Local structure of algebraic stacks and applications

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Contents

Stacks

Algebraic stacks

We view affine schemes X as representable functors $X\colon (\mathrm{AffSch})^{\mathrm{op}} \to (\mathrm{Set})$ where $X(T) = \mathrm{Hom}(T, X)$.

- An **algebraic space** is an étale sheaf $X\colon (\mathrm{AffSch})^{\mathrm{op}} \to (\mathrm{Set})$ together with an étale/smooth/flat atlas $\coprod \mathrm{Spec}(A_i) \to X$.
- An **algebraic stack** is an étale stack $\mathscr{X}\colon (\mathrm{AffSch})^{\mathrm{op}} \to (\mathrm{Grpd})$ together with a smooth/flat atlas $\coprod \mathrm{Spec}(A_i) \to \mathscr{X}$.

Features:

- "Underlying" topological space $|\mathscr{X}|$.
- A point $x\colon \mathrm{Spec}\, k \to \mathscr{X}$ has stabilizer $G_x = \mathrm{Aut}(x) = \mathrm{stab}(x)$, a group scheme over k.
- $\mathscr{X}$ is **Deligne–Mumford** if $\mathrm{Aut}(x)$ is finite and smooth for all x. (equivalently, exists an étale atlas)

Examples: moduli stacks

1. $\mathfrak{U}_{g,n} = \{\text{singular curves } C \text{ of genus } g \text{ with } n \text{ marked pts}\}$
2. $\mathfrak{M}_{g,n} = \{\text{nodal curves } C \text{ of genus } g \text{ with } n \text{ marked pts}\}$

DM 3. Stable curves: $\bar{\mathscr{M}}_{g,n} = \{C \in \mathfrak{M}_{g,n} \mid \operatorname{Aut}(C) \text{ finite}\}$

DM 4. Stable maps: $\bar{\mathscr{M}}_{g,n}(X) = \{C \in \mathfrak{M}_{g,n},\ f\colon C \to X \mid \operatorname{Aut}(f) \text{ finite}\}$

5. Stack of vector bundles/sheaves/complexes on a scheme X.
6. Stack of logarithmic structures $\mathscr{L}og(T)$.

Stacks of stable curves/maps are Deligne–Mumford. Other stacks have affine stabilizers (except curves with smooth genus 1 comp.).

Slogan: General stacks (or equivalently groupoids) are flexible and needed for general moduli problems.

Examples: quotient stacks

- Group G acting on scheme X gives **quotient stack** $\mathscr{X} = [X/G]$ with atlas $p\colon X \to [X/G]$. Well-behaved even if action not free.
- G-equivariant geometry on $X \iff$ geometry on $\mathscr{X} = [X/G]$:
  - $|\mathscr{X}|$ is space of G-orbits
  - $\mathrm{Aut}(p(x)) = G_x$, $x \in X$
  - $\Gamma(\mathscr{X}, F) = \Gamma(X, p^*F)^G$
  - $H^i(\mathscr{X}, F) = H^i_G(X, p^*F)$

Slogan: Quotient stacks (or equivalently group actions) are much easier to understand and have several tools in equivariant geometry.

Question

When is a general stack “locally” a quotient stack?

Local structure of Deligne–Mumford stacks

(Keel–Mori '97) A separated Deligne–Mumford stack $\mathscr{X}$ has a **coarse moduli space** $\pi\colon \mathscr{X} \to \mathbf{X}$ where $\mathbf{X}$ is an algebraic space such that

1. $|\mathscr{X}| = |\mathbf{X}|$ (π is a universal homeomorphism)
2. $\mathcal{O}_{\mathbf{X}} = \pi_* \mathcal{O}_{\mathscr{X}}$

Example

If G finite and $\mathscr{X} = [\operatorname{Spec} A/G]$, then $\mathbf{X} = [\operatorname{Spec} A^G]$.

