CANONICAL REDUCTION OF STABILIZERS FOR ARTIN STACKS WITH GOOD MODULI SPACES

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Abstract. We prove that if $\mathcal{X}$ is a smooth Artin stack with stable good moduli space $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$, then there is a canonical sequence of birational morphisms of smooth Artin stacks $\mathcal{X}_n \to \mathcal{X}_{n-1} \to \ldots \to \mathcal{X}_0 = \mathcal{X}$ with the following properties: (1) the maximum dimension of a stabilizer of a point of $\mathcal{X}_{k+1}$ is strictly smaller than the maximum dimension of a stabilizer of $\mathcal{X}_k$ and the final stack $\mathcal{X}_n$ has constant stabilizer dimension; (2) the morphisms $\mathcal{X}_{k+1} \to \mathcal{X}_k$ induce proper and birational morphisms of good moduli spaces $\mathbf{X}_{k+1} \to \mathbf{X}_k$; and (3) the algebraic space $\mathbf{X}_n$ has tame quotient singularities and is a partial desingularization of the good moduli space $\mathbf{X}$.

Combining our result with D. Bergh's recent destackification theorem for tame stacks, we obtain a full desingularization of $\mathbf{X}$.

1. Introduction

Consider the action of a reductive group G on a smooth projective variety X. For any ample G-linearized line bundle on X there is a corresponding projective geometric invariant theory (GIT) quotient $X/\!\!/G$. If $X^s = X^{ss}$ then $X/\!\!/G$ has finite quotient singularities. However, if $X^s \neq X^{ss}$ the singularities of $X/\!\!/G$ can be quite bad. In a classic paper, Kirwan [Kir85] used a careful analysis of stable and unstable points on blowups to prove that if $X^s \neq \emptyset$ there is a sequence of blowups along smooth centers $X_n \to X_{n-1} \to \ldots \to X_0 = X$ with the following properties:(1) The final blowup X_n is a smooth projective G-variety with $X_n^s = X_n^{ss}$. (2) The map of GIT quotients $X_n/\!\!/G \to X/\!\!/G$ is proper and

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birational. Since $X_n /\!\!/ G$ has only finite quotient singularities, we may view it as a partial resolution of the very singular quotient $X /\!\!/ G$.

Kirwan's result can be expressed in the language of algebraic stacks by noting that for linearly reductive groups, a GIT quotient $X /\!\!/ G$ can be interpreted as the good moduli space of the quotient stack $[X^{ss}/G]$. The purpose of this paper is to give a complete generalization of Kirwan's result to algebraic stacks. Precisely, we prove (Theorem 2.19) that if $\mathcal{X}$ is a smooth Artin stack with *stable* good moduli space $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$ then there is a canonical sequence of birational morphisms of smooth stacks $\mathcal{X}_n \to \mathcal{X}_{n-1} \ldots \to \mathcal{X}_0 = \mathcal{X}$ with the following properties: (1) The maximum dimension of a stabilizer of a point of $\mathcal{X}_{k+1}$ is strictly smaller than the maximum dimension of a stabilizer of $\mathcal{X}_k$ and the final stack $\mathcal{X}_n$ has constant stabilizer dimension. (2) The morphisms $\mathcal{X}_{k+1} \to \mathcal{X}_k$ induce proper and birational morphisms of good moduli spaces $\mathbf{X}_{k+1} \to \mathbf{X}_k$.

Since $\mathcal{X}_n$ has constant dimensional stabilizer we also prove (Proposition 2.6) that its moduli space $\mathbf{X}_n$ has only tame quotient singularities. Thus our theorem gives a canonical procedure to partially desingularize the good moduli space $\mathbf{X}$. Moreover, even in the special case of GIT quotients, our method allows us to avoid the intricate arguments used by Kirwan.

Our method can also be combined with the destackification results of Bergh [Ber17] to give a functorial resolution of the singularities of good moduli spaces of smooth Artin stacks in arbitrary characteristic (Corollary 7.2).

Outline of the proof of Theorem 2.19. In general, a blowup of an Artin stack with good moduli space need not have a good moduli space. However, we prove the following theorem about good moduli spaces and blowups (Theorem 3.5). Let $\mathcal{X}$ be a smooth Artin stack with good moduli space $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$ and $\mathcal{C} \subset \mathcal{X}$ be a closed, smooth substack. Let $\mathcal{X}'$ denote the complement of the strict transform of the saturation of $\mathcal{C}$ (with respect to the good moduli space map $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$) in $\mathrm{Bl}_{\mathcal{C}} \mathcal{X}$. Then $\mathcal{X}'$ has a good moduli space $\mathbf{X}'$ and the induced map $\mathbf{X}' \to \mathbf{X}$ is proper and an isomorphism over the complement of the image of $\mathcal{C}$ in $\mathbf{X}$. The proof of Theorem 3.5 makes use of the fact that Proj is a local construction and allows us to avoid invariant theoretic methods.

Given a closed substack $\mathcal{C} \subset \mathcal{X}$ the *Reichstein transform* $R(\mathcal{X}, \mathcal{C})$ of $\mathcal{X}$ along $\mathcal{C}$ is the complement of the strict transform of the saturation of $\mathcal{C}$ in the blowup $\mathrm{Bl}_{\mathcal{C}} \mathcal{X}$. The Reichstein transform was introduced in [EM12] where toric methods were used to prove that there is a canonical sequence of toric Reichstein transforms, called stacky star subdivisions, which turn an Artin toric stack into a Deligne–Mumford toric stack.

The term "Reichstein transform" was inspired by Reichstein's paper [Rei89] which contains the result that if $C \subset X$ is a smooth, closed G-invariant subvariety of a smooth, G-projective variety X then $(\mathrm{Bl}_C X)^{ss}$ is the complement of the strict transform of the saturation of $C \cap X^{ss}$ in the blowup of X^{ss} along $C \cap X^{ss}$.

With Theorem 3.5 in hand, the proof of Theorem 2.19 proceeds as follows. If $\mathcal{X}$ is a smooth Artin stack with good moduli space $\mathcal{X} \to \mathbf{X}$, then the substack $\mathcal{X}^{\max}$, corresponding to points with maximal dimensional stabilizer, is closed and smooth. Thus $\mathcal{X}' = R(\mathcal{X}, \mathcal{X}^{\max})$ is a smooth Artin stack whose good moduli space $\mathbf{X}'$ maps properly to $\mathbf{X}$ and is an isomorphism over the complement of $\mathbf{X}^{\max}$, the image of $\mathcal{X}^{\max}$ in $\mathbf{X}$. The stability hypothesis ensures that as long as the stabilizers are not all of constant dimension, $\mathbf{X}^{\max}$ is a proper closed substack of $\mathbf{X}$. Using the local structure theorem of [AHR15] we can show (Proposition 5.4) that the maximum dimension of the stabilizer of a point of $\mathcal{X}'$ is strictly smaller than the maximum dimension of the stabilizer of a point of $\mathcal{X}$. Theorem 2.19 then follows by induction.

Interestingly, the result fails if $\mathcal{X}$ is singular. We give an example (Example 4.7) showing that if $\mathcal{X}$ is singular, then the maximum stabilizer dimension of $R(\mathcal{X}, \mathcal{X}^{\max})$ need not drop. However, if $\mathcal{X}$ is singular and $\tilde{\mathcal{X}} \xrightarrow{f} \mathcal{X}$ is a resolution of singularities, then we can use Proposition 3.10 to show that there is an open substack $\mathcal{X}' \subset \tilde{\mathcal{X}}$ such that $\mathcal{X}'$ has a (stable) good moduli space $\mathbf{X}'$ and the induced morphism of moduli spaces is proper and birational. Applying our main theorem to $\mathcal{X}' \to \mathbf{X}'$ we can again obtain a reduction of stabilizers and partial desingularization of $\mathbf{X}$ (Corollary 7.5).

Conventions and Notation. All algebraic stacks are assumed to have affine diagonal and be of finite type over an algebraically closed field k.

A *point* of an algebraic stack $\mathcal{X}$ is an equivalence class of morphisms $\operatorname{Spec} K \xrightarrow{x} \mathcal{X}$ where K is a field, and $(x', K') \sim (x'', K'')$ if there is a k-field K containing K', K'' such that the morphisms $\operatorname{Spec} K \to \operatorname{Spec} K' \xrightarrow{x'} \mathcal{X}$ and $\operatorname{Spec} K \to \operatorname{Spec} K'' \xrightarrow{x''} \mathcal{X}$ are isomorphic. The set of points of $\mathcal{X}$ is denoted $|\mathcal{X}|$.

Since $\mathcal{X}$ is of finite type over a field it is noetherian. This implies that every point of $\xi \in |\mathcal{X}|$ is algebraic [LMB00, Théorème 11.3], [Ryd11, Appendix B]. This means that if $\operatorname{Spec} K \xrightarrow{x} \mathcal{X}$ is a representative for ξ, then the morphism x factors as $\operatorname{Spec} K \xrightarrow{\overline{x}} \mathcal{G}_\xi \to \mathcal{X}$, where $\overline{x}$ is faithfully flat and $\mathcal{G}_\xi \to \mathcal{X}$ is a representable monomorphism. Moreover, $\mathcal{G}_\xi$ is a gerbe over a field $k(\xi)$ which is called the *residue field* of the point ξ. The stack $\mathcal{G}_\xi$ is called the *residual gerbe* and is independent of the choice of representative $\operatorname{Spec} K \xrightarrow{x} \mathcal{X}$.

Given a morphism $\operatorname{Spec} K \xrightarrow{x} \mathcal{X}$, define the stabilizer group G_x to be the fiber product:

$$\begin{array}{ccc} G_x & \longrightarrow & \operatorname{Spec} K \\ \big\downarrow & & \big\downarrow{\scriptstyle (x,x)} \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times_k \mathcal{X} \end{array}$$

Since the diagonal is representable G_x is a K-group which we call the stabilizer of x.

Since we work over an algebraically closed field, any closed point is geometric and is represented by a morphism $\operatorname{Spec} k \xrightarrow{x} \mathcal{X}$. In this case the residual gerbe is BG_x where G_x is the stabilizer of x.

2. Stable good moduli spaces

2.1. Good moduli spaces.

Definition 2.1 ([Alp13, Definition 4.1])**.** A morphism $\pi\colon \mathcal{X} \to \mathbf{X}$ from an algebraic stack to an algebraic space is a *good moduli space* if

(1) π is *cohomologically affine*, meaning that the pushforward functor π_* on the category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules is exact.
(2) The natural map $\mathcal{O}_{\mathbf{X}} \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.

More generally, a morphism of Artin stacks $\phi\colon \mathcal{X} \to \mathcal{Y}$ satisfying conditions (1) and (2) is called a *good moduli space morphism*.

Remark 2.2. The morphism π is universal for maps to algebraic spaces, so the algebraic space $\mathbf{X}$ is unique up to isomorphism [Alp13, Theorem 6.6]. Thus, we can refer to $\mathbf{X}$ as *the* good moduli space of $\mathcal{X}$.

Remark 2.3. If $\mathcal{X} \to \mathbf{X}$ is a good moduli space then the stabilizer of any closed point of $\mathcal{X}$ is linearly reductive by [Alp13, Proposition 12.14].

Remark 2.4. Let $\mathcal{X}$ be a stack with finite inertia $I\mathcal{X} \to \mathcal{X}$. By the Keel–Mori theorem, there is a coarse moduli space $\pi\colon \mathcal{X} \to \mathbf{X}$. Following [AOV08] we say that $\mathcal{X}$ is *tame* if π is cohomologically affine. This happens precisely when the stabilizer groups are linearly reductive. In this case $\mathbf{X}$ is also the good moduli space of $\mathcal{X}$ by [Alp13, Example 8.1].

