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Stacky weighted blowups

D. Rydh

§ Introduction

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§ Cyclic covers and root stacks

§ Stacky blow-ups & applications

§ Weighted stacky blow-ups & applications

§ Resolution of singularities

§ Variation of GIT/wall-crossing

§ Introduction

toric / log geometry

Classical modifications

- normalization
- blow-ups
- weighted blow-ups

Stacky modifications

- root stacks, log root stacks
- stacky blow-ups (= blow-ups + root stacks)
- Kummer blow-ups (log geometry)
- stacky weighted blow-ups (generalizes notions above)

Most important case: blow-ups/root stacks w/ smooth centers on smooth varieties/stacks

Classical varieties

- weighted proj space $\mathbb{P}(d_0, d_1, \ldots, d_n)$ (singular)
- toric varieties (singular)
- $\mathrm{Proj}(R)$

Stacks

- weighted proj stacks (smooth)
- toric stacks (smooth)
- $\mathcal{P}\mathrm{roj}(R)$ "stacky Proj"

§ Applications

Classical blowups

char 0 [
- resolution of singularities (sing → smooth) [Hironaka 1964, ...]
- weak factorization ("res. of birational maps") [AKMW 2002]

- flattification (of maps) (non-flat → flat) [Raynaud–Gruson 1971, Hironaka 1975]

Stacky blowups

tame [
- étalification (ramified → unramified) [R'09]
- destackification (smooth stack → smooth scheme) [Bergh '17, Bergh–R '19]

char 0
- weak factorization of DM-stacks [R'15, Harper '17, Bergh '18]

Weighted stacky blowups

char 0 [
- res. of sings (easier + quicker) [Abramovich–Temkin–Włodarczyk '19]
- log. res. of sings / semistable reduction [Abramovich–Temkin–Włodarczyk '20 + '21]

- Cartierification ($\mathbb{Q}$-Cartier → Cartier) "Folklore"
- wallcrossing / variation of GIT. [R–Quek '21]

Stacky Proj

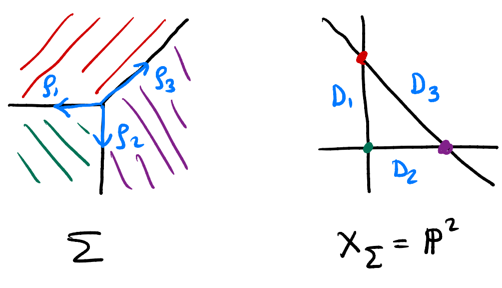
- "Gorensteinification" ($\mathbb{Q}$-Gorenstein → Gorenstein) (and $\mathbb{Q}$-invertible reflexive → invertible) [Abramovich–Hassett '09] (moduli of higher-dim varieties)

§ Toric varieties and stacks

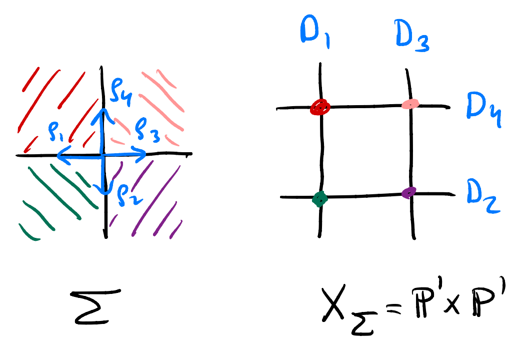
Input: A fan Σ (union of strongly convex polyhedral cones)

Output: A toric variety X_Σ

Example



Example



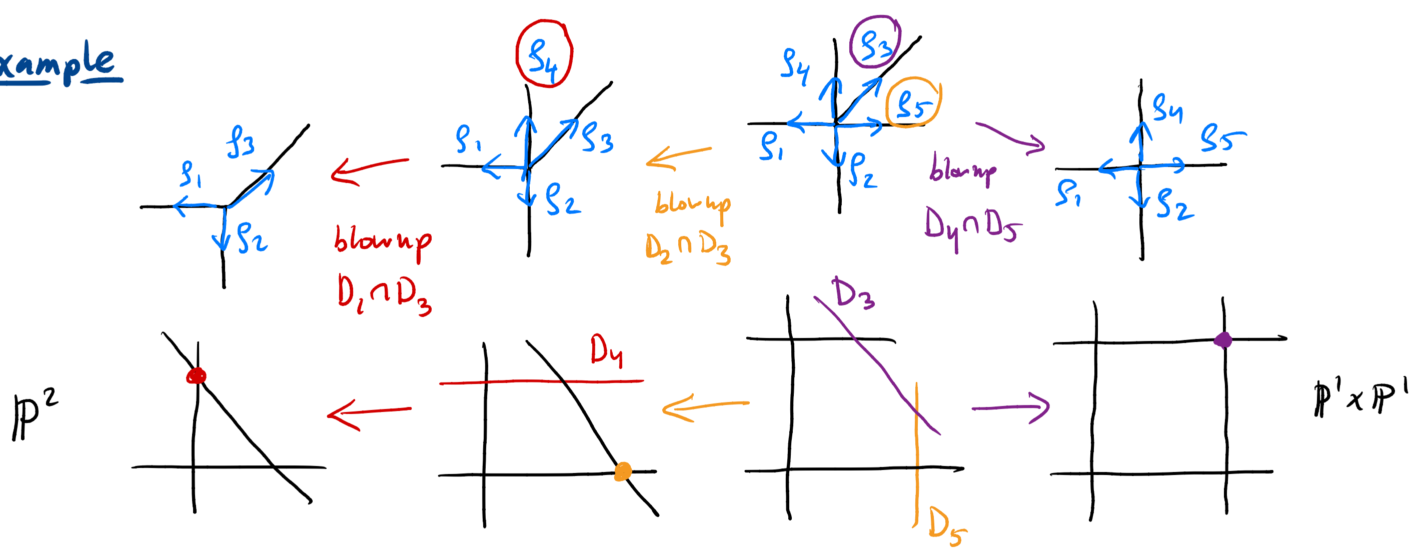
$\Sigma \subset \mathbb{R}^n \Rightarrow X_\Sigma \setminus \bigcup D_i \cong \mathbb{G}_m^n$