Orbifold description

If $\mathscr{X}$ has a coarse moduli space $\mathbf{X}$, then $\forall x \in |\mathscr{X}|$ there exists:

- U affine with action of $G_x = \operatorname{stab}(x)$
- $u \in U$ fix-point
- f étale, $f(u) = x$
- $\operatorname{stab}(u) \to \operatorname{stab}(x)$ isomorphism

$$\begin{array}{ccc} [U/G_x] & \xrightarrow{f} & \mathscr{X} \\ \downarrow & \square & \downarrow \pi \\ U/G_x & \longrightarrow & \mathbf{X} \end{array}$$

Statement of main theorem

Based on joint work with

Jarod-Alper.jpg

Jarod Alper

Jack-Hall.png

Jack Hall

Halpern_Leistner.jpg

Daniel Halpern-Leistner

[AHR1] *A Luna étale slice theorem for algebraic stacks*, 2015

[AHR2] *The étale local structure of algebraic stacks*, 2019

[AHHR3] *Artin algebraization for pairs with applications to the local structure of stacks and Ferrand pushouts*, 2020 (exp)

Main Theorem (AHR1 '15)

$\mathscr{X}$ algebraic stack of finite type over field $k = \bar{k}$, $x \in \mathscr{X}(k)$. Assume

1. *G_x is linearly reductive (e.g., GL_n in char 0 or a torus in char p),*
2. *G_y is affine for all $y \in |\mathscr{X}|$.*

Then there exists $f\colon [U/G_x] \to \mathscr{X}$ where

- *U affine with action of G_x, $u \in U$ fix-point*
- *f étale, $f(u) = x$ and $\mathrm{stab}(u) \to \mathrm{stab}(x)$ isomorphism*

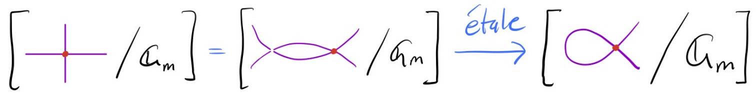
Remark:

Conditions 1+2 are necessary. Counter-examples: $\mathscr{X} = BG = [\mathbb{A}^1/G]$ where $G \to \mathbb{A}^1$ degeneration: (1) from $\mathbb{G}_m$ to $\mathbb{G}_a$, (2) from E to $\mathbb{G}_m$.

Examples and known cases

- (Sumihiro'74) If a torus T acts on a normal scheme X, then every point has an affine equivariant open neighborhood U. This gives an open immersion $[U/T] \to [X/T]$ (but $T \neq G_x$).
- (Luna'73) $\mathscr{X} = [X/G]$ where X affine and G linearly reductive: then Theorem holds with $U \hookrightarrow X$ locally closed.
- (Sumihiro+Luna) $\mathscr{X} = [X/G]$ where X normal scheme, G smooth affine, G_x linearly reductive.
- (Olsson'03) $\mathscr{X} = \mathscr{L}og$.
- (Alper–Kresch'14) $\mathscr{X} = \mathfrak{M}_{g,n}$.
- Let C nodal cubic with action of $G = \mathbb{G}_m$. Then Sumihiro fails.

  Local structure: $[U/\mathbb{G}_m] \xrightarrow{f} [C/\mathbb{G}_m], \quad U = \operatorname{Spec} k[x,y]/(xy)$



Refinements

- $f\colon [U/G_x] \to \mathscr{X}$ representable if $\Delta_{\mathscr{X}}$ separated, and $f\colon [U/G_x] \to \mathscr{X}$ affine if $\Delta_{\mathscr{X}}$ affine.
- If $\mathscr{X}$ is smooth, then exists

$$[\mathbb{A}^n/G_x] \xleftarrow{\;g\;} [U/G_x] \xrightarrow{\;f\;} \mathscr{X}$$

  where f and g are étale and $g(u) = 0$.
- If G_x not linearly reductive but $H \subset G_x$ linearly reductive:

$$f\colon [U/H] \longrightarrow \mathscr{X}$$

  syntomic/smooth/étale if G_x/H is arbitrary/smooth/étale.
- Version for derived stacks and quasi-smooth morphisms. (AHHR3)

