EXISTENCE AND PROPERTIES OF GEOMETRIC QUOTIENTS

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Abstract. In this paper, we study quotients of groupoids and coarse moduli spaces of stacks in a general setting. Geometric quotients are not always categorical, but we present a natural topological condition under which a geometric quotient is categorical. We also show the existence of geometric quotients of finite flat groupoids and give explicit local descriptions. Exploiting similar methods, we give an easy proof of the existence of quotients of flat groupoids with finite stabilizers. As the proofs do not use noetherian methods and are valid for general algebraic spaces and algebraic stacks, we obtain a slightly improved version of Keel and Mori's theorem.

Introduction

Quotients appear frequently in almost every branch of algebraic geometry, most notably in moduli problems. It has been known for a long time that the category of algebraic spaces is better suited for quotient problems than the category of schemes. In fact, even the easiest instances of quotients, such as quotients of schemes by free actions of finite groups, need not exist as schemes. In the category of algebraic spaces, on the other hand, quotients of free actions exist almost by definition.

In the category of schemes, it is easy to prove that geometric quotients are categorical and hence unique [GIT]. Surprisingly, this is not the case in the category of algebraic spaces as pointed out by Kollár [Kol97]. In particular, geometric quotients need not be unique.

The purpose of this paper is threefold. Firstly, we introduce a natural subclass of geometric quotients which we call *strongly geometric* and show that these quotients are categorical among algebraic spaces. Secondly, we show the existence of quotients by finite groups and give explicit étale-local descriptions. Thirdly, we show the existence of quotients of arbitrary flat groupoids with finite stabilizers, generalizing Keel and Mori's theorem [KM97].

A geometric quotient $X \to X/G$ is required to be *topological*, that is, the fibers should be the orbits and the quotient should have the quotient topology. A strongly geometric quotient is a geometric quotient that is *strongly topological*. This means that in addition to requiring that the quotient has the correct topology we also require that the equivalence relation determined by the quotient has the correct topology.

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Deligne has proved the existence of geometric quotients of separated algebraic spaces by arbitrary actions of finite groups, but without any published proof, cf. [Knu71, p. 183]. Deligne's idea was to use *fixed-point reflecting* étale covers to deduce the existence from the affine case. Matsuura independently proved the aforementioned result using the same method [Mat76]. Kollár developed Deligne's ideas in [Kol97] and showed that a geometric quotient of a proper group action is categorical in the category of algebraic spaces.

We extend Deligne and Kollár's results on fixed-point reflecting coverings. Using effective descent results for submersions [Ryd10] we then show that strongly geometric quotients are categorical in almost every situation (Theorem 3.16). We also settle Kollár's conjecture [Kol97, Rmk. 2.20] that geometric quotients are categorical among *locally separated* algebraic spaces.

We then proceed to show the existence of quotients of separated algebraic spaces by finite groups. Important to us is that we obtain an explicit étale-local description of the quotient space which does not follow from Keel and Mori's result. This local description is particularly nice for the symmetric product (Corollary 5.5) and such an explicit description is needed to deduce properties of $\mathrm{Sym}^n(X/S)$ from the affine case, as is done in [ES04, Ryd08b, RS10].

The techniques developed in this paper lead to a simple proof of Keel and Mori's theorem (Theorem 6.12).

Theorem. *Let S be an algebraic space and let $R \overset{s,t}{\rightrightarrows} X$ be a groupoid of algebraic spaces over S such that s and t are flat and locally of finite presentation and let $j=(s,t)\colon R \to X \times_S X$. If the stabilizer $j^{-1}\big(\Delta(X)\big) \to X$ is* finite*, then there is a uniform strongly geometric and categorical quotient $X \to X/R$ such that:*

(i) *$X/R \to S$ is separated (resp. quasi-separated) if and only if j is finite (resp. quasi-compact);*
(ii) *$X/R \to S$ is locally of finite type if S is locally noetherian and $X \to S$ is locally of finite type; and*
(iii) *$R \to X \times_{X/R} X$ is proper.*

Note that we are able to remove the finiteness hypotheses present in earlier proofs. In the original theorem [KM97, Thm. 1], it is assumed that the base scheme S is locally noetherian, that $X \to S$ is locally of finite type and that j is quasi-compact. In Conrad's treatment [Con05], the base scheme S is arbitrary, but $X \to S$ is locally of finite presentation and j is quasi-compact.

The hypothesis that the stabilizer is finite implies that the diagonal j is separated and locally quasi-finite [KM97, Lem. 2.7]. The stack $\mathscr{X} = [R \rightrightarrows X]$ is thus an algebraic stack with separated and locally quasi-finite diagonal. The quotient X/R is the *coarse moduli space* of $\mathscr{X}$. The stabilizer is a pull-back of the *inertia stack* $I_{\mathscr{X}} \to \mathscr{X}$. Rephrased in the language of stacks our generalization of the Keel–Mori theorem takes the following form.

Theorem. *Let $\mathscr{X}$ be an algebraic stack. A coarse moduli space $\pi\colon \mathscr{X} \to X$ such that π is separated exists if and only if $\mathscr{X}$ has finite inertia. In particular, any separated Deligne–Mumford stack has a coarse moduli space.*

This paper resulted from an attempt to understand fundamental questions about group actions and quotients. As a consequence, Sections 1–3 are written in a more general setting than needed for our generalization of the Keel–Mori theorem. For example, the results apply to non-flat equivalence relations [Kol11]. The impatient reader mainly interested in the Keel–Mori theorem is encouraged to go directly to §6.

The first step in the proof of the Keel–Mori theorem is to find a flat and locally quasi-finite presentation of a stack $\mathscr{X}$ with quasi-finite diagonal. This result is classical for noetherian stacks, cf. [Gab63, Lem. 7.2] and [KM97, Lem. 3.3]. The general case can be found in [Ryd11a, Thm. 7.1] and [SP, Lem. 04N0].

Assumptions and terminology. We work with general algebraic spaces and algebraic stacks as in [SP, Ryd11a, Ryd11b]. In particular, we do not assume, as in [Knu71, LMB00], that algebraic spaces are quasi-separated and that algebraic stacks have quasi-compact and separated diagonals. If the inertia stack is finite, then the diagonal is locally quasi-finite and separated but not necessarily quasi-compact. Nevertheless, the reader may, if so inclined, assume that all algebraic spaces and stacks have quasi-compact and separated diagonals. This would, however, necessitate tedious verifications that certain equivalence relations and groupoids have quasi-compact diagonals, e.g., in Theorem (3.19). The quasi-separatedness is typically better treated at subsequent stages using global methods.

We follow the terminology of EGA with one exception. As in [Ray70, LMB00] we mean by unramified a morphism locally of finite type and formally unramified but not necessarily locally of finite presentation. By a *presentation* of an algebraic stack $\mathscr{X}$, we always mean an algebraic space U together with a faithfully flat morphism $U \to \mathscr{X}$ locally of finite presentation.

Structure of the article. We begin with some general definitions and properties of quotients in §§1–2. Quotients are treated in full generality and we do not assume that the groupoids are flat. From Section 4 and on, all groupoids are flat and locally of finite presentation.

In §3 we generalize the results of Kollár [Kol97] on topological quotients, fixed-point reflecting morphisms and the *descent condition*. We show that universally open strongly geometric quotients satisfy the descent condition and hence are categorical. An important observation is that the fixed-point reflecting locus of a square étale morphism between groupoids is open if the stabilizer of the target groupoid is proper. This is the only place where the properness of the stabilizer is utilized.

In §4 we give an overview of well-known results on the existence and properties of quotients of affine schemes by finite flat groupoids. In this generality, the results are due to Grothendieck.

In §5 we use the results of §3 to deduce the existence of finite quotients for arbitrary algebraic spaces from the affine case treated in §4. The fixed-point reflecting étale cover with an essentially affine scheme that we need is constructed as a Weil restriction.

In §6 we restate the results of §3 in terms of stacks. We then deduce the existence of a coarse moduli space to any stack $\mathscr{X}$ with finite inertia stack, from the case where $\mathscr{X}$ has a finite flat presentation. Here, what we need is a fixed-point reflecting étale cover $\mathscr{W}$ of the stack $\mathscr{X}$ such that the cover $\mathscr{W}$ admits a finite flat presentation. After finding a quasi-finite flat presentation, this is accomplished using Hilbert schemes. We also give an example of a stack with quasi-finite, but not finite, inertia that does not have a coarse moduli space.

The existence results of §5 follow from the independent and more general results of §6 but the presentations of the quotients are quite different. In §5 we begin with an algebraic space X with an action of a finite groupoid and construct an essentially affine cover U with an action of the same groupoid. In §6 we begin by modifying the groupoid obtaining a quasi-finite groupoid action on an essentially affine scheme X. Then we take a covering which has an action of a finite groupoid.

In the appendices we have collected some results on effective descent of étale morphisms, AF-schemes, strong homeomorphisms and Weil restrictions.

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1. Groupoids and stacks

Let G be a group scheme over S, or more generally an algebraic group space (a group object in the category of algebraic spaces), and let X be an algebraic space over S. An action of G on X is a morphism $\sigma\colon G\times_S X \to X$ compatible with the group structure on G. The group action σ gives rise to a pre-equivalence relation $G\times_S X \underset{\pi_2}{\overset{\sigma}{\rightrightarrows}} X$, where π_2 is the second projection, i.e., a *groupoid in algebraic spaces*.

Definition (1.1). Let S be an algebraic space. An *S-groupoid in algebraic spaces* consists of two algebraic S-spaces, R and U, together with morphisms

(i) source and target $s,t\colon R\to U$,
(ii) composition $c\colon R\times_{t,U,s} R\to R$,
(iii) identity $e\colon U\to R$ and
(iv) inverse $i\colon R\to R$,

such that $\bigl(R(T),U(T),s,t,c,e,i\bigr)$ is a groupoid in sets for every affine S-scheme T in a functorial way. We will denote the groupoid by $R \underset{t}{\overset{s}{\rightrightarrows}} U$ or (R,U). A morphism of groupoids $f\colon (R,U)\to(R',U')$ consists of two morphisms $R\to R'$ and $U\to U'$, also denoted f, such that $f\colon (R(T),U(T))\to (R'(T),U'(T))$ is a morphism of groupoids in sets for every affine S-scheme T.

Remark **(1.2).** The inverse $i\colon R\to R$ is an involution such that $s=t\circ i$. Thus s has a property if and only if t has the same property. Let $G\to S$ be a group scheme acting on an algebraic space $X\to S$ and let $G\times_S X \underset{t}{\overset{s}{\rightrightarrows}} X$ be the associated groupoid. If $G\to S$ has a property stable under base change, then s and t have the same property.

Notation (1.3). By a *groupoid* we will always mean a groupoid in algebraic spaces. If $R \rightrightarrows U$ is a groupoid, then we let j be the diagonal morphism

$$j = (s,t)\colon R \to U \times_S U.$$

Let T be an S-space and let $g\colon T \to U$ be a morphism. The *stabilizer* of g, denoted $\operatorname{stab}(g)$, is the pull-back of j along $(g,g)\colon T \to U \times_S U$ and is a group space over T. If $h\colon T' \to T$, then $\operatorname{stab}(g \circ h) = \operatorname{stab}(g) \times_T T'$. The *stabilizer of the groupoid* is the universal stabilizer

$$\operatorname{stab}(U) := \operatorname{stab}(\mathrm{id}_U)\colon j^{-1}\big(\Delta(U)\big) \to U.$$

If $r\colon T \to R$ is a morphism, then conjugation by r induces an isomorphism $\operatorname{stab}(s \circ r) \to \operatorname{stab}(t \circ r)$. An inverse is given by conjugation by $i \circ r$.

We say that $R \rightrightarrows U$ is flat, locally of finite presentation, quasi-finite, etc. if s, or equivalently t, is flat, locally of finite presentation, quasi-finite, etc.

(1.4) *Topological equivalence relation induced by a groupoid* — Let $R \rightrightarrows U$ be a groupoid and let $u_1, u_2 \in |U|$ be points. We say that u_1 and u_2 are *R-equivalent*, written $u_1 \sim_R u_2$, if there exists a point $r \in |R|$ such that $s(r) = u_1$ and $t(r) = u_2$. It follows immediately from the groupoid axioms that $\sim_R$ is an equivalence relation on $|U|$ since $|R \times_U R| \to |R| \times_{|U|} |R|$ is surjective. The orbit $R(u)$ of $u \in |U|$ is the R-equivalence class of u. Explicitly we have that $R(u) = t(s^{-1}(u))$. The map of sets $|U| \to |U|/\sim_R$ is the coequalizer of $|R| \rightrightarrows |U|$. We say that a subset $W \subseteq |U|$ is *R-stable* if W is closed under $\sim_R$, or equivalently, if $s^{-1}(W) = t^{-1}(W)$ as subsets of $|R|$.

Remark **(1.5).** If $R \rightrightarrows U$ is a groupoid that is flat and locally of finite presentation, then the associated fppf stack $[R \rightrightarrows U]$ is algebraic [LMB00, Cor. 10.6] or [SP, Thm. 06DC]. In particular, when $G \to S$ is a flat group scheme of finite presentation acting on an algebraic space $X \to S$, the stack $[X/G] = [G \times_S X \rightrightarrows X]$ is algebraic.

Definition (1.6). Let $\mathscr{X}$ be an algebraic stack. The *inertia stack* $I_{\mathscr{X}} \to \mathscr{X}$ is the pull-back of the diagonal $\Delta_{\mathscr{X}/S}$ along the same morphism $\Delta_{\mathscr{X}/S}$. The inertia stack is independent of the base S.

Remark **(1.7).** Let $\mathscr{X}$ be an algebraic stack with a presentation $p\colon U \to \mathscr{X}$ and let $R = U \times_{\mathscr{X}} U$. Then $\mathscr{X} \cong [R \rightrightarrows U]$ and we have 2-cartesian diagrams

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S U \arrow[d, "p\times p"] \\ \mathscr{X} \arrow[r, "\Delta_{\mathscr{X}/S}"] & \mathscr{X} \times_S \mathscr{X} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{stab}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ I_{\mathscr{X}} \arrow[r] & \mathscr{X}. \end{tikzcd}$$

2. General remarks on quotients

2.1. **Topological, geometric and categorical quotients.** Recall that a morphism of topological spaces $f\colon X \to Y$ is *submersive* if it is surjective and Y has the quotient topology, i.e., a subset $Z \subseteq Y$ is open if and only if its inverse image $f^{-1}(Z)$ is open. Equivalently, $Z \subseteq Y$ is closed if and only if $f^{-1}(Z)$ is closed.