Input: set of toric divisors D_i $\rightsquigarrow$ ray $\rho_E = \sum \rho_i$ and $Z = \bigcap D_i$

Output: star subdivision Σ' = subdivide Σ by adding ray ρ_E.

Fact X_Σ smooth at Z $\Rightarrow$ $X_{\Sigma'} = \mathrm{Bl}_Z X_\Sigma$, $\rho_E \leftrightarrow$ exceptional divisor

Example



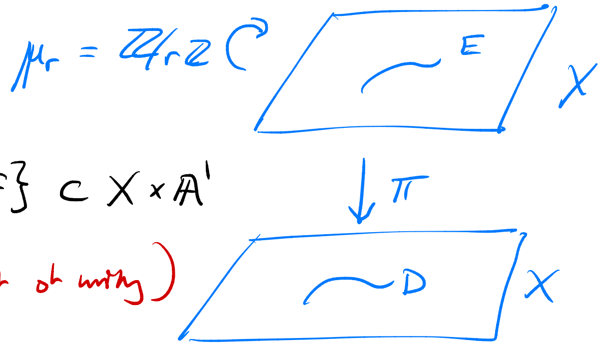
Input (simplicial) fan Σ

Output toric stack $\mathcal{X}_\Sigma$ (smooth with finite stabilizer groups)

X_Σ can be singular, D_i only Weil divisors ($\mathbb{Q}$-Cartier)

$\mathcal{X}_\Sigma$ smooth, D_i smooth and Cartier, star subdivision = blowup in smooth center.

§ Cyclic covers



Input: $D = \{f = 0\} \subset X$ variety, $r \in \mathbb{N}$

Output: cyclic cover $X' := \mathrm{Spec}_X\, \mathcal{O}_X[z]/(z^r - f) = \{z^r = f\} \subset X \times \mathbb{A}^1$

$\mathbb{Z}/r\mathbb{Z} \circlearrowright X'$, $a \in \mathbb{Z}/r\mathbb{Z}$: $z \longmapsto \xi^a z$ (ξ primitive rth root of unity)

Properties
- π finite and flat
- π ramified along D, $\pi^{-1}(D) = rE$, $E = \{z = 0\}$.

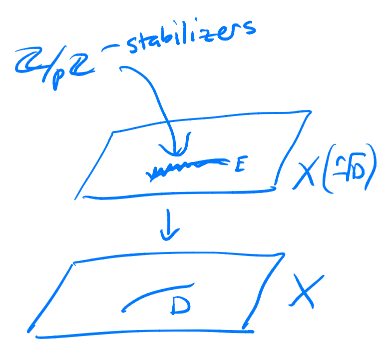
Slightly more general:

Input: $D = \{s = 0\}$, $s \in \Gamma(X, L^r)$ for some line bundle L, $r \in \mathbb{N}$

Drawbacks
- X' depends on f (or s) and not merely D. (e.g. if f non-vanishing, get $\mathbb{Z}/r\mathbb{Z}$-torsor)
- Need $L = \mathcal{O}(\frac{1}{r}D)$.

§ Root stacks

(Abramovich – Graber – Vistoli '08, Cadman '06, Matsuki – Olsson '05)



Input: • $D \hookrightarrow X$ Cartier divisor (section of line bundle), $r \in \mathbb{N}$

Output: root stack $X(\sqrt[r]{D}) \xrightarrow{\pi} X$

(more natural notation $X(\frac{1}{r}D)$ or $X(\sqrt[r]{(L, s)})$)

If $D = \{f = 0\}$, then $X(\sqrt[r]{D}) = X(\sqrt[r]{f=0}) = [X'/\mathbb{Z}/r\mathbb{Z}]$ only depends on D and doesn't need $\mathcal{O}(\frac{1}{r}D)$.

