EXISTENCE OF QUOTIENTS BY FINITE GROUPS AND COARSE MODULI SPACES

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Abstract. In this paper we prove the existence of several quotients in a very general setting. We consider finite group actions and more generally groupoid actions with finite stabilizers generalizing the results of Keel and Mori. In particular we show that any algebraic stack with finite inertia stack has a coarse moduli space. We also show that any algebraic stack with quasi-finite diagonal has a locally quasi-finite flat cover. The proofs do not use noetherian methods and are valid for general algebraic spaces and algebraic stacks.

Introduction

Let G be a finite group scheme acting on a separated algebraic space X. Then Deligne has proved that a geometric quotient X/G exists but without any published proof, cf. [Knu71, p.183]. The key tool in this proof is the usage of *fix-point reflecting* morphisms. Kollár developed Deligne's ideas in [Kol97] and applied these in two ways. Firstly, he showed that a geometric quotient of a proper group action is categorical in the category of algebraic spaces. Secondly, he showed the existence of a geometric quotient of the action of the symmetric group on X^n when S is locally noetherian and X is locally of finite type over S. More general results on the existence of quotients are then obtained by other methods. Under the same hypotheses on X and S, the general results of Keel and Mori [KM97] show the existence of X/G but the affine étale covering is not as explicit. We will prove a generalization of Keel and Mori's result and give more explicit affine étale coverings of finite quotients.

In the first part of this paper, we extend Deligne's and Kollár's ideas further. The setting is slightly more general, considering groupoid actions and not only group actions, and without finiteness assumptions on X and S. We replace the condition that the action should be proper with the weaker condition that the quotient should be *strongly geometric*. We show that under additional weak assumptions a strongly geometric quotient satisfies the descent condition [Kol97, 2.14] and is categorical. Under the same additional weak assumptions, we also positively answers Kollár's conjecture [Kol97, Rmk. 2.20] that any geometric quotient is categorical among locally separated algebraic spaces, cf. Theorem (3.15).

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In the second part, we show the existence of a geometric quotient X/G when G is a finite group and X is separated, cf. Theorem (5.4). We also give an explicit étale covering of X/G of the form U/G which is particularly nice for the symmetric product, cf. Theorem (5.5). Such an explicit cover is needed to obtain properties for $\operatorname{Sym}^n(X/S)$ from the affine case as is done in [ES04, Ryd07a, RS07]. Even if X is a separated scheme X/G need not be a scheme. In fact, we show that a necessary and sufficient condition for X/G to be a scheme is that every G-orbit is contained in an affine open subset of X, cf. Remark (4.9).

In the third part, we show that any algebraic stack with quasi-finite diagonal has a flat and locally quasi-finite presentation. This is well-known for finitely presented stacks [Gab63, Lem. 7.2] but a careful proof is needed for arbitrary stacks. Our proof is however remarkably simple after showing that every point is algebraic. The quasi-finite flat presentation is then obtained from a flat presentation by slicing, exactly as for schemes. We also obtain a new proof of the fact [LMB00, Thm. 8.1] that algebraic stacks with unramified diagonal are Deligne-Mumford.

Finally, we give a full generalization of the Keel-Mori theorem:

Theorem. *Let S be a scheme and X/S an algebraic space. Let $R \rightrightarrows X$ be a flat groupoid locally of finite presentation with quasi-compact diagonal $j : R \to X \times_S X$. If the stabilizer $j^{-1}\big(\Delta(X)\big) \to X$ is* finite *then there is a uniform geometric and categoric quotient $X \to X/R$ such that*

(i) *$X/R \to S$ is separated if and only if j is finite.*
(ii) *$X/R \to S$ is locally of finite type if S is locally noetherian and $X \to S$ is locally of finite type.*
(iii) *$R \to X \times_{X/R} X$ is proper.*

In the original theorem [KM97, Thm. 1], the base scheme S is assumed to be locally noetherian and X/S to be locally of finite presentation. This additional assumption on S was subsequently eliminated by Conrad in [Con05]. With the methods applied in this paper, no finiteness assumptions on X/S are needed.