Further refinements

- Not over a field, including mixed characteristic (AHR2). Sample theorem: given an algebraic stack $\mathscr{X}$, $x \in |\mathscr{X}|$ with linearly reductive stabilizer, there exists étale maps

$$\begin{array}{ccccc} [U/G] & \xrightarrow{h} & [V/\mathrm{GL}_n] & \xrightarrow{f} & \mathscr{X} \\ u & \longmapsto & v & \longmapsto & x \end{array}$$

  where U, V affine and $\mathrm{stab}(u) \subseteq \mathrm{stab}(v) = \mathrm{stab}(x)$. Here $G \to \mathrm{Spec}(\mathbb{Z})$ is either diagonalizable or split reductive.
- Locally around substack instead of point (AHHR3), also needed for syntomic case on previous slide.

Interlude: Good moduli spaces

Local structure of stacks with good moduli spaces

- In the local structure of a DM-stack $\mathscr{X}$, we had

$$\begin{array}{ccc} [U/G_x] & \xrightarrow{f} & \mathscr{X} \\ \downarrow & & \\ U/G_x & & \end{array} \quad \text{and} \quad \begin{array}{ccc} [U/G_x] & \xrightarrow{f} & \mathscr{X} \\ \downarrow & \square & \downarrow{\scriptstyle \pi} \\ U/G_x & \longrightarrow & \mathbf{X} \end{array} \quad \text{if } \mathscr{X} \text{ has a coarse space } \mathbf{X}.$$

- Similarly, in the main theorem, we have

$$\begin{array}{ccc} [U/G_x] & \xrightarrow{f} & \mathscr{X} \\ \downarrow & & \\ U/\!\!/G_x & & \end{array} \quad \text{and} \quad \begin{array}{ccc} [U/G_x] & \xrightarrow{f} & \mathscr{X} \\ \downarrow & \square & \downarrow{\scriptstyle \pi} \\ U/\!\!/G_x & \longrightarrow & \mathbf{X} \end{array} \quad \text{if } \mathscr{X} \text{ has a good moduli space } \mathbf{X}.$$

Good moduli spaces

Definition (Alper '08)

A **good moduli space** to $\mathscr{X}$ is a morphism $\pi: \mathscr{X} \to \mathbf{X}$ to an algebraic space $\mathbf{X}$ such that

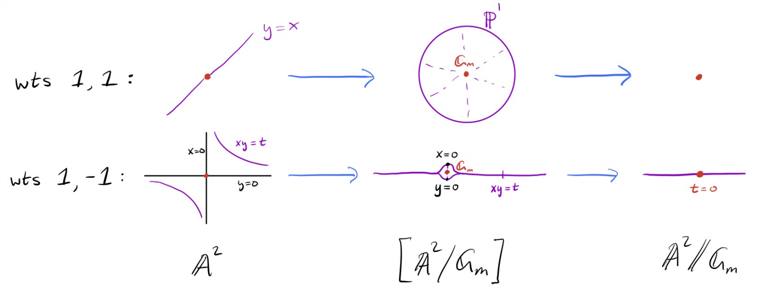
1. $\pi_*: \mathrm{QCoh}(\mathcal{O}_{\mathscr{X}}) \to \mathrm{QCoh}(\mathcal{O}_{\mathbf{X}})$ is exact, (π is cohomol. affine)
2. $\mathcal{O}_{\mathbf{X}} = \pi_* \mathcal{O}_{\mathscr{X}}$

Consequences:

- π is initial among maps to algebraic spaces
- π is universally closed and π_* preserves coherence
- Every fiber of π has a unique closed point and it has linearly reductive stabilizer

Examples of good moduli spaces

- (GIT) Let X be a scheme with an action of G linearly reductive and a G-linearized ample line bundle L. Then $\mathscr{X} = [X^{\mathrm{ss}}/G]$ has good moduli space $\mathbf{X} = X /\!\!/ G$.
- If $X = \operatorname{Spec} A$ affine, $L = \mathcal{O}_X$, then $X /\!\!/ G = \operatorname{Spec} A^G$.
- If X is projective, then $X /\!\!/ G = \operatorname{Proj}\left(\bigoplus_{n \geq 0} \Gamma(X, L^n)^G\right)$.
- $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $1, 1$ and invariant ring $k[x,y]_0 = k$
- $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $1, -1$ and invariant ring $k[x,y]_0 = k[xy]$