Properties
- $\exists$ Cartier divisor $E \subset X(\sqrt[r]{D})$ s.th. $\pi^{-1}(D) = rE$. (univ property: $\exists$ "$E = \frac{1}{r}D$")
- π flat, stacky modification: π proper and $X(\sqrt[r]{D}) \setminus E \longrightarrow X \setminus D$ isomorphism
- $X(\sqrt[r]{D})$ has $\mathbb{Z}/r\mathbb{Z}$-stabilizer groups along E.
- X, D smooth $\implies X(\sqrt[r]{D})$ smooth

§ Stacky blow-ups

Input: $Z \hookrightarrow X$ closed subscheme, $r \in \mathbb{N}$

Output: stacky blowup $Bl_{(Z,r)} X \xrightarrow{\pi} X$

Def $Bl_{(Z,r)} X = (Bl_Z X)\left(\sqrt[r]{\pi^{-1}(Z)}\right)$, exc. div. $E := \frac{1}{r}\pi^{-1}(Z)$

Properties
- π stacky modification ($\mathbb{Z}/r\mathbb{Z}$-stab along E)
- $\pi^{-1}(Z) = rE$
- $\pi|_{X \setminus Z}$ iso
- X, Z smooth $\Rightarrow$ $Bl_{(Z,r)} X$ smooth.

Ex
- $Bl_{(Z,1)} X = Bl_Z X$ (ordinary blowup)
- $Bl_{(D,r)} X = X(\sqrt[r]{D})$ (root stack)

Thm (Étalification, R'09) If $X' \xrightarrow{f} X$ generically étale (e.g. stacky modification or Galois cover) and tamely ramified (auto. in char 0) then $\exists$ sequences of stacky blowups

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{\text{sbu's}} & X' \\ \text{étale} \downarrow \tilde{f} & \circ & \downarrow f \\ \tilde{X} & \xrightarrow{\text{sbu's}} & X \end{array}$$

Cor Stacky blowups are cofinal among tame stacky modifications.

Thm (destackification, Bergh '17, Bergh–R '19) $\mathcal{X}$ smooth tame stack. $\exists$ "canonical"

$$\begin{array}{ccc} \underset{\text{smooth}}{\mathcal{X}'} & \xrightarrow{\text{sbu's}} & \underset{\text{smooth}}{\mathcal{X}} \\ \text{root stacks} \downarrow & \circ & \downarrow \text{coarse space} \\ \underset{\text{smooth}}{X'} & \xrightarrow{\text{modif}} & X \end{array}$$

s.th X' algebraic space (variety if X variety)

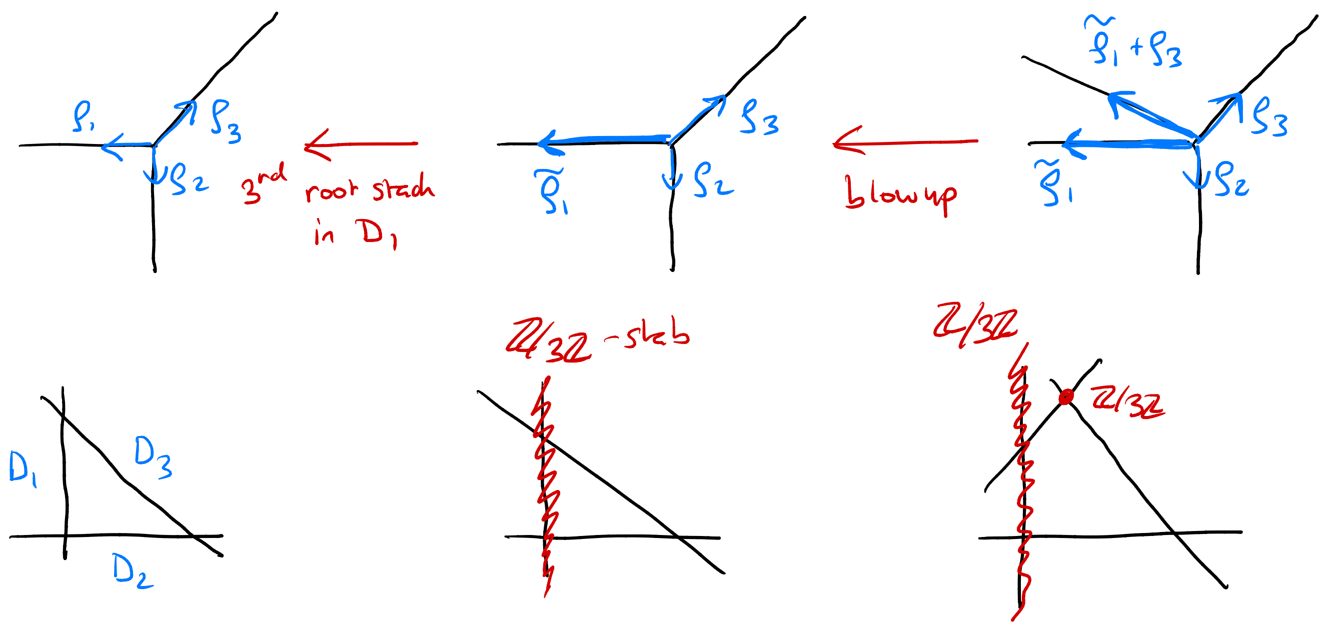
($\Rightarrow$ resolution of tame finite quotient singularities)

§ Toric stacks (cont.)