Proposition 2.5. *Let $\pi\colon \mathcal{X} \to \mathbf{X}$ be the good moduli space of a stack such that all stabilizers are 0-dimensional. Then $\mathcal{X}$ is a tame stack and $\mathbf{X}$ is also the coarse moduli space of $\mathcal{X}$. Moreover, $\mathcal{X}$ is separated if and only $\mathbf{X}$ is separated.*

Proof. By assumption, $\mathcal{X}$ has quasi-finite and separated diagonal (recall that our stacks have affine, hence separated, diagonals). Since $\mathcal{X}$ has a good moduli space, it follows that $\mathcal{X}$ has finite inertia [Alp14, Theorem 8.3.2], that is, $\mathcal{X}$ is tame and $\mathbf{X}$ is its coarse moduli space. Moreover, π is a proper universal homeomorphism, so $\mathcal{X}$ is separated if and only if $\mathbf{X}$ is separated [Con05, Theorem 1.1(2)]. □

We can generalize the previous proposition to stacks with constant dimensional stabilizers.

Proposition 2.6. *Let $\mathcal{X}$ be a reduced Artin stack with good moduli space $\pi\colon \mathcal{X} \to \mathbf{X}$. If the dimension of the stabilizers of points of $\mathcal{X}$ is constant then $\mathcal{X}$ is a gerbe over a tame stack $\mathcal{X}_{\mathrm{tame}}$ whose coarse space is $\mathbf{X}$. In particular, if $\mathcal{X}$ is smooth, then $\mathcal{X}_{\mathrm{tame}}$ is smooth and $\mathbf{X}$ has tame quotient singularities.*

To prove the proposition, we need some preliminary results on reduced identity components of group schemes.

Let G be an algebraic group of dimension n over a perfect field k. By [SGA3, Exposé VIa, Proposition 2.3.1] or [Sta16, Tag 0B7R] the identity component G^0 of G is an open and closed characteristic subgroup. Let $G_0 = (G^0)_{\mathrm{red}}$ (non-standard notation). Since the field is perfect, G_0 is a closed, smooth, subgroup scheme of G^0 [SGA3, Exposé VIa, 0.2] or [Sta16, Tag 047R]. Moreover, $\dim G_0 = \dim G = n$.

Remark 2.7. In general, G_0 is not normal in G^0, for example, take $G = \mathbb{G}_m \ltimes \boldsymbol{\alpha}_p$. But if G^0 is diagonalizable then $G_0 \subset G^0$ is characteristic, hence $G_0 \subset G$ is normal. Indeed, this follows from Cartier duality, since the torsion subgroup of an abelian group is a characteristic subgroup.

Lemma 2.8. *Let S be a scheme and let $G \to S$ be a group scheme of finite type such that $s \mapsto \dim G_s$ is locally constant. Let $H \subset G$ be a subgroup scheme such that $H_{\overline{s}} = G_{\overline{s},0}$ for every geometric point $\overline{s}\colon \operatorname{Spec} K \to S$. If S is reduced, then there is at most one such H and $H \to S$ is smooth.*

Proof. If S is reduced, then $H \to S$ is smooth [SGA3, Exposé VIb, Corollaire 4.4]. If H_1 and H_2 are two different subgroups as in the lemma, then so is $H_1 \cap H_2$. In particular, $H_1 \cap H_2$ is also flat. By the fiberwise criterion of flatness, it follows that $H_1 \cap H_2 = H_1 = H_2$. □

Note that the lemma is also valid if S is a reduced algebraic stack by passing to a smooth presentation.

Definition 2.9. If S is reduced and there exists a subgroup $H \subset G$ as in the lemma, then we say that H is the *reduced identity component* of G and denote it by G_0.

Proposition 2.10. *Let $\mathcal{X}$ be a reduced algebraic stack such that every stabilizer has dimension d. If either*

(1) $\operatorname{char} k = 0$*, or*
(2) $\mathcal{X}$ *admits a good moduli space,*

then there exists a unique normal subgroup $(I\mathcal{X})_0 \subset I\mathcal{X}$ such that $I\mathcal{X}_0 \to \mathcal{X}$ is smooth with connected fibers of dimension d. Moreover, when $\mathcal{X}$ admits a good moduli space, then $(I\mathcal{X})_0 \subset I\mathcal{X}$ is closed and $I\mathcal{X}/(I\mathcal{X})_0 \to \mathcal{X}$ is finite.

Proof. If $\operatorname{char} k = 0$, then the fibers of $I\mathcal{X} \to \mathcal{X}$ are smooth. It follows that the identity component $(I\mathcal{X})^0$ is represented by an open subgroup which is smooth over $\mathcal{X}$ [SGA3, Exposé VIb, Corollaire 4.4]. The identity component is always a normal subgroup.

If $\mathcal{X}$ instead admits a good moduli space, then we proceed as follows. By the local structure theorem of [AHR15, Theorem 2.9], for any closed point $x \in \mathcal{X}(k)$ there is an affine scheme $U = \operatorname{Spec} A$ and a cartesian diagram of stacks and good moduli spaces

$$\begin{array}{ccc} [\operatorname{Spec} A/G] & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec}(A^G) & \longrightarrow & \mathbf{X} \end{array}$$

where the horizontal maps are étale neighborhoods of x and $\pi(x)$ respectively and $G = G_x$ is the stabilizer at x. Since, the diagram is cartesian, the map $[U/G] \to \mathcal{X}$ is stabilizer preserving so the diagram of inertia groups

$$\begin{array}{ccc} I([U/G]) & \longrightarrow & I\mathcal{X} \\ \downarrow & & \downarrow \\ [U/G] & \longrightarrow & \mathcal{X} \end{array}$$

is also cartesian. Since $[U/G]$ is a quotient stack $I([U/G]) = [I_G U/G]$ where $I_G U = \{(g,u) : gu = u\} \subset G \times U$ is the relative inertia group for the action of G on U. Here G acts on $I_G U$ via $h(g,u) = (hgh^{-1}, hu)$.

Note that $[G_0 \times U/G]$, $I([U/G]) \subset [G \times U/G]$ are group schemes over $[U/G]$ with fibers of dimension d, and that $[G_0 \times U/G] \to [U/G]$

is smooth (a twisted form of G_0). It follows that $I([U/G])_0$ exists and equals $[G_0 \times U/G]$.

By Nagata's theorem, G^0 is diagonalizable [DG70, IV, §3, Theorem 3.6]. Thus $G_0 \subset G$ is normal (Remark 2.7), and hence so is $I([U/G])_0 \subset I([U/G])$.

Since $(-)_0$ is unique and commutes with étale base change, it follows by descent that $(I\mathcal{X})_0$ exists and is a normal closed subgroup.

Finally, we note that $I\mathcal{X}/(I\mathcal{X})_0$ is finite since $I_G U/(G_0 \times U) \subset (G/G_0) \times U$ is a closed subgroup of a finite group scheme. □

Note that we in the proof worked with the reduced stack $[U/G]$ rather than with the scheme U which perhaps is not reduced. If $\mathcal{X}$ is smooth, then one can arrange that U is smooth [AHR15, Theorem 1.1].

Proof of Proposition 2.6. We have seen that the inertia stack $I\mathcal{X} \to \mathcal{X}$ contains a closed, normal subgroup $I\mathcal{X}_0$ which is smooth over $\mathcal{X}$, such that $I\mathcal{X}/I\mathcal{X}_0 \to \mathcal{X}$ is finite with fibers that are linearly reductive finite groups (Proposition 2.10). By [AOV08, Appendix A], $\mathcal{X}$ is a gerbe over a stack $\mathcal{X} \,/\!\!/\, I\mathcal{X}_0$ which is the rigidification of $\mathcal{X}$ obtained by removing $I\mathcal{X}_0$ from the inertia. The stack $\mathcal{X}_{\mathrm{tame}} = \mathcal{X} \,/\!\!/\, I\mathcal{X}_0$ will be the desired tame stack. In the étale chart in the proof of Proposition 2.10, we have that $\mathcal{X}_{\mathrm{tame}} = [U/(G/G_0)]$.

The inertia of $\mathcal{X}_{\mathrm{tame}}$ is finite and linearly reductive because its pullback to $\mathcal{X}$ coincides with $I\mathcal{X}/I\mathcal{X}_0$ (or use the local description). Moreover, $\mathcal{X} \to \mathcal{X}_{\mathrm{tame}}$ has the universal property that a morphism $\mathcal{X} \to \mathcal{Y}$ factors (uniquely) through $\mathcal{X}_{\mathrm{tame}}$ if and only if $I\mathcal{X}_0 \to I\mathcal{Y}$ factors via the unit section $\mathcal{Y} \to I\mathcal{Y}$. In particular we obtain a factorization $\mathcal{X} \to \mathcal{X}_{\mathrm{tame}} \to \mathbf{X}$ and $\mathcal{X}_{\mathrm{tame}} \to \mathbf{X}$ is the coarse moduli space since it is initial among maps to algebraic spaces. □

Remark 2.11. If $\mathcal{X}$ is not reduced, then it need not be a gerbe over a tame stack. For example, take $\mathcal{X} = [\operatorname{Spec} k[x]/(x^n)/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts by multiplication.

Remark 2.12. If $\mathcal{X}$ is as in Proposition 2.10 and $\operatorname{char} k = p$, then in general there is no subgroup $(I\mathcal{X})_0$. For a counter-example, take $\mathcal{X} = BG$ with $G = \mathbb{G}_m \ltimes \boldsymbol{\alpha}_p$. Then $I(BG)$ is a reduced algebraic stack. Also, even if there is an open and closed subgroup $(I\mathcal{X})^0$ with connected fibers, this subgroup need not be flat. For a counter-example, take $\mathcal{X} = [\operatorname{Spec} k[x]/\boldsymbol{\mu}_p]$ where $\boldsymbol{\mu}_p$ acts with weight 1. Then $I\mathcal{X}$ has connected fibers but is not flat.

2.2. **Stable good moduli spaces.**

Definition 2.13. Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism. A point x of $\mathcal{X}$ is *stable* relative to π if $\pi^{-1}(\pi(x)) = \{x\}$ under the induced map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$. A point x of $\mathcal{X}$ is *properly stable* relative to π if it is stable and $\dim G_x = \dim G_{\pi(x)}$.

We say π is a stable (resp. properly stable) good moduli space morphism if the set of stable (resp. properly stable) points is dense.

Proposition 2.14. *The set of stable points defines an open (but possibly empty) substack $\mathcal{X}^s \subset \mathcal{X}$ which is saturated with respect to the morphism π. If $\mathcal{X}$ is irreducible then $\dim G_x - \dim G_{\pi(x)}$ is constant at all points of $\mathcal{X}^s$ and equals the minimum value of $\dim G_x - \dim G_{\pi(x)}$.*

Proof. If $Y \to \mathcal{Y}$ is any smooth or fppf morphism from a scheme then it follows from the definition of stable point that $(Y \times_{\mathcal{Y}} \mathcal{X})^s = Y \times_{\mathcal{Y}} \mathcal{X}^s$, so we can reduce to the case where $\mathcal{Y} = \mathbf{X}$ is a scheme.

If $\mathcal{Z}$ is an irreducible component of $\mathcal{X}$ then the map $\mathcal{Z} \stackrel{\pi|_{\mathcal{Z}}}{\to} \pi(\mathcal{Z})$ is a good moduli space morphism by [Alp13, Lemma 4.14].

If x is a point of $\mathcal{X}$ then $\pi^{-1}(\pi(x)) = \bigcup_{\mathcal{Z}\subset\mathcal{X}} (\pi|_{\mathcal{Z}})^{-1}(\pi(x))$ where the union is over the irreducible components of $\mathcal{X}$ which contain x. Thus a point x is stable if and only if $(\pi|_{\mathcal{Z}})^{-1}(\pi|_{\mathcal{Z}}(x)) = x$ for every irreducible component $\mathcal{Z}$ containing x. If we let $\mathcal{Z}^s$ be the set of stable points for the good moduli space morphism $\pi|_{\mathcal{Z}}$ then $\mathcal{X}^s = \left(\bigcup_{\mathcal{Z}} (\mathcal{Z} \smallsetminus \mathcal{Z}^s)\right)^c$ where the union is over all irreducible components of $\mathcal{X}$. Since we assume that $\mathcal{X}$ is noetherian there are only a finite number of irreducible components. Thus, it suffices to prove that $\mathcal{Z}^s$ is open for each irreducible component $\mathcal{Z}$. In other words, we are reduced to the case that $\mathcal{X}$ is irreducible.

