CANONICAL REDUCTION OF STABILIZERS FOR ARTIN STACKS WITH GOOD MODULI SPACES

DAN EDIDIN AND DAVID RYDH

ABSTRACT. We present a complete generalization of Kirwan's partial desingularization theorem on quotients of smooth varieties. Precisely, we prove that if $\mathcal{X}$ is an irreducible Artin stack with stable good moduli space $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$, then there is a canonical sequence of birational morphisms of Artin stacks $\mathcal{X}_n \to \mathcal{X}_{n-1} \to \ldots \to \mathcal{X}_0 = \mathcal{X}$ with the following properties: (1) the maximum dimension of a stabilizer of a point of $\mathcal{X}_{k+1}$ is strictly smaller than the maximum dimension of a stabilizer of $\mathcal{X}_k$ and the final stack $\mathcal{X}_n$ has constant stabilizer dimension; (2) the morphisms $\mathcal{X}_{k+1} \to \mathcal{X}_k$ induce proper and birational morphisms of good moduli spaces $\mathbf{X}_{k+1} \to \mathbf{X}_k$. If in addition the stack $\mathcal{X}$ is smooth, then each of the intermediate stacks $\mathcal{X}_k$ is smooth and the final stack $\mathcal{X}_n$ is a gerbe over a tame stack. In this case the algebraic space $\mathbf{X}_n$ has tame quotient singularities and is a partial desingularization of the good moduli space $\mathbf{X}$.

When $\mathcal{X}$ is smooth our result can be combined with D. Bergh's recent destackification theorem for tame stacks to obtain a full desingularization of the algebraic space $\mathbf{X}$.

CONTENTS

1. Introduction

Consider the action of a reductive group G on a smooth projective variety X. For any ample G-linearized line bundle on X there is a corresponding projective geometric invariant theory (GIT) quotient $X/\!\!/G$. If $X^s = X^{ss}$, then $X/\!\!/G$ has finite quotient singularities. However, if $X^s \neq X^{ss}$, then the singularities of $X/\!\!/G$ can be quite bad. In a classic paper, Kirwan [Kir85] used a careful analysis of stable and unstable points on blowups to prove that if $X^s \neq \emptyset$, then there is a sequence of blowups along smooth centers $X_n \to X_{n-1} \to \ldots \to X_0 = X$ with the following properties: (1) The final blowup X_n is a smooth projective G-variety with $X_n^s = X_n^{ss}$. (2) The map of GIT quotients $X_n/\!\!/G \to X/\!\!/G$ is proper and birational. Since $X_n/\!\!/G$ has only finite quotient singularities, we may view it as a partial resolution of the very singular quotient $X/\!\!/G$.

Kirwan's result can be expressed in the language of algebraic stacks by noting that for linearly reductive groups, a GIT quotient $X/\!\!/G$ can be interpreted as the good moduli space of the quotient stack $[X^{ss}/G]$. The purpose of this paper is to give a complete generalization of Kirwan's result to algebraic stacks.

Precisely, we prove (Theorem 2.11) that if $\mathcal{X}$ is a (not necessarily smooth) Artin stack with *stable* good moduli space $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$, then there is a canonical sequence of birational morphisms of stacks $\mathcal{X}_n \to \mathcal{X}_{n-1} \ldots \to \mathcal{X}_0 = \mathcal{X}$ with the following properties: (1) If $\mathcal{X}_k$ is connected[1], then the maximum dimension of a stabilizer of a point of $\mathcal{X}_{k+1}$ is strictly smaller than the maximum dimension of a stabilizer of $\mathcal{X}_k$ and the final stack $\mathcal{X}_n$ has constant stabilizer dimension. (2) The morphisms $\mathcal{X}_{k+1} \to \mathcal{X}_k$ induce proper and birational morphisms of good moduli spaces $\mathbf{X}_{k+1} \to \mathbf{X}_k$.

When the stack $\mathcal{X}$ is smooth, then each intermediate stack $\mathcal{X}_k$ is smooth. This follows because $\mathcal{X}_{k+1}$ is an open substack of the blowup of $\mathcal{X}_k$ along the closed smooth substack $\mathcal{X}^{\max}$ parametrizing points with maximal dimensional stabilizer. Since $\mathcal{X}_n$ has constant dimensional stabilizer it follows (Proposition A.2) that its moduli space $\mathbf{X}_n$ has only tame quotient singularities. Thus our theorem gives a canonical procedure to partially desingularize the good moduli space $\mathbf{X}$.

Even in the special case of GIT quotients, our method allows us to avoid the intricate arguments used by Kirwan. In addition, we are not restricted to characteristic 0. However, Artin stacks with good moduli spaces necessarily have linearly reductive stabilizers at closed points. In positive characteristic this imposes a strong condition on the stack. Indeed by Nagata's theorem if G is a linearly reductive group over a field of characteristic p, then G^0 is diagonalizable and $p \nmid [G : G^0]$.

Theorem 2.11 can be combined with the destackification results of Bergh [Ber17] to give a functorial resolution of the singularities of good moduli spaces of smooth Artin stacks in arbitrary characteristic (Corollary 7.2).

In the smooth case, our results were applied in [ES17] to study intersection theory on singular good moduli spaces. There, Theorem 2.11 is used to show that the pullback

---

[1] A sufficient condition for all of the $\mathcal{X}_k$ to be connected is that $\mathcal{X}$ is irreducible.

$A^*_{\mathrm{op}}(\mathbf{X})_{\mathbb{Q}} \to A^*(\mathcal{X})_{\mathbb{Q}}$ is injective, where A^*_{op} denotes the operational Chow ring defined by Fulton [Ful84].

When $\mathcal{X}$ is singular, but possesses a virtual smooth structure, a variant of Theorem 2.11 can be applied to define numerical invariants of the stack. In the papers [KL13, KLS17] the authors use a construction similar to Theorem 2.11 for GIT quotients to define generalized Donaldson–Thomas invariants of Calabi–Yau three-folds. For GIT quotient stacks, the intrinsic blowup of [KL13, KLS17] is closely related to the saturated blowup (Definition 3.2) of a stack $\mathcal{X}$ along the locus of maximal dimensional stabilizer (see 8.7). Similar ideas are also being considered in recent work in progress of Joyce and Tanaka.

Outline of the proof of Theorem 2.11. The key technical construction in our proof is the *saturated* blowup of a stack along a closed substack.

Saturated Proj and blowups. If $\mathcal{X}$ is an Artin stack with good moduli space morphism $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$, then the blowup of $\mathcal{X}$ along $\mathcal{C}$ does not necessarily have a good moduli space. The reason is that if $\mathcal{A}$ is any sheaf of graded $\mathcal{O}_{\mathcal{X}}$-modules, then $\mathrm{Proj}_{\mathcal{X}}(\mathcal{A})$ need not have a good moduli space. However, we prove (Proposition 3.4) that there is a canonical open substack $\mathrm{Proj}^{\pi}_{\mathcal{X}}(\mathcal{A}) \subset \mathrm{Proj}_{\mathcal{X}}(\mathcal{A})$ whose good moduli space is $\mathrm{Proj}_{\mathbf{X}}(\pi_*\mathcal{A})$. In general the morphism $\mathrm{Proj}^{\pi}_{\mathcal{X}}(\mathcal{A}) \to \mathcal{X}$ is not proper, but the natural morphism $\mathrm{Proj}_{\mathbf{X}}(\pi_*\mathcal{A}) \to \mathbf{X}$ is identified with the morphism of good moduli spaces induced from the morphism of stacks $\mathrm{Proj}^{\pi}_{\mathcal{X}}(\mathcal{A}) \to \mathcal{X}$. We call $\mathrm{Proj}^{\pi}_{\mathcal{X}}(\mathcal{A})$ the *saturated* Proj of $\mathcal{A}$ relative to the good moduli space morphism π (Definition 3.1).

If $\mathcal{C}$ is a closed substack of $\mathcal{X}$ with sheaf of ideals $\mathcal{I}$, then we call $\mathrm{Bl}^{\pi}_{\mathcal{C}}\mathcal{X} := \mathrm{Proj}^{\pi}_{\mathcal{X}}\left(\bigoplus \mathcal{I}^n\right)$ the *saturated blowup* of $\mathcal{X}$ along $\mathcal{C}$. When $\mathcal{X}$ and $\mathcal{C}$ are smooth, then $\mathrm{Bl}^{\pi}_{\mathcal{C}}\mathcal{X}$ has a particularly simple description (Proposition 4.5). It is the complement of the strict transform of the saturation of $\mathcal{C}$ with respect to the good moduli space morphism $\pi \colon \mathcal{X} \to \mathbf{X}$.

Given a closed substack $\mathcal{C} \subset \mathcal{X}$ the *Reichstein transform* $R(\mathcal{X}, \mathcal{C})$ of $\mathcal{X}$ along $\mathcal{C}$ is the complement of the strict transform of the saturation of $\mathcal{C}$ in the blowup $\mathrm{Bl}_{\mathcal{C}}\mathcal{X}$. The Reichstein transform was introduced in [EM12] where toric methods were used to prove that there is a canonical sequence of toric Reichstein transforms, called stacky star subdivisions, which turn an Artin toric stack into a Deligne–Mumford toric stack.

The term "Reichstein transform" was inspired by Reichstein's paper [Rei89] which contains the result that if $C \subset X$ is a smooth, closed G-invariant subvariety of a smooth, G-projective variety X then $(\mathrm{Bl}_C X)^{ss}$ is the complement of the strict transform of the saturation of $C \cap X^{ss}$ in the blowup of X^{ss} along $C \cap X^{ss}$.

Outline of the proof when $\mathcal{X}$ is smooth and connected. If $\mathcal{X}$ is a smooth Artin stack with good moduli space $\mathcal{X} \to \mathbf{X}$, then the substack $\mathcal{X}^{\mathrm{max}}$, corresponding to points with maximal dimensional stabilizer, is closed and smooth. Thus $\mathcal{X}' = R(\mathcal{X}, \mathcal{X}^{\mathrm{max}})$ is a smooth Artin stack whose good moduli space $\mathbf{X}'$ maps properly to $\mathbf{X}$ and is an isomorphism over the complement of $\mathbf{X}^{\mathrm{max}}$, the image of $\mathcal{X}^{\mathrm{max}}$ in $\mathbf{X}$. The stability hypothesis ensures that as long as the stabilizers are not all of constant dimension, $\mathbf{X}^{\mathrm{max}}$ is a proper closed substack of $\mathbf{X}$. Using the local structure theorem of [AHR15]

we can show (Proposition 6.1) that the maximum dimension of the stabilizer of a point of $\mathcal{X}'$ is strictly smaller than the maximum dimension of the stabilizer of a point of $\mathcal{X}$. The proof then follows by induction.

In the local case, Proposition 6.1 follows from Theorem 5.1 which states that if G is a connected, linearly reductive group acting on a smooth affine scheme, then the equivariant Reichstein transform $R_G(X, X^G)$ has no G-fixed points. The proof of Theorem 5.1 is in turn reduced to the case that $X = V$ is a representation of G, where the statement can be checked by direct calculation (Proposition 5.4).

The general case. For a singular stack $\mathcal{X}$, the strategy is essentially the same as in the smooth case. The locus of points $\mathcal{X}^{\max}$ of maximum dimensional stabilizer has a canonical substack structure but this need not be reduced. The proof is a bit technical, particularly in positive characteristic, and is given in the Appendix. When $\mathcal{X}$ is singular, the Reichstein transform $R(\mathcal{X}, \mathcal{X}^{\max})$ is not so useful: the maximum stabilizer dimension need not drop (cf. Example 5.7) and the Reichstein transform need not admit a good moduli space (cf. Example 5.8).

The saturated blowup, however, always admit a good moduli space. Moreover, we prove that if $\mathcal{X}^{\max} \neq \mathcal{X}$ and $\mathcal{X}$ is connected, then the saturated blowup $\mathrm{Bl}^{\tau}_{\mathcal{X}^{\max}} \mathcal{X}$ has strictly smaller dimensional stabilizers. This is again proved by reducing to the case that $\mathcal{X} = [X/G]$ where X is an affine scheme. Since, X can be embedded into a representation V of G, we can use the corresponding result for smooth schemes and functorial properties of saturated blowups (Proposition 3.11) to prove the result.

Conventions and Notation. All algebraic stacks are assumed to have affine diagonal and be of finite type over an algebraically closed field k.

A *point* of an algebraic stack $\mathcal{X}$ is an equivalence class of morphisms $\operatorname{Spec} K \xrightarrow{x} \mathcal{X}$ where K is a field, and $(x', K') \sim (x'', K'')$ if there is a k-field K containing K', K'' such that the morphisms $\operatorname{Spec} K \to \operatorname{Spec} K' \xrightarrow{x'} \mathcal{X}$ and $\operatorname{Spec} K \to \operatorname{Spec} K'' \xrightarrow{x''} \mathcal{X}$ are isomorphic. The set of points of $\mathcal{X}$ is denoted $|\mathcal{X}|$.

Since $\mathcal{X}$ is of finite type over a field it is noetherian. This implies that every point of $\xi \in |\mathcal{X}|$ is algebraic [LMB00, Théorème 11.3], [Ryd11, Appendix B]. This means that if $\operatorname{Spec} K \xrightarrow{x} \mathcal{X}$ is a representative for ξ, then the morphism x factors as $\operatorname{Spec} K \xrightarrow{\bar{x}} \mathcal{G}_\xi \to \mathcal{X}$, where $\bar{x}$ is faithfully flat and $\mathcal{G}_\xi \to \mathcal{X}$ is a representable monomorphism. Moreover, $\mathcal{G}_\xi$ is a gerbe over a field $k(\xi)$ which is called the *residue field* of the point ξ. The stack $\mathcal{G}_\xi$ is called the *residual gerbe* and is independent of the choice of representative $\operatorname{Spec} K \xrightarrow{x} \mathcal{X}$.

Given a morphism $\operatorname{Spec} K \xrightarrow{x} \mathcal{X}$, define the stabilizer group G_x to be the fiber product:

$$
\begin{array}{ccc}
G_x & \longrightarrow & \operatorname{Spec} K \\
\downarrow & & \downarrow {\scriptstyle (x,x)} \\
\mathcal{X} & \xrightarrow{\Delta_x} & \mathcal{X} \times_k \mathcal{X}
\end{array}
$$

Since the diagonal is representable G_x is a K-group which we call the stabilizer of x.

Since we work over an algebraically closed field, any closed point is geometric and is represented by a morphism $\operatorname{Spec} k \xrightarrow{x} \mathcal{X}$. In this case the residual gerbe is BG_x where G_x is the stabilizer of x.

2. Stable good moduli spaces and statement of the main theorem

2.1. Stable good moduli spaces.

Definition 2.1 ([Alp13, Definition 4.1]). A morphism $\pi \colon \mathcal{X} \to \mathbf{X}$ from an algebraic stack to an algebraic space is a *good moduli space* if

  (1) π is *cohomologically affine*, meaning that the pushforward functor π_* on the category of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules is exact.
  (2) The natural map $\mathcal{O}_\mathbf{X} \to \pi_* \mathcal{O}_\mathcal{X}$ is an isomorphism.

More generally, a morphism of Artin stacks $\phi \colon \mathcal{X} \to \mathcal{Y}$ satisfying conditions (1) and (2) is called a *good moduli space morphism.*

Remark 2.2. The morphism π is universal for maps to algebraic spaces, so the algebraic space $\mathbf{X}$ is unique up to isomorphism [Alp13, Theorem 6.6]. Thus, we can refer to $\mathbf{X}$ as *the* good moduli space of $\mathcal{X}$.

Remark 2.3. If $\mathcal{X} \to \mathbf{X}$ is a good moduli space, then the stabilizer of any closed point of $\mathcal{X}$ is linearly reductive by [Alp13, Proposition 12.14].

Remark 2.4. Let $\mathcal{X}$ be a stack with finite inertia $I\mathcal{X} \to \mathcal{X}$. By the Keel–Mori theorem, there is a coarse moduli space $\pi \colon \mathcal{X} \to \mathbf{X}$. Following [AOV08] we say that $\mathcal{X}$ is *tame* if π is cohomologically affine. This happens precisely when the stabilizer groups are linearly reductive. In this case $\mathbf{X}$ is also the good moduli space of $\mathcal{X}$ by [Alp13, Example 8.1]. Conversely, if $\pi \colon \mathcal{X} \to \mathbf{X}$ is the good moduli space of a stack such that all the stabilizers are 0-dimensional, then $\mathcal{X}$ is a tame stack with coarse moduli space $\mathbf{X}$ (Proposition A.1). More generally, if the stabilizers of $\mathcal{X}$ have constant dimension n and $\mathcal{X}$ is reduced, then $\mathcal{X}$ is a gerbe over a tame stack whose coarse space is $\mathbf{X}$ (Proposition A.2).

Definition 2.5. Let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism. A point x of $\mathcal{X}$ is *stable* relative to π if $\pi^{-1}(\pi(x)) = \{x\}$ under the induced map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$. A point x of $\mathcal{X}$ is *properly stable* relative to π if it is stable and $\dim G_x = \dim G_{\pi(x)}$.

We say π is a stable (resp. properly stable) good moduli space morphism if the set of stable (resp. properly stable) points is dense.

The dimension of the fibers of the relative inertia morphism $I\pi \to \mathcal{X}$ is an upper semi-continuous function [SGA3, Exposé VIb, Proposition 4.1]. Hence the set $\mathcal{X}_{>d} = \{x \in |\mathcal{X}| : \dim G_x - \dim G_{\pi(x)} > d\}$ is closed.

Proposition 2.6. *The set of stable points defines an open (but possibly empty) substack $\mathcal{X}^s \subset \mathcal{X}$ which is saturated with respect to the morphism π. If $\mathcal{X}$ is irreducible with*

generic point ξ, then

$$\mathcal{X}^s = \mathcal{X} \smallsetminus \pi^{-1}(\pi(\mathcal{X}_{>d}))$$

where $d = \dim G_\xi - \dim G_{\pi(\xi)}$ is the minimum dimension of the relative stabilizer groups. In particular, $\dim G_x - \dim G_{\pi(x)} = d$ at all points of $\mathcal{X}^s$.

Proof. If $Y \to \mathcal{Y}$ is any morphism, then $(Y \times_{\mathcal{Y}} \mathcal{X})^s = Y \times_{\mathcal{Y}} \mathcal{X}^s$. Indeed, if x is stable, then $\mathcal{G}_x \to \mathcal{G}_{\pi(x)}$ is a good moduli space [Alp13, Lemma 4.14], hence a gerbe, so $\pi^{-1}(\pi(x)) \to \pi(x)$ is universally injective. We can thus reduce to the case where $\mathcal{Y} = \mathbf{X}$ is a scheme.

If $\mathcal{Z}$ is an irreducible component of $\mathcal{X}$, then the map $\mathcal{Z} \overset{\pi|_{\mathcal{Z}}}{\to} \pi(\mathcal{Z})$ is a good moduli space morphism by [Alp13, Lemma 4.14].

If x is a point of $\mathcal{X}$, then $\pi^{-1}(\pi(x)) = \bigcup_{\mathcal{Z} \subset \mathcal{X}}(\pi|_{\mathcal{Z}})^{-1}(\pi(x))$ where the union is over the irreducible components of $\mathcal{X}$ which contain x. Thus a point x is stable if and only if $(\pi|_{\mathcal{Z}})^{-1}(\pi|_{\mathcal{Z}}(x)) = x$ for every irreducible component $\mathcal{Z}$ containing x. If we let $\mathcal{Z}^s$ be the set of stable points for the good moduli space morphism $\pi|_{\mathcal{Z}}$, then $\mathcal{X}^s = (\bigcup_{\mathcal{Z}}(\mathcal{Z} \smallsetminus \mathcal{Z}^s))^c$ where the union is over all irreducible components of $\mathcal{X}$. Since we assume that $\mathcal{X}$ is noetherian there are only a finite number of irreducible components. Thus, it suffices to prove that $\mathcal{Z}^s$ is open for each irreducible component $\mathcal{Z}$. In other words, we are reduced to the case that $\mathcal{X}$ is irreducible.