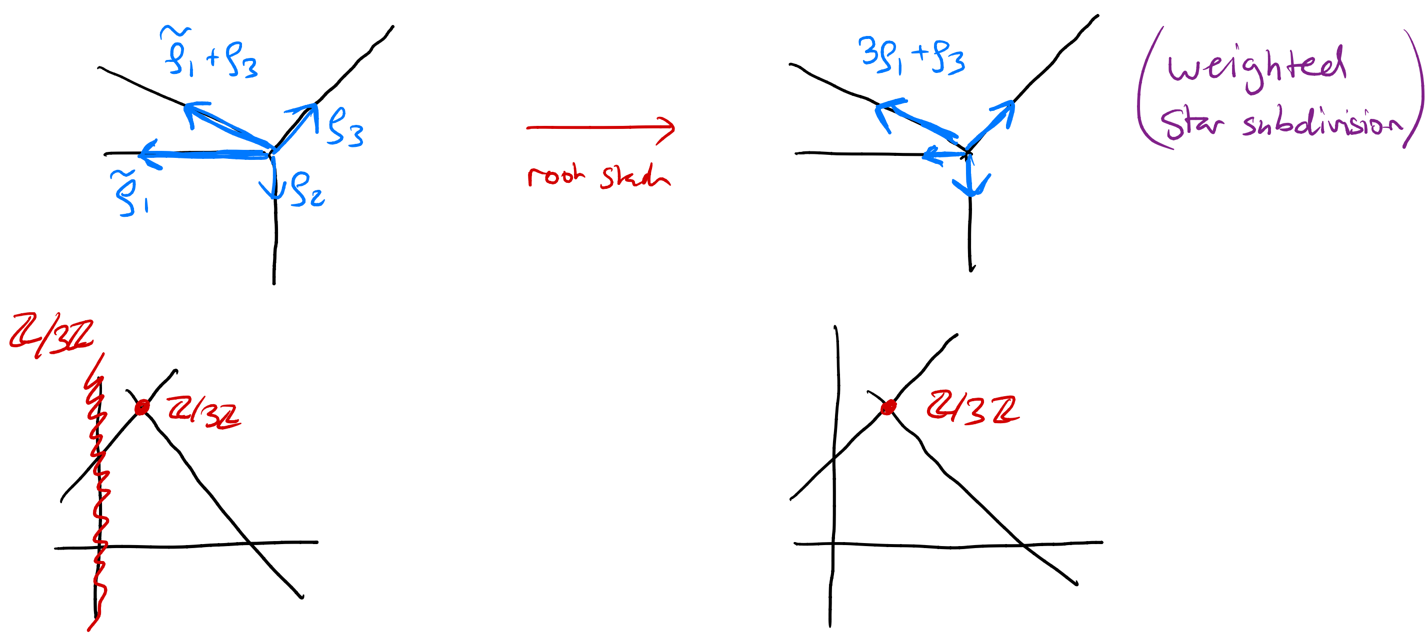
Input Σ fan, $\bar{d} = (d_1, \dots, d_n) \in \mathbb{Z}_+^n$ (replace ρ_i with $d_i \rho_i$)

Output $\mathcal{X}_{\Sigma, \bar{d}} = \mathcal{X}_\Sigma \left(\sqrt[d_1]{D_1}, \dots, \sqrt[d_n]{D_n} \right)$

Ex



Ex (cont) Can deroot $\tilde{\rho}_1$ after blowup



§ Weighted stacky blowups

Input: A filtration $\mathcal{I}_0 = \mathcal{O}_X \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \cdots$ of ideals $(\mathcal{I}_a \mathcal{I}_b \subseteq \mathcal{I}_{a+b})$

Output: weighted stacky blowup $Bl_{\mathcal{I}_\bullet} X \xrightarrow{\pi} X$

Def $Bl_{\mathcal{I}_\bullet} X = \mathcal{P}roj\left(\bigoplus_{d\geq 0} \mathcal{I}_d\right) = \left[Spec(\oplus \mathcal{I}_d) \setminus V(\mathcal{I}_{\geq 1}) \,/\, \mathbb{G}_m \right]$

Def "marked ideal" $(\mathcal{J}, d)$ = smallest filtration $\mathcal{I}_\bullet$ with $\mathcal{J} \subseteq \mathcal{I}_d$.

Ex • $Z \hookrightarrow X$, $\mathcal{I}_\bullet = (\mathcal{I}_Z, 1) \Rightarrow$ • $\mathcal{I}_n = \mathcal{I}_Z^n \ \forall n$ (usual $\mathcal{I}_Z$-adic filtration)

• $Bl_{\mathcal{I}_\bullet} X = Bl_Z X$ usual blowup.

• $Z \hookrightarrow X$, $\mathcal{I}_\bullet = (\mathcal{I}_Z, r) \Rightarrow$ • $\mathcal{I}_n = \mathcal{I}_Z^{\lceil n/r \rceil} \ \forall n$

• $Bl_{\mathcal{I}_\bullet} X = Bl_{(Z,r)} X$ stacky blowup.