The hypothesis that the stabilizer is finite implies that the diagonal j is separated and quasi-finite, cf. [KM97, Lemma. 2.7]. The stack $\mathscr{X} = [R \rightrightarrows X]$ is thus an Artin stack with quasi-finite diagonal. The quotient X/R is the *coarse moduli space* of $\mathscr{X}$. The stabilizer is a pull-back of the *inertia stack* $I_{\mathscr{X}} \to \mathscr{X}$. Rephrased in the language of stacks our generalization of the Keel-Mori theorem takes the following form:

Theorem. *Let $\mathscr{X}$ be an algebraic stack. Then a coarse moduli space $\pi : \mathscr{X} \to X$ such that π is separated exists if and only if $\mathscr{X}$ has finite inertia. In particular, any separated Deligne-Mumford stack has a coarse moduli space.*

The "only if" part follows from the observation that if a separated morphism π exists, then the inertia is proper.

This paper resulted from an attempt to understand basic questions about group actions and quotients. As a consequence, sections §§1–3 are written in a more general setting than needed for our generalization of the Keel-Mori theorem. The impatient reader mainly interested in the Keel-Mori theorem is encouraged to go directly to §7.

Assumptions and terminology. All schemes and algebraic spaces are assumed to be quasi-separated. We also require, as in [LMB00], that all algebraic stacks should have quasi-compact and separated diagonal. We will work over an arbitrary algebraic space S.

In practice, all groups schemes are flat, separated and of finite presentation over the base and many of the results will require one or several of these hypotheses. However, we will not make any general assumptions. Groups that are finite, flat and locally of finite presentation, or equivalently groups that are finite and locally free will be particularly frequent.

We follow the terminology of EGA with one exception. As in [Ray70] and [LMB00] we mean by unramified a morphism locally of finite type and formally unramified but not necessarily locally of finite presentation.

The usage of noetherian methods is limited to the proofs of the effective descent results for étale morphisms in the appendix.

Structure of the article. We begin with some general definitions and properties of quotients in §§1–2. Quotients are treated in full generality. In particular we do not assume that the groupoids are fppf. Most noteworthy is perhaps that we require topological quotients to be universally submersive in the constructible topology. This is a technical condition that is automatically satisfied in many cases, e.g. if the quotient is locally of finite presentation, quasi-compact or universally open. We also introduce the notion of strongly topological quotients. From section §4 and on all groupoids are fppf.

In §3 we generalize the results of Kollár [Kol97] on topological quotients, fix-point reflecting morphisms and the descent condition in two directions. Firstly, we replace the condition than that the group should act properly with the weaker condition that the quotient should be strongly topological. Secondly, we note that *integral* morphisms satisfies effective descent for étale morphisms. This allows us to treat the general case without finiteness assumptions. We denote (strongly) geometric quotients, which satisfy the descent condition, GC quotients and show that these are categorical quotients. Further, if a groupoid has a fix-point reflecting étale cover, the existence of a GC quotient is equivalent to the existence of a GC quotient for the cover.

In §4 we give an overview of well-known results on the existence and properties of quotients of affine schemes by finite flat groupoids. In this generality, the results are due to Grothendieck.

In §5 we use the results of §3 to deduce the existence of finite quotients for arbitrary algebraic space from the affine case. What is needed is a fix-point reflecting étale cover with an essentially affine scheme and this is accomplished using Hilbert schemes.

In §6 we show that every stack with quasi-finite diagonal has a locally quasi-finite flat presentation and that every point of such a stack is algebraic.

In §7 we restate the results of §3 in terms of stacks. We then deduce the existence of a coarse moduli space to any stack $\mathscr{X}$ with finite inertia stack, from the case where $\mathscr{X}$ has a finite flat presentation. Here, what we need is a fix-point reflecting étale cover $\mathscr{W}$ of the stack $\mathscr{X}$ such that the cover $\mathscr{W}$

admits a finite flat presentation. This is accomplished using the results of the previous section and using Hilbert schemes similar as in §5.

In the appendix the results needed on universal submersions, Hilbert schemes and descent are collected.