(2.1) *Constructible topology* — Let $f\colon X \to Y$ be a morphism of algebraic spaces. We say that f is submersive if the associated map of topological spaces $|f|\colon |X| \to |Y|$ is submersive [EGA$_{\mathrm{IV}}$, 15.7.8]. By slight abuse of notation we say that f^{cons} is submersive if the induced map of topological spaces $|f|^{\mathrm{cons}}\colon |X|^{\mathrm{cons}} \to |Y|^{\mathrm{cons}}$ is submersive. Here "cons" denotes the *constructible topology*, cf. [EGA$_{\mathrm{I}}$, 7.2.11] or [EGA$_{\mathrm{IV}}$, 1.9.13] and [Ryd10, §1]. If $\mathscr{X}$ is an algebraic stack (or an algebraic space) and $p\colon U \to \mathscr{X}$ is a presentation, then $W \subseteq |\mathscr{X}|$ is open in the constructible topology if and only if $p^{-1}(W)$ is open in the constructible topology. A subset $W \subseteq |\mathscr{X}|$ is constructible if and only if it is open and closed in the constructible topology.

The groupoid-averse reader may benefit from the comparison of the following definition with the corresponding Definition (6.1) for stacks.

Definition (2.2). Let $R \rightrightarrows X$ be an S-groupoid. A morphism $q\colon X \to Y$ is *equivariant* if $q \circ s = q \circ t$. Then $R \rightrightarrows X$ is also a Y-groupoid and $j\colon R \to X \times_S X$ factors through $X \times_Y X \hookrightarrow X \times_S X$. We denote the morphism $R \to X \times_Y X$ by $j_{/Y}$.

If $Y' \to Y$ is a morphism, we let $R' = R \times_Y Y'$ and $X' = X \times_Y Y'$. Then $q'\colon X' \to Y'$ is equivariant with respect to the groupoid $R' \rightrightarrows X'$. If a property of q is stable under flat base change $Y' \to Y$, then we say that the property is *uniform*. If it is stable under arbitrary base change, we say that it is *universal*. If q is an equivariant morphism, then we say that

(i) q is a *categorical quotient* (with respect to a full subcategory $\mathbf{C}$ of the category of algebraic spaces) if q is an initial object among equivariant morphisms (with target in $\mathbf{C}$). Concretely, this means that for any equivariant morphism $r\colon X \to Z$ (with $Z \in \mathbf{C}$) there is a unique morphism $Y \to Z$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "r"'] & Y \arrow[d, dashed] \\ & Z \end{tikzcd}$$

commutes.

(ii) q is a *Zariski quotient* if $|q|\colon |X| \to |Y|$ is the coequalizer of $|R| \rightrightarrows |X|$ in the category of topological spaces. Equivalently, the fibers of q are the R-orbits of $|X|$ and q is submersive.

(iii) q is a *constructible quotient* if $|q|^{\mathrm{cons}}\colon |X|^{\mathrm{cons}} \to |Y|^{\mathrm{cons}}$ is the coequalizer of $|R|^{\mathrm{cons}} \rightrightarrows |X|^{\mathrm{cons}}$ in the category of topological spaces. Equivalently the fibers of q are the R-orbits of $|X|$ and q^{cons} is submersive.

(iv) q is a *topological quotient* if it is both a universal Zariski quotient and a universal constructible quotient.

(v) q is a *strongly topological quotient* if it is a topological quotient and $j_{/Y}\colon R \to X \times_Y X$ is universally submersive.

(vi) q is a *geometric quotient* if it is a topological quotient and $\mathcal{O}_Y = (q_*\mathcal{O}_X)^R$, i.e., if the sequence of sheaves in the étale topology

$$\mathcal{O}_Y \longrightarrow q_*\mathcal{O}_X \overset{s^*}{\underset{t^*}{\rightrightarrows}} (q \circ s)_*\mathcal{O}_R \tag{2.2.1}$$

is exact.

(vii) q is a *strongly geometric quotient* if it is geometric and strongly topological.

Lemma (2.3). *Let $q\colon X \to Y$ be an equivariant morphism and let as before $j_{/Y}\colon R \to X \times_Y X$ denote the diagonal. The following are equivalent.*

(i) *For any field k and point $y\colon \mathrm{Spec}(k) \to Y$, the set $|X \times_Y \mathrm{Spec}(k)|$ has at least (resp. at most, resp. exactly) one $R \times_Y \mathrm{Spec}(k)$-orbit.*

(ii) *The morphism q is surjective (resp. the morphism $j_{/Y}$ is surjective, resp. the morphisms q and $j_{/Y}$ are surjective).*

If, in addition, q and $j_{/Y}$ are locally of finite type or integral, then these two conditions are equivalent to:

(iii) *For any algebraically closed field K, the map $q(K)\colon X(K)/R(K) \to Y(K)$ is surjective (resp. injective, resp. bijective).*

Proof. That (ii) $\Longrightarrow$ (i) follows from the definitions. We will show that (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (ii) without any assumptions on q and $j_{/Y}$. If $q(K)$ is surjective or if every fiber has at least one orbit, then clearly q is surjective. Assume instead that $q(K)$ is injective or that every fiber has at most one orbit. Choose a point $z\colon \mathrm{Spec}(K) \to X \times_Y X$ and let $y\colon \mathrm{Spec}(K) \to Y$ be the induced point of Y. We also let z denote the induced point $\mathrm{Spec}(K) \to X_y \times_{\mathrm{Spec}(K)} X_y$ and let $z_1, z_2\colon \mathrm{Spec}(K) \to X_y$ be the induced points of X_y. By assumption, there is a point $r \in |R_y|$, or even a point $r \in R_y(K)$, such that $s(r) = z_1$ and $t(r) = z_2$ in $|X_y|$. As the residue fields of z_1 and z_2 are K, we have that z is the unique point of $|X_y \times_{\mathrm{Spec}(K)} X_y|$ above $(z_1, z_2) \in |X_y| \times |X_y|$. It follows that the image of r along $j_y\colon R_y \to X_y \times_{\mathrm{Spec}(K)} X_y$ is z and we conclude that $j_{/Y}$ is surjective.

Finally, assume that q and $j_{/Y}$ are locally of finite type or integral and that K is an algebraically closed field. Then (ii) $\Longrightarrow$ (iii) since the surjectivity of q and $j_{/Y}$ implies the surjectivity of $q(K)$ and $j_{/Y}(K)$ respectively. □

The lemma gives the following alternative description of topological quotients:

$$\begin{aligned} q \text{ univ. Zariski} &\iff q \text{ univ. submersive and } j_{/Y} \text{ surjective} \\ q \text{ topological} &\iff q,\, q^{\mathrm{cons}} \text{ univ. submersive and } j_{/Y} \text{ surjective} \\ q \text{ strongly topological} &\iff q,\, q^{\mathrm{cons}},\, j_{/Y} \text{ univ. submersive.} \end{aligned}$$

Proposition (2.4). *Let $q\colon X \to Y$ be a topological quotient of the groupoid $R \rightrightarrows X$. If s has one of the properties: universally open, universally closed, quasi-compact; then so does q. For the first two properties, it is sufficient that q is a universal Zariski quotient.*

Proof. If s is universally open (resp. universally closed, resp. quasi-compact), then so is the projection $\pi_1\colon X \times_Y X \to X$ as $j_{/Y}$ is surjective. Since q and q^{cons} are universally submersive, we have that q is universally open (resp. universally closed, resp. quasi-compact) by [Ryd10, Prop. 1.7]. □

Remark **(2.5).** Examples of surjective morphisms that are universally submersive in the constructible topology are quasi-compact morphisms, universally open morphisms and morphisms locally of finite presentation [Ryd10, Prop. 1.6]. Therefore, by Proposition (2.4), a universal Zariski quotient is a topological quotient in the following cases:

(i) q is quasi-compact;
(ii) q is locally of finite presentation;
(iii) q is universally open;
(iv) s is proper (then q is universally closed with quasi-compact fibers, hence quasi-compact); or
(v) s is universally open (then q is universally open).

In most applications, s is universally open or even flat and locally of finite presentation so that q is also universally open.

Remark **(2.6).** The definitions of topological and geometric quotients given above are not standard but generalize other common definitions. In [Kol97], the conditions on topological and geometric quotients are slightly stronger: all algebraic spaces are assumed to be locally noetherian, and topological and geometric quotients are required to be locally of finite type.

In [GIT] the word topological is not defined explicitly but if we take topological to mean the first three conditions of a geometric quotient in [GIT, Def. 0.6], taking into account that iii) should be universally submersive, then a topological quotient is what here is called a universal Zariski quotient.

Remark **(2.7).** The condition, for a strongly topological quotient, that $j_{/Y}$ should be universally submersive is natural. Indeed, this ensures that the equivalence relation $X \times_Y X \hookrightarrow X \times_S X$ has the quotient topology induced from the groupoid. When Y is a *scheme*, or more generally a locally separated algebraic space, the monomorphism $X \times_Y X \hookrightarrow X \times_S X$ is an immersion. In this case, the topology on $X \times_Y X$ is induced by $X \times_S X$ and does not necessarily coincide with the quotient topology induced by $R \twoheadrightarrow X \times_Y X$.

If the groupoid has *proper* diagonal $j\colon R \to X \times_S X$, then every topological quotient is strongly topological. This explains why proper group actions are more tractable. An important class of strongly topological quotients are topological quotients such that $j_{/Y}\colon R \to X \times_Y X$ is proper.

Remark **(2.8).** A geometric quotient of schemes is always categorical in the category of schemes, cf. [GIT, Prop. 0.1, p. 4], but not necessarily in the category of algebraic spaces. As Kollár mentions in [Kol97, Rmk. 2.20] it is likely that every geometric quotient is categorical in the category of *locally separated* algebraic spaces. This is indeed the case, at least for universally open quotients, as shown in Theorem (3.16).

A natural condition, ensuring that a geometric quotient is categorical among all algebraic spaces, is that the *descent condition*, cf. Definition (3.6), is fulfilled. Universally open *strongly geometric* quotients satisfy the descent condition and are hence categorical, cf. Theorem (3.16).

Remark **(2.9).** Conversely a (strongly) topological and uniformly categorical quotient is (strongly) geometric. This is easily seen by considering equivariant maps $X \to \mathbb{A}^1_{\mathbb{Z}}$. Kollár has also shown that if G is an affine group, flat and locally of finite type over S acting *properly* on X, cf. Remark (2.11), such that a topological quotient exists, then a geometric quotient exists [Kol97, Thm. 3.13].

Proposition (2.10). *Let $R \rightrightarrows X$ be a groupoid and let $q\colon X \to Y$ be an equivariant morphism. Furthermore, let $f\colon Y' \to Y$ be a morphism and let $q'\colon X' \to Y'$ be the pull-back of q along f.*

(i) *If q is a topological quotient, then so is q'.*
(ii) *Assume that f is flat. If q is a geometric quotient, then so is q'.*
(iii) *Assume that f is covering in the fpqc topology (e.g., let f be faithfully flat and quasi-compact or locally of finite presentation). If q' is a topological (resp. geometric, resp. universal geometric) quotient, then so is q.*

In particular, a geometric quotient is always uniform. The statements remain valid if we replace "topological" and "geometric" with "strongly topological" and "strongly geometric".

Proof. Topological and strongly topological quotients are universal by definition. Part (iii) for topological and strongly topological quotients follows immediately as a morphism covering in the fpqc topology is submersive in both the Zariski and the constructible topology. As exactness of a sequence of sheaves is preserved by flat base change and can be checked fpqc locally, the statements on geometric quotients follow by considering the sequence (2.2.1). $\square$

2.2. **Separation properties.** Even if X is separated a quotient Y need not be. A sufficient criterion is that the groupoid has proper diagonal and a precise condition, for *schemes*, is that the image of the diagonal is closed.

Remark **(2.11).** Consider the following properties of the diagonal of a groupoid $R \rightrightarrows X$:

(i) the diagonal $j\colon R \to X \times_S X$ is proper;
(ii) the diagonal $j_{/Y}\colon R \to X \times_Y X$, with respect to an equivariant morphism $q\colon X \to Y$, is proper;
(iii) the diagonal $j\colon R \to X \times_S X$ is quasi-compact;
(iv) the stabilizer $\mathrm{stab}(X)\colon j^{-1}\big(\Delta(X)\big) \to X$ is proper; and
(v) the diagonal $j\colon R \to X \times_S X$ is a monomorphism.

If the groupoid is flat and locally of finite presentation, then, by the cartesian diagrams in Remark (1.7), these properties correspond to the following separation properties of the algebraic stack $\mathscr{X} = [R \rightrightarrows X]$:

(i) $\mathscr{X}$ is separated (over S);
(ii) $\mathscr{X} \to Y$ is separated;
(iii) $\mathscr{X}$ has quasi-compact diagonal (over S);
(iv) the inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ is proper; and
(v) $\mathscr{X}$ is an algebraic space.

If $q\colon X \to Y$ is a topological quotient, then, by Proposition (2.12) below, these properties imply that:

(i) Y is separated (over S); and
(iii) Y is quasi-separated (over S).

If (R, X) is the groupoid associated to a group action, then the group action is called *proper* if (i) holds. There is also the notion of a *separated* group action which means that the diagonal j has closed image.

Proposition (2.12). *Let $R \rightrightarrows X$ be a groupoid and let $q\colon X \to Y$ be a topological quotient.*

(i) *j is proper if and only if Y is separated and $j_{/Y}$ is proper.*
(ii) *j is quasi-compact if and only if Y is quasi-separated and $j_{/Y}$ is quasi-compact.*
(iii) *If $j_{/Y}$ is proper, then the stabilizer is proper.*
(iv) *If Y is locally separated, then Y is separated if and only if the image of j is closed.*

Proof. As q is a topological quotient we have that q and q^{cons} are universally submersive and that $j_{/Y}$ is surjective. The statements then easily follow from [Ryd10, Prop. 1.7] and the cartesian diagram

$$\begin{tikzcd} j\colon R \arrow[r, "j_{/Y}"] & X \times_Y X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_S X \arrow[d, "q\times q"] \\ & Y \arrow[r, hook, "\Delta_{Y/S}"] & Y \times_S Y. \end{tikzcd}$$

□

Remark **(2.13).** [Con05, Cor. 5.2] — Assume that there exists an equivariant morphism $q\colon X \to Y$ with respect to the groupoid (R, X) such that the diagonal $j_{/Y}$ is proper. Then the groupoid has a proper stabilizer. Moreover, if $r\colon X \to Z$ is a *categorical* quotient, then it follows that $j_{/Z}$ is proper as well. In particular, if $\mathscr{X}$ is an algebraic stack admitting a separated morphism $r\colon \mathscr{X} \to Y$ to an algebraic space Y, then $\mathscr{X}$ has a proper inertia stack and the categorical quotient $q\colon \mathscr{X} \to Z$, if it exists, is separated.