Properties
- $\exists$ exc. div $E \subset Bl_{\mathcal{I}_\bullet} X$, $nE \subseteq \pi^{-1}(V(\mathcal{I}_n))$ w/ equality for suff div n.
- π stacky modification, isomorphism outside $V(\mathcal{I}_1)$.
- $Bl_{\mathcal{I}_\bullet} X$ smooth if X smooth and $\mathcal{I}_\bullet$ smooth filtration

Def $\mathcal{I}_\bullet$ smooth if locally $\mathcal{I}_\bullet = (f_1, d_1) + (f_2, d_2) + \cdots + (f_m, d_m)$ where $V(f_1, \dots, f_m)$ smooth of codimension m.

"$\mathcal{I}_1 = (f_1^{1/d_1}, f_2^{1/d_2}, \dots, f_m^{1/d_m})$" $\quad \mathcal{I}_n = \left(f_1^{a_1} \cdots f_m^{a_m} : \sum_{i=1}^m a_i d_i \geq n\right)$

$\mathcal{I}_\bullet$ = integral closure of $\left((f_1^{N/d_1}, \dots, f_m^{N/d_m}), N\right)$ for N s.th. $d_i | N \ \forall i$.

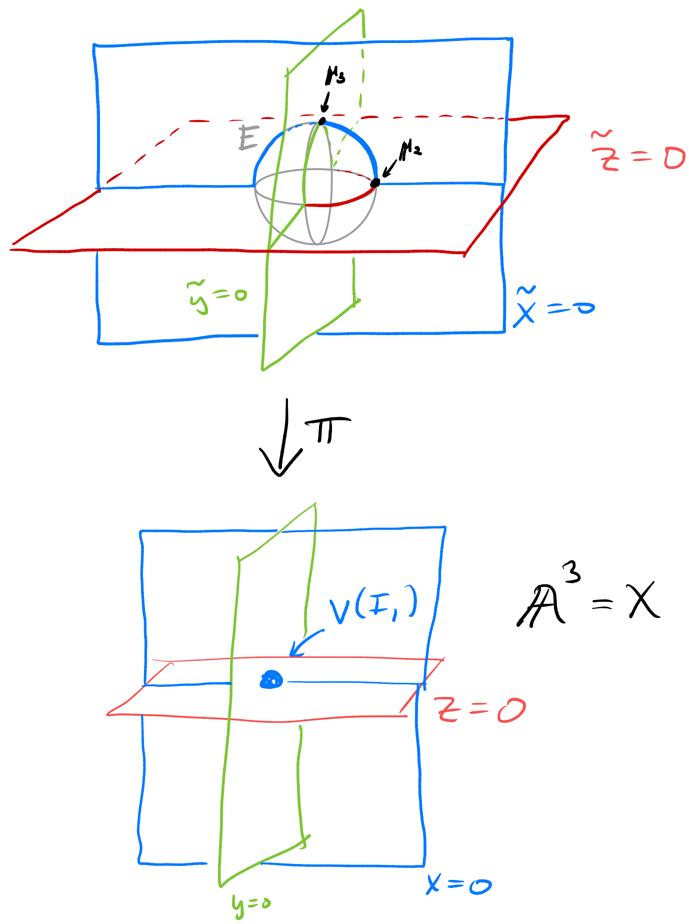
Coarse space of $Bl_{\mathcal{I}_\bullet} X$ is $Bl_{\mathcal{I}_N} X$ — a weighted blowup

Ex $\mathcal{X}_\Sigma$, $\mathcal{I}_\bullet = \sum (\mathcal{I}_{D_i}, a_i) \Rightarrow Bl_{\mathcal{I}_\bullet} \mathcal{X}_\Sigma = \mathcal{X}_{\Sigma'}$ $\quad \Sigma'$ = add ray $\sum a_i \rho_i$ to Σ

Ex $X = \mathbb{A}^3, \quad I_\bullet = (x,1) + (y,2) + (z,3)$

$$X' = Bl_{I_\bullet} X \xrightarrow{\pi} X$$

$$E = \mathcal{P}roj\Big(\bigoplus_{n\geq 0} I_n / I_{n+1}\Big) = \mathbb{P}(1,2,3)$$



x-chart: $\mathbb{A}^3$: coords $x, \frac{y}{x}, \frac{z}{x}$

y-chart: $[\mathbb{A}^3/\mu_2]$: coords $\frac{x}{y^{1/2}}, \; y^{1/2}, \; \frac{z}{(y^{1/2})^3}$
with weights $1, \; -1, \; 3=1$

z-chart: $[\mathbb{A}^3/\mu_3]$: coords $\frac{x}{z^{1/3}}, \; \frac{y}{(z^{1/3})^2}, \; z^{1/3}$
with weights $1, \; 2, \; -1$