Proof of main theorem

Overview

Theorem (Artin '69, '74)

A stack $\mathscr{X} : (\mathrm{AffSch})^{\mathrm{op}} \to (\mathrm{Grpd})$ *is algebraic if and only if ...*

is proven as follows:

- Let $x \in \mathscr{X}(k)$ be a point.
- Construct formal atlas $(\operatorname{Spec} A, u) \to (\mathscr{X}, x)$.
- Find atlas $(\operatorname{Spec} B, w) \to (\mathscr{X}, x)$ such that $A = \widehat{B}_w$.

The proof of the main theorem is similar but Spec A and Spec B are replaced with linearly fundamental stacks.

Definition

A stack $\mathscr{X}$ is **linearly fundamental** if it has an affine good moduli space and the resolution property, e.g., $\mathscr{X} = [\operatorname{Spec} A/G]$ where G is linearly reductive and embeddable in GL_N.

Outline of proof

$x \in |\mathscr{X}|$ closed point with ideal I. Infinitesimal neighborhoods:

$$BG_x = \mathscr{X}_x^{[0]} \hookrightarrow \mathscr{X}_x^{[1]} \hookrightarrow \ldots \hookrightarrow \mathscr{X}$$

Tangent stack: $\mathscr{T}_x := [T_x/G_x]$ smooth over k, where $T_x = \mathbb{V}(I/I^2)$.

Step 1 (Deformation theory) Lift $i_0 \colon \mathscr{X}_x^{[0]} \hookrightarrow \mathscr{T}_x$ to closed immersions $i_n \colon \mathscr{X}_x^{[n]} \hookrightarrow \mathscr{T}_x$. $(*)$

Step 2 (Completions) Completions $\widehat{\mathscr{T}_x}$ and $\widehat{\mathscr{X}_x} \hookrightarrow \widehat{\mathscr{T}_x}$ exist. $(*)$

Step 3 (Tannaka duality) Lift $\mathscr{X}_x^{[n]} \hookrightarrow \mathscr{X}$ to $\widehat{\imath} \colon \widehat{\mathscr{X}_x} \to \mathscr{X}$. $(\dagger)$

Step 4 (Equivariant Artin algebraization) $\exists \mathscr{W} \to \mathscr{X}$ finite type such that $\widehat{\mathscr{W}_w} \simeq \widehat{\mathscr{X}_x}$.

$*$ = uses linear reductivity

$\dagger$ = uses affine stabilizers

Step 1: Cohomological affineness

Obstruction to lifting $\mathscr{X}_x^{[n-1]} \to \mathscr{T}$ to $\mathscr{X}_x^{[n]} \to \mathscr{T}$ lies in

$$Ext^1_{\mathscr{X}_x^{[0]}}(L_{\mathscr{T}}, I^n/I^{n+1}) = H^1(\mathscr{X}_x^{[0]}, L_{\mathscr{T}}^\vee \otimes I^n/I^{n+1})$$

This obstruction group vanishes because

- $\mathscr{T}$ is smooth $\implies$ $L_{\mathscr{T}}$ perfect of Tor-amplitude $[0,1]$.
- $\mathscr{X}_x^{[0]} = BG_x$ is cohomologically affine: $\Gamma(\mathscr{X}_x^{[0]}, -)$ is exact.

Step 2: Complete stacks

Let $\mathscr{X}$ noetherian stack and $\mathscr{X}_0$ a closed substack.

Definition

We say that $(\mathscr{X}, \mathscr{X}_0)$ is **complete** if $\mathrm{Coh}(\mathscr{X}) \to \varprojlim_n \mathrm{Coh}(\mathscr{X}_n)$ is an equivalence of categories. ($\mathscr{X}_n$ is the nth inf. neighborhood of $\mathscr{X}_0$.)