The Keel–Mori theorem asserts that, conversely, if $R \rightrightarrows X$ is a flat groupoid locally of finite presentation such that the stabilizer map $j^{-1}\bigl(\Delta(X)\bigr) \to X$ is *finite*, then there exists a geometric and categorical quotient $X \to Z$ and $j_{/Z}$ is proper, cf. Theorem (6.12).

As we will be particularly interested in finite groupoids, we make the following observations.

Proposition (2.14). *Let $R \rightrightarrows X$ be a* proper *groupoid, i.e., s and t are proper, and let $q\colon X \to Y$ be a topological quotient.*

(i) *The diagonal j is proper (resp. quasi-compact) if and only if X is separated (resp. quasi-separated).*
(ii) *The diagonal $j_{/Y}$ is proper (resp. quasi-compact) if and only if q is separated (resp. quasi-separated).*

Proof. As s is separated the section $e\colon X \to R$ is closed. Thus $\Delta_{X/S} = j \circ e\colon X \hookrightarrow R \to X \times_S X$ is proper (resp. quasi-compact) if j is so. Conversely, if X is separated (resp. quasi-separated), then, as $s = \pi_1 \circ j$ is proper, it follows that j is proper (resp. quasi-compact). (ii) follows from (i) considering the S-groupoid as a Y-groupoid. □

2.3. **Free actions.**

Definition (2.15). We say that a groupoid $R \rightrightarrows X$ is an *equivalence relation* if $j\colon R \to X \times_S X$ is a monomorphism. Let X be an algebraic space over S with an action of a group scheme $G \to S$. We say that G acts *freely* if the associated groupoid is an equivalence relation, i.e., if the morphism $j\colon G \times_S X \to X \times_S X$ is a monomorphism.

Theorem (2.16). *Let $R \rightrightarrows X$ be a flat and locally finitely presented equivalence relation. Then there is a universal strongly geometric and categorical quotient $q\colon X \to X/R$ in the category of algebraic spaces. Furthermore, q is the quotient of the equivalence relation in the category of fppf sheaves. Hence, q is flat and locally of finite presentation.*

Proof. Let X/R be the quotient sheaf of the equivalence relation $R \rightrightarrows X$ in the fppf topology. Then X/R is an algebraic space by [Art74, Cor. 6.3] as explained in [LMB00, Cor. 10.4] and [SP, Thm. 04S6]. As X/R is a categorical quotient in the category of fppf sheaves, it is a categorical quotient in the category of algebraic spaces. As taking the quotient sheaf commutes with arbitrary base change, it is also a universal categorical quotient.

The quotient $q\colon X \to X/R$ is flat and locally of finite presentation and thus universally submersive in both the Zariski and the constructible topology. As $j_{/(X/R)}\colon R \to X \times_{X/R} X$ is an isomorphism, we thus have that q is a strongly topological quotient. It is then a universal strongly geometric quotient by Remark (2.9). □

Corollary (2.17). *Let $G \to S$ be a group scheme, flat and locally of finite presentation over S. Let $X \to S$ be an algebraic space with a free action of G. Then there is a universal strongly geometric and categorical quotient $q\colon X \to X/G$ in the category of algebraic spaces and q is flat and locally of finite presentation.*

Proof. This follows immediately from Theorem (2.16). □

3. Fixed-point reflecting morphisms and the descent condition

Let X and Y be algebraic spaces with actions of a group G such that geometric quotients X/G and Y/G exist. If $f\colon X \to Y$ is any étale G-equivariant morphism, then, in general, the induced morphism $f/G\colon X/G \to Y/G$ is not étale. The notion of fixed-point reflecting morphisms was introduced to remedy this problem. Under a mild assumption, f/G is étale if f is fixed-point reflecting. Moreover, the existence of X/G then follows from the existence of Y/G and $X = X/G \times_{Y/G} Y$, that is, f is *strongly étale.* The mild assumption that we need is that $q\colon Y \to Y/G$ satisfies effective descent for étale morphisms. This is almost always the case, cf. Appendix A.

An interpretation of this section in terms of stacks is given in §6.

Remark **(3.1).** Recall that the stabilizer of a point $x\colon \mathrm{Spec}(k(x)) \to X$ is the fiber $\mathrm{stab}(x) = j^{-1}(x,x) = \mathrm{stab}(X) \times_X \mathrm{Spec}(k(x))$, which is a group scheme over $k(x)$, cf. Notation (1.3). Assume that $R \rightrightarrows X$ is the groupoid associated to the action of a group G on X. Then $\mathrm{stab}(x) \subseteq G$ is the subgroup of elements $g \in G$ such that $g(x) = x$ (as morphisms $\mathrm{Spec}(k(x)) \to X$, not as points in $|X|$).

Definition (3.2). Let $f\colon (R_X,X)\to(R_Y,Y)$ be a morphism of groupoids. We say that f is *square* if the two commutative diagrams

$$\begin{tikzcd} R_X \arrow[r, "f"] \arrow[d, "s"] & R_Y \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} R_X \arrow[r, "f"] \arrow[d, "t"] & R_Y \arrow[d, "t"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

are cartesian.

Note that if one of the diagrams above is cartesian, then so is the other diagram, since the involution of the groupoid exchanges the two diagrams.

Definition (3.3). Let $f\colon (R_X,X)\to(R_Y,Y)$ be a morphism of groupoids. We say that f is *fixed-point reflecting* at $x\in|X|$, abbreviated fpr, if the canonical morphism of stabilizers $\mathrm{stab}(x)\to\mathrm{stab}(f\circ x)$ is an isomorphism for some, or equivalently every, representative $x\colon \mathrm{Spec}(k)\to X$. We let $\mathrm{fpr}(f)\subseteq|X|$ denote the subset over which f is fixed-point reflecting.

The fixed-point reflecting set $\mathrm{fpr}(f)\subseteq|X|$ is R_X-stable. Indeed, if $r\colon \mathrm{Spec}(k)\to R_X$ is a point, then there is a commutative diagram

$$\begin{tikzcd} \mathrm{stab}(s\circ r) \arrow[r] \arrow[d, "\cong"] & \mathrm{stab}(s\circ f\circ r) \arrow[d, "\cong"] \\ \mathrm{stab}(t\circ r) \arrow[r] & \mathrm{stab}(t\circ f\circ r) \end{tikzcd}$$

so that f is fpr at $s\circ r$ if and only if f is fpr at $t\circ r$. Some authors prefer the terminology *stabilizer preserving* rather than fixed-point reflecting.

Remark **(3.4).** If $X\to Z$ is an equivariant morphism with respect to the groupoid (R_X,X) and $Z'\to Z$ is any morphism, then $(R_X\times_Z Z', X\times_Z Z')\to(R_X,X)$ is fpr.

The following proposition sheds some light over the importance of proper stabilizer.

Proposition (3.5). *Let $f\colon (R_X,X)\to(R_Y,Y)$ be a square morphism of groupoids such that $f\colon X\to Y$ is* unramified. *If the stabilizer* $\mathrm{stab}(Y)\to Y$ *is* universally closed, *then the subset* $\mathrm{fpr}(f)$ *is open in X.*

Proof. There are cartesian diagrams

$$\begin{tikzcd} \mathrm{stab}(X) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{stab}(Y)\times_Y X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & R_X \arrow[r] \arrow[d, "j"] \arrow[ddr, phantom, "\square"] & R_Y \arrow[dd, "j"] \\ X \arrow[r, hook, "\Delta_{X/Y}"] & X\times_Y X \arrow[r, hook] \arrow[d, "\pi_2"] \arrow[dr, phantom, "\square"] & X\times_S X \arrow[d] & \\ & X \arrow[r, hook] & Y\times_S X \arrow[r] & Y\times_S Y. \end{tikzcd}$$

The morphism $\varphi\colon \mathrm{stab}(Y)\times_Y X\to X$, which is the second column above, is closed. A point $x\in|X|$ is fpr if and only if x is not in $\varphi\bigl(|\mathrm{stab}(Y)\times_Y X|\setminus|\mathrm{stab}(X)|\bigr)$. As f is unramified, $\Delta_{X/Y}$ is an open immersion and hence $|\mathrm{stab}(Y)\times_Y X|\setminus|\mathrm{stab}(X)|$ is closed. Thus $\mathrm{fpr}(f)$ is open. $\square$

Definition (3.6). Let $R_X \rightrightarrows X$ be a groupoid and let $q\colon X \to Z_X$ be a topological quotient. We say that the quotient q satisfies the *descent condition* if for any étale, square and fixed-point reflecting morphism of groupoids $f\colon (R_W, W) \to (R_X, X)$, there exists an algebraic space Z_W and a cartesian square

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "q"] \\ Z_W \arrow[r] & Z_X \end{tikzcd} \tag{3.6.1}$$

where $Z_W \to Z_X$ is *étale*. We say that q satisfies the *weak descent condition* if the descent condition holds when restricted to morphisms f such that there is a cartesian square

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "q"] \\ Q' \arrow[r] & Q \end{tikzcd}$$

where $X \to Q$ is an equivariant morphism to a *locally separated* algebraic space Q and $Q' \to Q$ is an étale morphism.

Moreover, we say that q satisfies the (weak) descent condition for $\acute{\mathbf{E}}\mathbf{t}_P(X)$ for $P \in \{\mathrm{qc}, \mathrm{qsep}, \mathrm{cons}, \mathrm{sep}, \mathrm{sep}+\mathrm{qc}, \mathrm{fin}\}$ if the (weak) descent condition holds when restricted to étale morphisms $W \to X$ with property P, cf. Appendix A.

Remark **(3.7).** If a space Z_W exists as above, then $W \to Z_W$ is a topological quotient. If q is a geometric quotient, then $W \to Z_W$ is also a geometric quotient by Proposition (2.10). A morphism $f\colon W \to X$ is *strongly étale* if the induced morphism on categorical quotients $Z_W \to Z_X$ is étale and the diagram (3.6.1) is cartesian [GIT, p. 198]. Thus, the descent condition ensures that every fixed-point reflecting étale square morphism is strongly étale. In [KM97, 2.4] the existence of Z_W is not a part of the descent condition.

For most purposes, such as in the following proposition, it is sufficient to restrict the descent condition to *separated* étale morphisms.

Proposition (3.8) ([Kol97, Cor. 2.15])**.** *Let $R \rightrightarrows X$ be a groupoid and let $q\colon X \to Z$ be a geometric quotient satisfying the descent condition (resp. the weak descent condition) for the category $\acute{\mathbf{E}}\mathbf{t}_{\mathrm{sep}}(X)$ of separated étale morphisms $W \to X$. Then q is a categorical quotient (resp. a categorical quotient among locally separated algebraic spaces).*

Moreover, it is enough that q satisfies the (weak) descent condition for $\acute{\mathbf{E}}\mathbf{t}_{\mathrm{sep}+\mathrm{qc}}(X)$ if we restrict the discussion to the category of quasi-separated algebraic spaces.

Proof. Let Q be an algebraic space (resp. a locally separated algebraic space) and let $r\colon X \to Q$ be an equivariant morphism. We have to prove that there is a unique morphism $g\colon Z \to Q$ such that $r = g \circ q$. As geometric quotients commute with open immersions, we can assume that Q is quasi-compact. Let $Q' \to Q$ be an étale presentation with Q' an affine scheme. Let $X' = X \times_Q Q'$ so that $X' \to X$ is separated and étale and also quasi-compact if Q is quasi-separated. As q satisfies the descent condition (resp. the weak descent condition), there is a geometric quotient $q' \colon X' \to Z'$ such that $X' = X \times_Z Z'$.

As Q' is affine, the morphism $X' \to Q'$ is determined by $\Gamma(Q') \to \Gamma(X')$. Moreover, as q' is geometric, we have that the image of $\Gamma(Q')$ lies in $\Gamma(Z') = \Gamma(X')^{R'}$. The induced homomorphism $\Gamma(Q') \to \Gamma(Z')$ gives a morphism $g' \colon Z' \to Q'$ such that $r' = g' \circ q'$ and this is the only morphism g' with this property. By étale descent, the morphism g' descends to a unique morphism $g \colon Z \to Q$ such that $r = g \circ q$. □

Definition (3.9). Let $f \colon X \to Y$ be a morphism of algebraic spaces. Let $W \subseteq |X|$ be a subset. We say that f is universally injective (resp. universally bijective, resp. a universal homeomorphism, resp. universally submersive) over W if for any cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

we have that the map $f'|_{g'^{-1}(W)} \colon g'^{-1}(W) \to |Y'|$ is injective (resp. bijective, resp. a homeomorphism, resp. submersive).

Lemma (3.10). *Let $f \colon X \to Y$ be an unramified morphism of algebraic spaces. Let $W \subseteq |X|$ be a subset such that f is a universal homeomorphism over W. Then $W \subseteq |X|$ is open.*

Proof. Let $\pi_1, \pi_2 \colon X \times_Y X \to X$ be the projections and let $W_1 = \pi_1^{-1}(W)$. Then, by assumption, we have that $\pi_2|_{W_1} \colon W_1 \to |X|$ is a homeomorphism. The diagonal $\Delta_f \colon X \to X \times_Y X$ is an open immersion. Thus $W_1 \cap \Delta_f(X)$ is open in W_1. It follows that $W = \pi_2\big(W_1 \cap \Delta_f(X)\big)$ is open in X. □

We will now describe the descent condition in terms of effective descent of étale morphisms.

