Taming wild ramification with stacks

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May 23, 2011 / New York City

KTH Mathematics

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6. Weak factorization conjecture for stacks.

State of the art

Good notion of stacky blow-ups in tame case (e.g., char. zero):

$$\begin{aligned}\text{tame stacky blow-up} &= \text{sequence of Kummer blow-ups}\\ &= \text{sequence of blow-ups and root stacks}\end{aligned}$$

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In pure characteristic p, have defined **Artin–Schreier stacks**:

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Stack of branched covers — side note

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Stack two more logs on top, running them in the opposite directions, and then fill the space between with long tree **branches**. **Cover** the **branches** with more . . .

Contents

Outline

Assumptions and terminology

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- A Deligne–Mumford stack is **tame** if $\forall x \in |X|$, $\operatorname{char} k(x) \nmid |\operatorname{stab}(x)|$.

Stacky modifications

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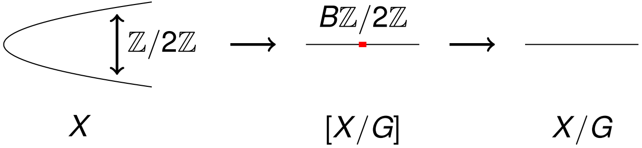
We let $\mathbf{Mod}(Y, U)$ denote the category of modifications of (Y, U) and $\mathbf{Mod}_{\text{stacky}}(Y, U)$ denote the 2-category of stacky modifications.

Examples of stacky modifications

Example

Let G be a finite group acting on a scheme X. Let $U \subset X$ be the locus where G acts freely. Then $([X/G], U/G) \to (X/G, U/G)$ is a stacky modification.



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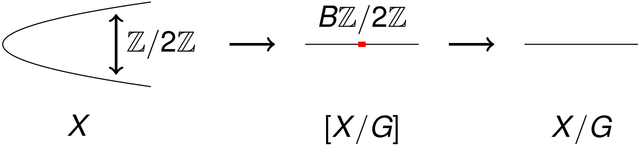

Example

Let X be an orbifold with coarse moduli space X_{cms}. Then $X \to X_{\mathrm{cms}}$ is a stacky modification.

Stacky modifications (cont.)

Lemma

Let $U \subseteq X$ be open dense.

- $\mathbf{Mod}_{\mathrm{stacky}}(X, U)$ *is equivalent to a directed* 1*-category.*
- $\mathbf{Mod}(X, U)$ *is equivalent to a partially ordered set.*

Stacky modifications (cont.)

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Remark: Every stacky modification $\pi\colon X \to Y$ factors as

$$X \xrightarrow{\text{stacky modification}} X_{\mathrm{cms}/Y} \xrightarrow{\text{modification}} Y$$

Blow-ups

A modification $p\colon (X, U) \to (Y, U)$ is a **blow-up** if there exists a closed substack $Z \hookrightarrow Y$ such that

1. $X = \mathrm{Bl}_Z Y = \mathrm{Proj}_Y \left(\bigoplus_{k \geq 0} \mathcal{I}^k \right)$ where $Z = V(\mathcal{I})$.
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In diagrams we will denote modifications with $\mathfrak{M}$ and blow-ups with $\mathfrak{B}$. Filled squares denote strict transforms.

Properties of blow-ups

1. **[Open extension]**

2. **[Closed extension]**

3. **[Étale quasi-extension]**

4. **[Strong cofinality]**

Properties of blow-ups

1. **[Open extension]** Blow-ups can be extended over an open immersion $Y \subseteq \overline{Y}$. [trivial]

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4. **[Strong cofinality]** Every modification is dominated by a blowup. [flatification]

$$\begin{tikzcd}
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- Wild modifications in mixed characteristic: sequences of Kummer blow-ups and Kummer–Artin–Schreier stacks ???

Outline

Generalized effective Cartier divisors

Recall that an effective Cartier divisor $D \hookrightarrow X$ can be described as a line bundle $\mathcal{L}$ together with a regular section $s \in \Gamma(X, \mathcal{L})$.

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We let $\mathrm{Div}(X) \hookrightarrow \mathrm{Div}_{\mathrm{gen}}(X)$ denote (generalized) effective Cartier divisors. If $D = (\mathcal{L}, s) \in \mathrm{Div}_{\mathrm{gen}}(X)$ we write $\mathcal{L} = \mathcal{O}(D)$, $s = s_D$ and $rD = (\mathcal{L}^r, s^r)$.