Examples

- If $S = \mathrm{Spec}(\varprojlim_n A/I^n)$, and $S_0 = V(I)$, then (S, S_0) is complete.
- If $p\colon X \to S$ is proper, $X_0 = p^{-1}(S_0)$, then (X, X_0) is complete.

Theorem (AHR1+AHR2)

Let $\pi\colon \mathscr{X} \to \mathbf{X}$ be a good moduli space, $\mathscr{X}_0 \subset \mathscr{X}$ a closed substack and $\mathbf{X}_0 = \pi(\mathscr{X}_0)$. Then if $\mathscr{X}_0$ is linearly fundamental

$$(\mathscr{X}, \mathscr{X}_0) \text{ complete} \iff (\mathbf{X}, \mathbf{X}_0) \text{ complete}$$

In particular, $\widehat{\mathscr{X}} := \mathscr{X} \times_{\mathbf{X}} \widehat{\mathbf{X}}$ is complete along $\mathscr{X}_0$.

Step 2′: Effectivity

Building upon the methods using the tangent stack, we also establish:

Theorem (AHR2)

Let $\mathscr{X}_0 \hookrightarrow \mathscr{X}_1 \hookrightarrow \mathscr{X}_2 \hookrightarrow \ldots$ be an adic system of noetherian stacks. If $\mathscr{X}_0$ is linearly fundamental, then there exists a linearly fundamental complete stack $(\widehat{\mathscr{X}}, \mathscr{X}_0)$ such that $\mathscr{X}_n$ is the nth infinitesimal neighborhood of $\mathscr{X}_0$.

A subtle problem in the proof is that the sequence of good moduli spaces $\mathbf{X}_0 \hookrightarrow \mathbf{X}_1 \hookrightarrow \ldots$ is not adic and a priori the completion of this sequence is not even noetherian. (It is noetherian by Godement'56.)

Step 3: Tannaka duality

Theorem (Lurie '04, Brandenburg–Chirvasitu '12, Hall–R '14)

Let $T, \mathscr{X}$ be noetherian algebraic stacks. The map of groupoids

$$\begin{aligned} \mathrm{Map}(T, \mathscr{X}) &\longrightarrow \mathrm{Hom}_{r\otimes}(\mathrm{Coh}(\mathscr{X}), \mathrm{Coh}(T)) \\ f &\longmapsto f^* \end{aligned}$$

is an equivalence if $\mathscr{X}$ has affine stabilizers and T is excellent.

($r\otimes$ = right-exact tensor functors. Derived analogues by Lurie, Bhatt '14 and Bhatt–Halpern-Leistner '15.)

In proof of main theorem:

$$\begin{aligned} \mathrm{Map}(\widehat{\mathscr{X}_x}, \mathscr{X}) &\overset{\mathrm{TD}}{=} \mathrm{Hom}_{r\otimes}(\mathrm{Coh}(\mathscr{X}), \mathrm{Coh}(\widehat{\mathscr{X}_x})) \\ &\overset{\mathrm{C}}{=} \varprojlim_n \mathrm{Hom}_{r\otimes}(\mathrm{Coh}(\mathscr{X}), \mathrm{Coh}(\mathscr{X}_x^{[n]})) \overset{\mathrm{TD}}{=} \varprojlim_n \mathrm{Map}(\mathscr{X}_x^{[n]}, \mathscr{X}) \end{aligned}$$

In particular: $\widehat{\mathscr{X}_x} = \varinjlim_n \mathscr{X}_x^{[n]}$.

Step 4: Equivariant Artin algebraization

Question (Algebraization)

Given $\overline{A} = \operatorname{Spec} k[[x_1, \ldots, x_n]]/I$, *when is* $\overline{A} = \widehat{A}$ *for a finite type k-algebra A?*

Yes, when $\overline{A}$ regular. No, in general.

