin{array}{ccc} [U/G_x] & \xrightarrow{f} & \mathscr{X} \\ \big\downarrow & & \\ U /\!\!/ G_x & & \end{array}$$

Remark: Conditions 1+2 are necessary.

Local structure theorem (smooth version)

We have the following more precise version when $\mathscr{X}$ is **smooth**.

$$[T_x/G_x] \longleftarrow [U/G_x] \longrightarrow \mathscr{X}$$

This is analogous to Weinstein's conjecture/Zung's theorem in differential geometry.

Known cases

1. Moduli stack of semistable curves, Alper–Kresch 2014 (separated diagonal)
2. Stack of log structures, Olsson 2003 (non-separated diagonal)

Outline of proof

0. $BG_x = \mathscr{X}_x^{[0]} \hookrightarrow \mathscr{X}_x^{[1]} \hookrightarrow \dots \hookrightarrow \mathscr{X}$
1. $\mathscr{T}_x := [T_x/G_x]$ smooth over k.
2. Lift $\mathscr{X}_x^{[0]} \hookrightarrow \mathscr{T}_x$ to closed immersions $\mathscr{X}_x^{[n]} \hookrightarrow \mathscr{T}_x$ (def thy)
3. Completions $\widehat{\mathscr{T}_x}$ and $\widehat{\mathscr{X}_x} \hookrightarrow \widehat{\mathscr{T}_x}$ exist (complete stacks)
4. Lift $\mathscr{X}_x^{[n]} \hookrightarrow \mathscr{X}$ to $\widehat{\mathscr{X}_x} \to \mathscr{X}$ (Tannaka duality)
5. $\exists \mathscr{W} \to \mathscr{X}$ finite type such that $\widehat{\mathscr{W}_w} \cong \widehat{\mathscr{X}_x}$ (equiv. Artin alg.)

Remarks

1. $\mathscr{T}_x = \mathbb{V}_{BG_x}(\mathcal{I}_x/\mathcal{I}_x^2)$.
3. Unobstructed b/c $H^n(BG_x, \mathcal{F}) = 0, \forall n > 0$ and $\mathscr{T}_x$ smooth.
5. $\mathscr{W} = [U/G_x]$: algebraization is done relative to BG_x.

Linear reductivity: (2)+(3). Affine stabilizers: (4).

Outline for schemes

If $\mathscr{X} = X$ is a scheme:

0. $X_x^{[n]} = \operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$
1. $T_x = \operatorname{Spec} \operatorname{Sym}_k(\mathfrak{m}/\mathfrak{m}^2)$.
2. $k[x_1, x_2, \ldots, x_n] = \operatorname{Sym}_k(\mathfrak{m}/\mathfrak{m}^2) \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^n$
3. $\widehat{\mathcal{O}}_{X,x} = \varprojlim_n \mathcal{O}_{X,x}/\mathfrak{m}^n$
4. $\operatorname{Spec}(\widehat{\mathcal{O}}_{X,x}) \to X$
5. $U \to X$ (for some open affine U containing x)

Question

What is $\widehat{\mathscr{X}_x} := \varinjlim_n \mathscr{X}_x^{[n]}$ for stacks?

Complete stacks

Definition

A noetherian stack $(\mathscr{X}, x)$ is **complete** if $\mathbf{Coh}(\mathscr{X}) \to \varprojlim_n \mathbf{Coh}(\mathscr{X}_x^{[n]})$ is an equivalence of categories.

- $(A, \mathfrak{m})$ complete local $\implies$ $(\operatorname{Spec} A, x)$ complete
- $(\mathscr{X}, x)$ complete $\implies$ $\mathscr{X} = \varinjlim_n \mathscr{X}_x^{[n]}$ (Tannaka duality)

Theorem (Alper–Hall–R 2015)

If $\pi\colon \mathscr{X} \to X$ is a good moduli space and (X, x_0) complete local scheme, then $(\mathscr{X}, x)$ is complete where x is the unique closed point above x_0.

- $\mathscr{T} \to T$ good moduli space $\implies$ $\widehat{\mathscr{T}} := \mathscr{T} \times_T \widehat{T}$
- $\exists \widehat{\mathscr{X}_x} \hookrightarrow \widehat{\mathscr{T}}$ (b/c $\widehat{\mathscr{T}}$ complete)

Equivariant Artin algebraization

Question

Given a complete local stack $\overline{\mathscr{W}}$, when is $\overline{\mathscr{W}} \cong \widehat{\mathscr{W}_w}$ for some stack $\mathscr{W}$ of finite type?

Theorem (Alper–Hall–R 2015)

*Let $\mathscr{X}$ be a stack of finite type, G linearly reductive group, and $(\overline{\mathscr{W}} = [\operatorname{Spec} \overline{A}/G], z)$ be a complete stack together with a **formally versal** map $\overline{\mathscr{W}} \to \mathscr{X}$. Then $\exists \mathscr{W} = [\operatorname{Spec} A/G] \to \mathscr{X}$ smooth and $\overline{\mathscr{W}} \cong \widehat{\mathscr{W}_w}$ over $\mathscr{X}$.*

In step 5 in proof of main theorem: $\overline{\mathscr{W}} = \widehat{\mathscr{X}_x}$.

Proof.

Refined Artin–Rees lemma and Artin approximation. □

Ten applications

1. **Luna's slice theorem** for non-normal schemes and algebraic spaces
2. **Sumihiro's theorem on torus actions** for non-normal schemes/alg.sp./DM-stacks
3. **BB-decompositions** for torus actions on smooth DM-stacks, Oprea 2006
4. **Equivariant versal deformation spaces** of sing. curves
5. Existence of **completion and henselization**: $\widehat{\mathscr{X}_x}$, $\mathscr{X}_x^h$.
6. $\widehat{\mathscr{X}_x} \cong \widehat{\mathscr{Y}_y} \quad \Longleftrightarrow \quad (\mathscr{X}, x) \xleftarrow{\text{ét}} (\mathscr{Z}, z) \xrightarrow{\text{ét}} (\mathscr{Y}, y)$.
7. **Compact generation** of $\mathbf{D}_{\mathrm{qc}}(\mathscr{X})$.
8. Criterion for **existence of good moduli space.**
9. Drinfeld's results on $\mathbb{G}_m$**-actions on algebraic spaces.**
10. **Global quotients and resolution property.**