Let d be the minimum of the dimensions of the stabilizers of the points of $\mathcal{X}$. The dimension of the fibers of the morphism $I\mathcal{X} \to \mathcal{X}$ is an upper semi-continuous function [SGA3, Exposé VIb, Proposition 4.1]. Hence the set $U = \{x \in |\mathcal{X}| : \dim G_x = d\}$ determines an open substack $\mathcal{U}$ which is dense since $\mathcal{X}$ is irreducible.

We claim that $\mathcal{X}^s = \left(\pi^{-1}(\pi(\mathcal{U}^c))\right)^c$. To see this we argue as follows.

By [Alp13, Proposition 9.1] if x is a point of $\mathcal{X}$ and $\pi^{-1}(\pi(x))$ is not a singleton then it contains a unique closed point y and $\dim G_y$ is greater than the dimension of any other stabilizer in $\pi^{-1}(\pi(x))$. Such a point is clearly not in the open set $\mathcal{U}$, so we conclude that $(\mathcal{X}^s)^c \subset \pi^{-1}(\pi(\mathcal{U}^c))$ or equivalently that $\mathcal{X}^s \supset \left(\pi^{-1}(\pi(\mathcal{U}^c)\right)^c$.

To obtain the reverse inclusion we need to show that if x is a point of $\mathcal{X}$ and $\pi^{-1}(\pi(x)) = x$ then $\dim G_x = d$. Consider the stack $\pi^{-1}(\pi(x))$ with its reduced stack structure. The monomorphism from the residual gerbe $\mathcal{G}_x \to \mathcal{X}$ factors through a monomorphism $\mathcal{G}_x \to \pi^{-1}(\pi(x)))$. Since $\pi^{-1}(\pi(x))$ has a single point the morphism $\mathcal{G}_x \to \pi^{-1}(\pi(x))_{\mathrm{red}}$ is an equivalence [Sta16, Tag 06MT]. Hence $\dim \pi^{-1}(\pi(x)) = \dim \mathcal{G}_x = -\dim G_x$.

Let ξ be the unique closed point in the generic fiber of π. Then $x \in \overline{\{\xi\}}$ so by upper semi-continuity $\dim G_x \geq \dim G_\xi$ and $\dim \pi^{-1}(\pi(x)) \geq \dim \pi^{-1}(\pi(\xi))$. Moreover, $\dim \pi^{-1}(\pi(\xi)) \geq -\dim G_\xi$ with equality if and only if $\pi^{-1}(\pi(\xi))$ is a singleton. It follows that

$$\dim \pi^{-1}(\pi(\xi)) \geq -\dim G_\xi \geq -\dim G_x = \dim \pi^{-1}(\pi(x))$$

is an equality so the generic fiber $\pi^{-1}(\pi(\xi))$ is a singleton and $\dim G_x = \dim G_\xi = d$. ☐

Let $\mathcal{X}$ be a reduced and irreducible Artin stack and let $\pi \colon \mathcal{X} \to \mathbf{X}$ be a good moduli space morphism with $\mathbf{X}$ an algebraic space and let $\mathbf{X}^s = \pi(\mathcal{X}^s)$. Since $\mathcal{X}^s$ is saturated, $\mathbf{X}^s$ is open in $\mathbf{X}$.

Proposition 2.15. *With notation as in the preceding paragraph $\mathcal{X}^s$ is a gerbe over a tame stack with coarse space $\mathbf{X}^s$. Moreover, $\mathcal{X}^s$ is the largest saturated open substack with this property.*

Proof. By Proposition 2.14 the dimension of the stabilizer G_x is constant at every point x of $\mathcal{X}^s$. Hence by Proposition 2.6, $\mathcal{X}^s$ is a gerbe over a tame stack whose coarse space is $\mathbf{X}^s$.

Conversely, if $\mathcal{U}$ is a saturated open substack which is a gerbe over a tame stack $\mathcal{U}_{\mathrm{tame}}$ then the good moduli space morphism $\mathcal{U} \to \mathbf{U}$ factors via $\mathcal{U}_{\mathrm{tame}}$. Since $|\mathcal{U}| \to |\mathcal{U}_{\mathrm{tame}}|$ and $|\mathcal{U}_{\mathrm{tame}} \to \mathbf{U}|$ are homeomorphisms, it follows that $\mathcal{U} \subset \mathcal{X}^s$ by definition. ☐

2.3. Examples.

Remark 2.16. If $\mathcal{X} = [X/G]$ is a quotient stack with $X = \operatorname{Spec} A$ an affine variety and G is a linearly reductive group, then the good moduli space morphism $\mathcal{X} \to \mathbf{X} = \operatorname{Spec} A^G$ is stable if and only if the action is stable in the sense of [Vin00]. This means that there is a closed orbit of maximal dimension. The morphism $\mathcal{X} \to \mathbf{X}$ is properly stable if the maximal dimension equals $\dim G$. Following [Vin00] we will say that a representation V of a linearly reductive group G is *stable* if the action of G on V is stable.

Example 2.17. If $X = \mathbb{A}^2$ where $G = \mathbb{G}_m$ acts by $\lambda(a,b) = (\lambda a, b)$ then the good moduli space morphism $[X/G] \to \mathbb{A}^1$ is not stable since the inverse image of every point under the quotient map $\mathbb{A}^2 \to \mathbb{A}^1$, $(a,b) \mapsto b$ contains a point with stabilizer of dimension 1. On the other hand, if we consider the action of $\mathbb{G}_m$ given by $\lambda(a,b) = (\lambda^d a, \lambda^{-e} b)$ with $d, e > 0$ then the good moduli space morphism $[X/G] \to \mathbb{A}^1$ is properly stable, since the inverse image of the open set $\mathbb{A}^1 \smallsetminus \{0\}$ is the Deligne–Mumford substack $[(\mathbb{A}^2 \smallsetminus V(xy))/\mathbb{G}_m]$.

Example 2.18. Consider the action of GL_n on $\mathfrak{gl}_n$ via conjugation. If we identify $\mathfrak{gl}_n$ with the space $\mathbb{A}^{n^2}$ of $n \times n$ matrices then the map $\mathfrak{gl}_n \to \mathbb{A}^n$ which sends a matrix to the coefficients of its characteristic polynomial is a good quotient, so the map $\pi\colon [\mathfrak{gl}_n/\operatorname{GL}_n] \to \mathbb{A}^n$ is a good moduli space morphism. The orbit of an $n \times n$ matrix is closed if and only if it is diagonalizable. Since the stabilizer of a matrix with distinct eigenvalues is a maximal torus T, such matrices have orbits of dimension $n^2 - n = \dim \operatorname{GL}_n - \dim T$ which is maximal.

If $U \subset \mathbb{A}^n$ is the open set corresponding to polynomials with distinct roots, then $\pi^{-1}(U)$ is a T-gerbe over the scheme U. Hence π is a stable good moduli space morphism, although it is not properly stable.

2.4. Statement of the main theorem.

Theorem 2.19. *Let $\mathcal{X}$ be a smooth Artin stack with stable good moduli space $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$. There is a canonical sequence of birational morphisms of smooth Artin stacks $\mathcal{X}_n \to \mathcal{X}_{n-1} \ldots \to \mathcal{X}_0 = \mathcal{X}$ and smooth closed substacks $\mathcal{C}_k \subset \mathcal{X}_k$ with the following properties.*

1. *Each $\mathcal{X}_k$ admits a stable good moduli space $\pi_k\colon \mathcal{X}_k \to \mathbf{X}_k$.*
2. *The morphism $\mathcal{X}_{k+1} \to \mathcal{X}_k$ is an open immersion over the complement of $\mathcal{C}_k$ and an isomorphism over the complement of $\pi_k^{-1}\big(\pi_k(\mathcal{C}_k)\big)$.*
3. *The morphism $\mathcal{X}_{k+1} \to \mathcal{X}_k$ induces a projective birational morphism of good moduli spaces $\mathbf{X}_{k+1} \to \mathbf{X}_k$.*
4. *The maximum dimension of the stabilizers of the points of $\mathcal{X}_{k+1}$ is strictly smaller than the maximum dimension of the stabilizers of the points of $\mathcal{X}_k$.*
5. *The stack $\mathcal{X}_n$ is a gerbe over a tame stack. Moreover, if $\mathcal{X} \to \mathbf{X}$ is properly stable then each of the good moduli space morphisms $\mathcal{X}_k \to \mathbf{X}_k$ is properly stable and $\mathcal{X}_n$ is a tame stack. The tame stack is separated if and only if $\mathbf{X}$ is separated.*

Remark 2.20. The birational morphisms $\mathcal{X}_{k+1} \to \mathcal{X}_k$ are *Reichstein transforms* in the centers $\mathcal{C}_k$. They are discussed in the next section.

3. Reichstein transforms

The following definition is straightforward extension of the one originally made in [EM12].

Definition 3.1. Let $\mathcal{X} \xrightarrow{\pi} \mathcal{Y}$ be a good moduli space morphism and let $\mathcal{C} \subset \mathcal{X}$ be a closed substack. The *Reichstein transform* with center $\mathcal{C}$, is the stack $R(\mathcal{X}, \mathcal{C})$ obtained by deleting the strict transform of the saturation $\pi^{-1}(\pi(\mathcal{C}))$ in the blowup of $\mathcal{X}$ along $\mathcal{C}$.

Recall that if $f\colon \operatorname{Bl}_{\mathcal{C}} \mathcal{X} \to \mathcal{X}$ is the blowup, then $E = f^{-1}(\mathcal{C})$ is the exceptional divisor and $\overline{f^{-1}(\mathcal{Z}) - E} = \operatorname{Bl}_{\mathcal{C}\cap\mathcal{Z}} \mathcal{Z}$ is the strict transform of $\mathcal{Z} \subset \mathcal{X}$.

Remark 3.2. Observe that if $\mathcal{X}$ and $\mathcal{C}$ are smooth then $R(\mathcal{X},\mathcal{C})$ is smooth since it is an open set in the blowup of a smooth stack along a closed smooth substack.

Remark 3.3. Let

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\psi} & \mathcal{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathcal{Y}' & \xrightarrow{\phi} & \mathcal{Y} \end{array}$$

be a cartesian diagram where the horizontal maps are flat and the vertical maps are good moduli morphisms. If $\mathcal{C} \subset \mathcal{X}$ is a closed substack then $R(\mathcal{X}', \psi^{-1}\mathcal{C}) = \mathcal{X}' \times_{\mathcal{X}} R(\mathcal{X},\mathcal{C})$. This follows because blowups commute with flat base change and the saturation of $\psi^{-1}(\mathcal{C})$ is the inverse image of the saturation of $\mathcal{C}$.

Definition 3.4 (Equivariant Reichstein transform)**.** If an algebraic group G acts on a scheme X with a good quotient $p\colon X \to X/\!\!/G$ and C is a G-invariant closed subscheme then we write $R_G(X,C)$ for the complement of the strict transform of $p^{-1}p(C)$ in the blowup of X along C. There is a natural G-action on $R_G(X,C)$ and $R([X/G],[C/G]) = [R_G(X,C)/G]$.

3.1. **Reichstein transforms and good moduli space morphisms.**

The goal of this section is to prove the following theorem.