Then $\mathcal{X}_{\leq d} = \mathcal{X}_d = \{x \in |\mathcal{X}| : \dim G_x = d\}$ is open and dense and to see that $\mathcal{X}^s = (\pi^{-1}(\pi(\mathcal{X}_{>d})))^c$ we argue as follows.

By [Alp13, Proposition 9.1] if x is a point of $\mathcal{X}$ and $\pi^{-1}(\pi(x))$ is not a singleton, then $\pi^{-1}(\pi(x))$ contains a unique closed point y and $\dim G_y$ is greater than the dimension of any other stabilizer in $\pi^{-1}(\pi(x))$. Such a point is clearly not in the open set $\mathcal{X}_d$, so we conclude that $(\mathcal{X}^s)^c \subset \pi^{-1}(\pi(\mathcal{X}_{>d}))$ or equivalently that $\mathcal{X}^s \supset (\pi^{-1}(\pi(\mathcal{X}_{>d})))^c$.

To obtain the reverse inclusion we need to show that if x is a point of $\mathcal{X}$ and $\pi^{-1}(\pi(x)) = x$, then $\dim G_x = d$. Consider the stack $\pi^{-1}(\pi(x))$ with its reduced stack structure. The monomorphism from the residual gerbe $\mathcal{G}_x \to \mathcal{X}$ factors through a monomorphism $\mathcal{G}_x \to \pi^{-1}(\pi(x)))$. Since $\pi^{-1}(\pi(x))$ has a single point the morphism $\mathcal{G}_x \to \pi^{-1}(\pi(x))_{\mathrm{red}}$ is an equivalence [Sta16, Tag 06MT]. Hence $\dim \pi^{-1}(\pi(x)) = \dim \mathcal{G}_x = -\dim G_x$.

Let ξ be the unique closed point in the generic fiber of π. Then $x \in \overline{\{\xi\}}$ so by upper semi-continuity $\dim G_x \geq \dim G_\xi$ and $\dim \pi^{-1}(\pi(x)) \geq \dim \pi^{-1}(\pi(\xi))$. Moreover, $\dim \pi^{-1}(\pi(\xi)) \geq -\dim G_\xi$ with equality if and only if $\pi^{-1}(\pi(\xi))$ is a singleton. It follows that

$$\dim \pi^{-1}(\pi(\xi)) \geq -\dim G_\xi \geq -\dim G_x = \dim \pi^{-1}(\pi(x))$$

is an equality so the generic fiber $\pi^{-1}(\pi(\xi))$ is a singleton and $\dim G_x = \dim G_\xi = d$. $\square$

Let $\mathcal{X}$ be a reduced and irreducible Artin stack and let $\pi \colon \mathcal{X} \to \mathbf{X}$ be a good moduli space morphism with $\mathbf{X}$ an algebraic space and let $\mathbf{X}^s = \pi(\mathcal{X}^s)$. Since $\mathcal{X}^s$ is saturated, $\mathbf{X}^s$ is open in $\mathbf{X}$.

Proposition 2.7. *With notation as in the preceding paragraph $\mathcal{X}^s$ is a gerbe over a tame stack with coarse space $\mathbf{X}^s$. Moreover, $\mathcal{X}^s$ is the largest saturated open substack with this property.*

Proof. By Proposition 2.6 the dimension of the stabilizer G_x is constant at every point x of $\mathcal{X}^s$. Hence by Proposition A.2, $\mathcal{X}^s$ is a gerbe over a tame stack whose coarse space is $\mathbf{X}^s$.

Conversely, if $\mathcal{U}$ is a saturated open substack which is a gerbe over a tame stack $\mathcal{U}_{\text{tame}}$, then the good moduli space morphism $\mathcal{U} \to \mathbf{U}$ factors via $\mathcal{U}_{\text{tame}}$. Since $|\mathcal{U}| \to |\mathcal{U}_{\text{tame}}|$ and $|\mathcal{U}_{\text{tame}} \to \mathbf{U}|$ are homeomorphisms, it follows that $\mathcal{U} \subset \mathcal{X}^s$ by definition. $\qquad\square$

2.2. Examples.

Remark 2.8. If $\mathcal{X} = [X/G]$ is a quotient stack with $X = \operatorname{Spec} A$ an affine variety and G is a linearly reductive group, then the good moduli space morphism $\mathcal{X} \to \mathbf{X} = \operatorname{Spec} A^G$ is stable if and only if the action is stable in the sense of [Vin00]. This means that there is a closed orbit of maximal dimension. The morphism $\mathcal{X} \to \mathbf{X}$ is properly stable if the maximal dimension equals $\dim G$. Following [Vin00] we will say that a representation V of a linearly reductive group G is *stable* if the action of G on V is stable.

Example 2.9. If $X = \mathbb{A}^2$ and $G = \mathbb{G}_m$ acts on X by $\lambda(a, b) = (\lambda a, b)$, then the good moduli space morphism $[X/G] \to \mathbb{A}^1$ is not stable since the inverse image of every point under the quotient map $\mathbb{A}^2 \to \mathbb{A}^1$, $(a, b) \mapsto b$ contains a point with stabilizer of dimension 1. On the other hand, if we consider the action of $\mathbb{G}_m$ given by $\lambda(a, b) = (\lambda^d a, \lambda^{-e} b)$ with $d, e > 0$, then the good moduli space morphism $[X/G] \to \mathbb{A}^1$ is properly stable, since the inverse image of the open set $\mathbb{A}^1 \smallsetminus \{0\}$ is the Deligne–Mumford substack $[(\mathbb{A}^2 \smallsetminus V(xy))/\mathbb{G}_m]$.

Example 2.10. Consider the action of GL_n on $\mathfrak{gl}_n$ via conjugation in characteristic zero. If we identify $\mathfrak{gl}_n$ with the space $\mathbb{A}^{n^2}$ of $n \times n$ matrices, then the map $\mathfrak{gl}_n \to \mathbb{A}^n$ which sends a matrix to the coefficients of its characteristic polynomial is a good quotient, so the map $\pi \colon [\mathfrak{gl}_n / \operatorname{GL}_n] \to \mathbb{A}^n$ is a good moduli space morphism. The orbit of an $n \times n$ matrix is closed if and only if it is diagonalizable. Since the stabilizer of a matrix with distinct eigenvalues is a maximal torus T, such matrices have orbits of dimension $n^2 - n = \dim \operatorname{GL}_n - \dim T$ which is maximal.

If $U \subset \mathbb{A}^n$ is the open set corresponding to polynomials with distinct roots, then $\pi^{-1}(U)$ is a T-gerbe over the scheme U. Hence π is a stable good moduli space morphism, although it is not properly stable.

2.3. Statement of the main theorem.

Theorem 2.11. *Let $\mathcal{X}$ be an Artin stack with stable good moduli space $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$ and let $\mathcal{E} \subset \mathcal{X}$ be an effective Cartier divisor (possibly empty). There is a canonical sequence of birational morphisms of Artin stacks $\mathcal{X}_n \to \mathcal{X}_{n-1} \ldots \to \mathcal{X}_0 = \mathcal{X}$, closed substacks $(\mathcal{C}_\ell \subset \mathcal{X}_\ell)_{0 \leq \ell \leq n-1}$, and effective Cartier divisors $(\mathcal{E}_\ell \subset \mathcal{X}_\ell)_{0 \leq \ell \leq n}$, $\mathcal{E}_0 = \mathcal{E}$, with the following properties for each $\ell = 0, 1, \ldots, n-1$.*

(1a) $|\mathcal{C}_\ell|$ is the locus in $\mathcal{X}_\ell \smallsetminus \mathcal{X}_\ell^s$ of points of maximal dimensional stabilizer.

(1b) $\mathcal{E}_{\ell+1}$ is the inverse image of $\mathcal{C}_\ell \cup \mathcal{E}_\ell$.

(1c) If $\mathcal{X}$ is smooth and $\mathcal{E}_0$ is snc, then $\mathcal{C}_{\ell+1}$ and $\mathcal{X}_{\ell+1}$ are smooth and $\mathcal{E}_{\ell+1}$ is snc.

(2a) There is a stable good moduli space $\pi_{\ell+1} \colon \mathcal{X}_{\ell+1} \to \mathbf{X}_{\ell+1}$. If π is properly stable, then so is $\pi_{\ell+1}$.

(2b) The morphism $f_\ell \colon \mathcal{X}_{\ell+1} \to \mathcal{X}_\ell$ induces an isomorphism $\mathcal{X}_{\ell+1} \smallsetminus f_\ell^{-1}(\mathcal{C}_\ell) \to \mathcal{X}_\ell \smallsetminus \pi_\ell^{-1}\big(\pi_\ell(\mathcal{C}_\ell)\big)$. In particular, we have an isomorphism $\mathcal{X}_{\ell+1}^s \smallsetminus \mathcal{E}_{\ell+1} \to \mathcal{X}_\ell^s \smallsetminus \mathcal{E}_\ell$.

(2c) The morphism $\mathcal{X}_{\ell+1} \to \mathcal{X}_\ell$ induces a projective birational morphism of good moduli spaces $\mathbf{X}_{\ell+1} \to \mathbf{X}_\ell$.

(3) The maximum dimension of the stabilizers of points of $\mathcal{X}_{\ell+1} \smallsetminus \mathcal{X}_{\ell+1}^s$ is strictly smaller than the maximum dimension of the stabilizers of points of $\mathcal{X}_\ell \smallsetminus \mathcal{X}_\ell^s$.

The final stack $\mathcal{X}_n$ has the following properties:

(4a) Every point of $\mathcal{X}_n$ is stable. In particular, π_n is a homeomorphism and the dimension of the stabilizers of $\mathcal{X}_n$ is locally constant.

(4b) $\mathcal{X}_n \to \mathcal{X}$ is an isomorphism over $\mathcal{X}^s$ and $\mathcal{X}_n \smallsetminus \mathcal{E}_n = \mathcal{X}^s \smallsetminus \mathcal{E}$. In particular, $\mathcal{X}^s \subset \mathcal{X}_n$ is schematically dense.

(4c) If $\mathcal{X}$ is properly stable, then $\mathcal{X}_n$ is a tame stack and $\mathbf{X}_n$ its coarse moduli space.

(4d) If $\mathcal{X}^s \smallsetminus \mathcal{E}$ is a gerbe over a tame stack (e.g., if $\mathcal{X}^s$ is reduced), then $\mathcal{X}_n$ is a gerbe over a tame stack.

The tame stack above is separated if and only if $\mathbf{X}$ is separated. The sequence $\mathcal{X}_n \to \mathcal{X}_{n-1} \to \cdots \to \mathcal{X}$ does not depend on $\mathcal{E}$.

Remark 2.12. The birational morphisms $\mathcal{X}_{\ell+1} \to \mathcal{X}_\ell$ are *Reichstein transforms* in the centers $\mathcal{C}_\ell$ if $\mathcal{X}$ is smooth and *saturated blowups* in the centers $\mathcal{C}_\ell$ in general. They are discussed in the next section. The closed substack $\mathcal{C}_\ell$ is the set of points in $\mathcal{X}_\ell \smallsetminus \mathcal{X}_\ell^s$ with maximal-dimensional stabilizer equipped with a canonical scheme structure which need not be reduced. In particular, $\mathcal{C}_\ell \cap \mathcal{X}_\ell^s = \emptyset$ and $\mathcal{X}_n \to \mathcal{X}$ is an isomorphism over the stable locus. Note that if $\mathcal{X}_\ell$ is connected, then no point with maximal dimensional stabilizer can be stable, so $\mathcal{C}_\ell$ is supported on the locus of points with maximal dimensional stabilizer in $\mathcal{X}_\ell$.

Remark 2.13. For singular $\mathcal{X}$, there are other possible sequences $\mathcal{X}_n \to \mathcal{X}_{n-1} \to \cdots \to \mathcal{X}$ that satisfy the conclusions: (1a), (2a)–(2c), (3), (4a) and (4c) but for which $f_\ell^{-1}(C_\ell)$ is not a Cartier divisor. In our sequence, using saturated blowups, $\mathcal{X}^s \subset \mathcal{X}_n$ is schematically dense and there are no new irreducible components appearing in the process. It is, however, possible to replace the saturated blowups with variants such as saturated symmetric blowups or intrinsic blowups, see Remark 8.7. For these variants, $\mathcal{X}^s$ is typically not schematically dense and $\mathcal{X}_n$ have additional irreducible components.

3. Saturated Proj and saturated blowups

In this section we study a variant of Proj and blowups that depends on a morphism π. In every result, π will be a good moduli space morphism but we define the constructions for more general morphisms.

Definition 3.1 (Saturated Proj). Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $\mathcal{A}$ be a (positively) graded sheaf of finitely generated $\mathcal{O}_{\mathcal{X}}$-algebras. Let $\pi^{-1}\pi_*\mathcal{A}^+$ denote the image of the natural homomorphism $\pi^*\pi_*\mathcal{A}^+ \to \mathcal{A}^+ \to \mathcal{A}$. Define $\operatorname{Proj}_{\mathcal{X}}^{\pi}\mathcal{A} = \operatorname{Proj}_{\mathcal{X}}\mathcal{A} \smallsetminus V(\pi^{-1}\pi_*\mathcal{A}^+)$. We call $\operatorname{Proj}_{\mathcal{X}}^{\pi}\mathcal{A}$ the *saturated* Proj of $\mathcal{A}$ relative to the morphism π.

Note that the morphism $\operatorname{Proj}_{\mathcal{X}}^{\pi}\mathcal{A} \to \mathcal{X}$ need not be proper. Also note that there is a canonical morphism $\operatorname{Proj}_{\mathcal{X}}^{\pi}\mathcal{A} \to \operatorname{Proj}_{\mathcal{Y}}\pi_*\mathcal{A}$: we are exactly removing the locus where $\operatorname{Proj}_{\mathcal{X}}\mathcal{A} \dashrightarrow \operatorname{Proj}_{\mathcal{Y}}\pi_*\mathcal{A}$ is not defined.

Definition 3.2 (Saturated blowups). Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $\mathcal{C} \subset \mathcal{X}$ be a closed substack with sheaf of ideals $\mathcal{I}$. We let $\operatorname{Bl}_{\mathcal{C}}^{\pi}\mathcal{X} = \operatorname{Proj}_{\mathcal{X}}^{\pi}\left(\bigoplus \mathcal{I}^n\right)$ and call it the *saturated blowup* of $\mathcal{X}$ in $\mathcal{C}$. The *exceptional divisor* of the saturated blowup is the restriction of the exceptional divisor of the blowup along the open substack $\operatorname{Bl}_{\mathcal{C}}^{\pi}\mathcal{X} \subset \operatorname{Bl}_{\mathcal{C}}\mathcal{X}$.

Example 3.3 (Saturated blowups and GIT). Let $\mathcal{X} = [X^{ss}/G]$ where $X^{ss} = X^{ss}(L)$ is the set of L-semistable points for the action of a reductive group G on a smooth projective variety X. Let $C \subset X$ be a smooth, closed G-invariant subscheme and let $\mathcal{C} = [(C \cap X^{ss}/G)]$. Let $p\colon X' \to X$ be the blowup of X along C and denote the exceptional divisor by E. Likewise let $\mathcal{X}' = \operatorname{Bl}_{\mathcal{C}}^{\pi}\mathcal{X}$ where $\pi\colon \mathcal{X} \to X/\!\!/G$ is the good moduli space morphism. Then by [Rei89, Theorem 2.4] and Proposition 4.5 below, $\mathcal{X}' = [(X')^{ss}(K)/G]$ where $K = p^*L^d(-E)$ for some $d \gg 0$.

We expect that a similar statement holds in general, but have not verified this.

Proposition 3.4. *If $\pi\colon \mathcal{X} \to \mathcal{Y}$ is a good moduli space morphism and $\mathcal{A}$ is a finitely generated graded $\mathcal{O}_{\mathcal{X}}$-algebra, then $\operatorname{Proj}_{\mathcal{X}}^{\pi}\mathcal{A} \to \operatorname{Proj}_{\mathcal{Y}}\pi_*\mathcal{A}$ is a good moduli space morphism and the morphism of good moduli spaces induced by the morphism $\operatorname{Proj}_{\mathcal{X}}^{\pi}\mathcal{A} \to \mathcal{X}$ is the natural morphism $\operatorname{Proj}_Y \pi_*\mathcal{A} \to \mathcal{Y}$.*

Proof. To show that the natural morphism

$$\operatorname{Proj}_{\mathcal{X}}^{\pi}\mathcal{A} \to \operatorname{Proj}_{\mathcal{Y}}\pi_*\mathcal{A}$$

is a good moduli space morphism, we may, by [Alp13, Proposition 4.9(ii)], work locally in the smooth or fppf topology on $\mathcal{Y}$ and assume that $\mathcal{Y}$ is affine. In this case $\operatorname{Proj}\pi_*\mathcal{A}$ is the scheme obtained by gluing the affine schemes $\operatorname{Spec}(\pi_*\mathcal{A})_{(f)}$ as f runs through elements $f \in \pi_*\mathcal{A}^+$. Likewise, $\operatorname{Proj}_{\mathcal{X}}^{\pi}\mathcal{A}$ is the open set in $\operatorname{Proj}_{\mathcal{X}}\mathcal{A}$ obtained by gluing the $\mathcal{X}$-affine stacks $\operatorname{Spec}_{\mathcal{X}}\mathcal{A}_{(f)}$ as f runs through π_*A^+. It is thus enough to prove that

$$\operatorname{Spec}_{\mathcal{X}}\mathcal{A}_{(f)} \to \operatorname{Spec}_{\mathcal{Y}}(\pi_*\mathcal{A})_{(f)}$$

is a good moduli space morphism.

By [Alp13, Lemma 4.14] if $\mathcal{A}$ is a sheaf of coherent $\mathcal{O}_{\mathcal{X}}$-algebras, then $\mathrm{Spec}_{\mathcal{X}} \mathcal{A} \to \mathrm{Spec}_{\mathcal{Y}} \pi_* \mathcal{A}$ is a good moduli space morphism and the diagram

$$
\begin{array}{ccc}
\mathrm{Spec}_{\mathcal{X}} \mathcal{A} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow {\scriptstyle \pi} \\
\mathrm{Spec}_{\mathcal{Y}} \pi_* \mathcal{A} & \longrightarrow & \mathcal{Y}
\end{array}
$$

is commutative. Since good moduli space morphisms are invariant under base change [Alp13, Proposition 4.9(i)] we see that

$$\mathrm{Spec}_{\mathcal{X}} \mathcal{A} \smallsetminus V(\pi^{-1}(\pi_* \mathcal{A}^+)) \to \mathrm{Spec}_{\mathcal{Y}} \pi_* \mathcal{A} \smallsetminus V(\pi_* \mathcal{A}^+)$$

is a good moduli space morphism. Now $\mathrm{Proj}_{\mathcal{Y}} \pi_* \mathcal{A}$ is the quotient of $\mathrm{Spec}_{\mathcal{Y}} \pi_* \mathcal{A} \smallsetminus V(\pi_* A^+)$ by the action of $\mathbb{G}_m$ on the fibers over $\mathcal{Y}$. It is a coarse quotient since $\pi_* \mathcal{A}$ is not necessarily generated in degree 1. Likewise, $\mathrm{Proj}_{\mathcal{X}}^{\pi} \mathcal{A}$ is the quotient of $\mathrm{Spec}_{\mathcal{X}} \mathcal{A} \smallsetminus V(\pi^{-1}(\pi_* \mathcal{A}^+))$ by the action of $\mathbb{G}_m$ on the fibers over $\mathcal{X}$.