Theorem (Artin '69)

Yes, when there exist a formally smooth $\operatorname{Spec} \overline{A} \to \mathscr{X}$ *where* $\mathscr{X}$ *is a stack of finite type over an excellent base scheme S. Then also have smooth* $\operatorname{Spec} A \to \mathscr{X}$*. ($\mathscr{X}$ need not be algebraic)*

Theorem (AHR1, AHHR3)

Given $(\overline{\mathscr{W}}, \mathscr{W}_0)$ *linearly fundamental and complete and formally smooth* $\overline{\mathscr{W}} \to \mathscr{X}$*. Then* $\exists \mathscr{W} \to \mathscr{X}$ *formally smooth with* $\overline{\mathscr{W}} = \widehat{\mathscr{W}}$*.*

Applications

Applications

Equivariant geometry

1. Sumihiro and Luna for $[X/G]$ with general X (AHR1)
2. Białynicki-Birula for Deligne–Mumford stacks (Oprea'06, AHR1)
3. Toric stacks: fans vs intrinsic (Geraschenko–Satriano'11)

Good moduli spaces

4. Kirwan desingularization of good moduli spaces (Edidin–R'17)
5. Existence of good moduli space (Alper–Halpern-Leistner–Heinloth'18)
6. Good moduli space vs adequate moduli spaces (AHR2)
7. Resolution by vector bundles (AHR2)
8. Étale-local embeddability of linearly reductive group schemes (AHR2)

Further applications

Moduli problems

9. Algebraicity of Hom-stacks etc (AHR1)
10. Generalized DT-invariants (Toda'16, Kiem–Li–Savvas'17)
11. Miniversal deformation spaces for singular curves (AHR1)

General results for stacks

12. Compact generation of derived categories (AHR1)
13. Existence of henselizations and completions (AHR1–AHHR3)
14. Existence of henselizations along affine closed subschemes (AHHR3)
15. Existence of Ferrand pushouts (AHHR3)
16. K-theory of stacks (Hoyois–Krishna'17)

A2. Białynicki-Birula decompositions for Deligne–Mumford stacks

Theorem (Oprea '06, Drinfeld '13, AHR1)

Let $\mathscr{X}$ be a proper Deligne–Mumford stack with a $\mathbb{G}_m$-action. Suppose that $\mathscr{X}$ is smooth and the coarse moduli space is a scheme. Then

1. *The fixed locus $\mathscr{X}^{\mathbb{G}_m} = \coprod_i \mathscr{F}_i$ is a disjoint union of smooth closed substacks.*
2. *There exists locally closed $\mathbb{G}_m$-equivariant substacks $\mathscr{X}_i \hookrightarrow \mathscr{X}$ and affine fibrations $\mathscr{X}_i \to \mathscr{F}_i$.*
3. *$\coprod_i |\mathscr{X}_i| \to |\mathscr{X}|$ is a bijection of sets.*

Apply main theorem to $[\mathscr{X}/\mathbb{G}_m]$ and reduce to a $\mathbb{G}_m$-representation. Then $\mathscr{F}_i$ and $\mathscr{X}_i$ become linear subspaces.

A4. Kirwan desingularization of good moduli spaces

Theorem (Kirwan '85, Reichstein '89, Edidin–R '17)

Let $\mathscr{X}$ be a noetherian stack with good moduli space $\pi : \mathscr{X} \to \mathbf{X}$. If π is generically a coarse moduli space, then there exists a canonical sequence of quasi-projective maps (saturated blow-ups)

$$\mathscr{X}_n \to \mathscr{X}_{n-1} \to \cdots \to \mathscr{X}_1 \to \mathscr{X}_0 = \mathscr{X}$$

such that each $\mathscr{X}_i$ has a good moduli space $\mathbf{X}_i$ and the $\mathbf{X}_{i+1} \to \mathbf{X}_i$ are blow-ups. The final moduli space $\mathscr{X}_n \to \mathbf{X}_n$ is a coarse moduli space. If $\mathscr{X}$ is smooth, then so is the $\mathscr{X}_i$ and $\mathbf{X}_n \to \mathbf{X}$ is a partial resolution of singularities.

Can be combined with functorial resolution of finite tame quotient singularities (Gabber'05, Bergh'14, Buonerba'15) to obtain a full resolution of $\mathbf{X}$, even in positive characteristic.