Theorem 3.5. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism and let $\mathcal{C} \subset \mathcal{X}$ be a closed substack with sheaf of ideals $\mathcal{I}$. If $\mathcal{X}$ and $\mathcal{C}$ are smooth, then $R(\mathcal{X},\mathcal{C}) \to \operatorname{Proj}(\oplus \pi_*(\mathcal{I}^n))$ is a good moduli space morphism.*

Remark 3.6. Note that since $\pi\colon \mathcal{X} \to \mathcal{Y}$ is a good moduli space morphism, $\pi_*(\mathcal{I}^n)$ is a coherent subsheaf of $\mathcal{O}_{\mathcal{Y}} = \pi_*(\mathcal{O}_{\mathcal{X}})$, that is, an ideal.

If the saturation of $\mathcal{C}$ with respect to the morphism π is a nowhere dense proper closed substack of $\mathcal{X}$, then the morphism $R(\mathcal{X},\mathcal{C}) \to \mathcal{X}$ is an isomorphism over the dense open subset of $\mathcal{X}$ which is the complement of the saturation of $\mathcal{C}$. In this case, $\pi(\mathcal{C})$ is also nowhere dense and the morphism $\operatorname{Proj}(\oplus \pi_*(\mathcal{I}^n)) \to \mathcal{Y}$ is an isomorphism over the dense open substack $\pi(\mathcal{C})^c$ of $\mathcal{Y}$.

The following example shows that Theorem 3.5 is false if $\mathcal{C}$ is singular.

Example 3.7. Let $U = \operatorname{Spec} k[x,y]$ where $\mathbb{G}_m$ acts by $\lambda(a,b) = (\lambda a, \lambda^{-1} b)$ and let $\mathcal{X} = [U/\mathbb{G}_m]$. Its good moduli space is $\mathbf{X} = \operatorname{Spec} k[xy]$. Let $Z = V(x^2y, xy^2) \subset U$ and $\mathcal{C} = [Z/\mathbb{G}_m]$. Its saturation is $\operatorname{sat}\mathcal{C} = V(x^3y^3)$ which has strict transform $\operatorname{Bl}_{\mathcal{C}} \operatorname{sat}\mathcal{C} = \emptyset$. Thus, the Reichstein transformation $R(\mathcal{X},\mathcal{C})$ equals $\operatorname{Bl}_{\mathcal{C}} \mathcal{X}$.

But $\operatorname{Bl}_{\mathcal{C}} \mathcal{X}$ has no good moduli space. To see this, note that $\operatorname{Bl}_{\mathcal{C}} \mathcal{X} = \operatorname{Bl}_P \mathcal{X}$ where $P = V(x,y)$. The exceptional divisor of the latter blowup is $[\mathbb{P}^1/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts by $\lambda[a:b] = [\lambda a : \lambda^{-1} b]$. This has no good moduli space since the closure of the open orbit contains the two fixed points $[0:1]$ and $[1:0]$.

The Reichstein transformation $R(\mathcal{X}, P)$, on the other hand, equals $\operatorname{Bl}_P \mathcal{X} \smallsetminus \{[0:1],[1:0]\}$ which is tame with coarse moduli space $\operatorname{Bl}_{\pi(P)} \mathbf{X} = \mathbf{X}$.

To prove Theorem 3.5 we will consider a more general construction that does behave well also in the singular situation.

Definition 3.8 (Saturated Proj)**.** Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $\mathcal{A}$ be a (positively) graded sheaf of finitely generated $\mathcal{O}_{\mathcal{X}}$-algebras. Let $\pi^{-1}\pi_*\mathcal{A}^+$ denote the image of the natural homomorphism $\pi^*\pi_*\mathcal{A}^+ \to \mathcal{A}^+ \to \mathcal{A}$. Define $\operatorname{Proj}^{\pi}_{\mathcal{X}} \mathcal{A} = \operatorname{Proj}_{\mathcal{X}} \mathcal{A} \smallsetminus V(\pi^{-1}\pi_*\mathcal{A}^+)$. We call $\operatorname{Proj}^{\pi}_{\mathcal{X}} \mathcal{A}$ the *saturated* Proj of $\mathcal{A}$ relative to the morphism π.

Note that the morphism $\operatorname{Proj}^{\pi}_{\mathcal{X}} \mathcal{A} \to \mathcal{X}$ need not be proper. Also note that there is a canonical morphism $\operatorname{Proj}^{\pi}_{\mathcal{X}} \mathcal{A} \to \operatorname{Proj}_{\mathcal{Y}} \pi_*\mathcal{A}$: we are exactly removing the locus where $\operatorname{Proj}_{\mathcal{X}} \mathcal{A} \dashrightarrow \operatorname{Proj}_{\mathcal{Y}} \pi_*\mathcal{A}$ is not defined.

Theorem 3.5 is now an immediate consequence of the following two results:

Proposition 3.9. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism and let $\mathcal{C} \subset \mathcal{X}$ be a closed substack with sheaf of ideals $\mathcal{I}$. If $\mathcal{X}$ and $\mathcal{C}$ are smooth, then $R(\mathcal{X},\mathcal{C}) = \operatorname{Proj}^{\pi}_{\mathcal{X}}\big(\bigoplus \mathcal{I}^n\big)$ as open substacks of* $\operatorname{Bl}_{\mathcal{C}} \mathcal{X}$*.*

Proposition 3.10. *If $\pi\colon \mathcal{X} \to \mathcal{Y}$ is a good moduli space morphism and $\mathcal{A}$ is a finitely generated graded $\mathcal{O}_{\mathcal{X}}$-algebra, then* $\operatorname{Proj}^{\pi}_{\mathcal{X}} \mathcal{A} \to \operatorname{Proj}_Y \pi_*\mathcal{A}$ *is a good moduli space morphism.*

Proof of Proposition 3.9. Let $\mathcal{A} = \oplus_{n\geq 0}\mathcal{I}^n$. The saturation of $\mathcal{C}$ is the subscheme defined by the ideal $\mathcal{J} = \pi_*\mathcal{I} \cdot \mathcal{O}_{\mathcal{X}}$ so the strict transform of the saturation is the blowup of the substack $V(\mathcal{J})$ along the ideal $\mathcal{I}/\mathcal{J}$, which is $\operatorname{Proj}_{\mathcal{C}}(\oplus_{n\geq 0}(\mathcal{I}^n/(\mathcal{I}^n \cap \mathcal{J}))$. Thus the ideal of the strict transform of the saturation is the graded ideal $\oplus_{n>0}(\mathcal{I}^n \cap \mathcal{J}) \subset \mathcal{A}$. We need to show that this ideal defines the same closed subset of the blowup as the ideal $\pi^{-1}\pi_*(\mathcal{A}^+)$.

Since $\mathcal{A}^+ = \oplus_{n>0}\mathcal{I}^n$ we have that

$$\pi^{-1}\pi_*(\mathcal{A}^+) = \pi_*(\mathcal{A}^+)\cdot\mathcal{A} = \oplus_{n\geq 0}\mathcal{K}_n$$

where $\mathcal{K}_n = \sum_{k>0}\pi_*(\mathcal{I}^k)\mathcal{I}^{n-k}$. We need to show that

$$\sqrt{\oplus_{n>0}\mathcal{I}^n\cap\mathcal{J}} = \sqrt{\oplus_{n>0}\mathcal{K}_n}$$

in $\mathcal{A}$. Observe that $\pi_*(\mathcal{I}^k)\cdot\mathcal{O}_{\mathcal{X}}\subset\mathcal{I}^k$ and $\pi_*(\mathcal{I}^k)\cdot\mathcal{O}_{\mathcal{X}}\subset\pi_*(\mathcal{I})\cdot\mathcal{O}_{\mathcal{X}} = \mathcal{J}$ so $\pi_*(\mathcal{I}^k)\mathcal{I}^{n-k}\subset\mathcal{I}^n\cap\mathcal{J}$. Hence $\mathcal{K}_n\subset\mathcal{I}^n\cap\mathcal{J}$.

To establish the opposite inclusion, we work smooth-locally on $\mathcal{Y}$. We may thus assume that $\mathcal{Y} = \operatorname{Spec} A$ and $\pi_*\mathcal{I} = (f_1, f_2, \ldots, f_a)\subset A$. The ideal $\mathcal{I}^n\cap\mathcal{J}$ can locally be described as all functions in $\mathcal{J} = (f_1, f_2, \ldots, f_a)\cdot\mathcal{O}_{\mathcal{X}}$ that vanish to order at least n along $\mathcal{C}$. If $\operatorname{ord}_{\mathcal{C}}(f_i) = d_i$, that is, if $f_i\in\mathcal{I}^{d_i}\smallsetminus\mathcal{I}^{d_i+1}$, then for any n greater than all the d_i's, we have that $\mathcal{I}^n\cap\mathcal{J} = \sum_{i=1}^a f_i\cdot\mathcal{I}^{n-d_i}$. Since $f_i\in\pi_*(\mathcal{I}^{d_i})$ it follows that

$$\mathcal{I}^n\cap\mathcal{J}\subset\sum_{i=1}^a\pi_*(\mathcal{I}^{d_i})\mathcal{I}^{n-d_i}\subset\mathcal{K}_n.$$

Thus $\oplus_{n>0}(\mathcal{I}^n\cap\mathcal{J})\subset\sqrt{\oplus_{n>0}\mathcal{K}_n}$ which completes the proof. □

Proof of Proposition 3.10. To show that the natural morphism

$$\operatorname{Proj}^{\pi}_{\mathcal{X}}\mathcal{A}\to\operatorname{Proj}_{\mathcal{Y}}\pi_*\mathcal{A}$$

is a good moduli space morphism, we may, by [Alp13, Proposition 4.9(ii)], work locally in the smooth or fppf topology on $\mathcal{Y}$ and assume that $\mathcal{Y}$ is affine. In this case $\operatorname{Proj}\pi_*\mathcal{A}$ is the scheme obtained by gluing the affine schemes $\operatorname{Spec}(\pi_*\mathcal{A})_{(f)}$ as f runs through elements $f\in\pi_*\mathcal{A}^+$. Likewise, $\operatorname{Proj}^{\pi}_{\mathcal{X}}\mathcal{A}$ is the open set in $\operatorname{Proj}_{\mathcal{X}}\mathcal{A}$ obtained by gluing the $\mathcal{X}$-affine stacks $\operatorname{Spec}_{\mathcal{X}}\mathcal{A}_{(f)}$ as f runs through π_*A^+. It is thus enough to prove that

$$\operatorname{Spec}_{\mathcal{X}}\mathcal{A}_{(f)}\to\operatorname{Spec}_{\mathcal{Y}}(\pi_*\mathcal{A})_{(f)}$$

is a good moduli space morphism.

By [Alp13, Lemma 4.14] if $\mathcal{A}$ is a sheaf of coherent $\mathcal{O}_{\mathcal{X}}$-algebras then $\operatorname{Spec}_{\mathcal{X}}\mathcal{A}\to\operatorname{Spec}_{\mathcal{Y}}\pi_*\mathcal{A}$ is a good moduli space morphism and the diagram

$$\begin{array}{ccc}\operatorname{Spec}_{\mathcal{X}}\mathcal{A} & \longrightarrow & \mathcal{X}\\ \downarrow & & \downarrow{\scriptstyle\pi}\\ \operatorname{Spec}_{\mathcal{Y}}\pi_*\mathcal{A} & \longrightarrow & \mathcal{Y}\end{array}$$

is commutative. Since good moduli space morphisms are invariant under base change [Alp13, Proposition 4.9(i)] we see that

$$\operatorname{Spec}_{\mathcal{X}}\mathcal{A}\smallsetminus V(\pi^{-1}(\pi_*\mathcal{A}^+))\to\operatorname{Spec}_{\mathcal{Y}}\pi_*\mathcal{A}\smallsetminus V(\pi_*\mathcal{A}^+)$$

is a good moduli space morphism. Now $\operatorname{Proj}_{\mathcal{Y}} \pi_*\mathcal{A}$ is the quotient of $\operatorname{Spec}_{\mathcal{Y}} \pi_*\mathcal{A} \smallsetminus V(\pi_* A^+)$ by the action of $\mathbb{G}_m$ on the fibers over $\mathcal{Y}$. It is a coarse quotient since $\pi_*\mathcal{A}$ is not necessarily generated in degree 1. Likewise, $\operatorname{Proj}^{\pi}_{\mathcal{X}} \mathcal{A}$ is the quotient of $\operatorname{Spec}_{\mathcal{X}} \mathcal{A} \smallsetminus V(\pi^{-1}(\pi_*\mathcal{A}^+))$ by the action of $\mathbb{G}_m$ on the fibers over $\mathcal{X}$.