Setup (3.11). Let $f \colon (R_W, W) \to (R_X, X)$ be an étale and square morphism of groupoids and let $X \to Z_X$ be a topological quotient. Consider the following diagram

(3.11.1)
$$\begin{tikzcd} & W \times_{Z_X} X \arrow[dd, dashed, "\varphi"] \arrow[dr, "f \times \mathrm{id}_X"] & \\ R_W \arrow[ur, "(s, f\circ t)"] \arrow[dr, "(f\circ s, t)"'] & & X \times_{Z_X} X. \\ & X \times_{Z_X} W \arrow[ur, "\mathrm{id}_X \times f"'] & \end{tikzcd}$$

As $X \to Z_X$ is topological, the two morphisms on the left are surjective. To give a $X \times_{Z_X} X$-morphism

$$\varphi \colon W \times_{Z_X} X \to X \times_{Z_X} W$$

is equivalent to specifying its graph

$$\Gamma_\varphi \hookrightarrow W \times_{Z_X} W \cong (W \times_{Z_X} X) \times_{X \times_{Z_X} X} (X \times_{Z_X} W),$$

which is an open subspace as $f\colon W \to X$ is étale. The morphism φ makes the diagram above commutative if and only if

$$j_{W/Z_X} = (s,t)\colon R_W \to W \times_{Z_X} W$$

factors through Γ_φ. Let $\Gamma_{W/Z_X} \subseteq |W \times_{Z_X} W|$ denote the image of j_{W/Z_X}. Then φ makes the diagram commutative if and only if $\Gamma_{W/Z_X} \subseteq \Gamma_\varphi$, or equivalently, if and only if $\Gamma_{W/Z_X} = \Gamma_\varphi$ since $(s, f \circ t)$ is surjective. In particular, there is at most one morphism φ making the diagram commutative.

Proposition (3.12). *Let $f\colon W \to X$ and $X \to Z_X$ be as in Setup* (3.11)*. There is a one-to-one correspondence between, on the one hand, topological quotients $W \to Z_W$ together with an étale morphism $Z_W \to Z_X$ such that the diagram* (3.6.1) *is cartesian and, on the other hand,* effective *descent data φ for $f\colon W \to X$ fitting into the diagram* (3.11.1)*. In particular, as there is at most one such morphism φ, there is at most one quotient Z_W in the descent condition* (3.6)*.*

Assume that f is fixed-point reflecting. Then the following holds.

(i) *The morphisms*

$$f \times \mathrm{id}_W\colon W \times_{Z_X} W \to X \times_{Z_X} W$$
$$\mathrm{id}_W \times f\colon W \times_{Z_X} W \to W \times_{Z_X} X$$

are universally bijective over the subset Γ_{W/Z_X}.

(ii) *The subset Γ_{W/Z_X} is open if and only if $f \times \mathrm{id}_W$ (or $\mathrm{id}_W \times f$) is universally submersive over Γ_{W/Z_X}.*

(iii) *An isomorphism φ fitting into the diagram* (3.11.1) *exists if and only if Γ_{W/Z_X} is open and then Γ_{W/Z_X} is the graph of φ.*

Proof. There is a one-to-one correspondence between étale morphisms $Z_W \to Z_X$ and étale morphisms $f\colon W \to X$ together with an effective descent datum φ, i.e., an isomorphism $\varphi\colon W \times_{Z_X} X \to X \times_{Z_X} W$ over $X \times_{Z_X} X$ satisfying the cocycle condition. This is because q is universally submersive and hence a morphism of descent for étale morphisms, cf. Proposition (A.1). Given a quotient Z_W of $R_W \rightrightarrows W$ together with an étale morphism $Z_W \to Z_X$ as in Definition (3.6), we have that the corresponding isomorphism φ is the composition of the canonical isomorphisms $W \times_{Z_X} X \cong W \times_{Z_W} W \cong X \times_{Z_X} W$ and that φ fits into the commutative diagram (3.11.1).

(i): By symmetry, it is enough to show that the first morphism is universally bijective over Γ_{W/Z_X}. Let $(x_1, w_2)\colon \mathrm{Spec}(k) \to X \times_{Z_X} W$ be a point. We have to show that there is exactly one lifting to $W \times_{Z_X} W$ in the image of R_W. As $R_W \to X \times_{Z_X} W$ is surjective, we can enlarge k and assume that there exists a lifting $r\colon \mathrm{Spec}(k) \to R_W$. We are free to enlarge k, so it is enough to show that any two liftings to R_W have the same image in $W \times_{Z_X} W$.

Let (w_1, w_2) be the image of r in $W \times_{Z_X} W$. The liftings of (w_1, w_2) to R_W are in bijection, via conjugation by r, to k-points of the stabilizer group scheme $\mathrm{stab}(w_1)$. On the other hand, as $f\colon W \to X$ is square, we have that liftings of (x_1, w_2) to R_W are in bijection with k-points of $\mathrm{stab}(x_1)$. As f is fixed-point reflecting, these two sets are canonically bijective so that every lifting of (x_1, w_2) to R_W comes from a lifting of (w_1, w_2) to R_W. Hence (w_1, w_2) is the unique lifting of (x_1, w_2) in the image of R_W.

(ii) follows from (i) and Lemma (3.10).

(iii): In the setup we have seen that if a morphism φ exists, then necessarily $\Gamma_{W/Z_X} = \Gamma_\varphi$ is open. Conversely, if Γ_{W/Z_X} is open, then $\Gamma_{W/Z_X} \to W \times_{Z_X} X$ and $\Gamma_{W/Z_X} \to X \times_{Z_X} W$ are universally bijective by (i) and hence isomorphisms [SGA$_1$, Exp. IX, Cor. 1.6]. It follows that Γ_{W/Z_X} is the graph of an isomorphism φ fitting into diagram (3.11.1). □

Corollary (3.13). *Let $R \rightrightarrows X$ be a groupoid and let $q\colon X \to Z$ be a topological quotient. Also let $g\colon Z' \to Z$ be a morphism and let $q'\colon X' \to Z'$ be the pull-back of q along g. Then q' is a topological quotient by Proposition* (2.10). *Let $P \in \{\mathrm{qc}, \mathrm{qsep}, \mathrm{cons}, \mathrm{sep}, \mathrm{sep} + \mathrm{qc}, \mathrm{fin}\}$ be a property of étale morphisms.*

(i) *Assume that g is étale and has property P. If q satisfies the descent condition with respect to $\mathbf{\acute{E}t}_P$, then so does q'.*
(ii) *Assume that g is covering in the fppf topology. Then q satisfies the descent condition for $\mathbf{\acute{E}t}_P$, if q' satisfies the descent condition for $\mathbf{\acute{E}t}_P$.*

Proof. To prove (i), let $f'\colon W' \to X'$ be an étale, square and fpr morphism of groupoids with property P. Then $W' \to X' \to X$ is also étale, square and fpr with property P. As q satisfies the descent condition there is a topological quotient $W' \to Z_{W'}$ and an étale morphism $Z_{W'} \to Z$.

We need to construct a Z-morphism $Z_{W'} \to Z'$ such that $W' = Z_{W'} \times_{Z'} X'$, that is, we need to descend the morphism f' along q. The local description of this morphism is $q' \circ f'\colon W' \to Z'$ and since q is a morphism of descent, it is enough to verify that $q' \circ f' \circ \pi_1, q' \circ f' \circ \pi_2\colon W' \times_{Z_{W'}} W' \to Z'$ coincide. Equivalently, it is enough to show that $f' \times f'\colon W' \times_{Z_{W'}} W' \to X' \times_Z X'$ factors through the open subspace $X' \times_{Z'} X'$. This follows from the commutative diagram

$$\begin{tikzcd}
R_{W'} \arrow[r] \arrow[d] & W' \times_{Z_{W'}} W' \arrow[r, "f' \times f'"] & X' \times_Z X' \\
R_{X'} \arrow[r] & X' \times_{Z'} X' \arrow[ur, hook] &
\end{tikzcd}$$

as $R_{W'} \to W' \times_{Z_{W'}} W'$ is surjective.

(ii) follows from an easy application of fppf descent, taking into account that, by Proposition (3.12), the quotient Z_W figuring in the descent condition is unique. □

The following theorem generalizes [Kol97, Thm. 2.14]. In Kollár's result, the group action has to be proper, i.e., the diagonal $j\colon R \to X \times_S X$ has to be proper, so that every topological quotient is strongly topological.

Theorem (3.14). *Let $R \rightrightarrows X$ be a groupoid and let $q\colon X \to Z$ be a topological quotient (resp. a strongly topological quotient) such that q satisfies* effective *descent for* $\acute{\mathbf{E}}\mathbf{t}_P(X)$*. Then* q *satisfies the weak descent condition (resp. the descent condition) for* $\acute{\mathbf{E}}\mathbf{t}_P(X)$*.*

Proof. Let $f\colon W \to X$ be an étale, square and fpr morphism with property P. As q satisfies effective descent for f it is, by Proposition (3.12), enough to show that the image $\Gamma_{W/Z}$ of $j_{W/Z}\colon R_W \to W \times_Z W$ is open, or equivalently, that $f \times \mathrm{id}_W\colon W \times_Z W \to X \times_Z W$ is universally submersive over $\Gamma_{W/Z}$. If q is a strongly topological quotient, then $(f \times \mathrm{id}_W) \circ j_{W/Z}$ is universally submersive since it is the pull-back of the universally submersive morphism $j_{/Z}\colon R_X \to X \times_Z X$. Thus $f \times \mathrm{id}_W$ is universally submersive over $\Gamma_{W/Z}$. This shows that q satisfies the descent condition.

Now, let $q\colon X \to Z$ be a topological quotient that satisfies effective descent for $\acute{\mathbf{E}}\mathbf{t}_P(X)$ and let $r\colon X \to Q$ be an equivariant morphism such that Q is a locally separated algebraic space. Then $X \times_Q X \hookrightarrow X \times_S X$ is an *immersion*. Thus

$$\pi_1\colon (X \times_Z X) \times_{X \times_S X} (X \times_Q X) \to X \times_Z X$$

is an immersion. Moreover, as $R_X \to X \times_Z X$ is surjective we have that π_1 is surjective. Hence we obtain a monomorphism $(X \times_Z X)_{\mathrm{red}} \to X \times_Q X$ over $X \times_S X$.

Let $Q' \to Q$ be an étale morphism with property P and let $W = X \times_Q Q'$ and $R_W = R_X \times_Q Q'$ so that $(R_W, W) \to (R_X, X)$ is fixed-point reflecting and square étale, cf. Remark (3.4). To show that q satisfies the weak descent condition, it is by Proposition (3.12) enough to show that $\Gamma_{W/Z} \subseteq |W \times_Z W|$ is open. We have a cartesian diagram

$$\begin{tikzcd}
(R_W)_{\mathrm{red}} \arrow[r, "(s,f\circ t)"'] \arrow[rr, bend left, "{(s,t)}"] \arrow[dd] & (W \times_Z X)_{\mathrm{red}} \arrow[r, hook] \arrow[d, hook, "\psi"] \arrow[dr, phantom, "\square"] & W \times_{Q'} W \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & Q' \arrow[d, hook, "\Delta_{Q'/Q}"] \\
\square & (W \times_Z W)_{\mathrm{red}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & W \times_Q W \arrow[r] \arrow[d] & Q' \times_Q Q' \\
(R_X)_{\mathrm{red}} \arrow[r, "j_{/Z}"] & (X \times_Z X)_{\mathrm{red}} \arrow[r, hook] & X \times_Q X.
\end{tikzcd}$$

As $\Delta_{Q'/Q}$ is an open immersion so is ψ and as $(s, f \circ t)$ is surjective it follows that the image of $(j_{W/Z})_{\mathrm{red}} = \psi \circ (s, f \circ t)$ is the open subspace $\Gamma_{W/Z} = \psi(W \times_Z X)$. □

Theorem (3.15). *Let* $R \rightrightarrows X$ *be a groupoid and let* $q\colon X \to Z$ *be a topological (resp. strongly topological) quotient.*

(i) *If* q *is universally open, then* q *satisfies the weak descent (resp. descent) condition for* $\acute{\mathbf{E}}\mathbf{t}_{\mathrm{qsep}}(X)$*.*
(ii) *If* q *is proper or integral, then* q *satisfies the weak descent (resp. descent) condition for* $\acute{\mathbf{E}}\mathbf{t}(X)$*.*

Proof. This follows from Theorem (3.14) as q is a morphism of effective descent by Theorem (A.2). □

The following theorem generalizes [Kol97, Cor. 2.15] and answers the conjecture in [Kol97, Rmk. 2.20].

Theorem (3.16). *Let $R \rightrightarrows X$ be a groupoid and let $q\colon X \to Z$ be a geometric (resp. strongly geometric) quotient. Consider the following conditions on the quotient q:*

(1a) *q is universally open;*
(1b) *q is proper or integral;*
(2a) *q is quasi-compact and Z is locally noetherian; and*
(2b) *q is quasi-compact and universally subtrusive (e.g., univ. closed).*

If (1a) *or* (1b) *holds, then q is a categorical quotient among locally separated algebraic spaces (resp. a categorical quotient). If* (2a) *or* (2b) *holds, then q is a categorical quotient among locally separated and quasi-separated algebraic spaces (resp. a categorical quotient among quasi-separated algebraic spaces).*

Proof. If (1a) or (1b) holds, then q is a morphism of effective descent for étale and separated morphisms, by Theorem (A.2). Likewise, if (2a) or (2b) holds, then q is a morphism of effective descent for étale, quasi-compact and separated morphisms. Therefore q satisfies the weak descent condition (resp. descent condition) for $\mathbf{\acute{E}t}_{\mathrm{sep}}(X)$ or $\mathbf{\acute{E}t}_{\mathrm{sep+qc}}(X)$, by Theorem (3.14), and is a categorical quotient by Proposition (3.8). $\square$

Definition (3.17). An equivariant morphism is called a GC quotient if it is a strongly geometric quotient that satisfies the descent condition for separated étale morphisms uniformly. As a GC quotient is categorical by Proposition (3.8), we will speak about *the* GC quotient when it exists.

Remark **(3.18).** The definition of GC quotient given by Keel and Mori differs slightly from ours. In [KM97] it simply means a geometric and uniform categorical quotient. However, every quotient $q\colon X \to Y$ appearing in [KM97] is such that $j_{/Y}\colon R \to X \times_Y X$ is proper, hence every quotient is strongly topological. As every groupoid in *loc. cit.* is flat and locally of finite presentation, it follows that every quotient is universally open. Every GC quotient in [KM97] thus satisfies the descent condition for $\mathbf{\acute{E}t}_{\mathrm{sep}}$ uniformly by Theorem (3.15).

Theorem (3.19) ([Kol97, Cor. 2.17])**.** *Let $f\colon (R_W, W) \to (R_X, X)$ be a surjective, étale, separated, square and fpr morphism of groupoids and let $Q = W \times_X W$. Assume that a* GC *quotient $W \to Z_W$ of (R_W, W) exists. Then* GC *quotients $Q \to Z_Q$ and $X \to Z_X$ exist and Z_X is the quotient of the étale equivalence relation $Z_Q \rightrightarrows Z_W$. Moreover, the natural squares of the diagram*

(3.19.1)
$$\begin{tikzcd} Q \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & W \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Z_Q \arrow[r, shift left] \arrow[r, shift right] & Z_W \arrow[r] & Z_X \end{tikzcd}$$

are cartesian.