Since the property of being a good moduli space is preserved by base change, $\mathrm{Spec}_{\mathcal{X}} \mathcal{A}_f \to \mathrm{Spec}(\pi_* \mathcal{A})_f$ is a good moduli space morphism. This gives us the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}_{\mathcal{X}} \mathcal{A}_f & \xrightarrow{q_{\mathcal{X}}} & \mathrm{Spec}_{\mathcal{X}} \mathcal{A}_{(f)} \\
\downarrow {\scriptstyle \pi_{\mathcal{A}_f}} & & \downarrow {\scriptstyle \pi_{\mathcal{A}_{(f)}}} \\
\mathrm{Spec}_{\mathcal{Y}}(\pi_* \mathcal{A})_f & \xrightarrow{q_{\mathcal{Y}}} & \mathrm{Spec}_{\mathcal{Y}}(\pi_* \mathcal{A})_{(f)}
\end{array}
$$

where $\pi_{\mathcal{A}_f}$ is a good moduli space morphism and $q_{\mathcal{X}}$ and $q_{\mathcal{Y}}$ are coarse $\mathbb{G}_m$-quotients. Note that the natural transformation $M \to (q_* q^* M)_0$ is an isomorphism for $q = q_{\mathcal{X}}$ and $q = q_{\mathcal{Y}}$. Since $(\pi_{\mathcal{A}})_*$ is compatible with the grading, it follows that

$$(\pi_{\mathcal{A}_{(f)}})_* M = ((q_{\mathcal{Y}})_* (\pi_{\mathcal{A}_f})_* (q_{\mathcal{X}})^* M)_0$$

is a composition of right-exact functors, hence exact. It follows that $\pi_{\mathcal{A}_{(f)}}$ is a good moduli space morphism. $\square$

3.1. **Variation of GIT.**

Remark 3.5. Let $f \colon \mathcal{X}' \to \mathcal{X}$ be a projective morphism and let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism. Choose an f-ample line bundle $\mathcal{L}$. Then $\mathcal{X}' = \mathrm{Proj}_{\mathcal{X}} \bigoplus_{n \geq 0} f_* \mathcal{L}^n$. We obtain an open substack $\mathcal{X}'_{\mathcal{L}} := \mathrm{Proj}_{\mathcal{X}}^{\pi} \bigoplus_{n \geq 0} f_* \mathcal{L}^n$ of $\mathcal{X}'$ and a good moduli space morphism

$$\mathcal{X}'_{\mathcal{L}} \to \mathrm{Proj}_{\mathcal{Y}} \bigoplus_{n \geq 0} \pi_* f_* \mathcal{L}^n.$$

The open substack $\mathcal{X}'_{\mathcal{L}}$ is the locus where $f^* \pi^* \pi_* f_* \mathcal{L}^n \to \mathcal{L}^n$ is surjective for all sufficiently divisible n, and this typically depends on $\mathcal{L}$, see Example 3.6. This can be interpreted as variation of GIT on the level of stacks.

Example 3.6. Let $\mathcal{X} = B\mathbb{G}_m$ and $f\colon \mathcal{X}' = \mathrm{Proj}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}} \oplus V^a) \to \mathcal{X}$ where $a > 0$ and V is the tautological line bundle on $B\mathbb{G}_m$. Then $\mathcal{X}'$ has three points: two closed points P_1 and P_2 corresponding to the projections $\mathcal{O}_{\mathcal{X}} \oplus V^a \to \mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{X}} \oplus V^a \to V^a$ and one open point in their complement. Let $\mathcal{O}(1)$ be the tautological f-ample line bundle and let $\mathcal{L} = \mathcal{O}(i) \otimes f^* V^j$ with $i \geq 1$ and $j \in \mathbb{Z}$. Then $\mathcal{X}'_{\mathcal{L}} = \mathrm{Proj}^{\pi}_{\mathcal{X}}\big(\oplus_{k \geq 0} \mathrm{Sym}^{ik}(\mathcal{O}_{\mathcal{X}} \oplus V^a) \otimes V^{jk}\big)$ and:

$$\mathcal{X}'_{\mathcal{L}} = \begin{cases} \mathcal{X}' \smallsetminus P_2 & \text{if } j/i = 0 \\ \mathcal{X}' \smallsetminus P_1 & \text{if } j/i = -a \\ \mathcal{X}' \smallsetminus \{P_1, P_2\} & \text{if } -a < j/i < 0 \\ \emptyset & \text{if } j/i > 0 \text{ or } j/i < -a. \end{cases}$$

3.2. **Properties of saturated Proj and saturated blowups.**

Proposition 3.7. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism and let $\mathcal{A}$ be a graded finitely generated $\mathcal{O}_{\mathcal{X}}$-algebra.*

(1) *If $\mathcal{A} \to \mathcal{B}$ is a surjection onto another graded $\mathcal{O}_{\mathcal{X}}$-algebra, then $\mathrm{Proj}^{\pi}_{\mathcal{X}} \mathcal{B} = \mathrm{Proj}^{\pi}_{\mathcal{X}} \mathcal{A} \times_{\mathrm{Proj}_{\mathcal{X}} \mathcal{A}} \mathrm{Proj}_{\mathcal{X}} \mathcal{B}$. In particular, there is a closed immersion $\mathrm{Proj}^{\pi}_{\mathcal{X}} \mathcal{B} \to \mathrm{Proj}^{\pi}_{\mathcal{X}} \mathcal{A}$.*

(2) *If $g\colon \mathcal{Y}' \to \mathcal{Y}$ is any morphism and $f\colon \mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ is the pull-back with good moduli space $\pi'\colon \mathcal{X}' \to \mathcal{Y}'$, then $\mathrm{Proj}^{\pi'}_{\mathcal{X}'} f^* \mathcal{A} = \mathrm{Proj}^{\pi}_{\mathcal{X}} \mathcal{A} \times_{\mathcal{X}} \mathcal{X}'$ as open substacks of $\mathrm{Proj}_{\mathcal{X}'} f^* \mathcal{A}$.*

Proof. (1) We have a closed immersion $\mathrm{Proj}_{\mathcal{X}} \mathcal{B} \to \mathrm{Proj}_{\mathcal{X}} \mathcal{A}$ and the saturated Proj's are the complements of $V(\pi^{-1}\pi_* \mathcal{B}^+)$ and $V(\pi^{-1}\pi_* \mathcal{A}^+)$ respectively. Since π is cohomologically affine, π_* preserves surjections. It follows that $\pi^{-1}\pi_* \mathcal{A}^+ \to \pi^{-1}\pi_* \mathcal{B}^+$ is surjective and the result follows.

(2) There is an isomorphism $\mathrm{Proj}_{\mathcal{X}'} f^* \mathcal{A} = \mathrm{Proj}_{\mathcal{X}} \mathcal{A} \times_{\mathcal{X}} \mathcal{X}'$ and the saturated Proj's are the complements of $V(\pi'^{-1}\pi'_* f^* \mathcal{A}^+)$ and $V(f^{-1}\pi^{-1}\pi_* \mathcal{A}^+)$ respectively. These are equal since $f^* \pi^* \pi_* = \pi'^* g^* \pi_* = \pi'^* \pi'_* f^*$ [Alp13, Proposition 4.7]. $\qquad\square$

Proposition 3.8. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism and let $\mathcal{C} \subset \mathcal{X}$ be a closed substack with sheaf of ideals $\mathcal{I}$. Let $f\colon \mathcal{X}' = \mathrm{Bl}^{\pi}_{\mathcal{C}} \mathcal{X} \to \mathcal{X}$ be the saturated blowup with good moduli space morphism $\pi'\colon \mathcal{X}' \to \mathcal{Y}'$ and exceptional divisor $\mathcal{E} \subset \mathcal{X}'$.*

(1) *There exists a positive integer d such that $\bigoplus_{n \geq 0} \pi_*(I^{dn})$ is generated in degree 1.*

(2) *For d as above, $\mathcal{Y}' = \mathrm{Bl}_{\pi_*(I^d)} \mathcal{Y}$ and $\pi'^{-1}(\mathcal{F}) = d\mathcal{E}$ where $\mathcal{F}$ is the exceptional divisor of $\mathcal{Y}' \to \mathcal{Y}$.*

(3) *As closed subsets, $\mathcal{E} = f^{-1}(\mathcal{C})$ and $f^{-1}\big(\pi^{-1}(\pi(\mathcal{C}))\big)$, coincide.*

(4) *The map f induces an isomorphism $\mathcal{X}' \smallsetminus \mathcal{E} \to \mathcal{X} \smallsetminus \pi^{-1}\big(\pi(\mathcal{C})\big)$. In particular, $\mathcal{X} \smallsetminus \pi^{-1}\big(\pi(\mathcal{C})\big) \subset \mathcal{X}'$ is schematically dense.*

Proof. (1) By Proposition 3.4, $\mathcal{Y}' = \mathrm{Proj}_{\mathcal{Y}} \pi_* \mathcal{A}$ where $\mathcal{A} = \bigoplus \mathcal{I}^n$. Since $\pi_* \mathcal{A}$ is a finitely generated algebra [AHR15, Lemma A.2], it is generated in degrees $\leq m$ for some m. If d is a multiple of the degrees of a set of generators, e.g., $d = m!$, then $\pi_* \mathcal{A}^{(d)} = \bigoplus_{n \geq 0} \pi_* I^{dn}$ is generated in degree 1.

(2) It follows from (1) that $\mathcal{Y}' = \mathrm{Proj}_{\mathcal{Y}} \pi_* \mathcal{A}^{(d)} = \mathrm{Bl}_{\pi_*(\mathcal{I}^d)} \mathcal{Y}$. To verify that $\pi'^{-1}(\mathcal{F}) = d\mathcal{E}$, we may work locally on $\mathcal{Y}$ and at the chart corresponding to an element $f \in \Gamma(\mathcal{X}, \mathcal{I}^d)$. Then $\mathcal{X}' = \mathrm{Spec}_{\mathcal{X}}(\bigoplus_{n \geq 0} \mathcal{I}^{nd})_{(f)}$, $\mathcal{Y}' = \mathrm{Spec}_{\mathcal{Y}}(\bigoplus_{n \geq 0} \pi_* \mathcal{I}^{nd})_{(f)}$, $\mathcal{F} = V(f)$ and $d\mathcal{E} = V(f)$.

(3) follows immediately from (2) since $f^{-1}\big(\pi^{-1}(\pi(\mathcal{C}))\big) = \pi'^{-1}(\mathcal{F})$ as sets.

(4) Since the saturated blowup commutes with flat base change on $\mathcal{Y}$, the map $f\colon \mathcal{X}' \to \mathcal{X}$ becomes an isomorphism after restricting to $\mathcal{X} \smallsetminus \pi^{-1}\big(\pi(\mathcal{C})\big)$. But $f^{-1}(\pi^{-1}\big(\pi(\mathcal{C})\big) = \mathcal{E}$ by (3). $\qquad\square$

Remark 3.9. Proposition 3.8 (2) generalizes [Kir85, Lemma 3.11] via Example 3.3.

Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism, let $\mathcal{C} \subset \mathcal{X}$ be a closed substack and consider the saturated blowup $p\colon \mathrm{Bl}_{\mathcal{C}}^{\pi} \mathcal{X} \to \mathcal{X}$. We have seen that it is an isomorphism outside $\mathcal{X} \smallsetminus \pi^{-1}\big(\pi(\mathcal{C})\big)$ and that $\mathcal{X} \smallsetminus \pi^{-1}\big(\pi(\mathcal{C})\big)$ is schematically dense in the saturated blowup.

Definition 3.10 (Strict transform of saturated blowups). If $\mathcal{Z} \subset \mathcal{X}$ is a closed substack, then we let the *strict transform* of $\mathcal{Z}$ along p denote the schematic closure of $\mathcal{Z} \smallsetminus \pi^{-1}\big(\pi(\mathcal{C})\big)$ in $p^{-1}(\mathcal{Z})$. Similarly, if $f\colon \mathcal{X}' \to \mathcal{X}$ is any morphism, then the strict transform of f along p is the schematic closure of $\mathcal{X}' \smallsetminus f^{-1}\pi^{-1}\big(\pi(\mathcal{C})\big)$ in $\mathcal{X}' \times_{\mathcal{X}} \mathrm{Bl}_{\mathcal{C}}^{\pi} \mathcal{X}$.

Proposition 3.11. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism, let $\mathcal{C} \subset \mathcal{X}$ be a closed substack and let $p\colon \mathrm{Bl}_{\mathcal{C}}^{\pi} \mathcal{X} \to \mathcal{X}$ be the saturated blowup.*

(1) *If $\mathcal{Z} \subset \mathcal{X}$ is a closed substack, then $\mathrm{Bl}_{\mathcal{C} \cap \mathcal{Z}}^{\pi} \mathcal{Z}$ is the strict transform of $\mathcal{Z}$ along p. In particular, there is a closed immersion $\mathrm{Bl}_{\mathcal{C} \cap \mathcal{Z}}^{\pi} \mathcal{Z} \to \mathrm{Bl}_{\mathcal{C}}^{\pi} \mathcal{X}$.*

(2) *If $g\colon \mathcal{Y}' \to \mathcal{Y}$ is any morphism, and $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ and $\mathcal{C}' = \mathcal{C} \times_{\mathcal{X}} \mathcal{X}'$, then $\mathrm{Bl}_{\mathcal{C}'}^{\pi'} \mathcal{X}'$ is the strict transform of $\mathcal{X}' \to \mathcal{X}$ along p. In particular, there is a closed immersion $\mathrm{Bl}_{\mathcal{C}'}^{\pi'} \mathcal{X}' \to \mathrm{Bl}_{\mathcal{C}}^{\pi} \mathcal{X} \times_{\mathcal{X}} \mathcal{X}'$ and this is an isomorphism if g is flat.*

Proof. (1) Let $\mathcal{A} = \bigoplus \mathcal{I}^n$ and $\mathcal{A}' = \bigoplus \mathcal{I}^n/(\mathcal{I}^n \cap \mathcal{J})$ where $\mathcal{I}$ defines $\mathcal{C}$ and $\mathcal{J}$ defines $\mathcal{Z}$. Then there is a closed immersion $\mathrm{Bl}_{\mathcal{C} \cap \mathcal{Z}}^{\pi} \mathcal{Z} = \mathrm{Proj}_{\mathcal{X}}^{\pi} \mathcal{A}' \to \mathrm{Proj}_{\mathcal{X}}^{\pi} \mathcal{A} = \mathrm{Bl}_{\mathcal{C}}^{\pi} \mathcal{X}$ by Proposition 3.7 (1). The result follows since $\mathrm{Bl}_{\mathcal{C} \cap \mathcal{Z}}^{\pi} \mathcal{Z}$ equals $p^{-1}(\mathcal{Z})$ outside $\pi^{-1}\big(\pi(\mathcal{C})\big)$ and is schematically dense.

(2) Let $\mathcal{A} = \bigoplus \mathcal{I}^n$ and $\mathcal{A}' = \bigoplus \mathcal{I}'^n$ where $\mathcal{I}$ defines $\mathcal{C}$ and $\mathcal{I}'$ defines $\mathcal{C}'$. Then $f^* \mathcal{A} \to \mathcal{A}'$ is surjective, so there is a closed immersion $\mathrm{Bl}_{\mathcal{C}'}^{\pi'} \mathcal{X}' \to \mathrm{Bl}_{\mathcal{C}}^{\pi} \mathcal{X} \times_{\mathcal{X}} \mathcal{X}'$ (Proposition 3.7). The result follows, since $\mathrm{Bl}_{\mathcal{C}'}^{\pi'} \mathcal{X}' \to \mathcal{X}'$ is an isomorphism outside $\pi^{-1}\big(\pi(\mathcal{C}')\big)$ and $\mathcal{X}' \smallsetminus \pi^{-1}\big(\pi(\mathcal{C}')\big)$ is schematically dense in $\mathrm{Bl}_{\mathcal{C}'}^{\pi'} \mathcal{X}'$ (Proposition 3.8 (4)). $\qquad\square$

Proposition 3.12. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a good moduli morphism and let $\mathcal{A}$ be a finitely generated graded $\mathcal{O}_{\mathcal{X}}$-algebra. Let $f\colon \mathcal{X}' := \mathrm{Proj}_{\mathcal{X}}^{\pi} \mathcal{A} \to \mathcal{X}$ be the saturated Proj and let $\pi'\colon \mathcal{X}' \to \mathcal{Y}' := \mathrm{Proj}_{\mathcal{Y}} \pi_* \mathcal{A}$ be its good moduli space morphism.*

(1) *If π is properly stable, then π' is properly stable.*

(2) *If π is stable and $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ for an ideal $\mathcal{I}$, then π' is stable.*

More precisely, in (1), or in (2) under the additional assumption that $\mathcal{X}$ is reduced, the inclusion $\mathcal{X}' \subset \mathrm{Proj}_{\mathcal{X}} \mathcal{A}$ is an equality over $\mathcal{X}^s$ and $\mathcal{X}'^s$ contains $f^{-1}(\mathcal{X}^s)$. In (2), $\mathcal{X}'^s$ always contains $f^{-1}(\mathcal{X}^s \smallsetminus V(\mathcal{I}))$.

Proof. The question is smooth-local on $\mathcal{Y}$ so we can assume that $\mathcal{Y}$ is affine. We can also replace $\mathcal{Y}$ with $\pi(\mathcal{X}^s)$ and assume that $\mathcal{X} = \mathcal{X}^s$, that is, every stabilizer of $\mathcal{X}$ has the same dimension.

In the first case, π is a coarse moduli space. The induced morphism $\pi_{\mathcal{A}} \colon \mathrm{Spec}_{\mathcal{X}} \mathcal{A} \to \mathrm{Spec}_{\mathcal{Y}} \pi_* \mathcal{A}$ is then also a coarse moduli space. The image along $\pi_{\mathcal{A}}$ of $V(\mathcal{A}^+)$ is $V(\pi_* \mathcal{A}^+)$. Since $\pi_{\mathcal{A}}$ is a homeomorphism, $\sqrt{\pi^{-1} \pi_* \mathcal{A}^+} = \sqrt{\mathcal{A}^+}$. It follows that $\mathcal{X}' = \mathrm{Proj}_{\mathcal{X}} \mathcal{A}$.

In the second case, if in addition $\mathcal{X}$ is reduced, then π factors through a gerbe $g \colon \mathcal{X} \to \mathcal{X}_{\mathrm{tame}}$ and a coarse moduli space $h \colon \mathcal{X}_{\mathrm{tame}} \to \mathcal{Y}$ (Proposition A.2). Since $\mathcal{I}^n = g^* g_* \mathcal{I}^n$, we conclude that $\mathrm{Proj}^{\pi}_{\mathcal{X}} \mathcal{A} = \left(\mathrm{Proj}^{h}_{\mathcal{X}_{\mathrm{tame}}} g_* \mathcal{A} \right) \times_{\mathcal{X}_{\mathrm{tame}}} \mathcal{X}$ and the question reduces to the first case.