Since the property of being a good moduli space is preserved by base change, $\operatorname{Spec}_{\mathcal{X}} \mathcal{A}_f \to \operatorname{Spec}(\pi_*\mathcal{A})_f$ is a good moduli space morphism. This gives us the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}_{\mathcal{X}} \mathcal{A}_f & \xrightarrow{q_{\mathcal{X}}} & \operatorname{Spec}_{\mathcal{X}} \mathcal{A}_{(f)} \\ \big\downarrow{\scriptstyle \pi_{\mathcal{A}_f}} & & \big\downarrow{\scriptstyle \pi_{\mathcal{A}_{(f)}}} \\ \operatorname{Spec}_{\mathcal{Y}}(\pi_*\mathcal{A})_f & \xrightarrow{q_{\mathcal{Y}}} & \operatorname{Spec}_{\mathcal{Y}}(\pi_*\mathcal{A})_{(f)} \end{array}$$

where $\pi_{\mathcal{A}_f}$ is a good moduli space morphism and $q_{\mathcal{X}}$ and $q_{\mathcal{Y}}$ are coarse $\mathbb{G}_m$-quotients. Note that the natural transformation $M \to (q_* q^* M)_0$ is an isomorphism for $q = q_{\mathcal{X}}$ and $q = q_{\mathcal{Y}}$. Since $(\pi_{\mathcal{A}})_*$ is compatible with the grading, it follows that

$$(\pi_{\mathcal{A}_{(f)}})_* M = ((q_{\mathcal{Y}})_* (\pi_{\mathcal{A}_f})_* (q_{\mathcal{X}})^* M)_0$$

is a composition of right-exact functors, hence exact. It follows that $\pi_{\mathcal{A}_{(f)}}$ is a good moduli space morphism. □

Remark 3.11. Let $f \colon \mathcal{X}' \to \mathcal{X}$ be a projective morphism and let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism. Choose an f-ample line bundle $\mathcal{L}$. Then $\mathcal{X}' = \operatorname{Proj}_{\mathcal{X}} \bigoplus_{n\geq 0} f_*\mathcal{L}^n$. We obtain an open substack $\mathcal{X}'_{\mathcal{L}} := \operatorname{Proj}^{\pi}_{\mathcal{X}} \bigoplus_{n\geq 0} f_*\mathcal{L}^n$ of $\mathcal{X}'$ and a good moduli space morphism

$$\mathcal{X}'_{\mathcal{L}} \to \operatorname{Proj}_{\mathcal{Y}} \bigoplus_{n\geq 0} \pi_* f_* \mathcal{L}^n.$$

The open substack $\mathcal{X}'_{\mathcal{L}}$ is the locus where $f^*\pi^*\pi_* f_*\mathcal{L}^n \to \mathcal{L}^n$ is surjective for all sufficiently divisible n, and this typically depends on $\mathcal{L}$, see Example 3.16. This can be interpreted as variation of GIT on the level of stacks.

Proposition 3.12. *Let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a good moduli morphism and let $\mathcal{A}$ be a finitely generated graded $\mathcal{O}_{\mathcal{X}}$-algebra. Let $f \colon \mathcal{X}' := \operatorname{Proj}^{\pi}_{\mathcal{X}} \mathcal{A} \to \mathcal{X}$ be the saturated proj and let $\pi' \colon \mathcal{X}' \to \mathcal{Y}' := \operatorname{Proj}_{\mathcal{Y}} \pi_*\mathcal{A}$ be its good moduli space morphism.*

(1) *If π is properly stable, then π' is properly stable.*
(2) *If π is stable and $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{I}^n$ for an ideal $\mathcal{I}$, then π' is stable.*

More precisely, in (1), or in (2) under the additional assumption that $\mathcal{X}$ is reduced, the inclusion $\mathcal{X}' \subset \operatorname{Proj}_{\mathcal{X}} \mathcal{A}$ is an equality over $\mathcal{X}^s$ and $\mathcal{X}'^s$ contains $f^{-1}(\mathcal{X}^s)$. In (2), $\mathcal{X}'^s$ always contains $f^{-1}(\mathcal{X}^s \smallsetminus V(\mathcal{I}))$.

Proof. The question is smooth-local on $\mathcal{Y}$ so we can assume that $\mathcal{Y}$ is affine. We can also replace $\mathcal{Y}$ with $\pi(\mathcal{X}^s)$ and assume that $\mathcal{X} = \mathcal{X}^s$, that is, every stabilizer of $\mathcal{X}$ has the same dimension.

In the first case, π is a coarse moduli space. The induced morphism $\pi_{\mathcal{A}}\colon \operatorname{Spec}_{\mathcal{X}} \mathcal{A} \to \operatorname{Spec}_{\mathcal{Y}} \pi_*\mathcal{A}$ is then also a coarse moduli space. The image along $\pi_{\mathcal{A}}$ of $V(\mathcal{A}^+)$ is $V(\pi_*\mathcal{A}^+)$. Since $\pi_{\mathcal{A}}$ is a homeomorphism, $\sqrt{\pi^{-1}\pi_*\mathcal{A}^+} = \sqrt{\mathcal{A}^+}$. It follows that $\mathcal{X}' = \operatorname{Proj}_{\mathcal{X}} \mathcal{A}$.

In the second case, if in addition $\mathcal{X}$ is reduced, then π factors through a gerbe $g\colon \mathcal{X} \to \mathcal{X}_{\text{tame}}$ and a coarse moduli space $h\colon \mathcal{X}_{\text{tame}} \to \mathcal{Y}$ (Proposition 2.6). Since $\mathcal{I}^n = g^*g_*\mathcal{I}^n$, we conclude that $\operatorname{Proj}^{\pi}_{\mathcal{X}} \mathcal{A} = \big(\operatorname{Proj}^{h}_{\mathcal{X}_{\text{tame}}} g_*\mathcal{A}\big) \times_{\mathcal{X}_{\text{tame}}} \mathcal{X}$ and the question reduces to the first case.

In the second case, without the additional assumption on $\mathcal{X}$, let $\mathcal{U} := \mathcal{X}^s \setminus V(\mathcal{I})$. Then $\operatorname{Proj}_{\mathcal{X}} \mathcal{A} \to \mathcal{X}$ is an isomorphism over $\mathcal{U}$ and $\mathcal{Y}' := \operatorname{Proj}_{\mathcal{Y}} \pi_*\mathcal{A} \to \mathcal{Y}$ is an isomorphism over $\pi(\mathcal{U})$ so $\mathcal{U} \subset \mathcal{X}'^s$. Moreover, $\mathcal{U} \subset \mathcal{X}'$ is dense so $\mathcal{X}'$ is stable. □

The condition that $\mathcal{A}$ is a Rees algebra in (2) is not superfluous. In Example 3.16 below, we have a stable, but not properly stable, good moduli space $\pi\colon \mathcal{X} = B\mathbb{G}_m \to \mathcal{Y} = \operatorname{Spec} k$ and a saturated proj $\mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is not stable: $\mathcal{X}' = [A^1/\mathbb{G}_m] \to \mathcal{Y}' = \mathcal{Y} = \operatorname{Spec} k$.

Remark 3.13 (Deligne–Mumford stacks)**.** Theorem 3.5 and Proposition 3.10 are non-trivial statements even when $\mathcal{X}$ is Deligne–Mumford. In this case they identify the coarse space of a blowup along a sheaf of ideals $\mathcal{I}$ as $\operatorname{Proj}(\oplus \pi_*\mathcal{I}^k)$ (Proposition 3.12).

Remark 3.14 (Adequate Reichstein transformations)**.** Theorem 3.5 and Propositions 3.9 and 3.10 also hold for stacks with adequate moduli spaces with essentially identical arguments.

Example 3.15. Let $\mathcal{X} = [\mathbb{A}^2/\boldsymbol{\mu}_2]$ where $\boldsymbol{\mu}_2$ acts by $-(a,b) = (-a,-b)$. The coarse space of $\mathcal{X}$ is the cone $Y = \operatorname{Spec}[x^2, xy, y^2]$. Theorem 3.5 says that if we let X' be the blowup of $\mathbb{A}^2$ at the origin then the quotient $X'/\boldsymbol{\mu}_2$ is Proj of the graded ring $\oplus S_i$ where S_i is the monomial ideal in the invariant ring $k[x^2, xy, y^2]$ generated by monomials of degree $\lceil i/2 \rceil$.

Example 3.16. Let $\mathcal{X} = B\mathbb{G}_m$ and $f\colon \mathcal{X}' = \operatorname{Proj}_{\mathcal{X}}(V^0 \oplus V^a) \to \mathcal{X}$ where $a > 0$ and V is the tautological line bundle on $B\mathbb{G}_m$. Then $\mathcal{X}'$ has three points: two closed points P_1 and P_2 corresponding to the projections $V^0 \oplus V^a \to V^0$ and $V^0 \oplus V^a \to V^a$ and one open point in their complement. Let $\mathcal{O}(1)$ be the tautological f-ample line bundle and let $\mathcal{L} = \mathcal{O}(i) \otimes f^*V^j$ with $i \geq 1$ and $j \in \mathbb{Z}$. Then $\mathcal{X}'_{\mathcal{L}} = \operatorname{Proj}^{\pi}_{\mathcal{X}}\big(\oplus_{k\geq 0} \operatorname{Sym}^{ik}(\mathcal{O}_{\mathcal{X}} \oplus V^a) \otimes V^{jk}\big)$ and:

$$\mathcal{X}'_{\mathcal{L}} = \begin{cases} \mathcal{X} \smallsetminus P_2 & \text{if } j/i = 0 \\ \mathcal{X} \smallsetminus P_1 & \text{if } j/i = -a \\ \mathcal{X} \smallsetminus \{P_1, P_2\} & \text{if } -a < j/i < 0 \\ \emptyset & \text{if } j/i > 0 \text{ or } j/i < -a. \end{cases}$$

4. Equivariant Reichstein transforms and fixed points

The goal of this section is to prove the following theorem.

Theorem 4.1. *Let $X = \operatorname{Spec} A$ be a smooth affine scheme with the action of a connected linearly reductive group G. Then $R_G(X, X^G)^G = \emptyset$.*

Remark 4.2. By [CGP10, Proposition A.8.10] the fixed locus X^G is a closed smooth subscheme of X. Note that if G acts trivially, then $X^G = X$ and $R_G(X, X^G) = \emptyset$.

Remark 4.3. Theorem 4.1 is false if we drop the assumption that X is smooth. See Example 4.7 below.

4.1. The case of a representation.

In this section we prove Theorem 4.1 when $X = V$ is a representation of G.

Proposition 4.4. *Let V be a representation of a connected linearly reductive group G. Then $R_G(V, V^G)^G = \emptyset$.*

Proof. Decompose $V = V^0 \oplus V^m$ such that V^0 is the trivial submodule and V^m is a sum of non-trivial irreducible G-modules. Viewing V as a variety we write $V = V^0 \times V^m$. The fixed locus for the action of G on V is $V^0 \times \{0\}$, so the blowup of V along V^G is isomorphic to $V^0 \times \tilde{V}^m$ where $\tilde{V}^m$ is the blowup of V^m at the origin. Also, the saturation of V^G is $V^0 \times \operatorname{sat}_G\{0\}$ where $\operatorname{sat}_G\{0\}$ is the G-saturation of the origin in the representation V^m. Thus $R_G(V, V^G) = V^0 \times R_G(V^m, 0)$ so to prove the proposition we are reduced to the case that $V = V^m$; that is, V is a sum of non-trivial irreducible representations and $\{0\}$ is the only G-fixed point.