Proof. Since $f\colon W \to X$ is square, étale and fixed-point reflecting so are the two projections $\pi_1, \pi_2\colon Q = W \times_X W \to W$. As $W \to Z_W$ satisfies the descent condition, there exists topological quotients $Q \to (Z_Q)_1$ and

$Q \to (Z_Q)_2$ induced by the two projections π_1 and π_2. Moreover, by Corollary (3.13) the quotients $Q \to (Z_Q)_1$ and $Q \to (Z_Q)_2$ satisfy the descent condition and it follows by Proposition (3.8) that $(Z_Q)_1 \cong (Z_Q)_2$ is the unique GC quotient. The two canonical morphisms $Z_Q \to Z_W$ are étale and the corresponding squares are cartesian.

We will now show that $Z_Q \rightrightarrows Z_W$ is an equivalence relation. First note that we have an equivalence relation $Q \rightrightarrows W$ that is described by étale fixed-point reflecting morphisms $e\colon W \to Q$ (the diagonal), $i\colon Q \to Q$ (switching the two factors) and $c\colon Q \times_W Q \to Q$ (projection onto the first and third factors of $W \times_X W \times_X W$). As the descent condition is satisfied, these morphisms descend to étale morphisms $e\colon Z_W \to Z_Q$, $i\colon Z_Q \to Z_Q$ and $c\colon Z_Q \times_{Z_W} Z_Q \to Z_Q$ so that $Z_Q \rightrightarrows Z_W$ is a groupoid in algebraic spaces. It remains to verify that $Z_Q \to Z_W \times Z_W$ is a monomorphism, or equivalently, that the stabilizer $Z_Q \times_{Z_W \times Z_W} Z_W \to Z_W$ is an isomorphism.

It is enough to show that the open immersion $e \times \mathrm{id}_{Z_W} : W \times_{Z_W \times Z_W} Z_W \to Q \times_{Z_W \times Z_W} Z_W$ is surjective. Let $q\colon W \to Z_W$ denote the quotient map and let $(w_1, w_2)\colon \mathrm{Spec}(k) \to Q = W \times_X W$ be a point such that $q \circ w_1 = q \circ w_2$. After replacing k with a field extension, we can assume that (w_1, w_2) is in the image of an element $r\colon \mathrm{Spec}(k) \to R_W$. Now consider the cartesian diagram

$$\begin{tikzcd}
\mathrm{stab}(W) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{stab}(X) \times_X W \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & R_W \arrow[d] \\
W \arrow[r, hook, "e"] & Q \arrow[r, hook] & W \times W.
\end{tikzcd}$$

As f is fixed-point reflecting, the leftmost arrow in the first row is an isomorphism and it follows that (w_1, w_2) factors through the diagonal $e\colon W \to Q$.

Let Z_X be the quotient sheaf of the equivalence relation in the étale topology. This is an algebraic space and there is a canonical morphism $X \to Z_X$ making the diagram (3.19.1) cartesian. As strongly geometric quotients and the descent condition are descended by étale base change by Proposition (2.10) and Corollary (3.13), it follows that $X \to Z_X$ is a GC quotient. $\square$

4. Finite quotients of affine and AF-schemes

In this section, we review the known results on quotients of finite locally free groupoids of affine schemes. These are then easily extended to groupoids of schemes such that every orbit is contained in an affine open subscheme. The general existence results were announced in [FGA, No. 212] by Grothendieck and proven in [Gab63] by Gabriel. An exposition of these results with full proofs can also be found in [DG70, Ch. III, §2].

Besides the existence results, a list of properties of the quotient when it exists is given in Proposition (4.7). This proposition is also valid for algebraic spaces.

Theorem (4.1) ([FGA, No. 212, Thm. 5.1]). *Let $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $R = \mathrm{Spec}(C)$ be affine schemes and let $R \rightrightarrows X$ be a finite locally free S-groupoid. Let $p_1, p_2\colon B \to C$ be the homomorphisms corresponding to s and t. Let $Y = \mathrm{Spec}\left(B^R\right)$ where B^R is the equalizer of the* *homomorphisms p_1 and p_2. Let $q\colon X \to Y$ be the morphism corresponding to the inclusion $B^R \hookrightarrow B$. Then q is integral and a* GC *quotient.*

Proof. That q is integral and a geometric quotient is proven in [Gab63, Thm. 4.1]. As $j_{/Y}$ is proper we have that q is a strongly geometric quotient. It satisfies the descent condition by Theorem (3.15) and is thus a GC quotient by definition. □

Remark **(4.2).** Theorem (4.1) has a long history and there are several proofs, e.g. [KM97, Prop. 5.1] and [Con05, §3]. In the classical situation, the groupoid is induced by a finite group or, more generally, a finite flat group scheme. A good exposition of the theorem for finite groups is given in [SGA_1, Exp. V]. For algebraic group schemes, the result can be found in [Mum70, Thm. 1, p. 111]. For general group schemes, a proof is given in [GM11, §4].

Lemma (4.3). *Let $R \rightrightarrows X$ be a finite locally free groupoid of schemes. Let $Z \subseteq |X|$ be a finite set of points and let $U \subseteq X$ be an affine open neighborhood of the orbit $R(Z)$. Then there exists an R-stable affine open neighborhood $U' \subseteq U$ of the orbit $R(Z)$.*

Proof. This is proved in [Gab63, 5b)], based on the proof of [SGA_1, Exp. VIII, Cor. 7.6]. □

Theorem (4.4) ([FGA, No. 212, Thm. 5.3])**.** *Let $R \rightrightarrows X$ be a finite locally free groupoid of* schemes*. Then a* GC *quotient $q\colon X \to Y$ with q affine and Y a* scheme *exists if and only if every R-orbit of $|X|$ is contained in an affine open subscheme.*

Proof. The necessity is obvious. To prove sufficiency, let $x \in |X|$. Then by assumption and Lemma (4.3), there is an affine open R-stable neighborhood U of the orbit $R(x)$. It is enough to prove the theorem after replacing (R, X) with $(R|_U, U)$ as a GC quotient is categorical. This is Theorem (4.1). □

Remark **(4.5).** In Theorem (5.3), we will show that if $X \to S$ is a *separated* algebraic space, then a GC quotient $q\colon X \to Y$ exists and is *affine*. Thus, if $X \to S$ is a separated scheme, then it follows, from Theorem (4.4), that a geometric quotient $Y = X/R$ exists as a *scheme* if and only if every R-orbit of $|X|$ is contained in an affine open subset.

Remark **(4.6).** When we replace the groupoid with a finite group, then Theorem (4.4) is a classical result. It can be traced back to Serre [Ser59, Ch. III, §12, Prop. 19] when X is an algebraic variety. Also see [Mum70, Thm. 1, p. 111] for the case when X is an algebraic scheme and [SGA_1, Exp. V §1] or [GM11, Thm. 4.16] for arbitrary schemes.

Recall that a scheme is AF if every finite set of points has an open affine neighborhood. Clearly, Theorem (4.4) applies to any AF-scheme X. For the definition and properties of AF morphisms we refer to Appendix B.

Proposition (4.7). *Let $R \rightrightarrows X$ be a finite locally free groupoid of algebraic spaces and assume that a geometric quotient $q\colon X \to Y$ exists and that q is* affine[1]. *Then q is an integral and universally open* GC *quotient and $j_{/Y}$ is proper. Consider the following properties of a morphism of algebraic spaces:*

- (A) *quasi-compact, universally closed, universally open, separated, quasi-separated, affine, quasi-affine,* AF*;*
- (B) *finite type, locally of finite type, proper;*
- (B′) *projective, quasi-projective.*

If $X \to S$ has one of the properties in (A)*, then $Y \to S$ has the corresponding property. The same holds for the properties in* (B) *if S is locally noetherian and for those in* (B′) *if S is noetherian.*

Proof. As s is universally open and proper, it follows by Proposition (2.4) that q is universally open and universally closed. In particular q is integral [EGA$_{\text{IV}}$, Prop. 18.2.8]. As $j_{/Y}$ is proper we have that q is strongly topological. Furthermore, q satisfies the descent condition by Theorem (3.15) and is thus a GC quotient.

The statement about the first three properties in (A) follows immediately as q is surjective. The properties "separated" and "quasi-separated" follow from Propositions (2.12) and (2.14). For properties "affine" and "quasi-affine" we can assume that S is affine and then the property "affine" follows from Theorem (4.1).

Assume that X is quasi-affine. To show that Y is quasi-affine it is enough to show that there is an affine open covering of the form $\{Y_f\}$ with $f \in \Gamma(Y)$. Let $y \in Y$ be a point and let $q^{-1}(y)$ be the corresponding orbit in X. Then, as X is quasi-affine, there is a global section $g \in \Gamma(X)$ such that X_g is an affine neighborhood of $q^{-1}(y)$, cf. [EGA$_{\text{II}}$, Cor. 4.5.4]. Let $f = \mathrm{N}_t(s^*g) \in \Gamma(X)$ be the norm of s^*g along t [EGA$_{\text{II}}$, 6.4.8]. Then f is invariant, i.e., $s^*f = t^*f$, and $X_f \subseteq X_g$ is an affine neighborhood of $q^{-1}(y)$, cf. the proof of [SGA$_1$, Exp. VIII, Cor. 7.6]. Thus $f \in \Gamma(Y)$ and $q(X_f) = Y_f$ is a geometric quotient of the groupoid $R_f \rightrightarrows X_f$. As X_f is affine so is Y_f by (A) for "affine". This shows that Y is quasi-affine.

Finally, assume that $X \to S$ is AF. Let S' be an affine scheme and let $S' \to S$ be a morphism so that $X' = X \times_S S'$ is an AF-scheme. Let $Y' = Y \times_S S'$ and $R' = R \times_S S'$ so that $q' \colon X' \to Y'$ is a topological quotient of $R' \rightrightarrows X'$. We will show that Y' is an AF-scheme. Let $Z \subseteq |Y'|$ be a finite subset. Then $q'^{-1}(Z)$ is a finite subset of X' and therefore admits an R'-stable affine open neighborhood $U' \subseteq X'$ according to Lemma (4.3). By Theorem (4.1), there is a GC quotient $U' \to V'$ where V' is an affine scheme. As q' is a topological quotient, the induced morphism $r \colon V' \to q'(U')$ is a separated universal homeomorphism. Then r is integral [Ryd10, Cor. 5.22] and it follows that $q'(U')$ is affine by Chevalley's theorem [Ryd09, Thm. 8.1].

Now assume that S is locally noetherian. As we have already shown the statement for quasi-compact, universally closed and separated, it is enough to show the statement for the property "locally of finite type" in group (B). Assume that $X \to S$ is locally of finite type. Then q is finite. As the quotient is uniform, we can, in order to prove that $Y \to S$ is locally of finite type, assume that S and Y are affine and hence also X. It is then easily

[1] The assumption that q is affine can be replaced with the condition that q is separated using [Ryd09, Thm. 8.5].

seen that $Y \to S$ is of finite type. For details, see the argument in [SGA$_1$, Exp. V, Cor. 1.5].

For the properties in (B′), we cannot assume that S is affine as projectivity and quasi-projectivity are not local on the base. The statement about (quasi-)projectivity is probably well-known but I am not aware of any full proof. A sketch is given in [Knu71, Ch. IV, Prop. 1.5] and is also discussed in [Ryd08c], but these proofs are for quotients by finite groups. We will now prove the general case.

Let $\mathcal{L}$ be an ample sheaf on X and let $\mathcal{L}' = \mathrm{N}_t(s^*\mathcal{L}) = \mathrm{N}_{t_*\mathcal{O}_R/\mathcal{O}_X}(t_*s^*\mathcal{L})$. This is an ample invertible sheaf by [EGA$_{\mathrm{II}}$, Cor. 6.6.2]. Moreover, it comes with a canonical R-linearization [GIT, Ch. 1, §3], i.e., a canonical isomorphism $\phi\colon s^*\mathcal{L}' \to t^*\mathcal{L}'$ satisfying a cocycle condition. This is obvious from the description $\mathcal{L}' = p^*\mathrm{N}_{X/\mathscr{X}}(\mathcal{L})$ where $p\colon X \to \mathscr{X}$ is the stack quotient of $R \rightrightarrows X$. Consider the graded $\mathcal{O}_X$-algebra $\mathcal{A} = \oplus_{n\geq 0}\mathcal{L}'^n$. As $\mathcal{L}'$ is ample, we have a canonical (closed) immersion $X \hookrightarrow \mathrm{Proj}(f_*\mathcal{A})$ where $f\colon X \to S$ is the structure morphism.

Let $(f_*\mathcal{A})^R$ be the invariant ring, where $(f_*\mathcal{L}'^n)^R$ is the equalizer of

$$\begin{tikzcd} f_*\mathcal{L}'^n \arrow[r, "s^*"] \arrow[rr, bend right, "t^*"'] & (f\circ s)_*s^*\mathcal{L}'^n \arrow[r, "\phi"] & (f\circ s)_*t^*\mathcal{L}'^n. \end{tikzcd}$$

It can then be shown, analogous to the case of a finite group action, that the quotient Y is a subscheme of $\mathrm{Proj}_S((f_*\mathcal{A})^R)$. As S is locally noetherian, it follows that $(f_*\mathcal{A})^R$ is a finitely generated $\mathcal{O}_S$-algebra, but it is not necessarily generated by elements of degree 1. As S is noetherian there is an integer m such that $(f_*\mathcal{A}^R)^{(m)} = \oplus_{n\geq 0} f_*(\mathcal{L}'^{mn})^R$ is generated in degree 1. Hence Y is (quasi-)projective. □

Remark **(4.8).** If S is of characteristic zero, i.e., a $\mathbb{Q}$-space, then the GC quotient $q\colon X \to Y$ of the proposition is *universal*, i.e., it commutes with any base change. In fact, there exists a Reynolds operator, i.e., an $\mathcal{O}_Y$-module retraction $\mathcal{R}\colon q_*\mathcal{O}_X \to \mathcal{O}_Y$ of the inclusion $\mathcal{O}_Y = (q_*\mathcal{O}_X)^R \hookrightarrow q_*\mathcal{O}_X$. The sequence (2.2.1) is thus split exact which shows that q is a universal geometric quotient. We construct the Reynolds operator as follows: the rank r of s is constant on each connected component and constant on R-orbits. Locally the Reynolds operator $\mathcal{R}$ is defined by $\frac{1}{r}\mathrm{Tr}_{s^*} \circ t^*$ where $s^*, t^*\colon q_*\mathcal{O}_X \to (q\circ s)_*\mathcal{O}_R$ are the $\mathcal{O}_Y$-homomorphisms induced by s and t.