In the second case, without the additional assumption on $\mathcal{X}$, let $\mathcal{U} := \mathcal{X}^s \smallsetminus V(\mathcal{I})$. Then $\mathrm{Proj}_{\mathcal{X}} \mathcal{A} \to \mathcal{X}$ is an isomorphism over $\mathcal{U}$ and $\mathcal{Y}' := \mathrm{Proj}_{\mathcal{Y}} \pi_* \mathcal{A} \to \mathcal{Y}$ is an isomorphism over $\pi(\mathcal{U})$ so $\mathcal{U} \subset \mathcal{X}'^s$. Moreover, $\mathcal{U} \subset \mathcal{X}'$ is dense so $\mathcal{X}'$ is stable. □

The condition that $\mathcal{A}$ is a Rees algebra in (2) is not superfluous. In Example 3.6 ($a = 1$, $i = 1$, $j = 0$), we have a stable, but not properly stable, good moduli space $\pi \colon \mathcal{X} = B\mathbb{G}_m \to \mathcal{Y} = \mathrm{Spec}\, k$ and a saturated Proj $\mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is not stable: $\mathcal{X}' = [A^1/\mathbb{G}_m] \to \mathcal{Y}' = \mathcal{Y} = \mathrm{Spec}\, k$.

Remark 3.13 (Deligne–Mumford stacks). Proposition 3.4 is a non-trivial statement even when $\mathcal{X}$ is Deligne–Mumford. In this case, the saturated Proj coincides with the usual Proj, see proof of Proposition 3.12. We can thus identify the coarse space of a blowup along a sheaf of ideals $\mathcal{I}$ as $\mathrm{Proj}(\oplus \pi_* \mathcal{I}^k)$ (Proposition 3.4) and as the blowup in $\pi_* \mathcal{I}^d$ for sufficiently divisible d (Proposition 3.8).

Example 3.14. Let $\mathcal{X} = [\mathbb{A}^2/\boldsymbol{\mu}_2]$ where $\boldsymbol{\mu}_2$ acts by $-(a, b) = (-a, -b)$. The coarse space of $\mathcal{X}$ is the cone $Y = \mathrm{Spec}[x^2, xy, y^2]$. Proposition 3.4 says that if we let X' be the blowup of $\mathbb{A}^2$ at the origin, then the quotient $X'/\boldsymbol{\mu}_2$ is Proj of the graded ring $\oplus S_i$ where S_i is the monomial ideal in the invariant ring $k[x^2, xy, y^2]$ generated by monomials of degree $\lceil i/2 \rceil$. This is isomorphic to the blowup of Y in (x^2, xy, y^2).

Remark 3.15 (Adequate stacks). Proposition 3.4 also holds for stacks with adequate moduli spaces with essentially identical arguments.

3.3. **Resolutions of singularities of stacks with good moduli spaces.** Recall that in characteristic zero, we have functorial resolution of singularities by blowups in smooth centers [BM08, Thm. 1.1]. To be precise, there is a functor BR which produces, for each reduced scheme X of finite type over a field of characteristic zero, a resolution of singularities $BR(X)$ which commutes with smooth morphisms. Here $BR(X) = \{X' \to \cdots \to X\}$ is a sequence of blowups in smooth centers with X' smooth. Also see [Kol07,

Theorem 3.36] although that algorithm may involve centers that are singular [Kol07, Example 3.106].

Artin stacks can be expressed as quotients $[U/R]$ of groupoid schemes $s, t \colon R \rightrightarrows U$ with s and t smooth morphisms. Thus, the resolution functor BR extends uniquely to Artin stacks. In particular, for every reduced Artin stack $\mathcal{X}$ of finite type over a field of characteristic zero, there is a projective morphism $\tilde{\mathcal{X}} \to \mathcal{X}$, a sequence of blowups, which is an isomorphism over a dense open set. Similarly, if X is a scheme with an action of a group scheme G, then there is a sequence of blowups in G-equivariant smooth centers that resolves the singularities of X.

In general if $\mathcal{X}$ is an Artin stack, then a resolution of singularities $\tilde{\mathcal{X}}$ need not have a good moduli space. However, the theory of saturated blowups implies that $\tilde{\mathcal{X}}$ contains an open set which has a good moduli space such that the induced map of good moduli spaces is proper and birational.

Proposition 3.16. *Let $\mathcal{X}$ be an integral Artin stack with stable good moduli space morphism $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$. Suppose that $\tilde{\mathcal{X}} \to \mathcal{X}$ is a projective birational morphism. Further assume that either*

(1) *$\mathcal{X}$ is properly stable, or*
(2) *$\tilde{\mathcal{X}} \to \mathcal{X}$ is a sequence of blowups.*

Then there exists an open substack $\mathcal{X}' \subset \tilde{\mathcal{X}}$ such that $\mathcal{X}'$ has a stable good moduli space $\mathcal{X}' \to \mathbf{X}'$ and the induced morphism of good moduli spaces is projective and birational.

Proof. Since $\tilde{\mathcal{X}} \to \mathcal{X}$ is projective we can write $\tilde{\mathcal{X}} = \mathrm{Proj}_{\mathcal{X}} \mathcal{A}$ for some graded sheaf $\mathcal{A}$ of finitely generated $\mathcal{O}_{\mathcal{X}}$-algebras. If $\tilde{\mathcal{X}} \to \mathcal{X}$ is a blowup, we choose $\mathcal{A}$ as the Rees algebra of this blowup. We treat a sequence of blowups by induction.

Let $\mathcal{X}' = \mathrm{Proj}_{\mathcal{X}}^{\pi} \mathcal{A}$. By Proposition 3.4, $\mathcal{X}' \to \mathbf{X}' = \mathrm{Proj}_{\mathbf{X}} \pi_* \mathcal{A}$ is a good moduli space morphism. By Proposition 3.12 it is stable. If $\tilde{\mathcal{X}} \to \mathcal{X}$ is an isomorphism over the open dense subset $U \subset \mathcal{X}$ (resp. a sequence of blowups with centers outside U), then $\mathbf{X}' \to \mathbf{X}$ is an isomorphism over the open dense subset $\pi(U \cap \mathcal{X}^s)$. $\qquad\square$

Corollary 3.17. *Let $\mathcal{X}$ be an integral Artin stack with stable good moduli space $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$ defined over a field of characteristic 0. There exists a quasi-projective birational morphism $\mathcal{X}' \to \mathcal{X}$ with the following properties.*

(1) *The stack $\mathcal{X}'$ is smooth and admits a good moduli space $\mathcal{X}' \xrightarrow{\pi'} \mathbf{X}'$.*
(2) *The induced map of moduli spaces $\mathbf{X}' \to \mathbf{X}$ is projective and birational.*

Proof. Follows immediately from functorial resolution of singularities by a sequence of blowups, and Proposition 3.16. $\qquad\square$

4. Reichstein transforms and saturated blowups

When $\mathcal{X}$ and $\mathcal{C}$ are smooth, then the saturated blowup of $\mathcal{X}$ along $\mathcal{C}$ has a particularly nice description in terms of *Reichstein transforms*.

The following definition is a straightforward extension of the one originally made in [EM12].

Definition 4.1. Let $\mathcal{X} \xrightarrow{\pi} \mathcal{Y}$ be a good moduli space morphism and let $\mathcal{C} \subset \mathcal{X}$ be a closed substack. The *Reichstein transform* with center $\mathcal{C}$, is the stack $R(\mathcal{X}, \mathcal{C})$ obtained by deleting the strict transform of the saturation $\pi^{-1}(\pi(\mathcal{C}))$ in the blowup of $\mathcal{X}$ along $\mathcal{C}$.

Recall that if $f \colon \mathrm{Bl}_{\mathcal{C}} \mathcal{X} \to \mathcal{X}$ is the blowup, then $E = f^{-1}(\mathcal{C})$ is the exceptional divisor and $\overline{f^{-1}(\mathcal{Z}) - E} = \mathrm{Bl}_{\mathcal{C} \cap \mathcal{Z}} \mathcal{Z}$ is the strict transform of $\mathcal{Z} \subset \mathcal{X}$.

Remark 4.2. Observe that if $\mathcal{X}$ and $\mathcal{C}$ are smooth, then $R(\mathcal{X}, \mathcal{C})$ is smooth since it is an open set in the blowup of a smooth stack along a closed smooth substack.

Remark 4.3. Let

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\psi} & \mathcal{X} \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
\mathcal{Y}' & \xrightarrow{\phi} & \mathcal{Y}
\end{array}
$$

be a cartesian diagram where the horizontal maps are flat and the vertical maps are good moduli morphisms. If $\mathcal{C} \subset \mathcal{X}$ is a closed substack, then $R(\mathcal{X}', \psi^{-1}\mathcal{C}) = \mathcal{X}' \times_{\mathcal{X}} R(\mathcal{X}, \mathcal{C})$. This follows because blowups commute with flat base change and the saturation of $\psi^{-1}(\mathcal{C})$ is the inverse image of the saturation of $\mathcal{C}$.

Definition 4.4 (Equivariant Reichstein transform)**.** If an algebraic group G acts on a scheme X with a good quotient $p \colon X \to X /\!\!/ G$ and C is a G-invariant closed subscheme, then we write $R_G(X, C)$ for the complement of the strict transform of $p^{-1}p(C)$ in the blowup of X along C. There is a natural G-action on $R_G(X, C)$ and $R([X/G], [C/G]) = [R_G(X, C)/G]$.

Proposition 4.5. *Let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism and let $\mathcal{C} \subset \mathcal{X}$ be a closed substack. If $\mathcal{X}$ and $\mathcal{C}$ are smooth, then $R(\mathcal{X}, \mathcal{C}) = \mathrm{Bl}_{\mathcal{C}}^{\pi} \mathcal{X}$ as open substacks of $\mathrm{Bl}_{\mathcal{C}} \mathcal{X}$. In general, $\mathrm{Bl}_{\mathcal{C}}^{\pi} \mathcal{X} \subset R(\mathcal{X}, \mathcal{C})$.*

Proof. Let $\mathcal{A} = \oplus_{n \geq 0} \mathcal{I}^n$. The saturation of $\mathcal{C}$ is the subscheme defined by the ideal $\mathcal{J} = \pi_* \mathcal{I} \cdot \mathcal{O}_{\mathcal{X}}$ so the strict transform of the saturation is the blowup of the substack $V(\mathcal{J})$ along the ideal $\mathcal{I}/\mathcal{J}$, which is $\mathrm{Proj}_{\mathcal{C}}(\oplus_{n \geq 0} (\mathcal{I}^n/(\mathcal{I}^n \cap \mathcal{J})))$. Thus the ideal of the strict transform of the saturation is the graded ideal $\oplus_{n > 0} (\mathcal{I}^n \cap \mathcal{J}) \subset \mathcal{A}$. We need to show that this ideal defines the same closed subset of the blowup as the ideal $\pi^{-1}\pi_*(\mathcal{A}^+)$.

Since $\mathcal{A}^+ = \oplus_{n > 0} \mathcal{I}^n$ we have that
$$
\pi^{-1}\pi_*(\mathcal{A}^+) = \pi_*(\mathcal{A}^+) \cdot \mathcal{A} = \oplus_{n \geq 0} \mathcal{K}_n
$$
where $\mathcal{K}_n = \sum_{k > 0} \pi_*(\mathcal{I}^k)\mathcal{I}^{n-k}$. We need to show that
$$
\sqrt{\oplus_{n > 0} \mathcal{I}^n \cap \mathcal{J}} = \sqrt{\oplus_{n > 0} \mathcal{K}_n}
$$

in $\mathcal{A}$. Observe that $\pi_*(\mathcal{I}^k)\cdot\mathcal{O}_\mathcal{X} \subset \mathcal{I}^k$ and $\pi_*(\mathcal{I}^k)\cdot\mathcal{O}_\mathcal{X} \subset \pi_*(\mathcal{I})\cdot\mathcal{O}_\mathcal{X} = \mathcal{J}$ so $\pi_*(\mathcal{I}^k)\mathcal{I}^{n-k} \subset$ $\mathcal{I}^n \cap \mathcal{J}$. Hence $\mathcal{K}_n \subset \mathcal{I}^n \cap \mathcal{J}$.

To establish the opposite inclusion, we work smooth-locally on $\mathcal{Y}$. We may thus assume that $\mathcal{Y} = \operatorname{Spec} A$ and $\pi_*\mathcal{I} = (f_1, f_2, \ldots, f_a) \subset A$. The ideal $\mathcal{I}^n \cap \mathcal{J}$ can locally be described as all functions in $\mathcal{J} = (f_1, f_2, \ldots, f_a)\cdot\mathcal{O}_\mathcal{X}$ that vanish to order at least n along $\mathcal{C}$. If $\operatorname{ord}_\mathcal{C}(f_i) = d_i$, that is, if $f_i \in \mathcal{I}^{d_i} \smallsetminus \mathcal{I}^{d_i+1}$, then for any n greater than all the d_i's, we have that $\mathcal{I}^n \cap \mathcal{J} = \sum_{i=1}^a f_i \cdot \mathcal{I}^{n-d_i}$. Since $f_i \in \pi_*(\mathcal{I}^{d_i})$ it follows that

$$\mathcal{I}^n \cap \mathcal{J} \subset \sum_{i=1}^a \pi_*(\mathcal{I}^{d_i})\mathcal{I}^{n-d_i} \subset \mathcal{K}_n.$$

Thus $\oplus_{n>0}(\mathcal{I}^n \cap \mathcal{J}) \subset \sqrt{\oplus_{n>0}\mathcal{K}_n}$ which completes the proof. $\square$

The following example shows that if $\mathcal{C}$ is singular, then the Reichstein transform need not equal the saturated blowup and could even fail to have a good moduli space. Also see Examples 5.7 and 5.8.

Example 4.6. Let $U = \operatorname{Spec} k[x, y]$ where $\mathbb{G}_m$ acts by $\lambda(a, b) = (\lambda a, \lambda^{-1}b)$ and let $\mathcal{X} = [U/\mathbb{G}_m]$. Its good moduli space is $\mathbf{X} = \operatorname{Spec} k[xy]$. Let $Z = V(x^2y, xy^2) \subset U$ and $\mathcal{C} = [Z/\mathbb{G}_m]$. Its saturation is $\operatorname{sat}\mathcal{C} = V(x^2y^2)$ which has strict transform $\operatorname{Bl}_\mathcal{C} \operatorname{sat}\mathcal{C} = \emptyset$. Thus, the Reichstein transform $R(\mathcal{X}, \mathcal{C})$ equals $\operatorname{Bl}_\mathcal{C} \mathcal{X}$.

We will show $\operatorname{Bl}_\mathcal{C} \mathcal{X}$ has no good moduli space. To see this, note that $(x^2y, y^2x) = (xy)\cdot(x, y)$. Since (xy) is invertible we conclude that $\operatorname{Bl}_\mathcal{C} \mathcal{X} = \operatorname{Bl}_P \mathcal{X}$ where $P = V(x, y)$. The exceptional divisor of the latter blowup is $[\mathbb{P}^1/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts by $\lambda[a : b] = [\lambda a : \lambda^{-1}b]$. This has no good moduli space since the closure of the open orbit contains the two fixed points $[0 : 1]$ and $[1 : 0]$.

The Reichstein transform $R(\mathcal{X}, P)$, on the other hand, equals $\operatorname{Bl}_P \mathcal{X} \smallsetminus \{[0 : 1], [1 : 0]\}$ which is tame with coarse moduli space $\operatorname{Bl}_{\pi(P)} \mathbf{X} = \mathbf{X}$.

5. Equivariant Reichstein transforms and fixed points

The goal of this section is to prove the following theorem.

Theorem 5.1. *Let $X = \operatorname{Spec} A$ be a smooth affine scheme with the action of a connected linearly reductive group G. Then $R_G(X, X^G)^G = \emptyset$.*

Remark 5.2. By [CGP10, Proposition A.8.10] the fixed locus X^G is a closed smooth subscheme of X. Note that if G acts trivially, then $X^G = X$ and $R_G(X, X^G) = \emptyset$.

Remark 5.3. Theorem 5.1 is false if we drop the assumption that X is smooth. See Example 5.7 below.

5.1. **The case of a representation.** In this section we prove Theorem 5.1 when $X = V$ is a representation of G.

Proposition 5.4. *Let V be a representation of a connected linearly reductive group G. Then $R_G(V, V^G)^G = \emptyset$.*

Proof. Decompose $V = V^0 \oplus V^m$ such that V^0 is the trivial submodule and V^m is a sum of non-trivial irreducible G-modules. Viewing V as a variety we write $V = V^0 \times V^m$. The fixed locus for the action of G on V is $V^0 \times \{0\}$, so the blowup of V along V^G is isomorphic to $V^0 \times \tilde{V}^m$ where $\tilde{V}^m$ is the blowup of V^m at the origin. Also, the saturation of V^G is $V^0 \times \text{sat}_G\{0\}$ where $\text{sat}_G\{0\}$ is the G-saturation of the origin in the representation V^m. Thus $R_G(V, V^G) = V^0 \times R_G(V^m, 0)$ so to prove the proposition we are reduced to the case that $V = V^m$; that is, V is a sum of non-trivial irreducible representations and $\{0\}$ is the only G-fixed point.

To prove the proposition we must show that every G-fixed point of the exceptional divisor $\mathbb{P}(V) \subset \tilde{V}$ is contained in the strict transform of

$$\text{sat}_G\{0\} = \{v \in V : 0 \in \overline{Gv}\}.$$

Let $x \in \mathbb{P}(V)$ be a G-fixed point. The fixed point x corresponds to a G-invariant line $L \subset V$, inducing a character χ of G. Since the origin is the only fixed point, the character χ is necessarily non-trivial. Let λ be a 1-parameter subgroup such that $\langle \lambda, \chi \rangle > 0$. Then λ acts with positive weight α on L and thus $L \subset \text{sat}_\lambda\{0\} = V_\lambda^+ \cup V_\lambda^-$ where

$$V_\lambda^+ = \{v \in V : \lim_{t \to 0} \lambda(t)v = 0\},$$
$$V_\lambda^- = \{v \in V : \lim_{t \to \infty} \lambda(t)v = 0\}$$

are the linear subspaces where λ acts with positive weights and negative weights respectively.

Since $\text{sat}_G\{0\} \supset \text{sat}_\lambda\{0\}$, it suffices to show that $x \in \mathbb{P}(V)$ lies in the strict transform of $\text{sat}_\lambda\{0\}$. The blowup of $\text{sat}_\lambda\{0\}$ in the origin intersects the exceptional divisor of $\tilde{V}$ in the (disjoint) linear subspaces $\mathbb{P}(V_\lambda^+) \cup \mathbb{P}(V_\lambda^-) \subset \mathbb{P}(V)$. Since $L \subset V_\lambda^+$ we see that our fixed point x is in $\mathbb{P}(V_\lambda^+)$ as desired. $\square$

5.2. **Completion of the proof of Theorem 5.1.** The following lemma is a special case of [Lun73, Lemma on p. 96] and Luna's fundamental lemma [Lun73, p. 94]. For the convenience of the reader, we include the first part of the proof.

Lemma 5.5 (Linearization)**.** *Let* $X = \text{Spec}\, A$ *be a smooth affine scheme with the action of a linearly reductive group* G. *If* $x \in X^G$ *is a closed fixed point, then there is a* G-saturated affine neighborhood U *of* x *and a* G-equivariant *strongly étale* morphism $\phi\colon U \to T_x X$, *with* $\phi(x) = 0$. *That is, the diagram*

$$\begin{CD} U @>{\phi}>> T_x X \\ @V{\pi_U}VV @VV{\pi}V \\ U /\!\!/ G @>{\psi}>> T_x X /\!\!/ G \end{CD}$$

is cartesian and the horizontal arrows are étale.