A5. Existence of good moduli spaces

Theorem (Alper–Halpern-Leistner–Heinloth '18)

Let $\mathscr{X}$ be an algebraic stack with affine diagonal. Then $\mathscr{X}$ admits a separated good moduli space (resp. a gms) if and only if

1. *$\mathscr{X}$ is Θ-reductive,*
2. *$\mathscr{X}$ is S-complete (resp. has "unpunctured inertia"), and*
3. *$\mathscr{X}$ has lin. red. stabilizers at closed point (auto. in char. zero).*

Θ-reductivity and S-completeness are lifting criteria for

- $\Theta_R = [\mathbb{A}^1/\mathbb{G}_m] \times \operatorname{Spec} R$
- $\mathrm{ST}_R = [\operatorname{Spec}(R[s,t]/(st-\pi))/\mathbb{G}_m]$

where R is a discrete valuation ring.

Corollary (Alper–Halpern-Leistner–Heinloth '18)

Let X be a projective scheme over a field of characteristic 0. Let σ be a stability condition (Bridgeland, Gieseker, Joyce–Song, ...) on $D^b(\mathrm{Coh}(X))$. Fix a vector $\gamma \in H^*(X)$. Then the moduli stack of σ-semistable objects with Chern character γ has a proper good moduli space.

They also give a semi-stable reduction theorem for stacks with θ-stratifications.

A6. Good vs adequate moduli spaces

Adequate moduli spaces (Alper'10) are the analogue of GIT-quotients in positive characteristic, allowing for geometrically reductive stabilizers. In particular, in the GIT setting $[X^{ss}/G] \to X /\!\!/ G$ is an adequate moduli space.

The following intuitive result is very non-obvious from the definitions.

Theorem (AHR2)

Let $\mathscr{X}$ be a noetherian stack with adequate moduli space $\pi : \mathscr{X} \to \mathbf{X}$ of finite type. Then π is a good moduli space if and only if every closed point has linearly reductive stabilizer.

A7–A8. Resolution property and embeddability

Theorem (AHR1, AHR2)

Let $\pi : \mathscr{X} \to \mathbf{X}$ be a good moduli space. Then there exists an étale surjective morphism $\mathbf{X}' \to \mathbf{X}$ such that $\mathscr{X}' = \mathscr{X} \times_{\mathbf{X}} \mathbf{X}'$ has the resolution property.

Previously not even known when $\mathbf{X} = \operatorname{Spec} k$.

Corollary

Let $G \to S$ be a flat affine linearly reductive group scheme. Then there exists an étale surjective morphism $S' \to S$ such that $G \times_S S'$ is a closed subgroup of $\mathrm{GL}_N \times S'$.

A12. Compact generation of derived categories

Theorem (Hall–R '14)

Let $\mathscr{X}$ be a qcqs stack. Let $f : \mathscr{W} \to \mathscr{X}$ be a quasi-finite faithfully flat representable and separated morphism $\mathscr{W} \to \mathscr{X}$ such that

1. *$\mathscr{W}$ has the resolution property, ($\mathscr{W} = [q\text{-affine}/\mathrm{GL}_N]$)*
2. *$\mathscr{W}$ has finite cohomological dimension (∗).*

Then $\mathrm{D}_{qc}(\mathscr{X})$ is compactly generated.

(∗) Char 0: always. Char p: no additive subgroups ($\mathbb{G}_a$, $\mathbb{Z}/p\mathbb{Z}$, $\boldsymbol{\alpha}_p$) of stabilizers.

Corollary (AHR1, AHHR3)

Let $\mathscr{X}$ be a qcqs algebraic stack with affine diagonal. $\mathrm{D}_{qc}(\mathscr{X})$ is compactly generated

- *(char p) if and only if $(G_x)^0_{\mathrm{red}}$ torus for all closed points $x \in |\mathscr{X}|$.*
- *(char 0) if G_x reductive for all closed points $x \in |\mathscr{X}|$.*

A13–A15. Existence of henselizations and Ferrand pushouts

Let $\mathscr{X}$ be an algebraic stack with affine stabilizers. If $x \in |\mathscr{X}|$ has linearly reductive stabilizer, then $\mathscr{X}_x^{\wedge}$ and $\mathscr{X}_x^h$ exist. Also similar results along closed substacks, in particular:

Theorem (AHHR3)

Let X be an algebraic space and $X_0 \hookrightarrow X$ a closed subspace that is an affine scheme. The henselization along X_0 exists and is affine.