The existence results of §5 follows from the independent and more general results of §7 but the presentations are not the same. In §5 we begin with an algebraic space X with an action of a finite groupoid and constructs an essentially affine cover U with an action of the same groupoid. In §7 we begin by modifying the groupoid obtaining a quasi-finite groupoid action on an essentially affine scheme X. Then we take a covering which has an action of a finite groupoid.

Comparison of methods. The main steps in proving the generalization of the Keel-Mori theorem are the following:

(i) We find a quasi-finite flat cover of $\mathscr{X}$ (Theorem 6.10).
(ii) We find an étale representable cover $\mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ has a finite flat cover $V \to \mathscr{W}$ with V a quasi-affine scheme (Proposition 7.12).
(iii) We show that $\mathscr{W} \to \mathscr{X}$ is fix-point reflecting over an open subset $\mathscr{W}|_{\mathrm{fpr}}$ and that $\mathscr{W}|_{\mathrm{fpr}} \to \mathscr{X}$ is surjective. (Proposition 7.5).
(iv) We deduce the existence of a coarse moduli space of $\mathscr{X}$ from the existence of a coarse moduli space of $\mathscr{W}$ (Theorem 7.11).

The assumption that $\mathscr{X}$ has finite inertia is only used in step (iii).

Keel and Mori [KM97] more or less proceed in the same way. However, using stacks, as in [Con05], instead of groupoids, as in [KM97], gives a more streamlined presentation and simplifies many proofs. In particular the reduction in (iv) from the quasi-finite case to the finite case becomes much more transparent. Using the descent condition in (iv) as we do also simplifies several of the proofs, in particular [Con05, Thm. 3.1 and Thm. 4.2]. We also avoid the somewhat complicated limit methods used in [Con05, §5].

1. Groupoids and stacks

Let G/S be a group scheme, or more generally an algebraic group space (a group object in the category of algebraic spaces), and X/S an algebraic space. An action of G on X is a morphism $\sigma : G \times_S X \to X$ compatible with the group structure on G. The group action σ gives rise to a pre-equivalence relation $G \times_S X \overset{\sigma}{\underset{\pi_2}{\rightrightarrows}} X$, where π_2 is the second projection, i.e. a *groupoid in algebraic spaces*:

Definition (1.1). Let S be an algebraic space. An *S-groupoid in algebraic spaces* consists of two algebraic S-spaces R and U together with morphisms

(i) source and target $s, t : R \to U$.
(ii) composition $c : R \times_{(s,t)} R \to R$.
(iii) identity $e : U \to R$.
(iv) inverse $i : R \to R$.

such that $\big(R(T), U(T), s, t, c, e, i\big)$ is a groupoid in sets for every affine S-scheme T in a functorial way. We will denote the groupoid by $R \underset{t}{\overset{s}{\rightrightarrows}} U$ or (R, U). A morphism of groupoids $f : (R, U) \to (R', U')$ consists of two morphisms $R \to R'$ and $U \to U'$, which we also denote f, such that $f : (R(T), U(T)) \to (R'(T), U'(T))$ is a morphism of groupoids in sets for every affine S-scheme T.

Remark **(1.2).** The inverse $i : R \to R$ is an isomorphism such that $i \circ i = \mathrm{id}_R$ and $s = t \circ i$. Thus s has a property if and only if t has the same property. Let G/S be a group scheme acting on an algebraic space X/S and let $G \times_S X \underset{t}{\overset{s}{\rightrightarrows}} X$ be the associated groupoid. If G/S has a property stable under base change, then s and t have the same property.

Notation (1.3). By a *groupoid* we will always mean a groupoid in algebraic spaces. If $R \rightrightarrows U$ is a groupoid then we let j be the diagonal morphism

$$j = (s, t) : R \to U \times_S U.$$

The *stabilizer* of the groupoid is the morphism $S(U) := j^{-1}\big(\Delta(U)\big) \to U$ which is an algebraic group space. We say that $R \rightrightarrows U$ is flat, locally of finite presentation, quasi-finite, etc. if s, or equivalently t, is flat, locally of finite presentation, quasi-finite, etc. As usual, we abbreviate "faithfully flat and locally of finite presentation" by fppf.