To prove the proposition we must show that every G-fixed point of the exceptional divisor $\mathbb{P}(V) \subset \tilde{V}$ is contained in the strict transform of

$$\operatorname{sat}_G\{0\} = \{v \in V : 0 \in \overline{Gv}\}.$$

Let $x \in \mathbb{P}(V)$ be a G-fixed point. The fixed point x corresponds to a G-invariant line $L \subset V$, inducing a character χ of G. Since the origin is the only fixed point, the character χ is necessarily non-trivial. Let λ be a 1-parameter subgroup such that $\langle \lambda, \chi \rangle > 0$. Then λ acts with positive weight α on L and thus $L \subset \mathrm{sat}_\lambda\{0\} = V_\lambda^+ \cup V_\lambda^-$ where

$$V_\lambda^+ = \{v \in V : \lim_{t \to 0} \lambda(t)v = 0\},$$
$$V_\lambda^- = \{v \in V : \lim_{t \to \infty} \lambda(t)v = 0\}$$

are the linear subspaces where λ acts with positive weights and negative weights respectively.

Since $\mathrm{sat}_G\{0\} \supset \mathrm{sat}_\lambda\{0\}$, it suffices to show that $x \in \mathbb{P}(V)$ lies in the strict transform of $\mathrm{sat}_\lambda\{0\}$. The blowup of $\mathrm{sat}_\lambda\{0\}$ in the origin intersects the exceptional divisor of $\tilde{V}$ in the (disjoint) linear subspaces $\mathbb{P}(V_\lambda^+) \cup \mathbb{P}(V_\lambda^-) \subset \mathbb{P}(V)$. Since $L \subset V_\lambda^+$ we see that our fixed point x is in $\mathbb{P}(V_\lambda^+)$ as desired. □

4.2. **Completion of the proof of Theorem 4.1.** The following lemma is a special case of [Lun73, Lemma on p. 96] and Luna's fundamental lemma [Lun73, p. 94]. For the convenience of the reader, we include the first part of the proof.

Lemma 4.5 (Linearization)**.** *Let $X = \operatorname{Spec} A$ be a smooth affine scheme with the action of a linearly reductive group G. If $x \in X^G$ is a closed fixed point, then there is a G-saturated affine neighborhood U of x and a G-equivariant* strongly étale *morphism $\phi\colon U \to T_xX$, with $\phi(x) = 0$. That is, the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\phi} & T_xX \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi} \\ U/\!\!/G & \xrightarrow{\psi} & T_xX/\!\!/G \end{array}$$

is cartesian and the horizontal arrows are étale.

Proof. Let $\mathfrak{m}$ be the maximal ideal corresponding to x. Since x is G-fixed the quotient map $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ is a map of G-modules. By the local finiteness of group actions there is a finitely generated G-submodule $V \subset \mathfrak{m}$ such that the restriction $V \to \mathfrak{m}/\mathfrak{m}^2$ is surjective. Since G is linearly reductive there is a summand $W \subset V$ such that $W \to \mathfrak{m}/\mathfrak{m}^2$ is an isomorphism of G-modules. Since $W \subset A$ we obtain a G-equivariant morphism $X \to T_xX = \mathrm{Spec}(\mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2))$ which is étale at x. Luna's fundamental lemma now gives an open saturated neighborhood U of x such that $U \to T_xX$ is strongly étale. □

Remark 4.6. Using Lemma 4.5 and arguing as in the proof of Proposition 4.4, we recover the result that X^G is smooth (Remark 4.2).

Completion of the Proof of Theorem 4.1. Every fixed point of $R_G(X, X^G)$ lies in the exceptional divisor $\mathbb{P}(N_{X^G}X)$. To show that $R_G(X, X^G)^G = \emptyset$ we can work locally in a neighborhood of a point $x \in X^G$. Thus we may assume (Lemma 4.5) that there is a strongly étale morphism $X \to T_xX$ yielding a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & T_xX \\ \downarrow & & \downarrow \\ X/\!\!/G & \longrightarrow & T_xX/\!\!/G. \end{array}$$

Hence $R_G(X, X^G)$ can be identified with the pullback of $R_G(T_xX, T_xX^G)$ along the morphism $X/\!\!/G \to T_xX/\!\!/G$ (Remark 3.3). By Proposition 4.4, $R_G(T_xX, T_xX^G)^G = \emptyset$ so therefore $R_G(X, X^G)^G = \emptyset$ as well. □

Example 4.7. Note that the conclusion of the theorem is false without the assumption that X is smooth. Let V be the 3-dimensional representation of $G = \mathbb{G}_m$ with weights $(-1, 1, 3)$. The polynomial $f = x_1x_3^2 + x_2^5$ is G-homogeneous of weight 5, so the subvariety $X = V(f)$ is G-invariant. Since all weights for the G-action are non-zero $X^G = (\mathbb{A}^3)^G = \{0\}$.

Let $\tilde{\mathbb{A}}^3$ be the blowup of the origin. The exceptional divisor is $\mathbb{P}(V)$ and has three fixed points $P_0 = [0 : 0 : 1], P_1 = [0 : 1 : 0], P_2 = [1 : 0 : 0]$. The exceptional divisor of $\tilde{X}$ is the projectivized tangent cone $\mathbb{P}(C_{\{0\}}X)$. Since $X = V(f)$ is a hypersurface and $x_1x_3^2$ is the sole term of lowest degree in f, we see that $\mathbb{P}(C_{\{0\}}X)$ is the subscheme $V(x_1x_3^2) \subset \mathbb{P}(V)$. This subvariety contains the 3 fixed points, so $\tilde{X}$ has 3 fixed points.

The saturation of 0 in X with respect to the G-action is $(X \cap V(x_1)) \cup (X \cap V(x_2, x_3))$. The intersection of the exceptional divisor with the strict transform of $X \cap V(x_1)$ is the projective subscheme $V(x_1, x_2^5)$ whose reduction is P_0.

The intersection of the exceptional divisor with the strict transform of $X \cap V(x_2, x_3)$ is the point $V(x_2, x_3) = P_2$. Thus the strict transform of the saturation of 0 in X does not contain all of the fixed points of $\tilde{X}$. Hence $R_G(X, X^G)^G \neq \emptyset$.

5. Reichstein transformations and the proof of Theorem 2.19

Lemma 5.1. *Let $\mathcal{X}$ be an Artin stack. Then the locus of points with maximal-dimensional stabilizer is a closed subset of $|\mathcal{X}|$.*

Proof. Since the representable morphism $I\mathcal{X} \to \mathcal{X}$ makes $I\mathcal{X}$ into an $\mathcal{X}$-group, the dimension of the fibers of the morphism is an upper semi-continuous function on $|\mathcal{X}|$. Thus the locus of points with maximal-dimensional stabilizer is closed. ☐

Proposition 5.2. *If $\mathcal{X}$ is smooth and admits a good moduli space, then the locus of points $\mathcal{X}^n$ with stabilizer of a fixed dimension n (with its reduced induced substack structure) is a locally closed smooth substack. In particular, the locus $\mathcal{X}^{\max}$ of points with maximal-dimensional stabilizer is a closed smooth substack.*

Proof. We may replace $\mathcal{X}$ with its open substack where every stabilizer has dimension at most n. Let x be a closed point of $\mathcal{X}^n = \mathcal{X}^{\max}$ and let G_x be its stabilizer group. Once again, by [AHR15, Theorem 2.9, Theorem 1.1] there is a smooth, affine scheme $U = \operatorname{Spec} A$ with an action of G_x and a cartesian diagram of stacks and moduli spaces

$$\begin{array}{ccc} [U/G_x] & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \pi \\ U/\!\!/G_x & \longrightarrow & \mathbf{X} \end{array}$$

where the horizontal arrows are étale and the image of $[U/G_x]$ contains x. It follows that $[U/G_x]^n$ is the inverse image of $\mathcal{X}^n$ under an étale morphism. In particular $[U/G_x]^n$ (with its reduced induced stack structure) is smooth if and only if $\mathcal{X}^n$ is smooth.

As in §2.1, let G_0 be the reduced identity component of G_x. Then $\dim G_0 = \dim G = n$ and any n-dimensional subgroup of G necessarily contains G_0.

Since G/G_0 is finite, hence affine, so is $BG_0 \to BG$. It follows that BG_0 is cohomologically affine, that is, G_0 is linearly reductive.

Since $n = \dim G_x$, a point of U has stabilizer dimension n if and only if it is fixed by the linearly reductive subgroup G_0. Thus $[U/G_x]^n = [U^{G_0}/G_x]$. By [CGP10, Proposition A.8.10] U^{G_0} is also smooth. Note that G_x acts on U^{G_0} because G_0 is a characteristic, hence normal, subgroup of G_x (Remark 2.7). ☐

Remark 5.3. Proposition 5.2 holds more generally for any smooth algebraic stack $\mathcal{X}$ such that the stabilizer of every closed point has linearly reductive identity component. Indeed, for any such point x there is an étale morphism $[U/G_x^0] \to \mathcal{X}$ and $[U/G_x^0]^n = [U^{G_0}/G_x]$ is smooth.

5.1. **Proof of Theorem 2.19.** Taking connected components, we may assume that $\mathcal{X}$ is irreducible. The proof of Theorem 2.19 then proceeds by descending induction on the maximum stabilizer dimension. Suppose that we constructed a stack $\mathcal{X}_k$ satisfying conclusions (1)–(4) of Theorem 2.19. By Proposition 5.2 $\mathcal{X}_k^{\max}$ is a closed smooth substack of $\mathcal{X}_k$. Hence $\mathcal{X}_{k+1} = R(\mathcal{X}_k, \mathcal{X}_k^{\max})$ is also smooth. If $\mathcal{X}_k^{\max} = \mathcal{X}_k$ then $\mathcal{X}_k$ is a gerbe over a tame stack by Proposition 2.6 and the process terminates. If this is the case, then $\mathcal{X}_k$ satisfies conclusion (5) by Remark 2.4. If $\mathcal{X}_k^{\max} \neq \mathcal{X}_k$ then the following Proposition shows that $\mathcal{X}_{k+1}$ also satisfies conclusions (1)–(4).

Proposition 5.4. *Let $\mathcal{X}$ be a smooth irreducible Artin stack with stable good moduli space morphism $\pi\colon \mathcal{X} \to X$. Let n be the maximum dimension of the fibers of $I\mathcal{X} \to \mathcal{X}$. If $\mathcal{X}^{\max} \neq \mathcal{X}$ then $\mathcal{X}' = R(\mathcal{X}, \mathcal{X}^{\max})$ is a smooth Artin stack with following properties.*

(1) *The morphism $f\colon \mathcal{X}' \to \mathcal{X}$ is an isomorphism over the dense open substack $\mathcal{X}^s$ and an open immersion over the complement of the smooth closed substack $\mathcal{X}^{\max}$.*
(2) *The stack $\mathcal{X}'$ has a good moduli space $\mathbf{X}'$ and the good moduli space morphism $\pi'\colon \mathcal{X}' \to \mathbf{X}'$ is stable.*
(3) *The induced morphism of good moduli spaces $\mathbf{X}' \to \mathbf{X}$ is proper and an isomorphism over the image of $\mathcal{X}^s$ in $\mathbf{X}$.*
(4) *Every point of $\mathcal{X}'$ has stabilizer of dimension strictly less than n.*

Proof. Assertion (1) follows from Remark 3.6 once we establish that the $\pi^{-1}(\pi(\mathcal{X}^{\max})) \subset (\mathcal{X}^s)^c$.