More generally, in any characteristic, the quotient is universal if the stack $[R \rightrightarrows X]$ is *tame* [AOV08].

Remark **(4.9).** When S is a *scheme* and $X \to S$ is AF, then any R-orbit of $|X|$ is contained in an affine open subscheme and the conclusion of Theorem (4.4) holds. In particular, this is true for $X \to S$ (quasi-)affine or (quasi-)projective. Furthermore, Proposition (4.7) shows that geometric quotients exist in the following categories:

(1) schemes affine over S;
(2) schemes quasi-affine over S;
(3) schemes that are AF over S;
(4) schemes that are projective over a noetherian base scheme S; and

(5) schemes that are quasi-projective over a noetherian base scheme S.

5. Finite quotients of algebraic spaces

Let $R_X \rightrightarrows X$ be a groupoid. For any étale morphism $U \to X$ we will construct a groupoid $R_W \rightrightarrows W$ with a square étale morphism $h\colon (R_W, W) \to (R_X, X)$. The construction requires that $R_X \rightrightarrows X$ is *proper*, flat and of finite presentation. If $R_X \rightrightarrows X$ is *finite* and $U \to X$ is surjective, then $h|_{\mathrm{fpr}}$ will be surjective. Using Theorem (3.19) and the results of Section 4, we then deduce the existence of a quotient X/R.

Proposition (5.1). *Let $R_X \rightrightarrows X$ be a groupoid that is proper, flat and of finite presentation and let $f\colon U \to X$ be an étale and separated morphism. Then there is a groupoid (R_W, W) together with a square separated étale morphism $h\colon (R_W, W) \to (R_X, X)$ and an étale and separated morphism $W \to U$. In particular, if U is an* AF*-scheme (e.g., a disjoint union of affine schemes), then so is W. If, in addition, $R_X \rightrightarrows X$ is* finite *and f is surjective, then $h|_{\mathrm{fpr}(h)}$ is surjective.*

Proof. We will construct W functorially such that a point in the fiber W_x over a point $x \in |X|$ corresponds to the choice of a point in the fiber $U_{x'}$ for every point x' in the orbit of x. More precisely, given an X-scheme T, an X-morphism $T \to W$ corresponds to a section of

$$\pi_{12}\colon T \times_{g,X,s} R_X \times_{t,X,f} U \to T \times_{g,X,s} R_X$$

where $g\colon T \to X$ is the structure morphism. Thus, W is the *Weil restriction* $\mathbf{R}_s(R_X \times_{t,X,f} U/R_X)$, cf. Appendix D. As $s\colon R_X \to X$ is proper, flat and of finite presentation, the functor W is an algebraic space, that is separated and locally of finite presentation over X. By Proposition (D.3), the morphism $W \to X$ is étale and separated. Furthermore, by the same proposition, the unit section of $s\colon R_X \to X$ gives rise to a factorization of $W \to X$ into an étale and separated morphism $W \to U$ followed by f. If U is an AF-scheme, then so is W since $W \to U$ is AF.

We obtain an easier description of W using the stack $\mathscr{X} = [R_X \rightrightarrows X]$. Then $W = \mathscr{W} \times_{\mathscr{X}} X$ where $\mathscr{W} = \mathbf{R}_{X/\mathscr{X}}(U/X)$. This induces a groupoid (R_W, W) with $R_W = W \times_{\mathscr{W}} W$ and the morphism $\mathscr{W} \to \mathscr{X}$ induces a square morphism $(R_W, W) \to (R_X, X)$. The morphism $W \to U$ is given by the adjunction formula $\mathrm{Hom}_X(T \times_{\mathscr{X}} X, U) = \mathrm{Hom}_{\mathscr{X}}(T, \mathscr{W})$ with $T = \mathscr{W}$.

Finally, we show that $h|_{\mathrm{fpr}(h)}$ is surjective when $R_X \to X$ is finite and $f\colon U \to X$ is surjective. Let $x\colon \mathrm{Spec}(\overline{k}) \to X$ be a geometric point of X. A lifting $w\colon \mathrm{Spec}(\overline{k}) \to W$ of x corresponds to a morphism $\varphi\colon s^{-1}(x) \to U$ such that $t = f \circ \varphi$. Let $R_x = s^{-1}(x)_{\mathrm{red}}$ which we consider as an X-scheme using t. As $U \to X$ is étale, any X-morphism $R_x \to U$ induces a unique morphism φ as above. If $R_X \to X$ is finite, then R_x is a finite set of points. We may then choose an X-morphism $R_x \to U$ such that its image contains at most one point in every fiber of $f\colon U \to X$. This corresponds to a point w in the fixed-point reflecting locus of $W \to X$. □

Remark **(5.2).** Let $G = \{g_1, g_2, \dots, g_n\}$ be a finite group acting on an algebraic space X and let $R_X = G \times X \rightrightarrows X$ be the induced groupoid. Let $f\colon U \to X$ be an étale and separated morphism. Then the étale cover

$W \to X$ of Proposition (5.1) is the fiber product of $g_1 \circ f, g_2 \circ f, \dots, g_n \circ f$. The morphism $W \to U$ is the projection on the factor corresponding to the identity element $g_i = e \in G$.

Theorem (5.3). *Let $R \rightrightarrows X$ be a finite locally free groupoid with finite stabilizer $\operatorname{stab}(X) = j^{-1}\big(\Delta(X)\big) \to X$. Then a* GC *quotient $q\colon X \to X/R$ exists and q is affine. Hence it has the properties of Proposition* (4.7)*.*

Proof. The question is étale-local over S so we can assume that S is affine. Let $\varphi\colon U \to X$ be an étale presentation such that U is a disjoint union of affine schemes. Let $h\colon W \to X$ be the étale cover constructed in Proposition (5.1) using U. As the stabilizer is finite, the subset $\operatorname{fpr}(h) \subseteq |W|$ is open by Proposition (3.5). Furthermore, $W|_{\mathrm{fpr}} \to X$ is surjective by Proposition (5.1). Thus $W|_{\mathrm{fpr}} \to X$ is an étale square fpr cover such that $W|_{\mathrm{fpr}}$ is an AF-scheme. By Theorem (4.4) and Remark (4.9), a GC quotient of $W|_{\mathrm{fpr}}$ exists. Hence a geometric quotient X/R exists by Theorem (3.19). □

Corollary (5.4) (Deligne)**.** *Let $G \to S$ be a finite locally free group scheme acting on a separated algebraic space $X \to S$. Then a* GC *quotient $q\colon X \to X/G$ exists, q is affine and $X/G \to S$ is separated.*

Proof. As X is separated, the finite locally free groupoid $G \times_S X \rightrightarrows X$ has finite diagonal. In particular, its stabilizer is finite. The existence of a GC quotient q thus follows from Theorem (5.3). □

For symmetric products we can find more explicit étale covers as follows.

Corollary (5.5). *Let $X \to S$ be a separated morphism of algebraic spaces. Then a* GC *quotient $\operatorname{Sym}^d(X/S) := (X/S)^d/\mathfrak{S}_d$ exists as a separated algebraic space. Let $\{S_\alpha \to S\}_\alpha$ and $\{U_\alpha \to X \times_S S_\alpha\}_\alpha$ be sets of étale morphisms of separated algebraic spaces. Then the diagram*

(5.5.1)
$$\begin{tikzcd} \coprod_\alpha (U_\alpha/S_\alpha)^d|_{\mathrm{fpr}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (X/S)^d \arrow[d] \\ \coprod_\alpha \operatorname{Sym}^d(U_\alpha/S_\alpha)|_{\mathrm{fpr}} \arrow[r] & \operatorname{Sym}^d(X/S) \end{tikzcd}$$

is cartesian and the horizontal morphisms are étale. If the U_α's are such that for every $s \in |S|$, any set of d points in X_s lies in the image of some U_α, then the horizontal morphisms are surjective. In particular, there is an étale cover of $\operatorname{Sym}^d(X/S)$ of the form $\big\{\operatorname{Sym}^d(U_\alpha/S_\alpha)|_{V_\alpha}\big\}$ with U_α and S_α affine and V_α an open subset.

Proof. Corollary (5.4) shows the existence of the GC quotient $\operatorname{Sym}^d(X/S)$. As $(U_\alpha/S_\alpha)^d \to (X/S)^d$ is étale, the diagram (5.5.1) is cartesian by the descent condition. Let $x = (x_1, x_2, \dots, x_d)\colon \operatorname{Spec}(k) \to (X/S)^d$ be a geometric point. If $x_1, x_2, \dots, x_d$ lies in the image of some U_α we can choose a lifting $u = (u_1, u_2, \dots, u_d)\colon \operatorname{Spec}(k) \to (U_\alpha/S_\alpha)^d$ such that $u_i = u_j$ if and only if $x_i = x_j$. Then u is a point in the fixed-point reflecting locus of $(U_\alpha/S_\alpha)^d \to (X/S)^d$. This shows the surjectivity of the horizontal morphisms. □

Remark **(5.6).** Note that the stabilizer of the permutation action of $\mathfrak{S}_d$ on $(X/S)^d$ is proper exactly when $X \to S$ is separated. Thus, the proof of Corollary (5.5) definitively fails unless $X \to S$ is separated.

6. Coarse moduli spaces of stacks

In this section, we will prove the existence of a coarse moduli space to any algebraic stack $\mathscr{X}$ with finite inertia. We proceed as follows.

(i) We find a quasi-finite flat presentation of $\mathscr{X}$ [Ryd11a, Thm. 7.1].
(ii) We find an étale representable cover $\mathscr{W} \to \mathscr{X}$ such that there exists an AF-scheme V and a finite flat presentation $V \to \mathscr{W}$ (Proposition 6.11).
(iii) We show that $\mathscr{W} \to \mathscr{X}$ is fixed-point reflecting over an open substack $\mathscr{W}|_{\mathrm{fpr}} \subseteq \mathscr{W}$ and that $\mathscr{W}|_{\mathrm{fpr}} \to \mathscr{X}$ is surjective. (Proposition 6.5).
(iv) We deduce the existence of a coarse moduli space to $\mathscr{X}$ from the existence of a coarse moduli space to $\mathscr{W}$ (Theorem 6.10).

The assumption that $\mathscr{X}$ has finite inertia, and not merely quasi-finite inertia, is only used in step (iii).

Keel and Mori [KM97] more or less proceed in the same way. Using stacks, as in [Con05], instead of groupoids, as in [KM97], gives a more streamlined presentation and simplifies many arguments. In particular, the reduction from the quasi-finite case to the finite case becomes much more transparent. In step (iv), we use the descent condition (Definition 6.6), which simplifies earlier proofs, cf. [Con05, Thm. 3.1 and Thm. 4.2]. We also avoid the somewhat complicated limit methods used in [Con05, §5] and obtain a slightly more general result.

We begin by rephrasing the results of §3 in stack language. If $U \to \mathscr{X}$ is a flat presentation and $R = U \times_{\mathscr{X}} U$, then there is a one-to-one correspondence between equivariant morphisms $U \to Z$ with respect to the groupoid $R \rightrightarrows U$ and morphisms $\mathscr{X} \to Z$. We say that a morphism $\mathscr{X} \to Z$ is a topological (resp. geometric, resp. categorical, ...) quotient if $U \to \mathscr{X} \to Z$ is such a quotient. This definition does not depend on the choice of presentation $U \to \mathscr{X}$ and can be rephrased as follows.

Definition (6.1). Let $\mathscr{X}$ be an algebraic stack, let Z be an algebraic space and let $q\colon \mathscr{X} \to Z$ be a morphism. Then q is:

(i) *categorical* if q is initial among morphisms from $\mathscr{X}$ to algebraic spaces;
(ii) *topological* if q is a universal homeomorphism;
(iii) *strongly topological* if q is a strong homeomorphism, cf. Appendix C;
(iv) *geometric* if q is a universal homeomorphism and $\mathcal{O}_Z \to q_*\mathcal{O}_{\mathscr{X}}$ is an isomorphism; and
(v) *strongly geometric* if q is a strong homeomorphism and $\mathcal{O}_Z \to q_*\mathcal{O}_{\mathscr{X}}$ is an isomorphism.

Remark **(6.2).** If $q'\colon \mathscr{X} \to Z'$ and $r\colon Z' \to Z$ are universal homeomorphisms (so that q' and $q = r \circ q'$ are topological quotients), then r need not be separated. If q is a strong homeomorphism (i.e., q is a strongly topological quotient), then r is necessarily separated by Corollary (C.3). Thus, a strongly topological quotient $q\colon \mathscr{X} \to Z$ is "maximally non-separated" among topological quotients.

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of stacks. There is then an induced morphism $\varphi\colon I_{\mathscr{X}} \to I_{\mathscr{Y}} \times_{\mathscr{Y}} \mathscr{X}$. If $x\colon \mathrm{Spec}(k) \to \mathscr{X}$ is a point and $y = f \circ x$, then φ_x is the natural morphism of k-groups $\mathcal{I}som_{\mathscr{X}}(x,x) \to \mathcal{I}som_{\mathscr{Y}}(y,y)$.

Definition (6.3). Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of stacks. We say that f is *fixed-point reflecting* (or stabilizer preserving), abbreviated fpr, at a point $x \in |\mathscr{X}|$ if $\varphi_x\colon I_{\mathscr{X}} \times_{\mathscr{X}} \mathrm{Spec}(k) \to I_{\mathscr{Y}} \times_{\mathscr{Y}} \mathscr{X} \times_{\mathscr{X}} \mathrm{Spec}(k)$ is an isomorphism for some representative $\mathrm{Spec}(k) \to \mathscr{X}$ of x. We say that f is fixed-point reflecting if f is fixed-point reflecting at every point. We let $\mathrm{fpr}(f) \subseteq |\mathscr{X}|$ be the subset over which f is fixed-point reflecting.

It is well-known that f is representable if and only if φ is a monomorphism. If f is unramified, then f is fixed-point reflecting at every point if and only if φ is an isomorphism.

Remark **(6.4).** Let $\mathscr{X}$ and $\mathscr{Y}$ be stacks with presentations $U \to \mathscr{X}$ and $V \to \mathscr{Y}$ and let $R_U = U \times_{\mathscr{X}} U$ and $R_V = V \times_{\mathscr{Y}} V$. Then there is a one-to-one correspondence between square morphisms $(R_U, U) \to (R_V, V)$ and morphisms $\mathscr{X} \to \mathscr{Y}$ together with an isomorphism $U \to V \times_{\mathscr{Y}} \mathscr{X}$. Under this correspondence, $U \to V$ is fixed-point reflecting if and only if $\mathscr{X} \to \mathscr{Y}$ is so.