Corollary

Let $X_0 \hookrightarrow X$ be a closed immersion of algebraic spaces/stacks and $X_0 \to Y_0$ an affine morphism. The pushout $X \amalg_{X_0} Y_0$ exists in the category of algebraic spaces/stacks.

This generalizes earlier results of Ferrand'70 (certain schemes) and Temkin–Tyomkin'13 (certain algebraic spaces).

Open questions

$\mathscr{X}$ algebraic stack, $x \in |\mathscr{X}|$.

- If G_x is geometrically reductive? Étale-locally $[U/\mathrm{GL}_N]$ with U affine?
- If G_x is not reductive, e.g., $\mathbb{G}_a$? Étale-locally $[U/\mathrm{GL}_N]$ with U quasi-affine?
- Non-reductive version of good moduli spaces (in progress)
- Version for $\mathscr{X}$ analytic stack? (Differential-geometric version: Weinstein'00, Zung'06)

General version of the main theorem

Linearly fundamental stacks

Definition

A stack $\mathscr{X}$ has the **resolution property** if every sheaf of finite type is the quotient of a vector bundle. Equivalently $\mathscr{X} = [\text{q-affine}/\mathrm{GL}_N]$.

A stack $\mathscr{X}$ is **linearly fundamental** if it has an affine good moduli space and the resolution property.

Examples and remarks

- If G is linearly reductive and embeddable in GL_N, then $\mathscr{X} = [\text{affine}/G]$ is linearly fundamental.
- If $x \in |\mathscr{X}|$ is a point with linearly reductive (geometric) stabilizer, then the residual gerbe $\mathscr{G}_x \hookrightarrow \mathscr{X}$ is linearly fundamental.
- A stack $\mathscr{X}$ is **fundamental** if it has an affine adequate moduli space and the resolution property. Equivalently $\mathscr{X} = [\text{affine}/\mathrm{GL}_N]$.

General version of the main theorem

Theorem (AHHR3)

Let $\mathscr{X}$ be a quasi-separated algebraic stack with affine stabilizers and (FC)=(finitely many different characteristics). Let

1. $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ *be a closed substack,*
2. $\mathscr{W}_0$ *be a linearly fundamental stack, and*
3. $f_0 : \mathscr{W}_0 \to \mathscr{X}_0$ *be an étale/smooth/syntomic morphism.*
4. *If f_0 is not smooth, assume that $\mathscr{X}_0$ has the resolution property.*

Then there exists

- *a linearly fundamental stack $\mathscr{W}$, and*
- *an étale/smooth/syntomic morphism $f : \mathscr{W} \to \mathscr{X}$ extending f_0.*

Without (FC) or other assumptions, can only conclude that $\mathscr{W}$ is fundamental. If $\mathscr{X}$ derived, can replace syntomic with quasi-smooth.

Nisnevich neighborhoods

Theorem

Let $\mathscr{X}$ be a quasi-separated algebraic stack with affine stabilizers and (FC). Let $x \in |\mathscr{X}|$ be a (not nec. closed) point with linearly reductive stabilizer. Then there exists a linearly fundamental stack $\mathscr{W}$ and an étale neighborhood $f : \mathscr{W} \to \mathscr{X}$ of $\mathscr{G}_x$.

To get a Nisnevich neighborhood we need splittings at every point.

Theorem

Let $\mathscr{X}$ be a quasi-separated algebraic stack with nice (=extension of finite tame étale group by multiplicative type) stabilizers. Then there exists a Nisnevich covering $f : \coprod_i [\mathrm{Spec}(A_i)/G_i] \to \mathscr{X}$ where $G_i \hookrightarrow \mathrm{GL}_N$ are nice (and defined over some affine scheme).

In both results, if $\Delta_{\mathscr{X}}$ is affine/separated, then f is affine/representable.

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