To do this we argue as follows. Since π is a stable good moduli space morphism $\mathcal{X}^s$ is a dense open substack of $\mathcal{X}$. By Proposition 2.14, $\mathcal{X}^{\max} \subset (\mathcal{X}^s)^c$. Since $\mathcal{X}^s$ is saturated, $(\mathcal{X}^s)^c$ is also saturated. Thus, $\pi^{-1}(\pi(\mathcal{X}^{\max})) \subset (\mathcal{X}^s)^c$.

That the stack $\mathcal{X}'$ has a good moduli space $\mathbf{X}'$ and that the morphism of good moduli spaces $\mathbf{X}' \to \mathbf{X}$ is projective and an isomorphism outside $\pi(\mathcal{X}^{\max})$ follow from Theorem 3.5. This proves (2) and (3) since we established that $\pi^{-1}(\pi(\mathcal{X}^{\max})) \subset (\mathcal{X}^s)^c$. Note that $\mathcal{X}'^s$ contains $f^{-1}(\mathcal{X}^s)$ since $\mathbf{X}' \to \mathbf{X}$ is an isomorphism over $\pi(\mathcal{X}^s)$.

We now prove assertion (4). By the local structure theorem [AHR15, Theorem 2.9] we may assume $\mathcal{X} = [U/G_x]$. Let G_0 be the reduced identity component of G_x. Then $[U/G_x]^{\max} = [U^{G_0}/G_x]$. To complete the proof we need to show that $R_{G_x}(U, U^{G_0})$ has no G_0-fixed point. By Theorem 4.1 we know that $R_{G_0}(U, U^{G_0})$ has no G_0-fixed points. We will prove (4) by showing that $R_{G_x}(U, U^{G_0}) = R_{G_0}(U, U^{G_0})$ as open subschemes of the blowup of U along U^{G_0}.

Consider the maps of quotients $U \xrightarrow{\pi_0} U/\!\!/G_0 \xrightarrow{q} U/\!\!/G_x$. If $U = \operatorname{Spec} A$ then these maps are induced by the inclusions of rings

$$A^{G_x} = (A^{G_0})^{(G_x/G_0)} \hookrightarrow A^{G_0} \hookrightarrow A.$$

Since the quotient group G_x/G_0 is a finite k-group scheme, $U/\!\!/G_0 = \operatorname{Spec} A^{G_0} \to U/\!\!/G_x = \operatorname{Spec}(A^{G_0})^{(G_x/G_0)}$ is a geometric quotient.

If $C \subset U$ is a G_x-invariant closed subset of U then its image in $U/\!\!/G_0$ is (G_x/G_0)-invariant, so it is saturated with respect to the quotient map $U/\!\!/G_0 \to U/\!\!/G_x$. Hence, as closed subsets of U, the saturations of C with respect to either the quotient map $U \to U/\!\!/G_0$ or to $U \to U/\!\!/G_x$ are the same[1]. It follows that if $C \subset U$ is G_x-invariant then $R_{G_x}(U, C)$ and $R_{G_0}(U, C)$ define the same open subset of the blowup of U along C. Since U^{G_0} is G_x-invariant we conclude that $R_{G_0}(U, U^{G_0}) = R_{G_x}(U, X^{G_0})$ as open subschemes of the blowup. □

Without the assumption that $\pi\colon \mathcal{X} \to \mathbf{X}$ is a stable good moduli space morphism, the conclusion in Proposition 5.4 that $\mathcal{X}' \to \mathcal{X}$ is birational can fail: it may happen that the saturation of $\mathcal{X}^{\max}$ equals $\mathcal{X}$ and thus that $\mathcal{X}' = \emptyset$. The following examples illustrate this.

Example 5.5. Let $\mathbb{G}_m$ acts on $X = \mathbb{A}^1$ with weight 1. The structure map $\mathbb{A}^1 \to \operatorname{Spec} k$ is a good quotient, so $\operatorname{Spec} k$ is the good moduli space of $\mathcal{X} = [\mathbb{A}^1/\mathbb{G}_m]$. The stabilizer of any point of $\mathbb{A}^1 - \{0\}$ is trivial, so $\mathcal{X}^{\max} = [\{0\}/\mathbb{G}_m]$ and the saturation of $\mathcal{X}^{\max}$ is all of $\mathcal{X}$. Hence $R(\mathcal{X}, \mathcal{X}^{\max}) = \emptyset$.

Example 5.6. Here is a non-toric example. Let $V = \mathfrak{sl}_2$ be the adjoint representation of $G = \mathrm{SL}_2(\mathbb{C})$. Explicitly, V can be identified with the vector space of traceless 2×2 matrices with SL_2-action given by conjugation. Let $V^{\mathrm{reg}} \subset V$ be the open set corresponding to matrices with non-zero determinant and set $X = V^{\mathrm{reg}} \times \mathbb{A}^2$. Let $\mathcal{X} = [X/G]$ where G acts by conjugation on the first factor and translation on the second factor. The map of affines $X \to \mathbb{A}^1 \smallsetminus \{0\}$ given by $(A, v) \mapsto \det A$ is a good quotient, so $\pi\colon \mathcal{X} \to \mathbb{A}^1 \smallsetminus \{0\}$ is a good moduli space morphism. However, the morphism π is not stable because the only closed orbits are the orbits of pairs $(A, 0)$.

The stabilizer of a point (A, v) with $v \neq 0$ is trivial and the stabilizer of $(A, 0)$ is conjugate to $T = \operatorname{diag}(t \;\; t^{-1})$ and $\mathcal{X}^{\max} = [(V^{\mathrm{reg}} \times \{0\})/G]$. Thus, $\pi^{-1}(\pi(\mathcal{X}^{\max})) = \mathcal{X}$ and $R(\mathcal{X}, \mathcal{X}^{\max}) = \emptyset$.

[1]The saturations with respect to the quotient maps come with natural scheme structures which are not the same. If $I \subset A$ is the ideal defining C in U then the saturation of C with respect to the quotient map $U \to U/\!\!/G_0$ is the ideal $I^{G_0}A$ while the ideal defining the saturation of C with respect to the quotient map $U \to U/\!\!/G_x$ is the ideal $I^{G_x}A$. While $I^{G_x}A \subset I^{G_0}A$, these ideals need not be equal.

6. Functoriality for strong morphisms

Let $\mathcal{X}$ and $\mathcal{Y}$ be Artin stacks with good moduli space morphisms, $\pi_{\mathcal{Y}}\colon \mathcal{Y} \to \mathbf{Y}$, $\pi_{\mathcal{X}}\colon \mathcal{X} \to \mathbf{X}$. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism and let $g\colon \mathbf{Y} \to \mathbf{X}$ be the induced morphism of good moduli spaces.

Definition 6.1. We say the morphism f is *strong* if the diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle \pi_{\mathcal{Y}}} & & \downarrow{\scriptstyle \pi_{\mathcal{X}}} \\ \mathbf{Y} & \xrightarrow{g} & \mathbf{X} \end{array}$$

is cartesian.

Note that a strong morphism is representable and stabilizer-preserving. A sharp criterion for when a morphism is strong can be found in [Ryd15].

Corollary 6.2. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a strong morphism of smooth Artin stacks with stable good moduli space morphisms $\mathcal{Y} \to \mathbf{Y}$ and $\mathcal{X} \to \mathbf{X}$. Let $\mathcal{Y}'$ and $\mathcal{X}'$ be the stacks produced by Theorem 2.19. Then there is a natural morphism $f'\colon \mathcal{Y}' \to \mathcal{X}'$ such that the diagram*

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

is cartesian.

Proof. The corollary follows by induction and the following proposition.

Proposition 6.3. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a strong morphism of smooth algebraic stacks with good moduli spaces. Then there is a natural morphism $f'\colon R(\mathcal{Y}, \mathcal{Y}^{\max}) \to R(\mathcal{X}, \mathcal{X}^{\max})$ such that the diagram*

$$\begin{array}{ccc} R(\mathcal{Y}, \mathcal{Y}^{\max}) & \xrightarrow{f'} & R(\mathcal{X}, \mathcal{X}^{\max}) \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array} \tag{6.3.1}$$

is cartesian.

Proof. We will prove that $f^{-1}(\mathcal{X}^{\max}) = \mathcal{Y}^{\max}$, that $(\mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X}) \times_{\mathcal{X}} \mathcal{Y} = \mathrm{Bl}_{\mathcal{Y}^{\max}} \mathcal{Y}$ and that the open subsets $R(\mathcal{X}, \mathcal{X}^{\max}) \times_{\mathcal{X}} \mathcal{Y}$ and $R(\mathcal{Y}, \mathcal{Y}^{\max})$ coincide. This gives a natural morphism f' such that (6.3.1) is cartesian.

These claims can be verified étale locally on $\mathbf{X}$ and $\mathbf{Y}$ at points of $\pi_{\mathcal{Y}}(\mathcal{Y}^{\max})$. Let $y \in |\mathcal{Y}^{\max}|$ and $x = f(y)$.

Since $\mathbf{Y} \to \mathbf{X}$ is of finite type we can, locally around $\pi_{\mathcal{Y}}(y)$, factor it as $\mathbf{Y} \hookrightarrow \mathbf{X} \times \mathbb{A}^n \to \mathbf{X}$ where the first map is a closed immersion and the second map is the smooth projection. By base change, this gives a local factorization of the morphism f as $\mathcal{Y} \overset{i}{\hookrightarrow} \mathcal{X} \times \mathbb{A}^n \overset{p}{\to} \mathcal{X}$.

Since $\mathcal{X} \times \mathbb{A}^n \to \mathcal{X}$ is flat and $(\mathcal{X} \times \mathbb{A}^n)^{\max} = \mathcal{X}^{\max} \times \mathbb{A}^n$ it follows from Remark 3.3 that $R(\mathcal{X} \times \mathbb{A}^n, (\mathcal{X} \times \mathbb{A}^n)^{\max}) = R(\mathcal{X}, \mathcal{X}^{\max}) \times \mathbb{A}^n$. We are therefore reduced to the case that the map f is a closed immersion. Since $\mathcal{X}$ and $\mathcal{Y}$ are smooth, f is necessarily a regular embedding.

We can apply Theorem [AHR15, Theorem 2.9 and Theorem 1.1] to reduce to the case that $\mathcal{X} = [X/G]$ where $G = G_y = G_x$ is a linearly reductive group and X is a smooth affine scheme. Let $Y = X \times_{\mathcal{X}} \mathcal{Y}$. Then $Y \to X$ is a regular closed immersion and $\mathcal{Y} = [Y/G]$. Since G is not necessarily smooth, it is not automatic that Y is smooth. But the fiber $Y \times_{\mathcal{Y}} BG_y = \operatorname{Spec} k$ is regular and $Y \times_{\mathcal{Y}} BG_y \to Y$ is a regular closed immersion since $\mathcal{Y}$ is smooth. It follows that Y is smooth over an open G-invariant neighborhood of the preimage of y. After replacing $\mathcal{Y}$ with an open saturated neighborhood of y [AHR15, Lemma 4.1], we can thus assume that Y is smooth.

Since $T_x X = T_y Y \times N_y(Y/X)$, we can slightly modify Lemma 4.5 to obtain the following commutative diagram of strong étale morphisms (after further shrinking of $\mathcal{X}$):

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ [T_y/G] & \longrightarrow & [T_x/G] \end{array}$$

Since f is strong, the action of the stabilizer G is trivial on the normal space N_y. Thus $(T_x)^{\max} = (T_y)^{\max} \times N_y$ and $T_x /\!\!/ G = T_y /\!\!/ G \times N_y$. The result is now immediate. □

7. Some corollaries

Recall that in characteristic zero, we have functorial resolution of singularities by blowups in smooth centers [BM08, Thm. 1.1]. To be precise, there is a functor BR which produces, for each reduced scheme X of finite type over a field of characteristic zero, a resolution of singularities $BR(X)$ which commutes with smooth morphisms. Here $BR(X) = \{X' \to \cdots \to X\}$ is a sequence of blowups in smooth centers with X' smooth. Also see [Kol07, Theorem 3.36] although that algorithm may involve centers that are singular [Kol07, Example 3.106].