The following is a reformulation of Proposition (3.5) for stacks.

Proposition (6.5). *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a representable and unramified morphism of stacks. If the inertia stack $I_{\mathscr{Y}} \to \mathscr{Y}$ is universally closed, then the subset* $\mathrm{fpr}(f) \subseteq |\mathscr{X}|$ *is open.*

Definition (6.6). Let $\mathscr{X}$ be an algebraic stack and let $q\colon \mathscr{X} \to X$ be a topological quotient. We say that q satisfies the *descent condition* if for any étale fixed-point reflecting morphism $f\colon \mathscr{W} \to \mathscr{X}$ there exists a topological quotient $\mathscr{W} \to W$ and a 2-cartesian square

$$\begin{tikzcd} \mathscr{W} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "q"] \\ W \arrow[r] & X \end{tikzcd}$$

where $W \to X$ is *étale.*

Proposition (6.7). *A strongly geometric quotient $\mathscr{X} \to X$ satisfies the descent condition for $\mathbf{\acute{E}t}_{\mathrm{qsep}}(\mathscr{X})$ and is categorical.*

Proof. See Theorem (3.15) and Proposition (3.8). $\square$

Definition (6.8). We say that $\mathscr{X} \to X$ is a *coarse moduli space* if $\mathscr{X} \to X$ is a strongly geometric quotient. As a strongly geometric quotient is categorical, we will speak about *the* coarse moduli space when it exists.

Remark **(6.9).** Let $\mathscr{X}$ be an algebraic stack with a flat presentation $U \to \mathscr{X}$ and let $R = U \times_{\mathscr{X}} U$. Let $q\colon \mathscr{X} \to X$ be a topological quotient. Then q satisfies the descent condition (resp. is a coarse moduli space) if and only if $U \to \mathscr{X} \to X$ satisfies the descent condition of Definition (3.6) (resp. is a GC quotient) for the groupoid (R, U).

Theorem (6.10). *Let $f\colon \mathscr{W} \to \mathscr{X}$ be a surjective, étale, separated and fpr morphism of algebraic stacks. Let $\mathscr{Q} = \mathscr{W} \times_{\mathscr{X}} \mathscr{W}$. If $\mathscr{W}$ has a coarse moduli space W, then coarse moduli spaces $\mathscr{Q} \to Q$ and $\mathscr{X} \to X$ exist. Furthermore, the diagram*

$$\begin{tikzcd}
\mathscr{Q} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{W} \arrow[r, "f"] \arrow[d] & \mathscr{X} \arrow[d] \\
Q \arrow[r, shift left] \arrow[r, shift right] & W \arrow[r] & X
\end{tikzcd}$$

is cartesian.

Proof. This follows immediately from Theorem (3.19) and Remark (6.9). $\square$

Proposition (6.11) ([KM97, §4], [Con05, Lem. 2.2], [Ryd11a, Thm. 6.3])**.** *Let $\mathscr{X}$ be an algebraic stack with locally quasi-finite and separated diagonal. Then there is a representable étale separated morphism $h\colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ has a* finite *flat presentation $V \to \mathscr{W}$ with V an* AF*-scheme. Moreover,* $\mathrm{fpr}(h) \subseteq |\mathscr{W}| \to |\mathscr{X}|$ *is surjective.*

Proof. By [Ryd11a, Thm. 7.1] (or [SP, Lem. 04N0] if $\Delta_{\mathscr{X}}$ is not quasi-compact) there is a locally quasi-finite flat presentation $p\colon U \to \mathscr{X}$ with U a scheme. Taking an open covering, we can assume that U is a disjoint union of affine schemes, hence AF. Let $\mathscr{H} = \mathrm{Hilb}(U/\mathscr{X}) \to \mathscr{X}$ be the Hilbert functor. As $U \to \mathscr{X}$ is locally of finite presentation and separated, it follows by fppf descent and the usual algebraicity result for Hilbert functors of algebraic spaces [Art69a, Cor. 6.2] that $\mathscr{H} \to \mathscr{X}$ is representable, locally of finite presentation and separated.

As $U \to \mathscr{X}$ is locally quasi-finite, a morphism $T \to \mathscr{H}$ corresponds to a closed subscheme $g\colon Z \hookrightarrow U \times_{\mathscr{X}} T$ such that the morphism $f = \pi_2 \circ g\colon Z \to T$ is finite, flat and of finite presentation over T. Note that g is an open immersion if and only if g is étale. As f is flat and f and $\pi_2\colon U \times_{\mathscr{X}} T \to T$ are both locally of finite presentation, we have that the morphism g is étale at z if and only if the fiber $g_{f(z)}$ is étale at z by [$\mathrm{EGA_{IV}}$, Rem. 17.8.3]. Let $Z_{\text{ét}}$ be the open subset of Z where g is étale. Then the open subset $T \setminus f(Z \setminus Z_{\text{ét}}) \subseteq T$ is the set of $t \in |T|$ such that g_t is open and closed. It follows that the substack $\mathscr{W} \subset \mathscr{H}$, parameterizing open and closed subschemes, is an open substack. It is also immediately clear that $\mathscr{W} \to \mathscr{X}$ is formally étale and hence étale. In fact, $\mathscr{W}$ can also be described as the étale sheaf of pointed sets $p_!\underline{\{0,1\}}_U$.

Let V be the universal family over $\mathscr{W}$. Then V is an AF-scheme. In fact, as $V \hookrightarrow U \times_{\mathscr{X}} \mathscr{W}$ is an open and closed immersion and $\mathscr{W} \to \mathscr{X}$ is étale and separated, we have that $V \to U$ is AF so that V is an AF-scheme, cf. Appendix B.

Moreover, we have a decomposition $\mathscr{W} = \coprod_{d\geq 0} \mathscr{W}_d$ into open and closed substacks where $\mathscr{W}_d$ parameterizes open and closed subschemes of rank d over the base. After replacing $\mathscr{W}$ with $\mathscr{W} \setminus \mathscr{W}_0$ we have that $V \to \mathscr{W}$ is surjective.

To see that $\mathscr{W}|_{\mathrm{fpr}} \to \mathscr{X}$ is surjective, let $x\colon \operatorname{Spec}(k) \to \mathscr{X}$ be a point. The stabilizer group scheme $\mathcal{I}som_{\mathscr{X}}(x,x)$ acts on the fiber U_x. If $Z \subseteq U_x$ is an open subscheme that is stable under this action and finite over $\operatorname{Spec}(k)$, then the corresponding point $[Z]\colon \operatorname{Spec}(k) \to \mathscr{W}$ is in the fixed-point reflecting locus of $\mathscr{W} \to \mathscr{X}$. As U_x is discrete and $\mathcal{I}som_{\mathscr{X}}(x,x)$ is finite, we can choose Z as any orbit of $|U_x|$. □

We are now ready to prove the full generalization of Keel and Mori's theorem [KM97].

Theorem (6.12). *Let $\mathscr{X}$ be an algebraic stack with finite inertia stack. Then $\mathscr{X}$ has a coarse moduli space $q\colon \mathscr{X} \to X$ and q is a* separated *universal homeomorphism. Let S be an algebraic space and let $\mathscr{X} \to S$ be a morphism. If $\mathscr{X} \to S$ is locally of finite type, then q is proper and quasi-finite. Consider the following properties:*

- (A) *quasi-compact, universally closed, universally open, separated, quasi-separated;*
- (B) *finite type, locally of finite type, proper.*

If $\mathscr{X} \to S$ has one of the properties in (A)*, then $X \to S$ has the corresponding property. If S is locally noetherian, the same holds for the properties in* (B)*.*

Proof. By Propositions (6.5) and (6.11), there is a representable, étale, fixed-point reflecting and surjective morphism $\mathscr{W} \to \mathscr{X}$ and a finite flat presentation $V \to \mathscr{W}$ with V an AF-scheme. A coarse moduli space $\mathscr{W} \to W$ exists by Theorem (4.4). It then follows, from Theorem (6.10), that a coarse moduli space $q\colon \mathscr{X} \to X$ exists, that the morphism $W \to X$ is étale and that $\mathscr{W} = \mathscr{X} \times_X W$.

As $\mathscr{W} \to W$ is separated, so is q. If $\mathscr{X} \to S$ is locally of finite type, then $\mathscr{X} \to X$ is locally of finite type and hence proper. We also then have that $V \to X$ is locally quasi-finite so that $\mathscr{X} \to X$ is quasi-finite.

Among the properties in (A), "separated" and "quasi-separated" follow from Proposition (2.12) and the rest are obvious. In (B), we only need to prove that if S is locally noetherian and $\mathscr{X} \to S$ is locally of finite type, then so is $X \to S$. As $\mathscr{W} \to S$ is locally of finite type, then so is $W \to S$ by Proposition (4.7). As $W \to X$ is étale and surjective, it follows that $X \to S$ is locally of finite type. □

Remark **(6.13).** If $\mathscr{X}$ is an algebraic stack such that the inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ is flat and locally of finite presentation, then the fppf sheaf X associated to the stack $\mathscr{X}$ is a coarse moduli space of $\mathscr{X}$. Indeed, X is an algebraic space and $\mathscr{X} \to X$ is an fppf gerbe [LMB00, Cor. 10.8]. It follows that $\mathscr{X} \to X$ is strongly geometric and that the formation of X commutes with arbitrary base change.

Thus, if $\mathscr{X}$ has finite or flat inertia, then it has a coarse moduli space. It is then easily seen that for the existence of a coarse moduli space, it is not necessary that $\mathscr{X}$ has proper inertia. In fact, if X is any algebraic space, $U \subseteq X$ is an open subset and G is a finite group, then there is an algebraic stack $\mathscr{X}$ with coarse moduli space X, stabilizer group G over U and trivial stabilizer group over $X \setminus U$. It is clear that the inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ is flat but not proper unless U is also closed.

The following example shows that if $\mathscr{X}$ is an algebraic stack with quasi-finite inertia, which is neither proper nor flat, then a coarse moduli space need not exist. In fact, the example does not admit a topological quotient, nor a categorical quotient. The proof hinges on the following lemma which is closely related to Artin's example of a non-algebraic functor [Art69b, Ex. 5.11].

Lemma (6.14). *Let $\mathbb{A}^2 = \operatorname{Spec}(k[x,y])$ be the affine plane, let $U = D(x)$ be the complement of the y-axis and let $Z = V(y)$ be the x-axis. Let X be an algebraic space and let $i\colon X \hookrightarrow \mathbb{A}^2$ be a monomorphism, locally of finite type, such that $i|_U$ and $i|_Z$ are isomorphisms. Then there is an open neighborhood $V \subseteq \mathbb{A}^2$ of $|U| \cup |Z|$ such that $i|_V$ is an isomorphism.*

Proof. We begin by noting that X is a scheme [LMB00, Thm. A.2] and that we can replace X with a quasi-compact open neighborhood of the quasi-compact subset $|U| \cup |Z| \subseteq |X|$ so that i becomes quasi-compact. Then i is quasi-finite and the lemma readily follows from Zariski's Main Theorem. $\square$

Example (6.15). Let k be a field of characteristic different from 2 and let $S = \operatorname{Spec}(k[x,y^2])$ be the affine plane. Let $U = \operatorname{Spec}(k[x,y])$ be another affine plane, seen as a ramified double covering of S. Let $\tau\colon U \to U$ be the S-involution on U given by $y \mapsto -y$. Finally, let W be the union of two copies of U glued along $x \neq 0$. On W we have an involution given by simultaneously applying τ and interchanging the two copies of U. This gives an action of $\mathbb{Z}/2\mathbb{Z}$ on W and we let $\mathscr{X} = [W/(\mathbb{Z}/2\mathbb{Z})]$.

For us, it will more convenient to study the presentation $U \to \mathscr{X}$ given by one of the copies of U in W. Then $U \times_{\mathscr{X}} U = G \times_S U$ where G is the group scheme $S \amalg (S \setminus \{x = 0\}) \subseteq (\mathbb{Z}/2\mathbb{Z})_S$ and the action of a non-trivial section of G on U is given by τ.

Let $S' = S \setminus \{y = 0\}$ and let $X' = \mathscr{X} \times_S S'$. Note that the stack X' has trivial inertia, so it is an algebraic space. The morphism $X' \to S'$ is étale but not separated. It is an isomorphism outside $\{x = 0\}$ and restricted to $x = 0$ it coincides with the étale double cover $U|_{S'} \to S'$.

Let Q be an algebraic space over S and let $\mathscr{X} \to Q$ be an S-morphism. Then there is an induced factorization

$$G \times_S U \to U \times_Q U \hookrightarrow U \times_S U.$$

We have that

$$U \times_S U = \operatorname{Spec}\bigl(k[x, y_1, y_2]/(y_1^2 - y_2^2)\bigr),$$

so that $U \times_S U$ is the union of two affine planes $U_i = \operatorname{Spec}(k[x, t_i])$ glued along the lines $t_i = 0$. In coordinates, we have that $t_1 = y_1 + y_2$ and $t_2 = y_1 - y_2$. The image of $G \times_S U \to U \times_S U$ is the union of U_1 and $U_2 \setminus \{x = 0\}$.

We now restrict everything to U_2. Then $(G \times_S U)|_{U_2}$ is the disjoint union of the open subscheme $D(x)$ and the closed subscheme $V(t_2)$. This observation, combined with Lemma (6.14), shows that $G\times_S U \to U\times_Q U$ is not surjective. Thus, the stack $\mathscr{X}$ has *no topological quotient*.

In addition, the stack $\mathscr{X}$ has no *categorical quotient*. In fact, for any closed point $s \in S$ on the y^2-axis but not on the x-axis, let $Q_s \to S$ be the non-separated algebraic space which is isomorphic to S outside s but an étale extension of degree 2 at s. To be precise, over S' the space Q_s is the quotient of $U|_{S'}$ by the group $S' \amalg (S' \setminus S'_s)$ where the second component acts by τ. If k is algebraically closed, then Q_s is even a scheme—the affine plane with a double point at s. It is clear that $\mathscr{X} \to S$ factors canonically through Q_s.

If a categorical quotient $\mathscr{X} \to Q$ existed, then, by definition, we would have morphisms $Q \to Q_s$ for every s as above. This shows that $U \times_Q U \hookrightarrow U \times_{Q_s} U \hookrightarrow U \times_S U$ would be contained in the union of U_1 and $U_2 \setminus B$ where B is the set of all closed points on the t_2-axis except the origin. This is again impossible according to the lemma.