Proof. Let $\mathfrak{m}$ be the maximal ideal corresponding to x. Since x is G-fixed the quotient map $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ is a map of G-modules. By the local finiteness of group actions there is a finitely generated G-submodule $V \subset \mathfrak{m}$ such that the restriction $V \to \mathfrak{m}/\mathfrak{m}^2$ is surjective. Since G is linearly reductive there is a summand $W \subset V$ such that $W \to \mathfrak{m}/\mathfrak{m}^2$ is an isomorphism of G-modules. Since $W \subset A$ we obtain a G-equivariant morphism $X \to T_x X = \operatorname{Spec}(\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2))$ which is étale at x. Luna's fundamental lemma now gives an open saturated neighborhood U of x such that $U \to T_x X$ is strongly étale. $\square$

Remark 5.6. Using Lemma 5.5 and arguing as in the proof of Proposition 5.4, we recover the result that X^G is smooth (Remark 5.2).

Completion of the Proof of Theorem 5.1. Every G-fixed point of $R_G(X, X^G)$ lies in the exceptional divisor $\mathbb{P}(N_{X^G}X)$. To show that $R_G(X, X^G)^G = \emptyset$ we can work locally in a neighborhood of a point $x \in X^G$. Thus we may assume (Lemma 5.5) that there is a strongly étale morphism $X \to T_x X$ yielding a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & T_x X \\ \downarrow & & \downarrow \\ X /\!\!/ G & \longrightarrow & T_x X /\!\!/ G. \end{array}$$

Hence $R_G(X, X^G)$ can be identified with the pullback of $R_G(T_x X, (T_x X)^G)$ along the morphism $X /\!\!/ G \to T_x X /\!\!/ G$ (Remark 4.3). By Proposition 5.4, $R_G(T_x X, (T_x X)^G)^G = \emptyset$ so therefore $R_G(X, X^G)^G = \emptyset$ as well. $\square$

Example 5.7. Note that the conclusion of Theorem 5.1 is false without the assumption that X is smooth. Let V be the 3-dimensional representation of $G = \mathbb{G}_m$ with weights $(-1, 1, 3)$. The polynomial $f = x_1 x_3^2 + x_2^5$ is G-homogeneous of weight 5, so the subvariety $X = V(f)$ is G-invariant. Since all weights for the G-action are non-zero $X^G = (\mathbb{A}^3)^G = \{0\}$.

Let $\tilde{\mathbb{A}}^3$ be the blowup of the origin. The exceptional divisor is $\mathbb{P}(V)$ and has three G-fixed points $P_0 = [0 : 0 : 1], P_1 = [0 : 1 : 0], P_2 = [1 : 0 : 0]$. The exceptional divisor of $\tilde{X}$ is the projectivized tangent cone $\mathbb{P}(C_{\{0\}}X)$. Since $X = V(f)$ is a hypersurface and $x_1 x_3^2$ is the sole term of lowest degree in f, we see that $\mathbb{P}(C_{\{0\}}X)$ is the subscheme $V(x_1 x_3^2) \subset \mathbb{P}(V)$. This subvariety contains the 3 fixed points, so $\tilde{X}$ has 3 fixed points.

The saturation of 0 in X with respect to the G-action is $(X \cap V(x_1)) \cup (X \cap V(x_2, x_3))$. The intersection of the exceptional divisor with the strict transform of $X \cap V(x_1)$ is the projective subscheme $V(x_1, x_2^5)$ whose reduction is P_0.

The intersection of the exceptional divisor with the strict transform of $X \cap V(x_2, x_3)$ is the point $V(x_2, x_3) = P_2$. Thus the strict transform of the saturation of 0 in X does not contain all of the fixed points of $\tilde{X}$. Hence $R_G(X, X^G)^G \neq \emptyset$.

The exceptional divisor of the saturated blowup of X in the origin is $\mathbb{P}(C_{\{0\}}X) \smallsetminus \big(V(x_1) \cup V(x_2, x_3)\big) = V(x_3) \smallsetminus \big(V(x_1) \cup V(x_2)\big)$ which has no G-fixed points.

Example 5.8. Consider the following variation of Example 5.7. Let V be the 5-dimensional representation of $G = \mathbb{G}_m$ with weights $(-1, 1, 3, 2, 7)$. The polynomials $f = x_1 x_3^2 + x_2^5$ and $g = x_1 x_5 + x_4^3$ are G-homogeneous of weights 5 and 6 so the subvariety $X = V(f, g)$ is G-invariant. As before we blow up the origin and the exceptional divisor of $\tilde{\mathbb{A}}^5$ is $\mathbb{P}(V)$ which has five G-fixed points. The exceptional divisor of $\tilde{X}$ is $\mathbb{P}(C_{\{0\}} X)$ which is given by $V(x_1 x_3^2, x_1 x_5) \subset \mathbb{P}(V)$. It contains the five fixed points of $\mathbb{P}(V)$. The saturation of 0 in X is the union of $X \cap V(x_1) = V(x_1, x_2^5, x_4^3)$ and $X \cap V(x_2, x_3, x_4, x_5) = V(x_2, x_3, x_4, x_5)$.

In particular, the exceptional divisor of $R_G(X, X^G)$ contains the closed subscheme $V(x_1, x_3, x_5) = \mathbb{P}(W)$ where W is the 2-dimensional representation with weights $(1, 2)$. But $\mathbb{P}(W)$ admits no good quotient by G since the closure of the open orbit contains both G-fixed points. It follows that $R_G(X, X^G)$ does not admit a good quotient.

The exceptional divisor of the saturated blowup of X in the origin is $\mathbb{P}(C_{\{0\}} X) \smallsetminus \big(V(x_1) \cup V(x_2, x_3, x_4, x_5) \big) = V(x_3, x_5) \smallsetminus \big(V(x_1) \cup V(x_2, x_4) \big)$ which has no G-fixed points and the saturated blowup admits a good moduli space.

6. The proof of Theorem 2.11 in the smooth case

In this section we prove the main theorem in the smooth case and prove that the algorithm is functorial with respect to strong morphisms.

6.1. **Proof of Theorem 2.11.** Let $\mathcal{X}$ be a smooth stack with a stable good moduli space and let $\mathcal{E}$ be an snc divisor (e.g., $\mathcal{E} = \emptyset$). Taking connected components, we may assume that $\mathcal{X}$ is irreducible. By Lemma B.1, for any stack $\mathcal{X}$ the locus $\mathcal{X}^{\max}$ of points of $\mathcal{X}$ with maximal dimensional stabilizer is a closed subset of $|\mathcal{X}|$. Moreover, if $\mathcal{X}$ is smooth, then Proposition B.2 implies that $\mathcal{X}^{\max}$ with its reduced induced substack structure is also smooth. When $\mathcal{X} = [X/G]$ with X smooth and G linearly reductive then $\mathcal{X}^{\max} = [X^{G_0}/G]$ where G_0 is the reduced identity component of G. For an arbitrary smooth stack $\mathcal{X}$ with good moduli space $\mathbf{X}$ the stack structure on $\mathcal{X}^{\max}$ can be étale locally described as follows.

If x is a closed point with stabilizer G_x, then by [AHR15, Theorem 2.9, Theorem 1.1] there is a cartesian diagram of stacks and good moduli spaces

$$
\begin{array}{ccc}
[U/G_x] & \longrightarrow & \mathcal{X} \\
\downarrow & \square & \downarrow {\scriptstyle \pi} \\
U /\!\!/ G_x & \longrightarrow & \mathbf{X}
\end{array}
$$

where the horizontal maps are étale. In this setup, the inverse image of $\mathcal{X}^{\max}$ in $[U/G_x]$ is $[U^{G_0}/G_x]$ where G_0 is the reduced identity component of G_x. See Appendix B for more details.

The proof of Theorem 2.11 proceeds by induction on the maximum stabilizer dimension. First suppose that the maximum stabilizer dimension equals the minimum stabilizer dimension. Then $\mathcal{X}^s = \mathcal{X}$ and $\mathcal{X}$ is a gerbe over a tame stack $\mathcal{X}_{\text{tame}}$ by

Proposition A.2. If $\mathcal{X}$ is properly stable, then every stabilizer has dimension zero and $\mathcal{X} = \mathcal{X}_{\text{tame}}$ is a tame stack by Proposition A.1. We have thus shown that the sequence of length 0, that is $\mathcal{X}_n = \mathcal{X}_0 = \mathcal{X}$, satisfies the conclusions (4a)–(4d) of the Theorem 2.11.

If the maximum stabilizer dimension is greater than the minimum stabilizer dimension, we let $\mathcal{X}_0 = \mathcal{X}$, $\mathcal{C}_0 = \mathcal{X}^{\max}$ and $f_1\colon \mathcal{X}_1 = R(\mathcal{X}_0, \mathcal{C}_0) \to \mathcal{X}_0$. The following Proposition shows that the conclusions in Theorem 2.11 hold for $\ell = 0$. In particular, $(\mathcal{X}_1, \mathcal{E}_1)$ satisfies the hypothesis of Theorem 2.11 and the maximal stabilizer dimension of $\mathcal{X}_1$ has dropped. By induction, we thus have $\mathcal{X}_n \to \cdots \to \mathcal{X}_1$ such that the conclusions also hold for $\ell = 1, \ldots, n$ and Theorem 2.11 for $\mathcal{X}$ follows.

Proposition 6.1. *Let $\mathcal{X}$ be a smooth irreducible Artin stack with stable (resp. properly stable) good moduli space morphism $\pi\colon \mathcal{X} \to X$ and let $\mathcal{E}$ be an snc divisor on $\mathcal{X}$. Let $f\colon \mathcal{X}' = R(\mathcal{X}, \mathcal{X}^{\max}) \to \mathcal{X}$ be the Reichstein transform. If $\mathcal{X}^{\max} \neq \mathcal{X}$ then:*

(1a) *$\mathcal{C} := \mathcal{X}^{\max}$ is smooth and meets $\mathcal{E}$ with normal crossings.*
(1b) *$\mathcal{X}'$ and $f^{-1}(\mathcal{C})$ are smooth and $\mathcal{E}' := f^{-1}(\mathcal{C}) \cup f^{-1}(\mathcal{E})$ is snc.*
(2a) *The stack $\mathcal{X}'$ has a good moduli space $\mathbf{X}'$ and the good moduli space morphism $\pi'\colon \mathcal{X}' \to \mathbf{X}'$ is stable (resp. properly stable).*
(2b) *f induces an isomorphism $\mathcal{X}' \smallsetminus f^{-1}(\mathcal{C}) \to \mathcal{X} \smallsetminus \pi^{-1}\big(\pi(\mathcal{C})\big)$. In particular, f induces an isomorphism $\mathcal{X}'^s \smallsetminus f^{-1}(\mathcal{C}) \to \mathcal{X}^s$.*
(2c) *The induced morphism of good moduli spaces $\mathbf{X}' \to \mathbf{X}$ is projective and an isomorphism over $\mathbf{X} \smallsetminus \pi(\mathcal{C})$. In particular, it is an isomorphism over $\mathbf{X}^s$.*
(3) *Every point of $\mathcal{X}'$ has a stabilizer of dimension strictly less than the maximum dimension of the stabilizers of points of $\mathcal{X}$.*

Proof. Assertion (1a) is Proposition B.2. The Reichstein transform $\mathcal{X}'$ is an open substack of $\text{Bl}_{\mathcal{X}^{\max}} \mathcal{X}$. Assertion (1b) thus follows from the corresponding properties of $\text{Bl}_{\mathcal{X}^{\max}}$.

Since $\mathcal{X}$ and $\mathcal{X}^{\max}$ are smooth, Proposition 4.5 implies that $R(\mathcal{X}, \mathcal{X}^{\max})$ is the saturated blowup of $\mathcal{X}$ along $\mathcal{X}^{\max}$. Assertions (2a)–(2c) then follow from the properties of the saturated blowup (Propositions 3.8 and 3.12) and the fact that $\mathcal{X}^{\max} \subset \mathcal{X} \smallsetminus \mathcal{X}^s$ by Proposition 2.6.

We now prove assertion (3). By the local structure theorem [AHR15, Theorem 2.9] we may assume $\mathcal{X} = [U/G_x]$. Let G_0 be the reduced identity component of G_x. Then $[U/G_x]^{\max} = [U^{G_0}/G_x]$. To complete the proof we need to show that $R_{G_x}(U, U^{G_0})$ has no G_0-fixed point. By Theorem 5.1 we know that $R_{G_0}(U, U^{G_0})$ has no G_0-fixed points. We will prove (3) by showing that $R_{G_x}(U, U^{G_0}) = R_{G_0}(U, U^{G_0})$ as open subschemes of the blowup of U along U^{G_0}.

Consider the maps of quotients $U \xrightarrow{\pi_0} U /\!\!/ G_0 \xrightarrow{q} U /\!\!/ G_x$. If $U = \operatorname{Spec} A$, then these maps are induced by the inclusions of rings

$$A^{G_x} = (A^{G_0})^{(G_x/G_0)} \hookrightarrow A^{G_0} \hookrightarrow A.$$

Since the quotient group G_x/G_0 is a finite k-group scheme, $U/\!\!/G_0 = \operatorname{Spec} A^{G_0} \to U/\!\!/G_x = \operatorname{Spec}(A^{G_0})^{(G_x/G_0)}$ is a geometric quotient.

If $C \subset U$ is a G_x-invariant closed subset of U, then its image in $U/\!\!/G_0$ is (G_x/G_0)-invariant, so it is saturated with respect to the quotient map $U/\!\!/G_0 \to U/\!\!/G_x$. Hence, as closed subsets of U, the saturations of C with respect to either the quotient map $U \to U/\!\!/G_0$ or to $U \to U/\!\!/G_x$ are the same[2]. It follows that if $C \subset U$ is G_x-invariant, then $R_{G_x}(U, C)$ and $R_{G_0}(U, C)$ define the same open subset of the blowup of U along C. Since U^{G_0} is G_x-invariant we conclude that $R_{G_0}(U, U^{G_0}) = R_{G_x}(U, U^{G_0})$ as open subschemes of the blowup. $\square$

Without the assumption that $\pi \colon \mathcal{X} \to \mathbf{X}$ is a stable good moduli space morphism, the conclusion in Proposition 6.1 that $\mathcal{X}' \to \mathcal{X}$ is birational can fail: it may happen that the saturation of $\mathcal{X}^{\mathrm{max}}$ equals $\mathcal{X}$ and thus that $\mathcal{X}' = \emptyset$. The following examples illustrate this.

Example 6.2. Let $\mathbb{G}_m$ acts on $X = \mathbb{A}^1$ with weight 1. The structure map $\mathbb{A}^1 \to \operatorname{Spec} k$ is a good quotient, so $\operatorname{Spec} k$ is the good moduli space of $\mathcal{X} = [\mathbb{A}^1/\mathbb{G}_m]$. The stabilizer of any point of $\mathbb{A}^1 - \{0\}$ is trivial, so $\mathcal{X}^{\mathrm{max}} = [\{0\}/\mathbb{G}_m]$ and the saturation of $\mathcal{X}^{\mathrm{max}}$ is all of $\mathcal{X}$. Hence $R(\mathcal{X}, \mathcal{X}^{\mathrm{max}}) = \emptyset$.

Example 6.3. Here is a non-toric example. Let $V = \mathfrak{sl}_2$ be the adjoint representation of $G = \operatorname{SL}_2(\mathbb{C})$. Explicitly, V can be identified with the vector space of traceless 2×2 matrices with SL_2-action given by conjugation. Let $V^{\mathrm{reg}} \subset V$ be the open set corresponding to matrices with non-zero determinant and set $X = V^{\mathrm{reg}} \times \mathbb{A}^2$. Let $\mathcal{X} = [X/G]$ where G acts by conjugation on the first factor and translation on the second factor. The map of affines $X \to \mathbb{A}^1 \smallsetminus \{0\}$ given by $(A, v) \mapsto \det A$ is a good quotient, so $\pi \colon \mathcal{X} \to \mathbb{A}^1 \smallsetminus \{0\}$ is a good moduli space morphism. However, the morphism π is not stable because the only closed orbits are the orbits of pairs $(A, 0)$.

The stabilizer of a point (A, v) with $v \neq 0$ is trivial and the stabilizer of $(A, 0)$ is conjugate to $T = \operatorname{diag}(t\ t^{-1})$ and $\mathcal{X}^{\mathrm{max}} = [(V^{\mathrm{reg}} \times \{0\})/G]$. Thus, $\pi^{-1}(\pi(\mathcal{X}^{\mathrm{max}})) = \mathcal{X}$ and $R(\mathcal{X}, \mathcal{X}^{\mathrm{max}}) = \emptyset$.

6.2. **Functoriality for strong morphisms.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Artin stacks with good moduli space morphisms, $\pi_{\mathcal{Y}} \colon \mathcal{Y} \to \mathbf{Y}$, and $\pi_{\mathcal{X}} \colon \mathcal{X} \to \mathbf{X}$. Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a morphism and let $g \colon \mathbf{Y} \to \mathbf{X}$ be the induced morphism of good moduli spaces.

---

[2]The saturations with respect to the quotient maps come with natural scheme structures which are not the same. If $I \subset A$ is the ideal defining C in U, then the saturation of C with respect to the quotient map $U \to U/\!\!/G_0$ is the ideal $I^{G_0} A$ while the ideal defining the saturation of C with respect to the quotient map $U \to U/\!\!/G_x$ is the ideal $I^{G_x} A$. While $I^{G_x} A \subset I^{G_0} A$, these ideals need not be equal.

Definition 6.4. We say the morphism f is *strong* if the diagram

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\ f\ } & \mathcal{X} \\
\downarrow{\scriptstyle \pi_{\mathcal{Y}}} & & \downarrow{\scriptstyle \pi_{\mathcal{X}}} \\
\mathbf{Y} & \xrightarrow{\ g\ } & \mathbf{X}
\end{array}
$$

is cartesian.

Note that a strong morphism is representable and stabilizer-preserving. We thus have an equality $\mathcal{Y}^{\mathrm{max}} = f^{-1}(\mathcal{X}^{\mathrm{max}})$ of closed substacks by Proposition B.4. A sharp criterion for when a morphism is strong can be found in [Ryd15].

Theorem 6.5. *Let* $f \colon \mathcal{Y} \to \mathcal{X}$ *be a strong morphism of smooth Artin stacks with stable good moduli space morphisms* $\mathcal{Y} \to \mathbf{Y}$ *and* $\mathcal{X} \to \mathbf{X}$. *Let* $\mathcal{Y}'$ *and* $\mathcal{X}'$ *be the stacks produced by Theorem 2.11. Then there is a natural morphism* $f' \colon \mathcal{Y}' \to \mathcal{X}'$ *such that the diagram*

$$
\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{\ f'\ } & \mathcal{X}' \\
\downarrow & & \downarrow \\
\mathcal{Y} & \xrightarrow{\ f\ } & \mathcal{X}
\end{array}
$$

is cartesian.

Proof. The theorem follows by induction and the following proposition.

Proposition 6.6. *Let* $f \colon \mathcal{Y} \to \mathcal{X}$ *be a strong morphism of smooth algebraic stacks with good moduli spaces. Then there is a natural morphism* $f' \colon R(\mathcal{Y}, \mathcal{Y}^{\mathrm{max}}) \to R(\mathcal{X}, \mathcal{X}^{\mathrm{max}})$ *such that the diagram*

(6.6.1)
$$
\begin{array}{ccc}
R(\mathcal{Y}, \mathcal{Y}^{\mathrm{max}}) & \xrightarrow{\ f'\ } & R(\mathcal{X}, \mathcal{X}^{\mathrm{max}}) \\
\downarrow & & \downarrow \\
\mathcal{Y} & \xrightarrow{\ f\ } & \mathcal{X}
\end{array}
$$

is cartesian.