Remark **(1.4).** When $R \rightrightarrows U$ is an fppf groupoid such that j is quasi-compact and separated, then the associated (fppf-)stack $[R \rightrightarrows U]$ is algebraic, i.e. an Artin stack [LMB00, Cor. 10.6]. In particular, when G/S is a flat and separated group scheme of finite presentation, the stack $[X/G] = [G \times_S X \rightrightarrows X]$ is algebraic.

Definition (1.5). Let $\mathscr{X}$ be an algebraic stack. The *inertia stack* $I_{\mathscr{X}} \to \mathscr{X}$ is the pull-back of the diagonal $\Delta_{\mathscr{X}/S}$ along the same morphism $\Delta_{\mathscr{X}/S}$. The inertia stack is independent of the base S.

Remark **(1.6).** Let $\mathscr{X}$ be an algebraic stack with a smooth or flat presentation $p : U \to \mathscr{X}$ with U an algebraic space and let $R = U \times_{\mathscr{X}} U$. Then $\mathscr{X} \cong [R \rightrightarrows U]$ and we have 2-cartesian diagrams

$$\text{(1.6.1)} \qquad \begin{array}{ccc} R & \xrightarrow{\ j\ } & U \times_S U \\ \downarrow & \square & \downarrow{\scriptstyle p\times p} \\ \mathscr{X} & \xrightarrow{\ \Delta_{\mathscr{X}/S}\ } & \mathscr{X} \times_S \mathscr{X} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S(U) & \longrightarrow & U \\ \downarrow & \square & \downarrow{\scriptstyle p} \\ I_{\mathscr{X}} & \longrightarrow & \mathscr{X}. \end{array}$$

2. General remarks on quotients

2.1. **Topological, geometric and categorical quotients.**

Definition (2.1). Let $R \rightrightarrows X$ be an S-groupoid. If $q : X \to Y$ is a morphism such that $q \circ s = q \circ t$ we say that q is a *quotient*. Then $R \rightrightarrows X$ is

also a Y-groupoid and $j : R \to X \times_S X$ factors through $X \times_Y X \hookrightarrow X \times_S X$. We denote the morphism $R \to X \times_Y X$ by j_Y.

If $Y' \to Y$ is a morphism we let $R' = R \times_Y Y'$ and $X' = X \times_Y Y'$. Then $q' : X' \to Y'$ is a quotient of the groupoid $R' \rightrightarrows X'$. If a property of q is stable under flat base change $Y' \to Y$ we say that the property is *uniform*. If it is stable under arbitrary base change, we say that it is *universal*. If q is a quotient then we say that

(i) q is a *categorical* quotient (with respect to a full subcategory $\mathcal{C}$ of the category of algebraic spaces) if q is an initial object among quotients (in $\mathcal{C}$). Concretely this means that for any quotient $r : X \to Z$ there is a unique morphism $Y \to Z$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ & \searrow^{r} & \downarrow \\ & & Z \end{array}$$

commutes.

(ii) q is a *Zariski* quotient if the underlying morphism of topological spaces $|q| : |X| \to |Y|$ is a quotient in the category of topological spaces. Equivalently the fibers of q are the orbits under R of X and q is submersive, i.e. $U \subseteq Y$ is open if and only if $q^{-1}(U)$ is open.

(iii) q is a *constructible* quotient if the morphism of topological spaces $q^{\mathrm{cons}} : X^{\mathrm{cons}} \to Y^{\mathrm{cons}}$, cf. §A.1, is a quotient in the category of topological spaces. Equivalently the fibers of q are the R-orbits of X and q^{cons} is submersive.

(iv) q is a *topological* quotient if it is both a universal Zariski quotient and a universal constructible quotient.

(v) q is a *strongly topological* quotient if it is a topological quotient and $j_Y : R \to X \times_Y X$ is universally submersive.

(vi) q is a *geometric* quotient if it is a topological quotient and $\mathcal{O}_Y = (q_*\mathcal{O}_X)^R$, i.e. the sequence of sheaves in the étale topology

$$\mathcal{O}_Y \longrightarrow q_*\mathcal{O}_X \underset{t^*}{\overset{s^*}{\rightrightarrows}} (q \circ s)_*\mathcal{O}_R \tag{2.1.1}$$

is exact.