Generalized constructions

- Weighted normal cone $N^\vee_{I_\bullet} = Spec(Gr_{I_\bullet} \mathcal{O}_X) = Spec\Big(\bigoplus_{n\geq 0} I_n/I_{n+1}\Big)$, $E = \mathcal{P}roj(N^\vee_{I_\bullet})$
- Deformation to wtd normal cone: $D_{I_\bullet} = Spec\Big(\bigoplus_{n\in\mathbb{Z}} I_n\Big) \quad (I_{-n} = \mathcal{O}_X)$

$$\begin{array}{ccccc}
Z & \hookrightarrow & Z\times\mathbb{A}^1 & \hookleftarrow & Z \\
\downarrow 0 & & \downarrow & & \downarrow I_1 \\
N_{I_\bullet} & \hookrightarrow & D_{I_\bullet} & \hookleftarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
X & \hookrightarrow & X\times\mathbb{A}^1 & \hookleftarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\{0\} & \hookrightarrow & \mathbb{A}^1 & \hookleftarrow & \{1\}
\end{array}$$

§ Resolution of singularities

Goal Resolve singular variety Z.

Strategy (Hironaka, Bierstone-Milman, Villamayor, Włodarczyk, ...)

1) (locally) embed $Z \hookrightarrow X$ w/ X smooth.
2) Resolve Z by blowups on X w/ smooth centers in Z.

} embedded resolution

3) Show that algorithm is smooth-functorial ($\Rightarrow$ does not depend on X)
4) Focus on I_Z.

Main invariant order of vanishing

$$\mathrm{ord}_p(I) = \max\{d : I \subset (m_p)^d\}$$

$$\mathrm{ord}_p(I) \geq d \iff p \in V(I, D(I), D^2(I), \dots, D^{d-1}(I))$$

$$\mathrm{ord}_p(I) = 1 \iff p \in V(I) \text{ and } \exists Z \hookrightarrow H \hookrightarrow X \text{ at } p$$

(H smooth hypersurface)

Ex Whitney umbrella $Z \subset \mathbb{A}^3$

$I = (\underbrace{x^2 - y^2 z}_{f})$

$\mathrm{Sing}(Z) = V\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = V(x, y^2, yz)$

$\mathrm{ord}_p(I) = 2 \quad \forall p \in S$

$H = \{x = 0\}$

Main ingredient (char 0!) $\exists$ (locally) hypersurface of maximal contact H

a smooth hypersurface containing locus of maximal order

Rough description

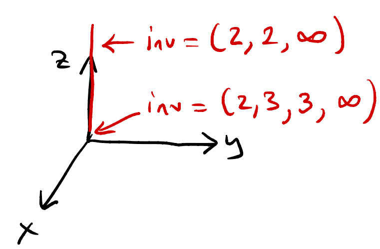
$d_1 = \max\{\mathrm{ord}_p I : p \in X\}$ $\qquad \{p : \mathrm{ord}_p I = d_1\} \subset H_1$

$d_2 \approx \max\{\mathrm{ord}_p I|_{H_1} : p \in H_1\}$ $\qquad \{p : \mathrm{ord}_p I|_{H_1} = d_2\} \subset H_2$

$d_3 \approx \quad \vdots$

$\mathrm{inv}(p) = (d_1 \leq d_2 \leq \dots \leq d_{n+1} = \infty)$. Blowup locus where invariant maximal $= H_1 \cap H_2 \cap \dots \cap H_n$

Ex (Whitney umbrella cont.)



at origin: $H_1 = V(x)$, $H_2 = V(y)$, $H_3 = V(z)$

$\Rightarrow$ blow up origin

chart z: $x' = \frac{x}{z}$, $y' = \frac{y}{z}$, $z' = z$

$$x^2 - y^2 z = (x'^2 - y'^2 z') z'^2$$

Let $\tilde{Z}$ = strict transform of Z $\Rightarrow I_{\tilde{Z}} = I_Z \cdot I_E^{-d_1}$

$I_{\tilde{Z}} = x'^2 - y'^2 z'$ same equation again! (and invariant has not dropped)

Reason: $d_1 \neq d_2$, Over H_1: $y^2 z = y'^2 z'^3$ and would like to remove 3 copies of E.
$d_1 = 2$, $d_2 = 3$

Solution: (Hironaka) Use history: remember exceptional divisors

Modified invariant $\approx (d_1, s_1, d_2, s_2, \ldots)$ $s_i = \#$ exc div through p.

Easier solution (Abramovich – Temkin – Włodarczyk '19, McQuillan '19)

Weighted blowup in $J_\bullet = (I_{H_1}, \frac{N}{d_1}) + (I_{H_2}, \frac{N}{d_2}) + (I_{H_3}, \frac{N}{d_3}) + \cdots$

$\Longrightarrow$ easy invariant drops! (N chosen s.th. $\frac{N}{d_i} \in \mathbb{Z}$ and relatively prime)