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Fact

$$\mathrm{Div}_{\mathrm{gen}}(X) = \mathrm{Mor}(X, [\mathbb{A}^1/\mathbb{G}_m])$$

Kummer extensions

Let K be a field. A Kummer extension K'/K is an extension of the form

$$K' = K[z]/z^r - s$$

where $s \in K^*$. They are in one-to-one correspondence with $\mathrm{H}^1(K, \mu_r) = K^*/(K^*)^r$ via the Kummer sequence

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The easiest globalization of a Kummer extension is a uniform cyclic covering $\pi\colon X' \to X$ specified by $D \in \mathrm{Div}(X)$ and an rth root of $\mathcal{O}(D)$ in $\mathrm{Pic}(X)$. Globally $X' \hookrightarrow \mathbb{V}(\mathcal{O}(-D)^{1/r})$. Locally,

$$X' = \mathrm{Spec}(A[z]/z^r - s) \to X = \mathrm{Spec}(A)$$

so that generically we obtain a Kummer extension $K(X')/K(X)$.

Root stacks

Definition

Let $D \in \mathrm{Div}_{\mathrm{gen}}(X)$ and $r \geq 1$ an integer. The **root stack** $X_{D,r}$ is the X-stack defined as

$$\mathrm{Mor}\left(T, X_{D,r}\right) = \left\{f \colon T \to X, E \in \mathrm{Div}_{\mathrm{gen}}(T) \mid f^* D = rE\right\}$$

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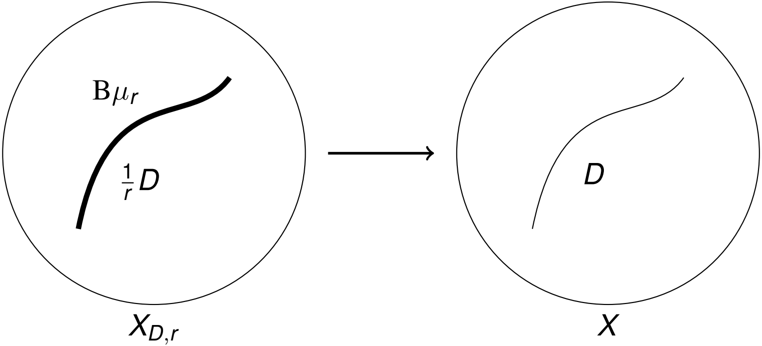
Facts

1. $X_{D,r}$ is a tame Artin stack and Deligne–Mumford if r is invertible along D.
2. $\pi\colon X_{D,r} \to X$ is a flat tame stacky modification.
3. $\frac{1}{r}D \to D$ is a μ_r-gerbe. Here $\frac{1}{r}D \in \mathrm{Div}_{\mathrm{gen}}(X_{D,r})$ is the tautological divisor such that $r\left(\frac{1}{r}D\right) = D$.
4. If $D = rE$ then $\left(X_{D,r}\right)^{\mathrm{norm}} = X^{\mathrm{norm}}$.

Root stacks (picture)



Locally a ramified μ_r-cover:

$$X = \mathrm{Spec}(A), \quad D = \{s = 0\}, \quad X_{D,r} = \left[\mathrm{Spec}(A[z]/z^r - s)/\mu_r\right]$$

Kummer blow-ups

Definition

Let $Z \hookrightarrow X$ be a closed subscheme and $r \geq 1$ an integer. The rth **Kummer blow-up** of Z is the stacky modification

$$\mathrm{Bl}_{Z,r}(X) := \mathrm{Bl}_Z(X)_{E,r} \to \mathrm{Bl}_Z(X) \to X.$$

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Explicitly, if $Z = V(\mathcal{I})$, then we have that

$$\mathrm{Bl}_{Z,r}(X) = \mathrm{Proj}_X(\mathcal{A}), \quad \text{(stacky proj)}$$

where $\mathcal{A} = \bigoplus_{k \in \mathbb{N}} \mathcal{I}^{\lceil k/r \rceil} = \underset{0}{\mathcal{O}} \oplus \underset{1}{\mathcal{I}} \oplus \underset{2}{\mathcal{I}} \oplus \cdots \oplus \underset{r}{\mathcal{I}} \oplus \underset{r+1}{\mathcal{I}^2} \oplus \underset{r+2}{\mathcal{I}^2} \oplus \ldots$

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Let $Z \hookrightarrow X$ be a closed subscheme and $r \geq 1$ an integer. The rth **Kummer blow-up** of Z is the stacky modification

$$\mathrm{Bl}_{Z,r}(X) := \mathrm{Bl}_Z(X)_{E,r} \to \mathrm{Bl}_Z(X) \to X.$$

where E is the exceptional divisor.