Appendix A. Descent of étale morphisms

In this appendix, we review and combine the descent results for étale morphisms found in [SGA$_1$, Exp. IX], [SGA$_4$, Exp. VIII, §9] and [Ryd10, §5]. For an algebraic space S, we let $\mathbf{\acute{E}t}(S) = \mathbf{\acute{E}t}_{\mathrm{all}}(S)$ denote the category of all étale morphisms $X \to S$ of algebraic spaces. We also consider the following full subcategories of $\mathbf{\acute{E}t}(S)$:

$$\begin{aligned}
\mathbf{\acute{E}t}_{\mathrm{qc}}(S) &= \{\text{étale and quasi-compact morphisms}\}\\
\mathbf{\acute{E}t}_{\mathrm{qsep}}(S) &= \{\text{étale and quasi-separated morphisms}\}\\
\mathbf{\acute{E}t}_{\mathrm{cons}}(S) &= \{\text{étale, quasi-compact and quasi-separated morphisms}\}\\
\mathbf{\acute{E}t}_{\mathrm{sep}}(S) &= \{\text{étale and separated morphisms}\}\\
\mathbf{\acute{E}t}_{\mathrm{sep+qc}}(S) &= \{\text{étale, separated and quasi-compact morphisms}\}\\
\mathbf{\acute{E}t}_{\mathrm{fin}}(S) &= \{\text{étale and finite morphisms}\}.
\end{aligned}$$

Note that $\mathbf{\acute{E}t}(S)$ is equivalent to the category of sheaves on the small étale site of S and that under this equivalence, $\mathbf{\acute{E}t}_{\mathrm{cons}}(S)$ is identified with the category of constructible sheaves, cf. [Mil80, Ch. V, Thm. 1.5] or [Art73, Ch. VII, §1].

For any morphism of algebraic spaces $f\colon S' \to S$ there is a pull-back functor $f^*\colon \mathbf{\acute{E}t}(S) \to \mathbf{\acute{E}t}(S')$ and this makes $\mathbf{\acute{E}t}(-)\colon \mathbf{AlgSp}^{\mathrm{op}} \to \mathbf{Cat}$ into a pseudo-functor. By the usual theory of descent [Gir64, Vis05] there is also a functor $f^*_D\colon \mathbf{\acute{E}t}(S) \to \mathbf{\acute{E}t}(S' \to S)$ where $\mathbf{\acute{E}t}(S' \to S)$ denotes the category of pairs $(X' \to S', \varphi)$ where $(X' \to S') \in \mathbf{\acute{E}t}(S')$ and $\varphi\colon X' \times_S S' \to S' \times_S X'$ is a descent datum. If $(X' \to S', \varphi)$ is in the essential image of f^*_D, then we say that φ is an *effective* descent datum.

For the definition of universally submersive morphisms and f^{cons}, see (2.1) or [Ryd10, §1]. For the definition of *universally subtrusive* morphisms, see [Ryd10, §2]. Examples of universally subtrusive morphisms are (i) morphisms that are covering in the fpqc topology, (ii) universally open and surjective morphisms and (iii) universally closed and surjective morphisms.

If S is locally noetherian, then $f\colon S' \to S$ is universally subtrusive if and only if f and f^{cons} are universally submersive (e.g., f quasi-compact and universally submersive).

Proposition (A.1) (Descent)**.** *Let $f\colon S' \to S$ be a universally submersive morphism. Then the functor $f_D^*\colon \mathbf{\acute{E}t}(S) \to \mathbf{\acute{E}t}(S' \to S)$ is fully faithful. Moreover, if f^{cons} is also universally submersive, $(X \to S) \in \mathbf{\acute{E}t}(S)$ and $P \in \{\mathrm{qc}, \mathrm{qsep}, \mathrm{cons}, \mathrm{sep}, \mathrm{sep}+\mathrm{qc}, \mathrm{fin}\}$, then $(X \to S) \in \mathbf{\acute{E}t}_P(S)$ if and only if $f^*(X \to S) \in \mathbf{\acute{E}t}_P(S')$.*

Proof. The first assertion is [SGA_4, Exp. VIII, Prop. 9.1]. The second assertion follows from [Ryd10, Prop. 1.7], cf. [Ryd10, Prop. 5.4]. □

Theorem (A.2) (Effective descent)**.** *Let $f\colon S' \to S$ be a surjective morphism of algebraic spaces. Then $f_D^*\colon \mathbf{\acute{E}t}_P(S) \to \mathbf{\acute{E}t}_P(S' \to S)$ is an equivalence of categories in the following cases:*

(1a) $P = \mathrm{all}$ *and f is proper or integral;*
(1b) $P = \mathrm{all}$ *and f is universally subtrusive and of finite presentation;*
(1c) $P = \mathrm{all}$ *and f is universally open and locally of finite presentation;*
(1d) $P = \mathrm{all}$ *and f is covering in the fppf topology;*
(2) $P = \mathrm{qsep}$ *and f is universally open;*
(3a) $P = \mathrm{cons}$ *and f is universally subtrusive and quasi-compact; or*
(3b) $P = \mathrm{cons}$ *and f is covering in the fpqc topology.*

Proof. (1a) and (1d) are [SGA_4, Exp. VIII, Thm. 9.4] and (1d) also follows from the more general descent result [LMB00, Cor. 10.4.2]. Using (1d) we can work Zariski-locally and then (1c) follows from (1b). To prove (1b) we can assume that S is affine and then f can be refined into an open covering followed by a proper and finitely presented surjective morphism [Ryd10, Thm. 3.12] and we conclude by (1a) and (1d). Finally (2) and (3a) are [Ryd10, Thm. 5.19 and Thm. 5.17] and (3b) is a special case of (3a). □

I do not know if effective descent holds for all étale morphisms when f is universally open or flat and quasi-compact. It is also possible that "universally subtrusive" can be replaced with "universally submersive", cf. [Ryd10, Ex. 5.24].

Appendix B. The AF condition

Definition (B.1). We say that a scheme X is AF (affine finie) if every finite set of points is contained in an affine open subscheme. We say that a morphism $f\colon X \to S$ of algebraic spaces is AF if $X \times_S T$ is an AF-scheme for every affine scheme T and morphism $T \to S$.

The AF condition is a natural condition for a wide range of problems. It guarantees the existence of many objects in the category of schemes such as finite quotients, cf. [SGA_1, Exp. V] and Theorem (4.4), push-outs [Fer03] and the Hilbert scheme of points [Ryd11b]. Moreover, on AF-schemes étale cohomology can be calculated using Čech cohomology [Art71, Cor. 4.2], [Sch03]. The AF condition (for 2 points) also guarantees the existence of embeddings into toric varieties [Wło93].

Proposition (B.2). *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of algebraic spaces.*

(i) *Let $Y' \to Y$ be arbitrary. If f is* AF*, then so is $f'\colon X \times_Y Y' \to Y'$.*
(ii) *An* AF*-scheme is separated.*
(iii) *An* AF*-morphism is strongly representable and separated.*
(iv) *If Y is* AF*, then f is* AF *if and only if X is* AF*.*
(v) *If there exists an f-ample invertible $\mathcal{O}_X$-module, then f is* AF*. In particular, (quasi-)affine and (quasi-)projective morphisms are* AF*.*
(vi) *If f is locally quasi-finite and separated, then f is* AF*.*
(vii) *If f and g are* AF*, then so is $g \circ f$.*
(viii) *If $g \circ f$ is* AF*, then so is f.*

Proof. (i) is obvious and (ii) is straightforward. (iii) and (iv) immediately follows from (ii) and the definitions. (v) follows from [$\mathrm{EGA}_{\mathrm{II}}$, Cor. 4.5.4] and (vi) follows from [LMB00, Thm. A.2] and (v). We deduce (vii) from (iv) and as the diagonal of g is locally quasi-finite and separated, we have that (viii) follows from (vi) and (vii). $\square$

Warning **(B.3).** The property AF for a morphism is *not local on the base in the Zariski topology.* For a counter-example, let Y be a projective three-fold with two smooth curves C and C' intersecting transversely at two points P and Q. Then let X be Hironaka's proper non-projective three-fold given by first blowing-up C and then the strict transform of C' in a neighborhood of P and by first blowing-up C' and then the strict transform of C in a neighborhood of Q. It is well-known that X is not AF, so the morphism $p\colon X \to Y$ is not AF. On the other hand, $p|_{Y \setminus P}$ and $p|_{Y \setminus Q}$ are compositions of two blow-ups and hence projective.

There is an analogue of Chevalley's theorem for AF-schemes: if $p\colon Z \to X$ is a finite and surjective morphism of algebraic spaces and Z is AF, then so is X, cf. [Gro10, Thm. 5.1.5] or Kollár [Kol11, Cor. 48].

The following criterion for projectivity was conjectured by Chevalley and proved by Kleiman [Kle66].

Theorem (B.4) (Chevalley–Kleiman criterion)**.** *Let X/k be a proper regular algebraic scheme. Then X is projective if and only if X is an* AF*-scheme.*

The AF condition is therefore also known as the Chevalley–Kleiman property [Kol11, Def. 47]. Note that proper singular AF-schemes need not be projective. In fact, there are singular, proper but non-projective AF-surfaces [Hor71, Nor78].

Appendix C. Strong homeomorphisms

If $f\colon X \to Y$ is a homeomorphism of *topological spaces*, then the diagonal map is a homeomorphism. If $f\colon X \to Y$ is a universal homeomorphism of *schemes*, then the diagonal morphism is a universal homeomorphism. Indeed, it is a surjective immersion, i.e., a nil-immersion.

If $f\colon X \to Y$ is a universal homeomorphism of *algebraic spaces*, however, then the diagonal is universally bijective but not necessarily a universal homeomorphism. A counterexample is given by Y as the affine line and X as a non-locally separated line. This motivates the following definition.

Definition (C.1). A morphism of algebraic stacks $f\colon \mathscr{X} \to \mathscr{Y}$ is a *strong homeomorphism* if f is a universal homeomorphism and the diagonal Δ_f is universally submersive.

Proposition (C.2). *Let $f\colon \mathscr{X} \to \mathscr{Y}$ and $g\colon \mathscr{Y} \to \mathscr{Z}$ be morphisms of algebraic stacks.*

(i) *A separated universal homeomorphism is a strong homeomorphism.*
(ii) *If f is a representable universal homeomorphism, then f is a strong homeomorphism if and only if f is locally separated, or equivalently, if and only if f is separated.*
(iii) *If f and g are strong homeomorphisms, then so is $g \circ f$.*
(iv) *If $g \circ f$ is a strong homeomorphism and f is universally submersive, then g is a strong homeomorphism.*
(v) *If $g \circ f$ is a strong homeomorphism and g is a representable strong homeomorphism, then f is a strong homeomorphism.*

Proof. Standard. □

Corollary (C.3). *Let $\mathscr{X}$ be an algebraic stack and let X and Y be algebraic spaces together with morphisms $f\colon \mathscr{X} \to X$ and $g\colon X \to Y$. If $g \circ f$ is a strong homeomorphism and f is a universal homeomorphism, then g is separated.*

Appendix D. Weil restriction

In Section 5, we will have use of the *Weil restriction*, cf. [BLR90, §7.6], which is defined as follows.

Definition (D.1). Let $X \to S$ and $Z \to X$ be morphisms of algebraic spaces. The Weil restriction of Z along $X \to S$ is the contravariant functor $\mathbf{R}_{X/S}(Z)$ which takes an S-scheme T to the set $\mathrm{Hom}_{X\times_S T}(X \times_S T, Z \times_S T)$, that is, the set of sections of $Z \times_S T \to X \times_S T$.

Remark **(D.2).** When the functor $\mathbf{R}_{X/S}(Z)$ is representable by a scheme, it is denoted by $\Pi_{X/S}Z$ in [FGA, No. 195, §C 2]. Let $X \to S$ and $Y \to S$ be algebraic spaces. Then $\mathbf{R}_{X/S}(Z)$ is a generalization of the functor $T \mapsto \mathrm{Hom}_T(X_T, Y_T)$ where $X_T = X \times_S T$ and $Y_T = Y \times_S T$. In fact, $\mathrm{Hom}_T(X_T, Y_T) = \mathbf{R}_{X/S}(X \times_S Y)(T)$.

It is easily seen that if $X \to S$ is flat, proper and of finite presentation and $Z \to X$ is separated, then $\mathbf{R}_{X/S}(Z)$ is an open subfunctor of the Hilbert functor $\mathrm{Hilb}(Z/S)$. If in addition $Z \to S$ is locally of finite presentation, then $\mathbf{R}_{X/S}(Z)$ is an algebraic space, locally of finite presentation and separated over S. In fact, Artin has shown the algebraicity of $\mathrm{Hilb}(Z/S)$ under these hypotheses [Art69a, Cor. 6.2]. If $X \to S$ is flat, *finite* and of finite presentation and $Z \to X$ is *arbitrary*, then $\mathbf{R}_{X/S}(Z)$ is also an algebraic space [Ryd11b].

Proposition (D.3). *Let $f\colon X \to S$ be a flat and proper morphism of finite presentation between algebraic spaces and let $Z \to X$ be an étale and separated morphism. Then $\mathbf{R}_{X/S}(Z) \to S$ is étale and separated. If $X \to S$ has a section, then there is an étale and separated morphism $\mathbf{R}_{X/S}(Z) \to Z \times_X S$.*

Proof. As $\mathbf{R}_{X/S}(Z) \to S$ is locally of finite presentation and separated, it is enough to show that $\mathbf{R}_{X/S}(Z) \to S$ is also formally étale. This follows easily from the functorial description of $\mathbf{R}_{X/S}(Z)$. Actually, if we identify étale morphisms with sheaves of sets in the small étale site, then $\mathbf{R}_{X/S}(Z)$ is nothing but the étale sheaf f_*Z. If $X \to S$ has a section $s\colon S \to X$ and T is any S-scheme, then there is a natural map

$$\mathrm{Hom}_X(X \times_S T, Z) \to \mathrm{Hom}_X(T, Z) = \mathrm{Hom}_S(T, Z \times_X S)$$

which induces an S-morphism $\mathbf{R}_{X/S}(Z) \to Z \times_X S$. As $\mathbf{R}_{X/S}(Z)$ and $Z \times_X S$ are étale over S, it follows that $\mathbf{R}_{X/S}(Z) \to Z \times_X S$ is étale. $\square$

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