(vii) q is a *strongly geometric* quotient if it is geometric and strongly topological.

Remark **(2.2).** Propositions (A.3) and (2.9) shows that a universal Zariski quotient is a universal constructible quotient in the following cases:

(i) q is quasi-compact.
(ii) q is locally of finite presentation.
(iii) q is universally open.
(iv) s is proper (then q is universally closed and quasi-compact).
(v) s is universally open (then q is universally open).

In most applications s will be fppf and thus universally open. This also implies that q is universally open, cf. Proposition (2.9).

Remark **(2.3).** The definitions of topological and geometric quotients given above are not standard but generalize other common definitions. In [Kol97], the conditions on topological and geometric quotients are slightly stronger: All algebraic spaces are assumed to be locally noetherian and topological and geometric quotients are required to be locally of finite type.

In [MFK94] the word topological is not defined explicitly but if we take topological to mean the three first conditions of a geometric quotient in [MFK94, Def. 0.6], taking into account that iii) should be universally submersive, then a topological quotient is what here is a universal Zariski quotient.

We have the following alternative descriptions of a topological quotient:

Lemma (2.4). *A quotient $q : X \to Y$ is topological if and only if q and q^{cons} are universally submersive, and the following equivalent conditions hold*

(i) *For any field k and point $y : \mathrm{Spec}(k) \to Y$ the fiber of q over y is an R-orbit.*
(ii) *The diagonal $j_Y : R \to X \times_Y X$ is surjective.*

If in addition q is locally of finite type or integral then these two conditions are equivalent to

(iii) *For any algebraically closed field K the map $q(K) : X(K)/R(K) \to Y(K)$ is a bijection.*

A quotient $q : X \to Y$ is strongly topological if and only if q, q^{cons} and j_Y are universally submersive.

Remark **(2.5).** The condition, for a strongly topological quotient, that j_Y should be universally submersive is natural. Indeed, this ensures that the equivalence relation $X \times_Y X \hookrightarrow X \times_S X$ has the quotient topology induced from the groupoid. When Y is a *scheme*, or more generally a locally separated algebraic space, the monomorphism $X \times_Y X \hookrightarrow X \times_S X$ is an immersion. In this case, the topology on $X \times_Y X$ is induced by $X \times_S X$ and does not necessarily coincide with the quotient topology induced by $R \twoheadrightarrow X \times_Y X$.

If the groupoid has *proper* diagonal $j : R \to X \times_S X$ then every topological quotient is strongly topological. This explains why proper group actions are more amenable. An important class of strongly topological quotients are quotients such that $j_Y : R \to X \times_Y X$ is proper.

Remark **(2.6).** A geometric quotient of schemes is always categorical in the category of schemes but not necessarily in the category of algebraic spaces. As Kollár mentions in [Kol97, Rmk. 2.20] it is likely that every geometric quotient is categorical in the category of locally separated algebraic spaces. This is indeed the case, at least for universally open quotients, as shown in Theorem (3.15).

A natural condition, ensuring that a geometric quotient is categorical among all algebraic spaces, is that the *descent condition*, cf. Definition (3.6), should be fulfilled. Universally open and *strongly geometric* quotients satisfy the descent condition, cf. Theorem (3.15).

Remark **(2.7).** Conversely a (strongly) topological and uniformly categorical quotient is (strongly) geometric. This is easily seen by considering quotients $X \to \mathbb{A}^1_{\mathbb{Z}}$. Kollár has also shown that if G is an affine group, flat and locally of finite type over S acting *properly* on X, cf. Remark (2.10), such that a topological quotient exists, then a geometric quotient exists [Kol97, Thm. 3.13].

Proposition (2.8). *Let $R \rightrightarrows X$ be a groupoid and let $q : X \to Y$ be a quotient. Further, let $f : Y' \to Y$ be a* flat *morphism and let $q' : X' \to Y'$ be the pull-back of q along f. Then*

(i) *If q is a geometric quotient then so is q'.*
(ii) *Assume that f is fpqc or fppf, i.e. faithfully flat and quasi-compact or faithfully flat and locally of finite presentation. Then q is a topological (resp. geometric, resp. universal geometric) quotient if and only if q' is a topological (resp. geometric, resp. universal geometric) quotient.*

In particular, a geometric quotient is always uniform. The statements remain valid if we replace "topological" and "geometric" with "strongly topological" and "strongly geometric".