Explicitly, if $Z = V(\mathcal{I})$, then we have that

$$\mathrm{Bl}_{Z,r}(X) = \mathrm{Proj}_X(\mathcal{A}), \quad \text{(stacky proj)}$$

where $\mathcal{A} = \bigoplus_{k \in \mathbb{N}} \mathcal{I}^{\lceil k/r \rceil} = \underset{0}{\mathcal{O}} \oplus \underset{1}{\mathcal{I}} \oplus \underset{2}{\mathcal{I}} \oplus \cdots \oplus \underset{r}{\mathcal{I}} \oplus \underset{r+1}{\mathcal{I}^2} \oplus \underset{r+2}{\mathcal{I}^2} \oplus \ldots$

- If Z and X are regular, then so is $\mathrm{Bl}_{Z,r}(X)$.

Tame stacky blow-ups

Definition

A **tame stacky blow-up** $\pi\colon (X', U) \to (X, U)$ is a sequence of Kummer blow-ups

$$X' = X_n \to X_{n-1} \to \cdots \to X_1 = X$$

where $X_{k+1} = \mathrm{Bl}_{Z_k, r_k} X_k$ for some $Z_k \hookrightarrow X_k$ disjoint from $U = X_k \times_X U$ such that r_k is invertible along Z_k.

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- If $x' \in |X'|$ then $\mathrm{stab}(x') \hookrightarrow \mathrm{stab}(\pi(x)) \times A$ for an abelian group A.
- If X is a toric stack, then any subdivision can be realized as a tame stacky blow-up with smooth equivariant centers.

Tame stacky blow-ups (cont.)

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Proof of cofinality: tame étalification!

- A flat modification is an isomorphism.
- An étale stacky modification is an isomorphism.

Generalized Abhyankar lemma

Theorem (Generalized Abhyankar lemma — smooth tame étalification)

Let X be a regular scheme of characteristic zero. Let $\pi\colon X' \to X$ be a finite covering that is generically étale. Then there is a sequence of Kummer blow-ups with smooth centers $\widetilde{X} \to X$ such that $\mathrm{norm}(X' \times_X \widetilde{X})$ is étale over $\widetilde{X}$.

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Proof.

Let $D \hookrightarrow X$ be the branch divisor. We can assume that D has simple normal crossings. For every component D_i of D, choose $a_i \in \mathbb{N}$ such that the ramification index e_Z divides a_i for every component $Z \hookrightarrow X'$ above D_i. Then $\widetilde{X} = X_{D_1,a_1} \times_X X_{D_2,a_2} \times_X \cdots \times_X X_{D_n,a_n}$ does the job (it is a sequence of n smooth root stacks). $\square$

Outline

1. Stacky modifications and blow-ups
2. Root stacks and Kummer blow-ups
3. Artin–Schreier stacks
   - Artin–Schreier coverings
   - Artin–Schreier stacks
   - Higher rank Artin–Schreier stacks
   - Extension problem
4. Results and applications

Artin–Schreier extensions

Let K be a field of characteristic p. An Artin–Schreier extension K'/K is an extension of the form

$$K' = K[z]/z^p - z - a, \quad (\mathbb{Z}/p\mathbb{Z} \text{ acts via } z \mapsto z+1)$$

where $a \in K$.

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where $a \in K$. They are in one-to-one correspondence with $\mathrm{H}^1(K, \mathbb{Z}/p\mathbb{Z}) = K/\wp(K)$ via the Artin–Schreier sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \longrightarrow 0$$

where $\wp(x) = x^p - x$ is the Artin–Schreier operator.

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where $\wp(x) = x^p - x$ is the Artin–Schreier operator.
Global versions of Artin–Schreier extensions are more subtle than cyclic coverings. Let us first study the case where the base is a DVR.