Proof. We will prove that $f^{-1}(\mathcal{X}^{\mathrm{max}}) = \mathcal{Y}^{\mathrm{max}}$, that $(\mathrm{Bl}_{\mathcal{X}^{\mathrm{max}}} \mathcal{X}) \times_{\mathcal{X}} \mathcal{Y} = \mathrm{Bl}_{\mathcal{Y}^{\mathrm{max}}} \mathcal{Y}$ and that the open subsets $R(\mathcal{X}, \mathcal{X}^{\mathrm{max}}) \times_{\mathcal{X}} \mathcal{Y}$ and $R(\mathcal{Y}, \mathcal{Y}^{\mathrm{max}})$ coincide. This gives a natural morphism f' such that (6.6.1) is cartesian. These claims can be verified étale locally on $\mathbf{X}$ and $\mathbf{Y}$ at points of $\pi_{\mathcal{Y}}(\mathcal{Y}^{\mathrm{max}})$. Let $y \in |\mathcal{Y}^{\mathrm{max}}|$ and $x = f(y)$.

Since $\mathbf{Y} \to \mathbf{X}$ is of finite type we can, locally around $\pi_{\mathcal{Y}}(y)$, factor it as $\mathbf{Y} \hookrightarrow \mathbf{X} \times \mathbb{A}^n \to \mathbf{X}$ where the first map is a closed immersion and the second map is the smooth projection. By base change, this gives a local factorization of the morphism f as $\mathcal{Y} \xhookrightarrow{i} \mathcal{X} \times \mathbb{A}^n \xrightarrow{p} \mathcal{X}$.

Since $\mathcal{X} \times \mathbb{A}^n \to \mathcal{X}$ is flat and $(\mathcal{X} \times \mathbb{A}^n)^{\mathrm{max}} = \mathcal{X}^{\mathrm{max}} \times \mathbb{A}^n$ it follows from Remark 4.3 that $R(\mathcal{X} \times \mathbb{A}^n, (\mathcal{X} \times \mathbb{A}^n)^{\mathrm{max}}) = R(\mathcal{X}, \mathcal{X}^{\mathrm{max}}) \times \mathbb{A}^n$. We are therefore reduced to the

case that the map f is a closed immersion. Since $\mathcal{X}$ and $\mathcal{Y}$ are smooth, f is necessarily a regular embedding.

We can apply Theorem [AHR15, Theorem 2.9 and Theorem 1.1] to reduce to the case that $\mathcal{X} = [X/G]$ where $G = G_y = G_x$ is a linearly reductive group and X is a smooth affine scheme. Let $Y = X \times_{\mathcal{X}} \mathcal{Y}$. Then $Y \to X$ is a regular closed immersion and $\mathcal{Y} = [Y/G]$. Since G is not necessarily smooth, it is not automatic that Y is smooth. But the fiber $Y \times_{\mathcal{Y}} BG_y = \operatorname{Spec} k$ is regular and $Y \times_{\mathcal{Y}} BG_y \to Y$ is a regular closed immersion since $\mathcal{Y}$ is smooth. It follows that Y is smooth over an open G-invariant neighborhood of the preimage of y. After replacing $\mathcal{Y}$ with an open saturated neighborhood of y [AHR15, Lemma 4.1], we can thus assume that Y is smooth.

Since $T_x X = T_y Y \times N_y(Y/X)$, we can slightly modify Lemma 5.5 to obtain the following commutative diagram of strong étale morphisms (after further shrinking of $\mathcal{X}$):

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\ f\ } & \mathcal{X} \\
\downarrow & & \downarrow \\
[T_y/G] & \longrightarrow & [T_x/G].
\end{array}
$$

Since f is strong, the action of the stabilizer G is trivial on the normal space N_y. Thus $(T_x)^{\max} = (T_y)^{\max} \times N_y$ and $T_x /\!\!/ G = T_y /\!\!/ G \times N_y$. The result is now immediate. $\qquad \square$

7. Corollaries of Theorem 2.11 in the smooth case

7.1. **Reduction to quotient stacks.** Suppose that $\mathcal{X}$ is a smooth Artin stack such that the good moduli space morphism $\pi\colon \mathcal{X} \to \mathbf{X}$ is properly stable. The end result of our canonical reduction of stabilizers (Theorem 2.11) is a smooth tame stack $\mathcal{X}_n$.

Proposition 7.1. *Let $\mathcal{X}$ be a smooth Artin stack with properly stable good moduli space. Suppose that $\mathcal{X}_n$ is Deligne–Mumford (automatic if $\operatorname{char} k = 0$) and that either $\mathcal{X}$ has generically trivial stabilizer or $\mathbf{X}$ is quasi-projective. Then*

(1) *$\mathcal{X}_n$ is a quotient stack $[U/\operatorname{GL}_m]$ where U is an algebraic space.*
(2) *If, in addition, $\mathbf{X}$ is separated, then U is separated and the action of GL_m on U is proper.*
(3) *If, in addition, $\mathbf{X}$ is a scheme, then so is U.*
(4) *If, in addition, $\mathbf{X}$ is a separated scheme, then we can take U to be quasi-affine.*
(5) *If, in addition, $\mathbf{X}$ is projective, then there is a projective variety X with a linearized action of a GL_n such that $X^s = X^{ss} = U$. Moreover, if $\operatorname{char} k = 0$, we can take X to be smooth.*

Proof. If the generic stabilizer of $\mathcal{X}$ is trivial, so is the generic stabilizer of $\mathcal{X}_n$. Hence by [EHKV01, Theorem 2.18] (trivial generic stabilizer) or [KV04, Theorem 2] (quasi-projective coarse space), $\mathcal{X}_n$ is a quotient stack. This proves (1).

If $\mathbf{X}$ is separated, then $\mathcal{X}_n$ is a separated quotient stack so GL_m must act properly. This proves (2). (Note that if GL_m acts properly on U, then U is necessarily separated. This also follows immediately since $U \to \mathcal{X}_n$ is affine.)

The morphism $U \to \mathbf{X}_n$ is affine. Indeed, there is a finite surjective morphism $V \to \mathcal{X}_n$ [EHKV01, Theorem 2.7] where V is a scheme and $V \to \mathbf{X}_n$ is finite and surjective, hence affine. It follows that $U \times_{\mathcal{X}_n} V \to \mathbf{X}_n$ is affine and hence $U \to \mathbf{X}_n$ is affine as well (Chevalley's theorem). One can also deduce this directly from $U \to \mathbf{X}_n$ being representable and cohomologically affine (Serre's theorem).

In particular, if $\mathbf{X}$ is a scheme, then so is $\mathbf{X}_n$ and U. This proves (3). Similarly, if $\mathbf{X}$ is a separated scheme, then so is $\mathbf{X}_n$ and U. But U is a smooth separated scheme and thus has a G-equivariant ample family of line bundles. It follows that $\mathcal{X}_n$ has the resolution property and that we can choose U quasi-affine, see [Tot04, Theorems 1.1, 1.2] for further details. This proves (4).

We now prove (5). Since U is quasi-affine, it is also quasi-projective. By [Sum74, Theorem 1] there is an immersion $U \subset \mathbb{P}^N$ and a representation $\mathrm{GL}_m \to \mathrm{PGL}_{N+1}$ such that the GL_m-action on U is the restriction of the PGL_{N+1}-action on $\mathbb{P}^N$. Let X be the closure of U in $\mathbb{P}^N$. The action of G on X is linearized with respect to the line bundle $\mathcal{O}_X(1)$. Our statement follows from [MFK94, Converse 1.13].

Finally, if $\mathrm{char}\, k = 0$, then by equivariant resolution of singularities we can embed U into a non-singular projective G-variety X. $\square$

Note that we only used that $\mathbf{X}_n$ is Deligne–Mumford to deduce that $\mathcal{X}_n$ is a quotient stack.

7.2. **Resolution of good quotient singularities.** Combining the main theorem with destackification of tame stacks [Ber17, BR14], we obtain the following results, valid in any characteristic.

Corollary 7.2 (Functorial destackfication of stacks with good moduli spaces)**.** *Let* $\mathcal{X}$ *be a smooth Artin stack with stable good moduli space morphism* $\pi\colon \mathcal{X} \to \mathbf{X}$. *Then there exists a sequence* $\mathcal{X}_n \to \cdots \to \mathcal{X}_1 \to \mathcal{X}_0 = \mathcal{X}$ *of birational morphisms of smooth Artin stacks such that*

(1) *Each* $\mathcal{X}_k$ *admits a stable good moduli space* $\pi_k\colon \mathcal{X}_k \to \mathbf{X}_k$.
(2) *The morphism* $\mathcal{X}_{k+1} \to \mathcal{X}_k$ *is either a Reichstein transform in a smooth center, or a root stack in a smooth divisor.*
(3) *The morphism* $\mathcal{X}_{k+1} \to \mathcal{X}_k$ *induces a projective birational morphism of good moduli spaces* $\mathbf{X}_{k+1} \to \mathbf{X}_k$.
(4) $\mathbf{X}_n$ *is a smooth algebraic space.*
(5) $\mathcal{X}_n \to \mathbf{X}_n$ *is a composition of a gerbe* $\mathcal{X}_n \to (\mathcal{X}_n)_{\mathrm{rig}}$ *and a root stack* $(\mathcal{X}_n)_{\mathrm{rig}} \to \mathbf{X}_n$ *in an snc divisor* $D \subset \mathbf{X}_n$.

Moreover, the sequence is functorial with respect to strong smooth morphisms $\mathcal{X}' \to \mathcal{X}$, *that is, if* $\mathbf{X}' \to \mathbf{X}$ *is smooth and* $\mathcal{X}' = \mathcal{X} \times_{\mathbf{X}} \mathbf{X}'$, *then the sequence* $\mathcal{X}'_n \to \cdots \to \mathcal{X}'$ *is obtained as the pull-back of* $\mathcal{X}_n \to \cdots \to \mathcal{X}$ *along* $\mathcal{X}' \to \mathcal{X}$.

Proof. We first apply Theorem 2.11 to $\mathcal{X}$ and can thus assume that $\mathcal{X}$ is a gerbe over a tame stack $\mathcal{X}_{\text{tame}}$. We then apply destackification to $\mathcal{Y} := \mathcal{X}_{\text{tame}}$. This gives a sequence of smooth stacky blowups $\mathcal{Y}_n \to \mathcal{Y}_{n-1} \to \cdots \to \mathcal{Y}_1 \to \mathcal{Y}_0 = \mathcal{Y}$, such that $\mathbf{Y}_n$ is smooth and $\mathcal{Y}_n \to \mathbf{Y}_n$ factors as a gerbe $\mathcal{Y}_n \to (\mathcal{Y}_n)_{\text{rig}}$ followed by a root stack $(\mathcal{Y}_n)_{\text{rig}} \to \mathbf{Y}_n$ in an snc divisor. A smooth stacky blowup is either a root stack along a smooth divisor or a blowup in a smooth center. A blowup on a tame stack is the same thing as a Reichstein transform.

We let $\mathcal{X}_k = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_k$. Then $\mathcal{X}_n \to \mathcal{Y}_n \to \mathcal{Y}_{\text{rig}}$ is a gerbe and $\mathbf{X}_n = \mathbf{Y}_n$. $\qquad\square$

Corollary 7.3 (Resolution of good quotient singularities)**.** *If X is a stable good moduli space of a smooth stack, then there exists a projective birational morphism $p \colon X' \to X$ where X' is a smooth algebraic space. The resolution is functorial with respect to smooth morphisms.*

8. The proof of Theorem 2.11 in the singular case

Recall that the set of points $|\mathcal{X}^{\text{max}}| \subset |\mathcal{X}|$ which has maximal stabilizer dimension is closed. This set has a canonical structure as a closed substack that we denote $\mathcal{X}^{\text{max}}$ (Appendix B). If $\mathcal{X}$ is smooth, then $\mathcal{X}^{\text{max}}$ is smooth. If $f \colon \mathcal{X} \to \mathcal{Y}$ is a stabilizer-preserving morphism, for example a closed immersion or a strong morphism, then $\mathcal{X}^{\text{max}} = f^{-1}(\mathcal{Y}^{\text{max}})$. When $\mathcal{X} = [U/G]$ and G has the same dimension as the maximal stabilizer dimension, then $\mathcal{X}^{\text{max}} = [U^{G_0}/G]$.

The locus of stable points $\mathcal{X}^s$ may have connected components of different stabilizer dimensions. A complication in the singular case is that the closures of these components may intersect. The following lemma takes care of this problem.

Lemma 8.1. *Let $\mathcal{X}$ be an Artin stack with stable good moduli space morphism $\pi \colon \mathcal{X} \to X$. Let N (resp. n) be the maximum dimension of a stabilizer of a point of $\mathcal{X}$ (resp. a point of $\mathcal{X} \smallsetminus \mathcal{X}^s$). Let $\mathcal{X}_k^s \subset \mathcal{X}^s$ denote the subset of points that are stable with stabilizer of dimension k. Let $\mathcal{X}_{\leq n} \subset \mathcal{X}$ denote the subset of points with stabilizer of dimension at most n. Then*

(1) $\mathcal{X}^s$ is the disjoint union of the $\mathcal{X}_k^s$.
(2) We have a partition of $\mathcal{X}$ into open and closed substacks:

$$(8.1.1) \qquad\qquad \mathcal{X} = \mathcal{X}_{\leq n} \amalg \mathcal{X}_{n+1}^s \amalg \cdots \amalg \mathcal{X}_N^s.$$

In $\mathcal{X}_{\leq n}$ we have the following two open substacks

$$\mathcal{X}_n^s \quad and \quad \mathcal{X}^* = \mathcal{X}_{\leq n} \smallsetminus \overline{\mathcal{X}_n^s}.$$

We let $\overline{\mathcal{X}_n^s}$ and $\overline{\mathcal{X}^*}$ denote their schematic closures.

(3) Every point of $\overline{\mathcal{X}_n^s}$ has stabilizer of dimension n and every point of $\overline{\mathcal{X}^*}$ has stabilizer of dimension at most n.
(4) $\mathcal{X}_n^s = \mathcal{X}_{\leq n} \smallsetminus \overline{\mathcal{X}^*}$. Thus, every point of $\mathcal{X} \smallsetminus \mathcal{X}^s$ with stabilizer of dimension n is contained in $\overline{\mathcal{X}^*}$. In particular, $|\overline{\mathcal{X}^*}^{\text{max}}| = |\mathcal{X} \smallsetminus \mathcal{X}^s|^{\text{max}}$.

Proof. (1) Since the stabilizer dimension is locally constant on $\mathcal{X}^s$, we have that $\mathcal{X}^s = \coprod_k \mathcal{X}^s_k$.

(2) The subset of points of stabilizer dimension $\geq n+1$ is closed by upper semi-continuity. By assumption this set is also contained in $\mathcal{X}^s$ and hence open. This gives the decomposition in (8.1.1).

(3) Every point of either $\overline{\mathcal{X}^s_n}$ or $\overline{\mathcal{X}^*}$ lies in $\mathcal{X}_{\leq n}$ and thus has stabilizer of dimension at most n. Every point of $\overline{\mathcal{X}^s_n}$ has stabilizer of dimension at least n by upper semi-continuity.

(4) Since $\mathcal{X}^s$ is open, $\mathcal{X}^s_n \subset \mathcal{X}_{\leq n} \smallsetminus \overline{\mathcal{X}^*}$. Suppose that $x \in |\mathcal{X}_{\leq n}|$ is not stable. If x has stabilizer of dimension $< n$, then $x \in \overline{\mathcal{X}^*}$. If x has stabilizer of dimension n, then x is the unique closed point in $\pi^{-1}(\pi(x))$ and there exists a generization y with stabilizer of dimension $< n$ [Alp13, Proposition 9.1]. Thus $y \in \overline{\mathcal{X}^*}$ and so $x \in \overline{\mathcal{X}^*}$. $\qquad\square$

Remark 8.2. If all points of $\mathcal{X}^s$ have stabilizers of the same dimension and $\mathcal{X} \neq \mathcal{X}^s$, then $\mathcal{X} = \mathcal{X}^* = \overline{\mathcal{X}^*}$. Two notable examples are irreducible stacks and properly stable stacks. When this is the case, the proof in the singular case simplifies quite a bit. In general, note that the closed immersion $\overline{\mathcal{X}^s_n} \cup \overline{\mathcal{X}^*} \to \mathcal{X}_{\leq n}$ is surjective but not necessarily an isomorphism if $\mathcal{X}_{\leq n}$ has embedded components.

Example 8.3. Let $U = \operatorname{Spec} k[x,y,z]/(xz,yz)$ and let $\mathbb{G}_m$ act with weights $(1,-1,0)$. Then $\mathcal{X} = [U/\mathbb{G}_m]$ has two irreducible components: the first component $\overline{\mathcal{X}^s_1} = V(x,y) = \mathcal{X}^{\mathrm{max}}$ has stabilizer $\mathbb{G}_m$ at every point, the second component $\overline{\mathcal{X}^*} = V(z) = \overline{\mathcal{X}^s_0}$ has a single point with stabilizer $\mathbb{G}_m$. The algorithm will blow up $\overline{\mathcal{X}^*}^{\mathrm{max}} = V(x,y,z)$.

We prove Theorem 2.11 in the singular case using induction on the maximum dimension of the stabilizer of a point of $\mathcal{X} \smallsetminus \mathcal{X}^s$ and the smooth case to verify that the maximum dimension drops.

First suppose $\mathcal{X} = \mathcal{X}^s$ and let us see that Theorem 2.11 holds with $\mathcal{X}_n = \mathcal{X}_0 = \mathcal{X}$. Note that $\mathcal{X}$ has locally constant stabilizer dimensions so it is a disjoint union of stacks with constant stabilizer dimensions. If in addition $\mathcal{X}$ is properly stable, then the stabilizer dimensions are all equal to zero and $\mathcal{X}$ is a tame stack (Proposition A.1). If instead $\mathcal{X} \smallsetminus \mathcal{E}$ is a gerbe over a tame stack (automatic if $\mathcal{X} \smallsetminus \mathcal{E}$ is reduced by Proposition A.2), then $(\mathcal{X} \smallsetminus \mathcal{E})^{\mathrm{max}} = \mathcal{X} \smallsetminus \mathcal{E}$ on every connected component of $\mathcal{X}$ (Corollary B.7). Since $\mathcal{X} \smallsetminus \mathcal{E}$ is schematically dense in $\mathcal{X}$ it follows that $\mathcal{X}^{\mathrm{max}} = \mathcal{X}$ on every connected component of $\mathcal{X}$. Thus $\mathcal{X}$ is a gerbe over a tame stack (Corollary B.7). We have now proven Theorem 2.11 when $\mathcal{X} = \mathcal{X}^s$.

Now suppose $\mathcal{X} \neq \mathcal{X}^s$. Let n be the maximal dimension of a stabilizer of $\mathcal{X} \smallsetminus \mathcal{X}^s$ and assume that the theorem has been proven for smaller n. Let $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{C}_0 = \overline{\mathcal{X}^*}^{\mathrm{max}}$ in the notation of Lemma 8.1. Let $f_1 \colon \mathcal{X}_1 = \mathrm{Bl}^{\pi}_{\mathcal{C}_0} \mathcal{X}_0$ be the saturated blowup and $\mathcal{E}_1$ its exceptional divisor. That the conclusion of the main theorem holds for $\ell = 0$ follows by the following proposition. In particular, the maximal dimension of a stabilizer of $\mathcal{X}_1 \smallsetminus \mathcal{X}^s_1$ is strictly less than n so by induction we have $\mathcal{X}_n \to \cdots \to \mathcal{X}_1$ such that the conclusions also hold for $\ell = 1, \ldots, n$ and the theorem follows for $\mathcal{X}$.