Artin stacks can be expressed as quotients $[U/R]$ of groupoid schemes $s, t\colon R \rightrightarrows U$ with s and t smooth morphisms. Thus, the resolution functor BR extends uniquely to Artin stacks. In particular, for every reduced Artin stack $\mathcal{X}$ of finite type over a field of characteristic zero, there is a projective morphism $\tilde{\mathcal{X}} \to \mathcal{X}$, a sequence of blowups, which is an isomorphism over a dense open set. Similarly, if X is a scheme with an action of a group scheme G, then there is a sequence of blowups in G-equivariant smooth centers that resolves the singularities of X.

7.1. **Reduction to quotient stacks.** Suppose that $\mathcal{X}$ is a smooth Artin stack such that the good moduli space morphism $\pi\colon \mathcal{X} \to \mathbf{X}$ is properly stable. The end result of our canonical reduction of stabilizers (Theorem 2.19) is a smooth tame stack $\mathcal{X}_n$.

Corollary 7.1. *Let $\mathcal{X}$ be a smooth Artin stack with properly stable good moduli space. Suppose that $\mathcal{X}_n$ is Deligne–Mumford (automatic if* $\operatorname{char} k = 0$*) and that either $\mathcal{X}$ has generically trivial stabilizer or $\mathbf{X}$ is quasi-projective. Then*

1. *$\mathcal{X}_n$ is a quotient stack $[U/\operatorname{GL}_m]$ where U is an algebraic space.*
2. *If in addition, $\mathbf{X}$ is separated then U is separated and the action of* GL_m *on U is proper.*
3. *If in addition, $\mathbf{X}$ is a scheme, then so is U.*
4. *If in addition, $\mathbf{X}$ is a separated scheme then we can take U to be quasi-affine.*
5. *If in addition, $\mathbf{X}$ is projective then there is a projective variety X with a linearized action of a* GL_n *such that $X^s = X^{ss} = U$. Moreover, if* $\operatorname{char} k = 0$ *we can take X to be smooth.*

Proof. If the generic stabilizer of $\mathcal{X}$ is trivial, so is the generic stabilizer of $\mathcal{X}_n$. Hence by [EHKV01, Theorem 2.18] (trivial generic stabilizer) or [KV04, Theorem 2] (quasi-projective coarse space), $\mathcal{X}_n$ is a quotient stack. This proves (1).

If $\mathbf{X}$ is separated then $\mathcal{X}_n$ is a separated quotient stack so GL_m must act properly. This proves (2). (Note that if GL_m acts properly on U then U is necessarily separated. This also follows immediately since $U \to \mathcal{X}_n$ is affine.)

The morphism $U \to \mathbf{X}_n$ is affine. Indeed, there is a finite surjective morphism $V \to \mathcal{X}_n$ [EHKV01, Theorem 2.7] where V is a scheme and $V \to \mathbf{X}_n$ is finite and surjective, hence affine. It follows that $U \times_{\mathcal{X}_n} V \to \mathbf{X}_n$ is affine and hence $U \to \mathbf{X}_n$ is affine as well (Chevalley's theorem). One can also deduce this directly from $U \to \mathbf{X}_n$ being representable and cohomologically affine (Serre's theorem).

In particular, if $\mathbf{X}$ is a scheme, then so is $\mathbf{X}_n$ and U. This proves (3). Similarly, if $\mathbf{X}$ is a separated scheme then so is $\mathbf{X}_n$ and U. But U is a smooth separated scheme and thus has a G-equivariant ample family of line bundles. It follows that $\mathcal{X}_n$ has the resolution property and that we can choose U quasi-affine, see [Tot04, Theorems 1.1, 1.2] for further details. This proves (4).

We now prove (5). Since U is quasi-affine, it is also quasi-projective. By [Sum74, Theorem 1] there is an immersion $U \subset \mathbb{P}^N$ and a representation $\mathrm{GL}_m \to \mathrm{PGL}_{N+1}$ such that the GL_m-action on U is the restriction of the PGL_{N+1}-action on $\mathbb{P}^N$. Let X be the closure of U in $\mathbb{P}^N$. The action of G on X is linearized with respect to the line bundle $\mathcal{O}_X(1)$. Our statement follows from [MFK94, Converse 1.13].

Finally, if $\operatorname{char} k = 0$ then by equivariant resolution of singularities we can embed U into a non-singular projective G-variety X. □

Note that we only used that $\mathbf{X}_n$ is Deligne–Mumford to deduce that $\mathcal{X}_n$ is a quotient stack.

7.2. **Resolution of good quotient singularities.** Combining the main theorem with destackification of tame stacks [Ber17, BR14], we obtain the following result, valid in any characteristic.

Corollary 7.2 (Functorial destackfication of stacks with good moduli spaces)**.** *Let $\mathcal{X}$ be a smooth Artin stack with stable good moduli space morphism $\pi\colon \mathcal{X} \to \mathbf{X}$. Then there exists a sequence $\mathcal{X}_n \to \cdots \to \mathcal{X}_1 \to \mathcal{X}_0 = \mathcal{X}$ of birational morphisms of smooth Artin stacks such that*

1. *Each $\mathcal{X}_k$ admits a stable good moduli space $\pi_k\colon \mathcal{X}_k \to \mathbf{X}_k$.*
2. *The morphism $\mathcal{X}_{k+1} \to \mathcal{X}_k$ is either a Reichstein transform in a smooth center, or a root stack in a smooth divisor.*
3. *The morphism $\mathcal{X}_{k+1} \to \mathcal{X}_k$ induces a projective birational morphism of good moduli spaces $\mathbf{X}_{k+1} \to \mathbf{X}_k$.*
4. *$\mathbf{X}_n$ is a smooth algebraic space.*
5. *$\mathcal{X}_n \to \mathbf{X}_n$ is a composition of a gerbe $\mathcal{X}_n \to (\mathcal{X}_n)_{\mathrm{rig}}$ and a root stack $(\mathcal{X}_n)_{\mathrm{rig}} \to \mathbf{X}_n$ in an snc divisor $D \subset \mathbf{X}_n$.*

Moreover, the sequence is functorial with respect to strong smooth morphisms $\mathcal{X}' \to \mathcal{X}$, that is, if $\mathbf{X}' \to \mathbf{X}$ is smooth and $\mathcal{X}' = \mathcal{X} \times_{\mathbf{X}} \mathbf{X}'$, then the sequence $\mathcal{X}'_n \to \cdots \to \mathcal{X}'$ is obtained as the pull-back of $\mathcal{X}_n \to \cdots \to \mathcal{X}$ along $\mathcal{X}' \to \mathcal{X}$.

Proof. We first apply Theorem 2.19 to $\mathcal{X}$ and can thus assume that $\mathcal{X}$ is a gerbe over a tame stack $\mathcal{X}_{\mathrm{tame}}$. We then apply destackification to $\mathcal{Y} := \mathcal{X}_{\mathrm{tame}}$. This gives a sequence of smooth stacky blowups $\mathcal{Y}_n \to \mathcal{Y}_{n-1} \to \cdots \to \mathcal{Y}_1 \to \mathcal{Y}_0 = \mathcal{Y}$, such that $\mathbf{Y}_n$ is smooth and $\mathcal{Y}_n \to \mathbf{Y}_n$

factors as a gerbe $\mathcal{Y}_n \to (\mathcal{Y}_n)_{\mathrm{rig}}$ followed by a root stack $(\mathcal{Y}_n)_{\mathrm{rig}} \to \mathbf{Y}_n$ in an snc divisor. A smooth stacky blowup is either a root stack along a smooth divisor or a blowup in a smooth center. A blowup on a tame stack is the same thing as a Reichstein transformation.

We let $\mathcal{X}_k = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_k$. Then $\mathcal{X}_n \to \mathcal{Y}_n \to \mathcal{Y}_{\mathrm{rig}}$ is a gerbe and $\mathbf{X}_n = \mathbf{Y}_n$. □

Corollary 7.3 (Resolution of good quotient singularities)**.** *If X is a stable good moduli space of a smooth stack, then there exists a projective birational morphism $p\colon X' \to X$ where X' is a smooth algebraic space. The resolution is functorial with respect to smooth morphisms.*

7.3. **The case of singular stacks.** If $\mathcal{X} \to \mathbf{X}$ is a stable good moduli space morphism with $\mathcal{X}$ singular, Example 4.7 shows that we cannot expect to reduce the dimension of the stabilizers of $\mathcal{X}$ by Reichstein transforms. However, if $\mathcal{X}$ has a resolution of singularities then the following proposition implies that we can find a birational morphism $\mathcal{X}' \to \mathcal{X}$ from a non-singular stack $\mathcal{X}'$ with stable good moduli space morphism $\mathcal{X}' \to \mathbf{X}'$.

Proposition 7.4. *Let $\mathcal{X}$ be an integral Artin stack with stable good moduli space morphism $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$. Suppose that $\tilde{\mathcal{X}} \to \mathcal{X}$ is a projective birational morphism. Further assume that either*

(1) *$\mathcal{X}$ is properly stable, or*
(2) *$\tilde{\mathcal{X}} \to \mathcal{X}$ is a sequence of blowups.*

Then there exists an open substack $\mathcal{X}' \subset \tilde{\mathcal{X}}$ such that $\mathcal{X}'$ has a stable good moduli space $\mathcal{X}' \to \mathbf{X}'$ and the induced morphism of good moduli spaces is projective and birational.

Proof. Since $\tilde{\mathcal{X}} \to \mathcal{X}$ is projective we can write $\tilde{\mathcal{X}} = \operatorname{Proj}_{\mathcal{X}} \mathcal{A}$ for some graded sheaf $\mathcal{A}$ of finitely generated $\mathcal{O}_{\mathcal{X}}$-algebras. If $\tilde{\mathcal{X}} \to \mathcal{X}$ is a blowup, we choose $\mathcal{A}$ as the Rees algebra of this blowup. We treat a sequence of blowups by induction.

Let $\mathcal{X}' = \operatorname{Proj}^{\pi}_{\mathcal{X}} \mathcal{A}$. By Proposition 3.10, $\mathcal{X}' \to \mathbf{X}' = \operatorname{Proj}_{\mathbf{X}} \pi_*\mathcal{A}$ is a good moduli space morphism. By Proposition 3.12 it is stable. If $\tilde{\mathcal{X}} \to \mathcal{X}$ is an isomorphism over the open dense subset $U \subset \mathcal{X}$ (resp. a sequence of blowups with centers outside U), then $\mathbf{X}' \to \mathbf{X}$ is an isomorphism over the open dense subset $\pi(U \cap \mathcal{X}^s)$. □

Corollary 7.5. *Let $\mathcal{X}$ be an integral Artin stack with stable good moduli space $\mathcal{X} \xrightarrow{\pi} \mathbf{X}$ defined over a field of characteristic 0. There exists a quasi-projective birational morphism $\mathcal{X}' \to \mathcal{X}$ with the following properties.*

(1) *The stack $\mathcal{X}'$ is smooth and admits a good moduli space $\mathcal{X}' \xrightarrow{\pi'} \mathbf{X}'$.*
(2) *The stabilizers of $\mathcal{X}'$ have constant dimension equal to the minimum of the dimensions of the stabilizers of $\mathcal{X}$.*
(3) *The induced map of moduli spaces $\mathbf{X}' \to \mathbf{X}$ is projective and birational.*

Proof. Follows immediately from functorial resolution of singularities by a sequence of blowups, Proposition 7.4 and Theorem 2.19. $\square$

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