Artin–Schreier covers of DVRs

Let $X = \mathrm{Spec}(A)$ be the spectrum of a DVR A with uniformizer $t \in A$. Every separable extension $K'/K(X)$ determines a finite generically étale cover $\pi\colon X' = \mathrm{norm}_{K'} X \to X$.

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type	cover data: $X' \to X$	stack data: $[X'/G] \to X$
Kummer	$r \in \mathbb{Z}_+$, $u \in A^*/(A^*)^r$	$r \in \mathbb{Z}_+$
Artin–Schreier	$a \in \mathbb{Z}_+$, $f \in A/t^a\wp(A)$	$a \in \mathbb{Z}_+$, $f \in A/t^a$

(a is the jump in the higher ramification series of π.)

Artin–Schreier stacks

The data of an **Artin–Schreier stack** over X consists of:

- $a \in \mathbb{Z}_+$,
- $D \in \mathrm{Div}_{\mathrm{gen}}(X)$,
- A non-vanishing section $f \in \Gamma(aD, \mathcal{O}(aD)|_{aD})$.

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Such data is in one-to-one correspondence with morphisms $X \to [\mathbb{P}(a,1)/\mathbb{G}_a]$ — local coordinates given by $(f : s_D)$. The corresponding $\mathbb{G}_a$-bundle is given by $\delta(f)$ in:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s_{aD}} \mathcal{O}(aD) \longrightarrow \mathcal{O}(aD)|_{aD} \longrightarrow 0$$

$$\Gamma(X, \mathcal{O}_X) \xrightarrow{s_{aD}} \Gamma(X, \mathcal{O}(aD)) \longrightarrow \Gamma(aD, \mathcal{O}(aD)|_{aD}) \xrightarrow{\delta} \mathrm{H}^1(X, \mathcal{O}_X)$$

$$f \longmapsto \delta(f)$$

Interlude: Universal root stack

One-to-one correspondence

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We have a cartesian square

$$\begin{tikzcd} X_{D,r} \arrow[r, "\frac{1}{r}D"] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\hat{}r"] \\ X \arrow[r, "D"] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

Universal Artin–Schreier stack

Recall that we had a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Data of an Artin–Schreier} \\ \text{stack } (D, a, f) \text{ on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{morphisms} \\ X \to [\mathbb{P}(a,1)/\mathbb{G}_a] \end{array} \right\}.$$

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$$\begin{tikzcd} X_{D,a,f} \arrow[r, "(\frac{1}{p}D,a,v)"] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathbb{P}(a,1)/\mathbb{G}_a]} \arrow[d, "\Psi"] \\ X \arrow[r, "(D,a,f)"] & {[\mathbb{P}(a,1)/\mathbb{G}_a]} \end{tikzcd} \qquad \begin{tikzcd} (v : w) \arrow[d, mapsto] \\ (v^p - vw^{a(p-1)} : w^p) \end{tikzcd}$$

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where Ψ is the universal Artin–Schreier stack and

$$v \in \Gamma\left(\frac{a}{p}D, \mathcal{O}\left(\frac{a}{p}D\right)\Big|_{\frac{a}{p}D}\right)$$

is a non-vanishing function such that $f = v^p - vs_{\frac{a}{p}D}^{(p-1)}$.

Properties of Artin–Schreier stacks

Let (D, a, f) be the data of an Artin–Schreier stack over X:

$$\pi\colon X_{D,a,f} \to X$$

- Let $U = X \setminus D$. Then $(X_{D,a,f}, U) \to (X, U)$ is a flat, wild, stacky modification.
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- If X and D are regular (plus extra condition if $p \mid a$) then $X_{D,a,f}$ is regular.

First problem

Problem: Wild ramification is more complicated than Artin–Schreier stacks!

Example of a restricted Artin–Schreier stack

Example

Consider the Artin–Schreier covering of $\mathrm{Spec}\big(\mathbb{F}_p(\sqrt{2})[t]\big)$

$$z^p - z - \frac{\sqrt{2}}{t} = 0.$$

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The Weil-restriction along $\mathbb{F}_p(\sqrt{2})/\mathbb{F}_p$ of the corresponding Artin–Schreier stack is a complicated stacky modification related to the twisted Artin–Schreier sequence:

$$0 \longrightarrow G \longrightarrow \mathbb{G}_a^2 \xrightarrow{\wp'} \mathbb{G}_a^2 \longrightarrow 0$$

where $\wp'(x, y) = (x^p - x, 2^{\frac{p-1}{2}} y^p - y)$ and G is a twisted version of $(\mathbb{Z}/p\mathbb{Z})^2$ if $\sqrt{2} \notin \mathbb{F}_p$.