Proposition 8.4. *Let $\mathcal{X}$ be an Artin stack with stable good moduli space morphism $\pi \colon \mathcal{X} \to X$. Let n be the maximum dimension of a stabilizer of a point of $\mathcal{X} \smallsetminus \mathcal{X}^s$. Let $\mathcal{C} = (\overline{\mathcal{X}^*})^{\max} = (\overline{\mathcal{X}^*})^n$ have the scheme structure of Proposition B.4 and let $\mathcal{X}' = \mathrm{Bl}^\pi_{\mathcal{C}}(\mathcal{X})$ be the saturated blowup. Then $\mathcal{X}'$ is an Artin stack with the following properties.*

- (2a) *The stack $\mathcal{X}'$ has a good moduli space $\mathbf{X}'$ and the good moduli space morphism $\pi' \colon \mathcal{X}' \to \mathbf{X}'$ is stable (properly stable if π is properly stable).*
- (2b) *The morphism $f \colon \mathcal{X}' \to \mathcal{X}$ induces an isomorphism $\mathcal{X}' \smallsetminus f^{-1}(\mathcal{C}) \to \mathcal{X} \smallsetminus \pi^{-1}\big(\pi(\mathcal{C})\big)$.*
- (2c) *The induced morphism of good moduli spaces $\mathbf{X}' \to \mathbf{X}$ is projective and an isomorphism over the image of $\mathcal{X}^s$ in $\mathbf{X}$.*
- (3) *Every point of $\mathcal{X}' \smallsetminus (\mathcal{X}')^s$ has stabilizer of dimension strictly less than n.*

Proof. Assertions (2a)–(2c) follow from the properties of the saturated blowup (Propositions 3.8 and 3.12).

Since $\mathcal{X}^s_n \amalg \mathcal{X}^*$ is open in $\mathcal{X}_{\leq n}$ and its complement $\mathcal{X}_{\leq n} \smallsetminus (\mathcal{X}^s_n \amalg \mathcal{X}^*)$ is contained in $\mathcal{C}$, we conclude that $f^{-1}(\mathcal{X}^s_n \amalg \mathcal{X}^*)$ is schematically dense in $\mathrm{Bl}_{\mathcal{C}}(\mathcal{X}_{\leq n})$ and thus

$$\mathrm{Bl}_{\mathcal{C}}(\mathcal{X}_{\leq n}) = \mathrm{Bl}_{\mathcal{C}}(\overline{\mathcal{X}^s_n}) \cup \mathrm{Bl}_{\mathcal{C}}(\overline{\mathcal{X}^*})$$

and similarly for saturated blowups. We note that every point of $\mathrm{Bl}_{\mathcal{C}}(\overline{\mathcal{X}^s_n})$ has stabilizer of dimension n by upper semi-continuity. If every point of $\mathrm{Bl}_{\mathcal{C}}(\overline{\mathcal{X}^*})$ has stabilizer of dimension $< n$, then the two components are necessarily disjoint and every point of $\mathrm{Bl}_{\mathcal{C}}(\overline{\mathcal{X}^s_n})$ is stable. To prove (3) we may thus replace $\mathcal{X}$ with $\overline{\mathcal{X}^*}$. Then $\mathcal{C} = \mathcal{X}^{\max} = \mathcal{X}^n$ and every stable point has dimension strictly less than n.

By the local structure theorem [AHR15, Theorem 1.2] we may assume that $\mathcal{X} = [U/G]$ where G is the stabilizer of a point in $\mathcal{X}^n$ and $U = \mathrm{Spec}\, A$ is affine. By the local finiteness of group actions, there is a finite-dimensional G-submodule $V \subset A$ such that $\mathrm{Sym}(V) \to A$ is surjective. We thus have a closed G-equivariant embedding $U \to V^\vee$. Consequently, we have a closed embedding of stacks $\mathcal{X} = [U/G] \to \mathcal{Y} = [V^\vee/G]$. Since $\mathcal{X}^{\max} = \mathcal{Y}^{\max} \cap \mathcal{X}$ we obtain a closed embedding

$$\mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X} \to \mathrm{Bl}_{\mathcal{Y}^{\max}} \mathcal{Y}.$$

Indeed, $\mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X}$ is the strict transform, that is, the closure of $\mathcal{X} \smallsetminus \mathcal{X}^{\max}$ in $\mathrm{Bl}_{\mathcal{Y}^{\max}} \mathcal{Y}$. This also holds for saturated blowups by Proposition 3.11.

From the smooth case (Proposition 6.1 (3)), we know that $\mathrm{Bl}^\pi_{\mathcal{Y}^{\max}} \mathcal{Y}$ has no points with stabilizer of dimension n. Hence, neither does $\mathrm{Bl}^\pi_{\mathcal{X}^{\max}} \mathcal{X}$. This proves (3). $\qquad\square$

8.1. Functoriality for strong morphisms of singular stacks.

Theorem 6.5 can be extended to strong morphisms of singular stacks with stable good moduli spaces. Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a strong morphism of stacks with stable good moduli space morphisms $\pi_{\mathcal{X}} \colon \mathcal{X} \to \mathbf{X}$, $\pi_{\mathcal{Y}} \colon \mathcal{Y} \to \mathbf{Y}$. Assume that every point of both $\mathcal{X}^s$ and $\mathcal{Y}^s$ has stabilizer of a fixed dimension n; e.g., that $\mathcal{X}$ and $\mathcal{Y}$ are connected or properly stable. Let $\mathcal{X}' \to \mathcal{X}$ and $\mathcal{Y}' \to \mathcal{Y}$ be the canonical morphisms produced by Theorem 2.11.

Proposition 8.5 (Functoriality for strong morphisms). *Under the assumptions above,* $\mathcal{Y}'$ *is the schematic closure of* $\mathcal{Y}^s$ *in* $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$.

Proof. Under our assumptions $\mathcal{Y}'$ and $\mathcal{X}'$ are produced by taking repeated saturated blowups in $\mathcal{X}^{\max}$ and $\mathcal{Y}^{\max}$ respectively. If $f^{-1}(\mathcal{X}^{\max}) = \emptyset$, then $\mathcal{Y} \times_{\mathcal{X}} \mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X} = \mathcal{Y}$. If $f^{-1}(\mathcal{X}^{\max}) \neq \emptyset$, then the maximal stabilizer dimensions of $\mathcal{X}$ and $\mathcal{Y}$ coincide and $\mathcal{Y}^{\max} = f^{-1}(\mathcal{X}^{\max})$ since f is strong. Hence by Proposition 3.11, the saturated blowup $\mathrm{Bl}_{\mathcal{Y}^{\max}}^{\pi_{\mathcal{Y}}} \mathcal{Y} \to \mathcal{Y}$ is the the strict transform of $\mathrm{Bl}_{\mathcal{X}^{\max}}^{\pi_{\mathcal{X}}} \mathcal{X} \to \mathcal{X}$ in the sense of Definition 3.10; that is, it is the schematic closure of $\mathcal{Y} \smallsetminus \mathcal{Y}^{\max}$ in $\mathcal{Y} \times_{\mathcal{X}} \mathrm{Bl}_{\mathcal{X}^{\max}}^{\pi_{\mathcal{X}}} \mathcal{X}$. After the final blowup, $\mathcal{Y}^s$ becomes schematically dense in $\mathcal{Y}'$ and the result follows. $\qquad\square$

Example 8.6. The following example shows that if $\mathcal{Y}$ is singular, then $\mathcal{Y}'$ need not equal the fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ even if the morphism $\mathcal{Y} \to \mathcal{X}$ is lci.

Let $\mathcal{X} = [\mathbb{A}^3/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts by weights $(1, 1, -1)$. Then $\mathcal{X}^{\max}$ is the origin. Let $\mathcal{Y} \subset \mathcal{X}$ be the closed substack defined by the ideal (xz). Since xz is invariant, $f \colon \mathcal{Y} \to \mathcal{X}$ is a strong regular embedding. The induced morphism of good moduli spaces is the closed immersion $\mathbb{A}^1 \to \mathbb{A}^2$.

Since the maximum dimension of a stabilizer is one, the canonical reduction of stabilizers is obtained by a single saturated blowup along $\mathcal{X}^{\max}$. Since $\mathcal{X}$ is smooth, this is also the Reichstein transform. The saturation of $\mathcal{X}^{\max}$ is the substack defined by the ideal (xz, yz) which is the union of two irreducible substacks: the divisor $V(z)$ and the codimension-two substack $V(x, y)$. We conclude that $\mathcal{X}' = \left[\operatorname{Spec} k[\frac{x}{z}, \frac{y}{z}, z] \smallsetminus V(\frac{x}{z}, \frac{y}{z})/\mathbb{G}_m\right]$. and thus $\mathcal{X}' \times_{\mathcal{X}} \mathcal{Y} \subset \mathcal{X}'$ is the closed substack defined by the ideal $(z^2 \frac{x}{z})$. On the other hand, the strict transform of $\mathcal{Y}$ is $\mathcal{Y}' = \mathrm{Bl}_{\mathcal{Y}^{\max}}^{\pi_{\mathcal{Y}}} \mathcal{Y}$ which is cut out by $(\frac{x}{z})$.

Remark 8.7 (Symmetric and intrinsic blowups). Let $\mathcal{X}$ be a stack with a properly stable good moduli space. Then our algorithm repeatedly makes saturated blowups in $\mathcal{X}^{\max}$. An alternative would be to replace the saturated blowup by the saturated *symmetric blowup*: if $\mathcal{X}^{\max}$ is defined by the ideal sheaf $\mathcal{J}_{\mathcal{X}}$, we may take the saturated Proj of the symmetric algebra $\operatorname{Sym}(\mathcal{J}_{\mathcal{X}})$ instead of the saturated blowup which is the saturated Proj of the Rees algebra $\bigoplus_{k \geq 0} \mathcal{J}_{\mathcal{X}}^k$. When $\mathcal{X}$ is smooth, then so is $\mathcal{X}^{\max}$ and the symmetric algebra of $\mathcal{J}_{\mathcal{X}}$ coincides with the Rees algebra of $\mathcal{J}_{\mathcal{X}}$. If $\mathcal{X} \to \mathcal{Y}$ is a closed embedding into a smooth stack with a good moduli space, then $\bigoplus_{k \geq 0} \mathcal{J}_{\mathcal{Y}}^k = \operatorname{Sym}(\mathcal{J}_{\mathcal{Y}}) \to \operatorname{Sym}(\mathcal{J}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}) \to \operatorname{Sym}(\mathcal{J}_{\mathcal{X}}) \to \bigoplus_{k \geq 0} \mathcal{J}_{\mathcal{X}}^k$ are surjective. We thus have closed embeddings

$$\mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X} \to \operatorname{SymBl}_{\mathcal{X}^{\max}} \mathcal{X} \to (\mathrm{Bl}_{\mathcal{Y}^{\max}} \mathcal{Y}) \times_{\mathcal{Y}} \mathcal{X} \to \mathrm{Bl}_{\mathcal{Y}^{\max}} \mathcal{Y}.$$

In particular, the maximum stabilizer dimension of $\operatorname{SymBl}_{\mathcal{X}^{\max}} \mathcal{X}$ also drops.

For a quasi-projective scheme U over $\mathbb{C}$ with an action of a reductive group G, Kiem, Li and Savvas [KL13, KLS17] have defined the *intrinsic blowup* $\mathrm{Bl}^G U$ of U. If $U \to V$ is a G-equivariant embedding into a smooth scheme, then we have an induced embedding of stacks $\mathcal{X} = [U^{ss}/G] \to \mathcal{Y} = [V^{ss}/G]$ that admit good moduli spaces. The stack $[\mathrm{Bl}^G U/G]$ is a closed substack of $(\mathrm{Bl}_{\mathcal{Y}^{\max}} \mathcal{Y}) \times_{\mathcal{Y}} \mathcal{X}$ that is slightly larger than both the blowup and the symmetric blowup of $\mathcal{X}$ in $\mathcal{X}^{\max}$ and is independent on the choice

of a smooth embedding. It would be interesting to find a definition of the intrinsic blowup for a stack with a good moduli space that (1) does not use smooth embeddings and (2) does not use a presentation $\mathcal{X} = [U^{ss}/G]$. The intrinsic blowup also seems to be related to derived stacks and blowups of such and it would be interesting to describe this relationship.

Appendix A. Gerbes and good moduli spaces

Let $\pi\colon \mathcal{X} \to \mathbf{X}$ be a good moduli space morphism. In this appendix, we study when π factors as $\mathcal{X} \to \mathcal{X}_{\mathrm{tame}} \to \mathbf{X}$ where $\mathcal{X} \to \mathcal{X}_{\mathrm{tame}}$ is a gerbe and $\mathcal{X}_{\mathrm{tame}} \to \mathbf{X}$ is a coarse moduli space. A necessary condition is that the stabilizers of $\mathcal{X}$ have locally constant dimension. We prove that this is sufficient when $\mathcal{X}$ is reduced (Proposition A.2) and that $\mathcal{X}_{\mathrm{tame}}$ is a tame stack. When $\mathcal{X}$ is not reduced, we give a precise condition in Corollary B.7. When there is a factorization $\mathcal{X} \to \mathcal{X}_{\mathrm{tame}} \to \mathbf{X}$ as above, then $\mathcal{X} \to \mathbf{X}$ is a homeomorphism and in fact a coarse moduli space in the sense of [Ryd13, Definition 6.8].

Proposition A.1. *Let $\pi\colon \mathcal{X} \to \mathbf{X}$ be the good moduli space of a stack such that all stabilizers are 0-dimensional. Then $\mathcal{X}$ is a tame stack and $\mathbf{X}$ is also the coarse moduli space of $\mathcal{X}$. Moreover, $\mathcal{X}$ is separated if and only $\mathbf{X}$ is separated.*

Proof. By assumption, $\mathcal{X}$ has quasi-finite and separated diagonal (recall that our stacks have affine, hence separated, diagonals). Since $\mathcal{X}$ has a good moduli space, it follows that $\mathcal{X}$ has finite inertia [Alp14, Theorem 8.3.2], that is, $\mathcal{X}$ is tame and $\mathbf{X}$ is its coarse moduli space. Moreover, π is a proper universal homeomorphism, so $\mathcal{X}$ is separated if and only if $\mathbf{X}$ is separated [Con05, Theorem 1.1(2)]. $\square$

We can generalize the previous proposition to stacks with constant dimensional stabilizers.

Proposition A.2. *Let $\mathcal{X}$ be a reduced Artin stack with good moduli space $\pi\colon \mathcal{X} \to \mathbf{X}$. If the dimension of the stabilizers of points of $\mathcal{X}$ is constant, then $\mathcal{X}$ is a gerbe over a tame stack $\mathcal{X}_{\mathrm{tame}}$ whose coarse space is $\mathbf{X}$. In particular, if $\mathcal{X}$ is smooth, then $\mathcal{X}_{\mathrm{tame}}$ is smooth and $\mathbf{X}$ has tame quotient singularities.*

A.1. **Reduced identity components.** To prove the proposition, we need some preliminary results on reduced identity components of group schemes in positive characteristic.

Let G be an algebraic group of dimension n over a perfect field k. By [SGA3, Exposé VIa, Proposition 2.3.1] or [Sta16, Tag 0B7R] the identity component G^0 of G is an open and closed characteristic subgroup. Let $G_0 = (G^0)_{\mathrm{red}}$ (non-standard notation). Since the field is perfect, G_0 is a closed, smooth, subgroup scheme of G^0 [SGA3, Exposé VIa, 0.2] or [Sta16, Tag 047R]. Moreover, $\dim G_0 = \dim G = n$.

Remark A.3. In general, G_0 is not normal in G^0, for example, take $G = \mathbb{G}_m \ltimes \boldsymbol{\alpha}_p$. But if G^0 is diagonalizable, then $G_0 \subset G^0$ is characteristic, hence $G_0 \subset G$ is normal. Indeed,

this follows from Cartier duality, since the torsion subgroup of an abelian group is a characteristic subgroup.

Lemma A.4. *Let S be a scheme and let $G \to S$ be a group scheme of finite type such that $s \mapsto \dim G_s$ is locally constant. Let $H \subset G$ be a subgroup scheme such that $H_{\overline{s}} = G_{\overline{s},0}$ for every geometric point $\overline{s} \colon \operatorname{Spec} K \to S$. If S is reduced, then there is at most one such H and $H \to S$ is smooth.*

Proof. If S is reduced, then $H \to S$ is smooth [SGA3, Exposé VIb, Corollaire 4.4]. If H_1 and H_2 are two different subgroups as in the lemma, then so is $H_1 \cap H_2$. In particular, $H_1 \cap H_2$ is also smooth. By the fiberwise criterion of flatness, it follows that $H_1 \cap H_2 = H_1 = H_2$.   □

Note that the lemma is also valid if S is a reduced algebraic stack by passing to a smooth presentation.

Definition A.5. If S is reduced and there exists a subgroup $H \subset G$ as in the lemma, then we say that H is the *reduced identity component* of G and denote it by G_0.

Proposition A.6. *Let $\mathcal{X}$ be a reduced algebraic stack such that every stabilizer has dimension d. If either*

(1) *char $k = 0$, or*
(2) *$\mathcal{X}$ admits a good moduli space,*

then there exists a unique normal subgroup $(I\mathcal{X})_0 \subset I\mathcal{X}$ such that $I\mathcal{X}_0 \to \mathcal{X}$ is smooth with connected fibers of dimension d. Moreover, when $\mathcal{X}$ admits a good moduli space, then $(I\mathcal{X})_0 \subset I\mathcal{X}$ is closed and $I\mathcal{X}/(I\mathcal{X})_0 \to \mathcal{X}$ is finite.

Proof. If char $k = 0$, then the fibers of $I\mathcal{X} \to \mathcal{X}$ are smooth. It follows that the identity component $(I\mathcal{X})^0$ is represented by an open subgroup which is smooth over $\mathcal{X}$ [SGA3, Exposé VIb, Corollaire 4.4]. The identity component is always a normal subgroup.

If $\mathcal{X}$ instead admits a good moduli space, then we proceed as follows. By the local structure theorem of [AHR15, Theorem 2.9], for any closed point $x \in \mathcal{X}(k)$ there is an affine scheme $U = \operatorname{Spec} A$ and a cartesian diagram of stacks and good moduli spaces

$$
\begin{array}{ccc}
[\operatorname{Spec} A/G] & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow {\scriptstyle \pi} \\
\operatorname{Spec}(A^G) & \longrightarrow & \mathbf{X}
\end{array}
$$

where the horizontal maps are étale neighborhoods of x and $\pi(x)$ respectively and $G = G_x$ is the stabilizer at x. Since, the diagram is cartesian, the map $[U/G] \to \mathcal{X}$ is stabilizer preserving so the diagram of inertia groups

$$
\begin{array}{ccc}
I([U/G]) & \longrightarrow & I\mathcal{X} \\
\downarrow & & \downarrow \\
[U/G] & \longrightarrow & \mathcal{X}
\end{array}
$$

is also cartesian. Since $[U/G]$ is a quotient stack $I([U/G]) = [I_G U/G]$ where $I_G U = \{(g, u) : gu = u\} \subset G \times U$ is the relative inertia group for the action of G on U. Here G acts on $I_G U$ via $h(g, u) = (hgh^{-1}, hu)$.

Note that $[G_0 \times U/G]$, $I([U/G]) \subset [G \times U/G]$ are group schemes over $[U/G]$ with fibers of dimension d, and that $[G_0 \times U/G] \to [U/G]$ is smooth (a twisted form of G_0). It follows that $I([U/G])_0$ exists and equals $[G_0 \times U/G]$.