Proof. Topological and strongly topological quotients are always universal. Part (ii) for topological and strongly topological quotients follows immediately as fpqc and fppf morphisms are submersive in both the Zariski and the constructible topology. What remains to be shown concerns the exactness of (2.1.1). In [MFK94, Rem. (7), p. 9] the case when G/S is a group scheme acting on X and $R = G \times_S X$ is handled. The general case is proven similarly. □

Proposition (2.9). *Let $q : X \to Y$ be a quotient of the groupoid $R \rightrightarrows X$. If q is topological and s has one of the properties: universally open, universally closed, quasi-compact; then so does q.*

Proof. Note that j_Y is surjective as q is topological. If s is universally open (resp. universally closed, resp. quasi-compact) then so is the projection $\pi_1 : X \times_Y X \to X$. As q and q^{cons} are universally submersive we have that q is universally open (resp. universally closed, resp. quasi-compact) by Propositions (A.1) and (A.2). □

2.2. **Separation properties.** Even if X is separated a quotient Y need not be. A sufficient criterion is that the groupoid has proper diagonal and a precise condition, for *schemes*, is that the image of the diagonal is closed.

Remark **(2.10).** Consider the following properties of the diagonal of a groupoid $R \rightrightarrows X$.

(i) The diagonal $j : R \to X \times_S X$ is proper.
(ii) The diagonal $j_Y : R \to X \times_Y X$, with respect to a quotient $q : X \to Y$, is proper.
(iii) The diagonal $j : R \to X \times_S X$ is quasi-compact.
(iv) The stabilizer morphism $S(X) = j^{-1}\big(\Delta(X)\big) \to X$ is proper.
(v) The diagonal $j : R \to X \times_S X$ is a monomorphism.

If s and t are fppf, then by the cartesian diagrams in (1.6.1) these properties correspond to the following separation properties of the stack $\mathscr{X} = [R \rightrightarrows X]$.

(i) $\mathscr{X}$ is separated (over S).
(ii) $\mathscr{X} \to Y$ is separated.
(iii) $\mathscr{X}$ is quasi-separated.
(iv) The inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ is proper.
(v) The stack $\mathscr{X}$ is a sheaf (and representable by an algebraic space if it is quasi-separated).

If $q : X \to Y$ is a topological quotient, then these properties imply that

(i) Y is separated (over S).
(iii) Y is quasi-separated.

If R is the groupoid associated to a group action, the group action is called *proper* if (i) holds. There is also the notion of a *separated* group action which means that the diagonal j has closed image.

Proposition (2.11). *Let $R \rightrightarrows X$ be a groupoid and let $q : X \to Y$ be a topological quotient. Then*

(i) *If Y is locally separated, then Y is separated if and only if the image of j is closed.*
(ii) *The diagonal j is proper if and only if Y is separated and j_Y is proper.*
(iii) *If j_Y is proper then j is quasi-compact and the stabilizer is proper.*

Proof. As q is a topological quotient we have that q is universally submersive and that the image of j is $X \times_Y X$. The statements then follow easily from Proposition (A.1) and the cartesian diagram

$$\begin{array}{ccc} X \times_Y X & \hookrightarrow & X \times_S X \\ \downarrow & \square & \downarrow{\scriptstyle q\times q} \\ Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y \end{array}$$

noting that $\Delta_{Y/S}$ is quasi-compact as Y is quasi-separated. □

Remark **(2.12).** [Con05, Cor. 5.2] — If there exists a quotient $q : X \to Y$ such that the diagonal j_Y is proper then the groupoid has a proper stabilizer. Further, if j_Y is proper and $r : X \to Z$ is a *categorical* quotient, then j_Z is proper. Thus if $\mathscr{X}$ is a stack such that there is a separated morphism $r : \mathscr{X} \to Z$ to an algebraic space Z, then $\mathscr{X}$ has a proper inertia stack and a categorical quotient $q : \mathscr{X} \to Y$, if it exists, is separated.