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Weil restrictions of Artin–Schreier stacks have ramification that cannot be handled by Artin–Schreier stacks!

F-bundles

Definition

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Let $f\colon X' \to X$ be a finite flat morphism and let $\mathcal{E} = f_*\mathcal{O}_{X'}$. The geometric Frobenius gives a homomorphism $F_{X'/X}\colon F^*\mathcal{E} \to \mathcal{E}$. This is an isomorphism if and only if f is étale.

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An F-bundle $(\mathcal{E}, \varphi)$ gives a twisted Artin–Schreier sequence:

$$0 \longrightarrow G \longrightarrow \mathbb{V}(\mathcal{E}^\vee) \xrightarrow{\wp} \mathbb{V}(\mathcal{E}^\vee) \longrightarrow 0$$

where $\wp(x) = x^p - x$ and $x \mapsto x^p$ is defined by:

$$\mathcal{E}^\vee \xrightarrow{\varphi^\vee} F^*\mathcal{E}^\vee \xrightarrow{can} \mathrm{Sym}^p(\mathcal{E}^\vee)$$

G is a twisted version of $(\mathbb{Z}/p\mathbb{Z})^{\operatorname{rk}\mathcal{E}}$.

Higher rank Artin–Schreier stacks

The data of a **higher rank Artin–Schreier stack** over X consists of:

- $a \in \mathbb{Z}_+$,
- $D \in \mathrm{Div}_{\mathrm{gen}}(X)$,
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From this data we can construct an Artin–Schreier stack $\pi : X_{D,a,\mathcal{E},f} \to X$.

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From this data we can construct an Artin–Schreier stack $\pi\colon X_{D,a,\mathcal{E},f} \to X$.

- There is a canonical divisor $\frac{1}{p}D \hookrightarrow X_{D,a,\mathcal{E},f}$ such that $p\frac{1}{p}D = \pi^*D$.
- Let $U = X \setminus D$. Then $(X_{D,a,\mathcal{E},f}, U) \to (X, U)$ is a flat, wild, stacky modification.
- If $p \nmid a$, then $\pi|_{\frac{1}{p}D}\colon \frac{1}{p}D \to D$ is a trivial G-gerbe.

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Higher rank Artin–Schreier stacks are powerful enough to capture all wild ramification. . .

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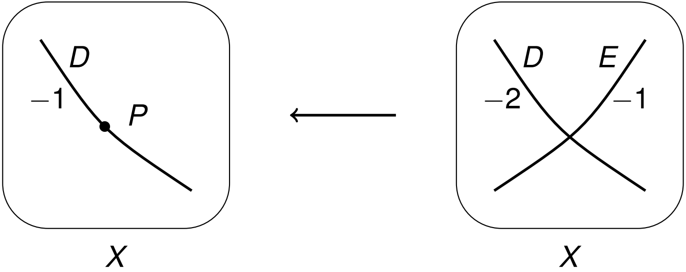
- Is this related to the fact that the gerbe over D always is trivial?
- Are quasi-Deligne–Mumford stacks needed? ($\mathbb{Z}/p\mathbb{Z}$ degenerates to α_p, cf. work of S. Maugeais et al.)

Example demonstrating second problem

Let X be a smooth projective surface (e.g., $X = \mathrm{Bl}_0(P^2)$) with a smooth (-1)-divisor D and let $P \in D$ be a point. Assume we are given an Artin–Schreier stack over $X \setminus P$ with stacky structure along $D \setminus P$. We would like to extend this over X, possibly after replacing $(X, X \setminus P)$ with a stacky modification. As $\Gamma(D, \mathcal{O}_D(aD)) = 0$ we need to blow-up P.



Now $\Gamma(D, \mathcal{O}_D(aD + abE)) \neq 0$ for sufficiently large b but then $\Gamma(E, \mathcal{O}_E(aD + abE)) = 0$ instead.