Ex: $I = (x^2 - y^2 z)$, $J_\bullet = (x, \frac{6}{2}) + (y, \frac{6}{3}) + (z, \frac{6}{3}) = (x, 3) + ((y,z), 2)$

chart z: $z' = z^{1/2}$, $y' = \frac{y}{z'^2}$, $x' = \frac{x}{z'^3}$

$$x^2 - y^2 z = x'^2 z'^6 - y'^2 z'^4 z'^2 = \underbrace{(x'^2 - y'^2)}_{} \cdot z'^6$$

strict tfm has invariant $(2, 2, \infty) < (2, 3, 3, \infty)$

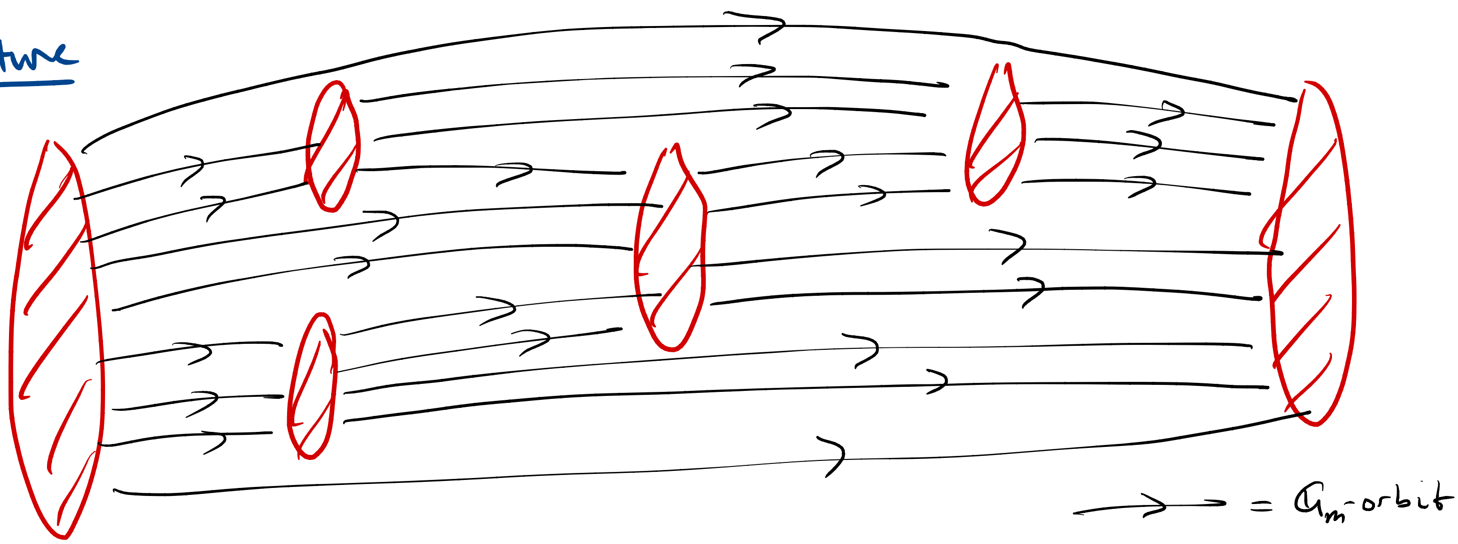
In general: $\pi^{-1} I = \tilde{I} \cdot I_E^N$ and $\operatorname{inv}(\tilde{I}) < \operatorname{inv}(I)$.

§ Variation of GIT

Input: • $\mathbb{G}_m \circlearrowright X$ projective
• $\mathcal{L}$ ample $\mathbb{G}_m$-equivariant l.b.

($\Leftrightarrow$ $X \hookrightarrow \mathbb{P}(V)$
$\mathbb{G}_m \circlearrowright V$ vector space.)

Picture

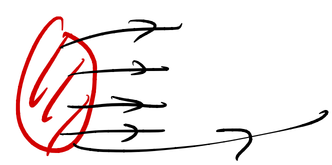


$d_1 < d_2 < d_3 < d_4 < d_5$

$\{d_1, d_2, \dots, d_n\}$ = weights of $\mathcal{L} / V$.

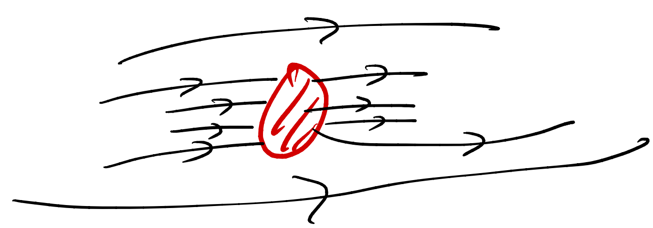
Fixpoints $X^{\mathbb{G}_m} = \coprod X^\circ_{d_i}$

Def $X^+_{d_i} = \{ x \in X : \lim_{t \to 0} tx \in X^\circ_{d_i} \}$



$X^-_{d_i} = \{ x \in X : \lim_{t \to \infty} tx \in X^\circ_{d_i} \}$

$X_d = X \setminus \bigcup_{d_j < d} X^-_{d_j} \cup \bigcup_{d_j > d} X^+_{d_j}$



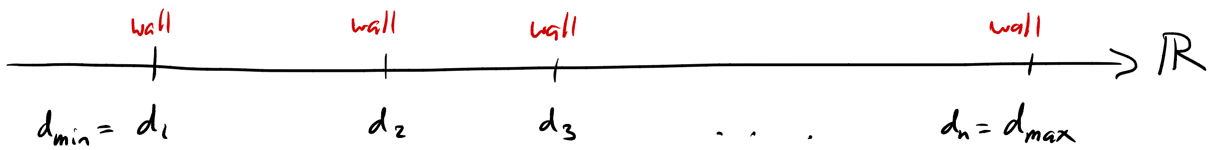
Fact (GIT) • $X_d = X^{ss, \mathcal{L} \otimes \chi^d}$