The Keel-Mori theorem (7.13) asserts that conversely, if the stabilizer map $j^{-1}\big(\Delta(X)\big) \to X$ is *finite* then there exists a geometric and categorical quotient X/R and $j_{X/R}$ is proper.

As we will be particulary interested in finite groupoids we make the following observation:

Proposition (2.13). *If $R \rightrightarrows X$ is a* proper *groupoid, i.e. s and t are proper, and $q : X \to Y$ is a topological quotient then*

(i) *The diagonal j is proper if and only if X is separated.*

(ii) *The diagonal j_Y is proper if and only if q is separated.*

Proof. As s is separated the section $e : X \to R$ is closed. Thus $\Delta_{X/S} = j \circ e : X \hookrightarrow R \to X \times_S X$ is closed if j is proper. Conversely if X is separated then as $s = \pi_1 \circ j$ is proper it follows that j is proper. (ii) follows from (i) if we consider the S-groupoid as a Y-groupoid. □

2.3. **Free actions.**

Definition (2.14). We say that the groupoid $R \rightrightarrows X$ is an *equivalence relation* if $j : R \to X \times_S X$ is a monomorphism. Let X/S be an algebraic space with an action of a group G. We say that G acts *freely* if the associated groupoid is an equivalence relation, i.e. if the morphism $j : G \times_S X \to X \times_S X$ is a monomorphism.

Theorem (2.15). *Let $R \rightrightarrows X$ be an fppf equivalence relation with quasi-compact diagonal $j : R \to X \times_S X$. Then there is a universal geometric and categorical quotient $q : X \to X/R$ in the category of algebraic spaces. Furthermore q is the quotient of the equivalence relation in the fppf topology and hence q is fppf.*

Proof. Let X/R be the quotient sheaf of the equivalence relation $R \rightrightarrows X$ in the fppf topology. Then X/R is an algebraic space by [Art74, Cor. 6.3] as explained in [LMB00, Cor. 10.4]. As X/R is a categorical quotient in the category of sheaves on the fppf topology, it is a categorical quotient in the category of algebraic spaces. As taking the quotient sheaf commutes with arbitrary base change, it is further a universal categorical quotient.

The quotient q is fppf and thus universally submersive in both the Zariski and constructible topology. As the fibers are clearly the orbits it is thus a topological quotient. It is then a universal geometric quotient by Remark (2.7). □

Corollary (2.16). *Let G/S be a group scheme, flat and of finite presentation over S. Let X/S be an algebraic space with a free action of G. Then there is a universal geometric and categorical quotient $q : X \to X/G$ in the category of algebraic spaces and q is fppf.*

Proof. Follows immediately from Theorem (2.15) as the properties of G implies that the associated groupoid $G \times_S X \rightrightarrows X$ is an fppf equivalence relation with quasi-compact diagonal. □

3. Fix-point reflecting morphisms and the descent condition

Let X and Y be algebraic spaces with an action of a group G such that geometric quotients X/G and Y/G exists. If $f : X \to Y$ is any étale G-equivariant morphism, then in general the induced morphism $f/G : X/G \to Y/G$ is not étale. The notion of fix-point reflecting morphisms was introduced to remedy this problem. Under mild hypotheses f/G is étale if f is fix-point reflecting. Moreover it is then also possible to assert the existence of X/G from the existence of Y/G and furthermore $X = X/G \times_{Y/G} Y$.

An interpretation of this section in terms of stacks is given in §7.

Remark **(3.1).** Recall, cf. Notation (1.3), that the stabilizer of a groupoid $R \rightrightarrows X$ is the morphism $S(X) = j^{-1}\big(\Delta(X)\big) \to X$. The stabilizer of a point $x \in X$ is the fiber $S(x) = j^{-1}(x, x)$ which is a group scheme over $k(x)$. Assume that $R \rightrightarrows X$ is the groupoid associated to the action of a group G on X. Set-theoretically the stabilizer of x is then all group elements $g \in G$ such that $g(x) = x$ and g acts trivially on the residue field $k(x)$.