Outline

Flatification

Theorem (Raynaud–Gruson '71)

Let (Y, U) be a schemepair and let $f\colon X \to Y$ be a morphism such that $f|_U$ is flat. Then $\exists$ blow-up $(\widetilde{Y}, U) \to (Y, U)$ such that the strict transform $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is flat.

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$$\begin{tikzcd} & X \arrow[d, "f"] \\ \widetilde{Y} = \mathrm{Bl}_Z(Y) \arrow[r] & Y \end{tikzcd} \qquad (Z \cap U = \emptyset)$$

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$$\begin{tikzcd} \widetilde{X} = \mathrm{Bl}_{f^{-1}(Z)}(X) \arrow[r] \arrow[d, "\widetilde{f}"'] \arrow[dr, phantom, "\blacksquare"] & X \arrow[d, "f"] \\ \widetilde{Y} = \mathrm{Bl}_Z(Y) \arrow[r] & Y \end{tikzcd} \qquad (Z \cap U = \emptyset)$$

Flatification

Theorem (R.)

*Let (Y, U) be a **quasi-Deligne–Mumford** stackpair and let $f\colon X \to Y$ be a morphism such that $f|_U$ is flat. Then $\exists$ blow-up $(\widetilde{Y}, U) \to (Y, U)$ such that the strict transform $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is flat.*

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Proof.

Étale dévissage and Raynaud–Gruson's theorem (alternatively Riemann–Zariski spaces). □

Étalification

Theorem (R. '09)

Let $f\colon (X, U) \to (Y, V)$ be a morphism of Deligne–Mumford stacks such that $f|_V$ is étale and tamely ramified. Then $\exists$ a tame stacky blow-up $(\widetilde{Y}, V) \to (Y, V)$ and a blow-up $(\widetilde{X}, U) \to (X \times_Y \widetilde{Y}, U)$ such that $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is étale.

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Proof.

Riemann–Zariski spaces and étale dévissage. $\square$

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Conjecture

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The proof works in the non-tame case subject to the existence of a cofinal category of stacky blow-ups with good properties.

Compactification of Deligne–Mumford stacks

Theorem (R. '09)

Let $f\colon X \to S$ be a separated morphism between tame Deligne–Mumford stacks.

- *There is a factorization $f = \bar{f} \circ j\colon X \to \overline{X} \to S$ where j is an open immersion and $\bar{f}$ is proper, tame and Deligne–Mumford.*

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This was proved for smooth X by Reichstein and Youssin in characteristic zero using resolutions of singularities (2000). Granting embedded functorial resolution of singularities we can also arrange so that $(X', U) \to (X, U)$ is a tame stacky blow-up with smooth centers.

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Let (X, U) be a Deligne–Mumford stackpair. Then

$$\varinjlim_{\substack{(\widetilde{X},U)\to(X,U)\\ \text{tame stacky blow-up}}} \mathbf{F\acute{E}T}(\widetilde{X}) \to \mathbf{F\acute{E}T}_{tame}(X, U)$$

is an equivalence of categories. In particular, if $u \in |U|$ then we have an isomorphism of pro-finite groups

$$\pi_1^{tame}(U; u) \to \varprojlim_{(\widetilde{X},U)} \pi_1(\widetilde{X}; u).$$

Perhaps generalizes to étale homotopy theory a'la Artin–Mazur.

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Stacky semi-stable reduction

Theorem (de Jong '97)

Let (S, U) be an integral and excellent schemepair.
Let $\pi\colon C \to S$ be a proper flat family of curves such that $\pi|_U$ is a semi-stable family.
Then $\exists$ a generically étale alteration $S' \to S$ and a modification $C' \to C \times_S S'$ such that π' is a semi-stable family.

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NB! $S' \to S$ need not be étale over U.

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Stacky semi-stable reduction

Theorem (Temkin '10)

Let (S, U) be a normal schemepair.
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Let (S, U) be a normal Deligne–Mumford stackpair.
Let $\pi\colon C \to S$ be a, not necessarily proper, flat family of curves such that $\pi|_U$ is a semi-stable family.
Assume that over every valuation ring, semi-stable reduction can be obtained after a tame extension.
Then $\exists$ a tame stacky blow-up $(S', U) \to (S, U)$ and a modification $(C', \pi^{-1}(U)) \to (C \times_S S', \pi^{-1}(U))$ such that π' is a semi-stable family.