• $\exists\, X_d \longrightarrow X /\!\!/_{\mathcal{L} \otimes \chi^d} \mathbb{G}_m$ good quotient

Rmk $X^{+}_{d_i}, X^{\circ}_{d_i}, X^{-}_{d_i} \subset X_{d_i}$

Def $\mathcal{M}(d) = [X_d / \mathbb{G}_m]$

Rmk
- $X_d^{\mathbb{G}_m} = \emptyset \iff d \notin \{d_1, \dots, d_n\} \iff \mathcal{M}(d)$ Deligne–Mumford
- $X_d = X_{d'} \iff d, d' \in (d_i, d_{i+1})$
- $X_d = \emptyset \iff d < d_{min}$ or $d > d_{max}$



- $X_{d_i+\varepsilon} = X_{d_i} \setminus X^{-}_{d_i}$
  $X_{d_i-\varepsilon} = X_{d_i} \setminus X^{+}_{d_i}$

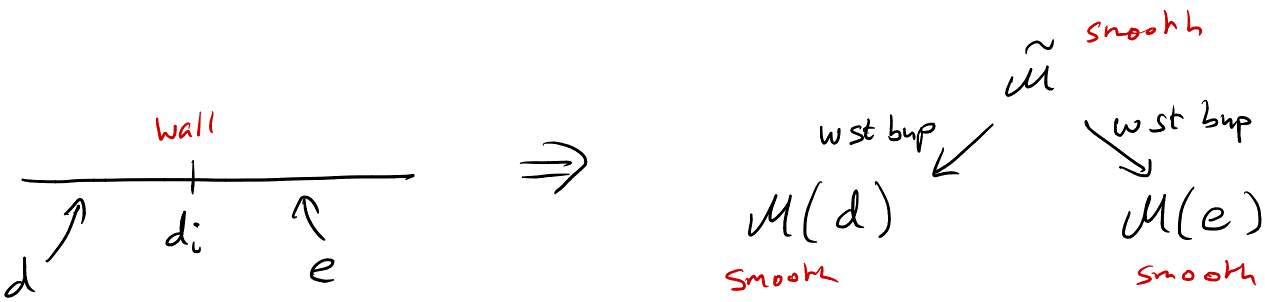
§ Wall-crossing

Thm (wall-crossing, R-Quek '21) X smooth. Fix a wall d_i.

(1) $\exists$ canonical smooth filtration $I^{+}_{\bullet}$ on X_{d_i} with $X^{+}_{d_i} = V(I^{+}_1)$

(2) ——— " ——— $I^{-}_{\bullet}$ —— " —— $X^{-}_{d_i} = V(I^{-}_1)$

(3) $Bl_{I^{+}_{\bullet}} \mathcal{M}(d_i+\varepsilon) \cong Bl_{I^{-}_{\bullet}} \mathcal{M}(d_i - \varepsilon)$



Rmk Very easy result. Wellknown that one gets

$$\begin{array}{ccccc} & & \tilde{M} \ (\text{singular}) & & \\ & \overset{\text{w bup}}{\swarrow} & & \overset{\text{w bup}}{\searrow} & \\ M(d) & & & & M(e) \\ \text{singular} & & & & \text{singular} \end{array}$$

Brion–Procesi '90
Thaddeus '96
Dolgachev–Hu '98

Generalizations

- X smooth DM-stack.
- replace $\mathbb{G}_m$ with reductive group G and $G \xrightarrow{\chi} \mathbb{G}_m$ (with limitations)
- replace $G \circlearrowright X$, χ with $\mathcal{X}$, $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ $(\mathcal{X} \leftrightarrow [X/G])$
  $\mathcal{X} \downarrow_{\text{gms}} X$

§ Weak factorization

Input: X, Y smooth proper birational varieties/DM-stacks in char 0

Thm (weak factorization) $\exists$ "canonical"

$$\begin{array}{ccccccccc} & X_1 & & X_3 & & & X_{n-1} & & \\ \swarrow & & \searrow \ \swarrow & & \searrow & \cdots & \swarrow & & \searrow \\ X = X_0 & & X_2 & & & & & & X_n = Y \end{array}$$

[AKMW '02] (varieties) blowups in smooth centers
[R '15, Harper '17, Bergh '18] (stacks) stacky blowups in smooth centers
[R '21] (stacks) weighted stacky blowups in smooth centers

Step 1 $\exists$ canonical cobordism $\mathbb{G}_m \circlearrowright B$ smooth, $B_{d_1+\varepsilon} \cong X$, $B_{d_n-\varepsilon} \cong Y$

Step 2 Wallcrossing.