By Nagata's theorem, G^0 is diagonalizable [DG70, IV, §3, Theorem 3.6]. Thus $G_0 \subset G$ is normal (Remark A.3), and hence so is $I([U/G])_0 \subset I([U/G])$.

Since $(-)_0$ is unique and commutes with étale base change, it follows by descent that $(I\mathcal{X})_0$ exists and is a normal closed subgroup.

Finally, we note that $I\mathcal{X}/(I\mathcal{X})_0$ is finite since $I_G U/(G_0 \times U) \subset (G/G_0) \times U$ is a closed subgroup of a finite group scheme. $\square$

Note that we in the proof worked with the reduced stack $[U/G]$ rather than with the scheme U which perhaps is not reduced. If $\mathcal{X}$ is smooth, then one can arrange that U is smooth [AHR15, Theorem 1.1].

Proof of Proposition A.2. We have seen that the inertia stack $I\mathcal{X} \to \mathcal{X}$ contains a closed, normal subgroup $I\mathcal{X}_0$ which is smooth over $\mathcal{X}$, such that $I\mathcal{X}/I\mathcal{X}_0 \to \mathcal{X}$ is finite with fibers that are linearly reductive finite groups (Proposition A.6). By [AOV08, Appendix A], $\mathcal{X}$ is a gerbe over a stack $\mathcal{X} \mathbin{\!/\mkern-5mu/\!} I\mathcal{X}_0$ which is the rigidification of $\mathcal{X}$ obtained by removing $I\mathcal{X}_0$ from the inertia. The stack $\mathcal{X}_{\text{tame}} = \mathcal{X} \mathbin{\!/\mkern-5mu/\!} I\mathcal{X}_0$ will be the desired tame stack. In the étale chart in the proof of Proposition A.6, we have that $\mathcal{X}_{\text{tame}} = [U/(G/G_0)]$.

The inertia of $\mathcal{X}_{\text{tame}}$ is finite and linearly reductive because its pull-back to $\mathcal{X}$ coincides with $I\mathcal{X}/I\mathcal{X}_0$ (or use the local description). Moreover, $\mathcal{X} \to \mathcal{X}_{\text{tame}}$ has the universal property that a morphism $\mathcal{X} \to \mathcal{Y}$ factors (uniquely) through $\mathcal{X}_{\text{tame}}$ if and only if $I\mathcal{X}_0 \to I\mathcal{Y}$ factors via the unit section $\mathcal{Y} \to I\mathcal{Y}$. In particular we obtain a factorization $\mathcal{X} \to \mathcal{X}_{\text{tame}} \to \mathbf{X}$ and $\mathcal{X}_{\text{tame}} \to \mathbf{X}$ is the coarse moduli space since it is initial among maps to algebraic spaces. $\square$

Remark A.7. If $\mathcal{X}$ is not reduced, then it need not be a gerbe over a tame stack. For example, take $\mathcal{X} = [\operatorname{Spec} k[x]/(x^n)/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts by multiplication.

Remark A.8. If $\mathcal{X}$ is as in Proposition A.6 and char $k = p$, then in general there is no subgroup $(I\mathcal{X})_0$. For a counter-example, take $\mathcal{X} = BG$ with $G = \mathbb{G}_m \ltimes \boldsymbol{\alpha}_p$. Then $I(BG)$ is a reduced algebraic stack. Also, even if there is an open and closed subgroup $(I\mathcal{X})^0$ with connected fibers, this subgroup need not be flat. For a counter-example, take $\mathcal{X} = [\operatorname{Spec} k[x]/\boldsymbol{\mu}_p]$ where $\boldsymbol{\mu}_p$ acts with weight 1. Then $I\mathcal{X}$ has connected fibers but is not flat.

Appendix B. Fixed loci of Artin stacks

Lemma B.1. *Let $\mathcal{X}$ be an Artin stack. Then the locus of points with maximal-dimensional stabilizer is a closed subset of $|\mathcal{X}|$.*

Proof. Since the representable morphism $I\mathcal{X} \to \mathcal{X}$ makes $I\mathcal{X}$ into an $\mathcal{X}$-group, the dimension of the fibers of the morphism is an upper semi-continuous function on $|\mathcal{X}|$. Thus the locus of points with maximal-dimensional stabilizer is closed. $\qquad\square$

If a group scheme G/k acts on X, then the fixed locus X^G can be given a canonical scheme structure: X^G represents the functor of G-equivariant maps $T \to X$ where T is equipped with the trivial action. The functor X^G is represented by a closed sub-scheme [CGP10, Proposition A.8.10 (1)] because the coordinate ring of G is a projective (even free) k-module. Moreover, if G is linearly reductive and X is smooth, then X^G is also smooth [CGP10, Proposition A.8.10 (2)]

Proposition B.2. *If $\mathcal{X}$ is smooth and admits a good moduli space, then the locus of points $\mathcal{X}^n$ with stabilizer of a fixed dimension n (with its reduced induced substack structure) is a locally closed smooth substack. In particular, the locus $\mathcal{X}^{\max}$ of points with maximal-dimensional stabilizer is a closed smooth substack. Moreover, if $\mathcal{E}$ is an snc divisor, then $\mathcal{X}^n$ meets $\mathcal{E}$ with normal crossings.*

Proof. We may replace $\mathcal{X}$ with its open substack where every stabilizer has dimension at most n. Let x be a closed point of $\mathcal{X}^n = \mathcal{X}^{\max}$ and let G_x be its stabilizer group. By the local structure theorem [AHR15, Theorem 2.9, Theorem 1.1] there is a smooth, affine scheme $U = \operatorname{Spec} A$ with an action of G_x and a cartesian diagram of stacks and moduli spaces

$$
\begin{array}{ccc}
[U/G_x] & \longrightarrow & \mathcal{X} \\
\downarrow & \square & \downarrow{\scriptstyle \pi} \\
U /\!\!/ G_x & \longrightarrow & \mathbf{X}
\end{array}
$$

where the horizontal arrows are étale and $u \in U$ is a fixed point above x. It follows that $[U/G_x]^n$ is the inverse image of $\mathcal{X}^n$ under an étale morphism. In particular $[U/G_x]^n$ (with its reduced induced stack structure) is smooth at u if and only if $\mathcal{X}^n$ is smooth at x.

As in §A.1, let G_0 be the reduced identity component of G_x. Then $\dim G_0 = \dim G = n$ and any n-dimensional subgroup of G necessarily contains G_0.

Since G/G_0 is finite, hence affine, so is $BG_0 \to BG$. It follows that BG_0 is cohomologically affine, that is, G_0 is linearly reductive.

Since $n = \dim G_x$, a point of U has stabilizer dimension n if and only if it is fixed by the linearly reductive subgroup G_0. Thus $[U/G_x]^n = [U^{G_0}/G_x]$. By [CGP10, Proposition A.8.10(2)] U^{G_0} is also smooth. Note that G_x acts on U^{G_0} because G_0 is a characteristic, hence normal, subgroup of G_x (Remark A.3).

If $\mathcal{E} = \sum_{i=1}^n \mathcal{E}_i$ is an snc divisor on $\mathcal{X}$, let $D = \sum_{i=1}^n D_i$ denote the pull-back to U. For $I \subset \{1, 2, \ldots, n\}$, let $D_I = \cap_{i \in I} D_i$. First note that D is snc at u. Indeed, since $BG_x \to [D_I/G_x]$ is a regular embedding, so is $u \to D_I$, for every I. Since $D_I^{G_0} = U^{G_0} \cap D_I$ is smooth at u [CGP10, Proposition A.8.10(2)], it follows that U^{G_0} meets D with normal crossings at u. By flat descent, $\mathcal{X}^n$ meets $\mathcal{E}$ with normal crossings. $\qquad\square$

Remark B.3. Proposition B.2 holds more generally for any smooth algebraic stack $\mathcal{X}$ such that the stabilizer of every closed point has linearly reductive identity component. Indeed, for any such point x there is an étale morphism $[U/G_x^0] \to \mathcal{X}$ and $[U/G_x^0]^n = [U^{G_0}/G_x^0]$ is smooth.

If X is an arbitrary algebraic space and $\mathcal{X} = [X/G]$, then we can equip $\mathcal{X}^{\mathrm{max}}$ with the substack structure of $[X^{G_0}/G]$. The next proposition shows that this substack structure on $\mathcal{X}^{\mathrm{max}}$ is independent on the presentation $\mathcal{X} = [X/G]$. Combining this fact with the local structure theorem we can conclude that if $\mathcal{X}$ is an arbitrary stack with a good moduli space, then $\mathcal{X}^{\mathrm{max}}$ has a canonical scheme structure. To achieve this, we start with a slightly different definition of $\mathcal{X}^{\mathrm{max}}$, not referring to G.

Proposition B.4. *Let $\mathcal{X}$ be an Artin stack that admits a good moduli space. Let n be the maximal dimension of the stabilizer groups. Consider the functor $F \colon (\mathrm{Sch}_{/\mathcal{X}})^{\mathrm{op}} \to (\mathrm{Set})$ where $F(T \to \mathcal{X})$ is the set of closed subgroups $H_0 \subset I\mathcal{X} \times_{\mathcal{X}} T$ that are smooth over T with connected fibers of dimension n. Then the functor is represented by a closed substack $\mathcal{X}^{\mathrm{max}} = \mathcal{X}^n$. In particular, for any $T \to \mathcal{X}$, there is at most one such subgroup H_0 and it is characteristic. Moreover,*

(1) *$|\mathcal{X}^{\mathrm{max}}|$ is the set of points with stabilizer of dimension n.*
(2) *If $\mathcal{X} = [X/G]$ where G is linearly reductive, then $\mathcal{X}^n = [X^{G_0}/G]$.*
(3) *The following are equivalent (a) $\mathcal{X} = \mathcal{X}^{\mathrm{max}}$, (b) $I\mathcal{X}$ contains a closed subgroup, smooth over $\mathcal{X}$ with connected fibers of dimension n, and (c) $I\mathcal{X}$ contains a closed subgroup, flat over $\mathcal{X}$ with fibers of dimension n.*
(4) *When $\mathcal{X}$ is smooth, $\mathcal{X}^{\mathrm{max}}$ is smooth.*

Note that from the functorial description of $\mathcal{X}^{\mathrm{max}}$, it follows that if $\mathcal{X}' \to \mathcal{X}$ is a stabilizer-preserving morphism, then $\mathcal{X}'^{\mathrm{max}} = \mathcal{X}^{\mathrm{max}} \times_{\mathcal{X}} \mathcal{X}'$.

Proof of Proposition B.4. (1) is an immediate consequence of the main claim since if $T = \mathrm{Spec}\, k$ with k perfect, then $H_0 = (I\mathcal{X} \times_{\mathcal{X}} T)_0$ is the unique choice of H_0. (3a) $\Longrightarrow$ (3b) follows by definition and (3b) $\Longrightarrow$ (3c) is trivial.

To prove the main claim and (3c) $\Longrightarrow$ (3a) we may work fppf-locally around a point with stabilizer of dimension n. Using the local structure theorem of [AHR15], we can thus assume that $\mathcal{X} = [X/G]$ where G is linearly reductive of dimension n. The main claim thus follows from (2). The following lemma applied to $H = I\mathcal{X} \times_{\mathcal{X}} T$ implies (2) and (3c) $\Longrightarrow$ (3a). If $\mathcal{X}$ is smooth, then X^{G_0} is smooth [CGP10, Proposition A.8.10 (2)] and (4) follows.

Lemma B.5. *Let G be a linearly reductive group scheme of dimension n over k. Let T be a scheme and let $H \subset G \times T$ be a closed subgroup. Then the following are equivalent:*

(1) *$G_0 \times T \subset H$.*
(2) *There exists a closed subgroup $H_0 \subset H$ that is smooth over T with connected fibers of dimension n.*
(3) *There exists a closed subgroup $H_1 \subset H$ that is flat over T with fibers of dimension n.*

and under these conditions H_0 is unique and equals $G_0 \times T$.

In particular, the functor F representing closed subgroups $H_0 \subset H$ that are smooth with connected fibers of dimension n coincides with the functor representing that $G_0 \times T \subset H$ and this is represented by a closed subscheme T^n of T.

Proof. We can replace G with G^0 and H with $H \cap G^0 \times T$ and thus assume that G is connected. We can also assume that T is affine.

If char $k = 0$, then G is reduced and connected so any closed subgroup $H_1 \subset G \times T$ that is flat with fibers of dimension n necessarily equals $G \times T$ by the fiberwise criterion of flatness.

If char $k = p$, then G is diagonalizable by Nagata's theorem [DG70, IV, §3, Theorem 3.6] so $G_0 \times T$ is the unique closed subgroup of $G \times T$ that is smooth of dimension n. Indeed, we have seen uniqueness when T is reduced (Lemma A.4) and the uniqueness in general follows from rigidity of groups of multiplicative type [SGA3, Exposé X, Corollaire 2.3, Exposé IX, Corollaire 3.4bis].

Thus, the functor F represents when $G_0 \times T \subset H$, or equivalently, when $H \cap (G_0 \times T) \subset G_0 \times T$ is the identity, that is, the Weil restriction

$$F = \prod_{G_0 \times T/T} (H \cap G_0 \times T)/(G_0 \times T).$$

This functor is represented by a closed subscheme since $G_0 \times T \to T$ is pure/essentially free, see [SGA3, Exposé VIII, Théorème 6.4], [RG71, Théorème 4.1.1] or [CGP10, Proposition A.8.10 (1)].   □

Remark B.6. Similarly, we have canonical locally closed substacks $\mathcal{X}^n$ for any integer $n \geq 0$.

We can now generalize Proposition A.2 to non-reduced stacks.

Corollary B.7. *Let $\mathcal{X}$ be a stack with good moduli space $\mathbf{X}$. Then $\mathcal{X}$ is a gerbe over a tame stack if and only if $\mathcal{X}^n \subset \mathcal{X}$ is a closed and open substack for every n.*

Proof. We can assume that $\mathcal{X}$ is connected. If $\mathcal{X} = \mathcal{X}^n$, then $I\mathcal{X}$ contains a smooth normal subgroup $(I\mathcal{X})_0$ with connected fibers of dimension n and $\mathcal{X} \to \mathcal{X} /\!\!/ I\mathcal{X}_0$ is the requested gerbe, see proof of Proposition A.2.

Conversely, if $f\colon \mathcal{X} \to \mathcal{X}_{\text{tame}}$ is an fppf-gerbe, then $If \subset I\mathcal{X}$ is a flat normal subgroup and $I\mathcal{X}/If = f^*I\mathcal{X}_{\text{tame}}$ is finite. Thus If is closed with fibers of the same dimension as the stabilizers of $\mathcal{X}$ and it follows that $\mathcal{X}^n = \mathcal{X}$ by Proposition B.4.   □

References

[AHR15]   Jarod Alper, Jack Hall, and David Rydh, *A Luna étale slice theorem for algebraic stacks*, Preprint, Apr 2015, arXiv:1504.06467. 3, 11, 19, 20, 23, 27, 30, 31, 32, 33

[Alp13]   Jarod Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402. 5, 6, 9, 10, 11, 26

[Alp14]   Jarod Alper, *Adequate moduli spaces and geometrically reductive group schemes*, Algebr. Geom. **1** (2014), no. 4, 489–531. 29

[AOV08]   Dan Abramovich, Martin Olsson, and Angelo Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 4, 1057–1091. 5, 31

[Ber17]   Daniel Bergh, *Functorial destackification of tame stacks with abelian stabilisers*, Compos. Math. **153** (2017), no. 6, 1257–1315. 2, 24

[BM08]   Edward Bierstone and Pierre D. Milman, *Functoriality in resolution of singularities*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 609–639. 13

[BR14]   Daniel Bergh and David Rydh, *Functorial destackification and weak factorization of orbifolds*, In preparation, 2014, p. 13. 24

[CGP10]   Brian Conrad, Ofer Gabber, and Gopal Prasad, *Pseudo-reductive groups*, New Mathematical Monographs, vol. 17, Cambridge University Press, Cambridge, 2010. 16, 32, 33, 34

[Con05]   Brian Conrad, *The Keel–Mori theorem via stacks*, 2005, preprint available at `math.stanford.edu/~conrad`, p. 12. 29

[DG70]   Michel Demazure and Pierre Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970, Avec un appendice *Corps de classes local* par Michiel Hazewinkel. 31, 34

[EHKV01]   Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777. 23, 24

[EM12]   Dan Edidin and Yogesh Moore, *Partial desingularizations of good moduli spaces of Artin toric stacks*, Michigan Math Journal **61** (2012), no. 3, 451–474. 3, 15

[ES17]   Dan Edidin and Matthew Satriano, *Towards an Intersection Chow Cohomology Theory for GIT Quotients*, ArXiv e-prints (2017), arXiv:1707.05890. 2

[Ful84]   William Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984. 3

[Kir85]   Frances Clare Kirwan, *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers*, Ann. of Math. (2) **122** (1985), no. 1, 41–85. 2, 12

[KL13]   Young-Hoon Kiem and Jun Li, *A wall crossing formula of Donaldson-Thomas invariants without Chern-Simons functional*, Asian J. Math. **17** (2013), no. 1, 63–94. 3, 28

[KLS17]   Y.-H. Kiem, J. Li, and M. Savvas, *Generalized Donaldson-Thomas Invariants via Kirwan Blowups*, ArXiv e-prints (2017), arXiv:1712.02544. 3, 28

[Kol07]   János Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007. 14

[KV04]   Andrew Kresch and Angelo Vistoli, *On coverings of Deligne–Mumford stacks and surjectivity of the Brauer map*, Bull. London Math. Soc. **36** (2004), no. 2, 188–192. 23

[LMB00]   Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000. 4

[Lun73]   Domingo Luna, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33. 17

[MFK94]   D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Springer-Verlag, Berlin, 1994. 24

[Rei89]   Zinovy Reichstein, *Stability and equivariant maps*, Invent. Math. **96** (1989), no. 2, 349–383. 3, 9

[RG71]   Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), no. 1, 1–89. 34

[Ryd11]   David Rydh, *Étale dévissage, descent and pushouts of stacks*, J. Algebra **331** (2011), 194–223. 4

[Ryd13]   David Rydh, *Existence and properties of geometric quotients*, J. Algebraic Geom. **22** (2013), no. 4, 629–669. 29

[Ryd15] David Rydh, *A generalization of Luna's fundamental lemma for stacks with good moduli spaces*, In preparation, Oct 2015, p. 16. 22

[SGA3]   M. Demazure and A. Grothendieck, *Schemas en groupes*, Re-edition of SGA3, edited by Phillipe Gille and Patrick Polo.
`https://webusers.imj-prg.fr/~patrick.polo/SGA3`. 5, 29, 30, 34

[Sta16] The Stacks Project Authors, *Stacks project*, `http://stacks.math.columbia.edu`, 2016. 6, 29

[Sum74] Hideyasu Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28. 24

[Tot04] Burt Totaro, *The resolution property for schemes and stacks*, J. Reine Angew. Math. **577** (2004), 1–22. 24

[Vin00] Ernest Vinberg, *On stability of actions of reductive algebraic groups*, Lie Algebras, Rings and Related Topics: Proceedings of the Second International Tainan-Moscow Algebra Workshop (Tainan, Jan 11–17, 1997), Springer, 2000, pp. 188–202. 7

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-COLUMBIA, COLUMBIA, MISSOURI 65211
E-mail address: `edidind@missouri.edu`

KTH ROYAL INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICS, SE-100 44 STOCK-HOLM, SWEDEN
E-mail address: `dary@math.kth.se`