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Stacky semi-stable reduction

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Summary

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- Cofinality together with these properties gives étalification and a bunch of applications (e.g., compactification of DM-stacks).
- Kummer case (=tame case) well understood and Artin–Schreier case partly understood.

End of talk

The end

Outline

Ramification vs stacky modifications

We say that $X_1 \to X$ is "more ramified" than $X_2 \to X$ if there is a morphism $X_1 \to X_2$ up to an étale morphism.

$$\begin{tikzcd} X_1' \arrow[r, "\text{étale}"] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

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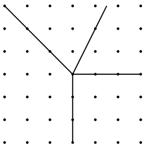
("ramification type of $X_1 \to X$" $>$ "ramification type of $X_2 \to X$")
For every generically étale morphism $X' \to X$ there is a stacky modification $\widetilde{X} \to X$ that is more ramified.

$$\begin{tikzcd} \text{“Galois” closure} \arrow[r, dashed] & X'' \arrow[r, "\mathfrak{S}_d\text{-torsor}"] \arrow[d, "\text{finite}"'] \arrow[dr, phantom, "\circ"] & \widetilde{X} = [X''/\mathfrak{S}_d] \arrow[d, "\mathfrak{M}_{\mathrm{st}}"] \\ & X' \arrow[r, "\text{finite}"] & X \end{tikzcd}$$

Outline

Toric stacks

Let $N = \mathbb{Z}^d$ and let $\Sigma \subseteq N_{\mathbb{Q}}$ be a rational simplicial fan.

Σ 

To Σ we associate the toric variety X_Σ.

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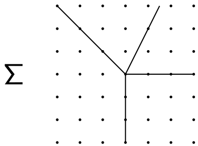

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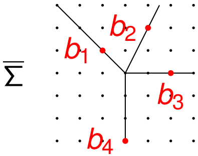

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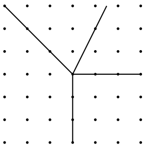

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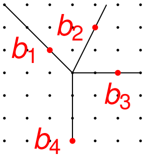
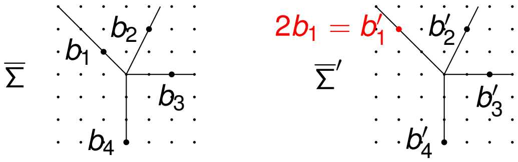


To the **stacky fan** $\overline{\Sigma} = (\Sigma, \mathbf{b})$ we associate a **toric stack** $\mathscr{X}_{\overline{\Sigma}}$.
Toric stacks are always **regular** and tame.

Toric stacks and root stacks

Let D_i be the toric divisor corresponding to the ray ρ_i. Taking the r^{th} root stack of D_i results in the toric stack with stacky fan $\overline{\Sigma}' = \{\Sigma', \mathbf{b}'\}$ where $\Sigma' = \Sigma$ and $b'_j = b_j$ for $j \neq i$ and $b'_i = rb_i$:



2^{d} root stack of D_1

Star subdivisions

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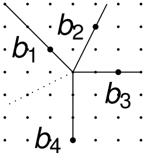

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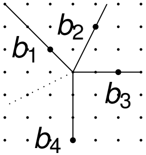

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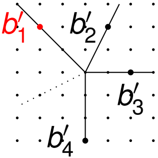

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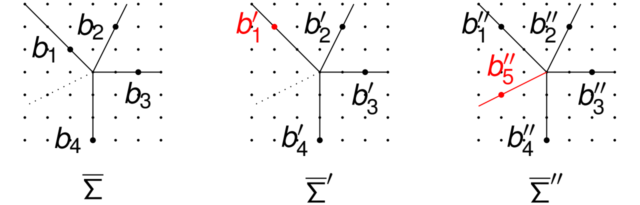

$$\mathscr{X}_{\overline{\Sigma}} \xleftarrow{\mathrm{Bl}_{D_1,2}} \mathscr{X}_{\overline{\Sigma}'} \xleftarrow{\mathrm{Bl}_{D'_1 \cap D'_4,1}} \mathscr{X}_{\overline{\Sigma}''}$$

Weak factorization of toric stacks

In the language of toric stacks and stacky blow-ups we have:

Theorem (Włodarczyk '98)

1. *A proper birational map $\mathscr{X}_{\underline{\Sigma}} \dashrightarrow \mathscr{X}_{\underline{\Sigma}'}$ between toric stacks factors as a sequence of stacky blow-ups and stacky blow-downs with smooth equivariant centers.*
2. *A proper birational map $X_{\Sigma} \dashrightarrow X_{\Sigma'}$ between regular toric varieties factors as a sequence of blow-ups and blow-downs with smooth equivariant centers.*

Weak factorization

Theorem (Abramovich–Karu–Matsuki–Włodarczyk '02, W '03)

A proper birational map $X \dashrightarrow Y$ between regular varieties over a field of characteristic zero, factors as a sequence of blow-ups and blow-downs with smooth centers.

Weak factorization

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A proper birational map $X \dashrightarrow Y$ between regular varieties over a field of characteristic zero, factors as a sequence of blow-ups and blow-downs with smooth centers.

Conjecture (R.)

A proper birational map $\mathscr{X} \dashrightarrow \mathscr{Y}$ between regular ***DM-stacks*** *over a field of characteristic zero, factors as a sequence of* ***stacky blow-ups*** *and* ***stacky blow-downs*** *with smooth centers.*

Outline

Simultaneous desingularization

Let X be an variety and $K'/K(X)$ a finite field extension. It is well-known (example by Abhyankar) that it is sometimes impossible to find a resolution of singularities $\widetilde{X} \to X$ such that $\mathrm{norm}_{K'} \widetilde{X}$ also is regular. However:

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Theorem

Let X be a regular variety and let $K'/K(X)$ a finite separable field extension. Assume that functorial embedded resolution of singularities exists for X (e.g., X of characteristic zero) and that $K'/K(X)$ is tamely ramified over X. Then there exists a sequence of Kummer blow-ups with smooth centers

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Proof.

First blow-up so that the branch divisor has simple normal crossings. Then the theorem easily follows from the generalized Abhyankar lemma (use Zariski–Nagata purity). $\square$

Simultaneous desingularization (wild case)

Conjecture

Let X be a regular variety and let $K'/K(X)$ a finite separable field extension. Assume that functorial embedded resolution of singularities exists for X. Then there exists a sequence of "stacky blow-ups with smooth centers"

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such that $\operatorname{norm}_{K'} X_n$ *is a regular stack that is étale over* X_n.

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- a Deligne–Mumford stack is **tame** if $\forall x \in |X|$, char $k(x) \nmid |\operatorname{stab}(x)|$;

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- (mixed characteristic) a morphism of Deligne–Mumford stacks $f\colon X \to Y$ is **strictly tame** if $\forall x \in |X|$, the order $|\operatorname{stab}_Y(x)|$ is invertible along $\overline{f(x)}$.

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- a Deligne–Mumford stack is **tame** if $\forall x \in |X|$, $\operatorname{char} k(x) \nmid |\operatorname{stab}(x)|$;
- a morphism of Deligne–Mumford stacks $f\colon X \to Y$ is **tame** if every fiber is tame;
- (mixed characteristic) a morphism of Deligne–Mumford stacks $f\colon X \to Y$ is **strictly tame** if $\forall x \in |X|$, the order $|\operatorname{stab}_Y(x)|$ is invertible along $\overline{f(x)}$.

In characteristic zero, every stack is tame. In equal characteristic "strictly tame" and "tame" coincide.

Outline

Quasi-projective varieties and stacks

Let X/k be a variety. The following are equivalent:

1. X is quasi-projective.
2. $\exists$ open embedding $X \subseteq \overline{X}$ with $\overline{X}$ projective.
3. $\exists$ embedding $X \hookrightarrow \mathbb{P}^n_k$.

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Definition (char. 0)

Let X/k be a separated DM-stack of finite type over a field k of characteristic zero. The stack X is **(quasi-)projective** if:

1. X is a **global quotient stack**, i.e., $X = [U/\mathrm{GL}_n]$ for some algebraic space U.
2. The coarse moduli space X_{cms} is (quasi-)projective.

Quasi-projective varieties and stacks (cont.)

Theorem (Kresch '09)

Let X/k be a DM-stack of characteristic zero. The following are equivalent:

1. *X is quasi-projective.*
2. *$\exists$ an open embedding $X \subseteq \overline{X}$ into a projective stack.*
3. *$\exists$ an embedding $X \hookrightarrow P$ where P is a smooth projective DM-stack.*

Moreover, every smooth DM-stack with (quasi-)projective cms is (quasi-)projective.