Definition (3.2). Let $f : (R_X, X) \to (R_Y, Y)$ be a morphism of groupoids. We say that f is *square* if the two commutative diagrams

$$\begin{array}{ccc} R_X & \xrightarrow{f} & R_Y \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} R_X & \xrightarrow{f} & R_Y \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{f} & Y \end{array}$$

are cartesian.

Definition (3.3). Let $f : (R_X, X) \to (R_Y, Y)$ be a morphism of groupoids. We say that f is *fix-point reflecting*, abbreviated fpr, if the canonical morphism of stabilizers $S(x) \to S(f(x)) \times_{k(f(x))} \operatorname{Spec}(k(x))$ is an isomorphism for every point $x \in X$. We let $\operatorname{fpr}(f) \subseteq X$ be the subset over which f is fix-point reflecting. This is an R_X-invariant subset.

Remark **(3.4).** If $X \to Z$ is a quotient of the groupoid (R_X, X) and $W \to Z$ is any morphism, then $W \times_Z X \to X$ is fpr.

The following proposition sheds some light over the importance of proper stabilizer.

Proposition (3.5). *Let $f : (R_X, X) \to (R_Y, Y)$ be a square morphism of groupoids such that $f : X \to Y$ is* unramified. *If the stabilizer $S(Y) \to Y$ is* proper *then the subset* $\operatorname{fpr}(f)$ *is open in X.*

Proof. There are cartesian diagrams

$$\begin{array}{ccccccc} S(X) & \hookrightarrow & S(Y) \times_Y X & \hookrightarrow & R_X & \longrightarrow & R_Y \\ \downarrow & \square & \downarrow & \square & \downarrow{\scriptstyle j} & & \\ X & \xrightarrow{\Delta_{X/Y}} & X \times_Y X & \hookrightarrow & X \times_S X & \square & \downarrow{\scriptstyle j} \\ & & \downarrow{\scriptstyle \pi_2} & \square & \downarrow & & \\ & & X & \hookrightarrow & Y \times_S X & \longrightarrow & Y \times_S Y. \end{array}$$

A point $x \in X$ is fpr if and only if x is not in the image of $S(Y) \times_Y X \setminus S(X)$ by the proper morphism $S(Y) \times_Y X \to X$, which is the second column in the diagram. As f is unramified $\Delta_{X/Y}$ is an open immersion and hence $S(Y) \times_Y X \setminus S(X)$ is closed. Thus $\operatorname{fpr}(f)$ is open. □

Definition (3.6). Let $R_X \rightrightarrows X$ be a groupoid and let $q : X \to Z_X$ be a topological quotient. We say that q satisfies the *descent condition* if for any étale, separated, square and fix-point reflecting morphism of groupoids $f : (R_W, W) \to (R_X, X)$ there exists an algebraic space Z_W and a cartesian

square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ \downarrow & \square & \downarrow q \\ Z_W & \longrightarrow & Z_X \end{array} \tag{3.6.1}$$

where $Z_W \to Z_X$ is *étale*. We say that q satisfies the *weak descent condition* if the descent condition holds when restricted to morphisms f such that there is a cartesian square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ \downarrow & \square & \downarrow q \\ Q' & \longrightarrow & Q \end{array}$$

where $X \to Q$ is a quotient to a *locally separated* algebraic space Q and $Q' \to Q$ is an étale morphism.

Remark **(3.7).** If a space Z_W exists as above, then $W \to Z_W$ is a topological quotient and if q is a geometric quotient then $W \to Z_W$ is also a geometric quotient by Proposition (2.8). In [KM97, 2.4] the existence of Z_W is not a part of the descent condition and f need not be separated.

Proposition (3.8). *Let $f : (R_W, W) \to (R_X, X)$ be an étale, square and fix point reflecting morphism and let $X \to Z_X$ be a topological quotient. Consider the following diagram*

$$\begin{array}{ccccc} & & W \times_{Z_X} X & & \\ & \overset{(s,f\circ t)}{\nearrow} & & \overset{f\times \mathrm{id}_X}{\searrow} & \\ R_W & & \downarrow \varphi & & X \times_{Z_X} X. \\ & \underset{(f\circ s,t)}{\searrow} & & \underset{\mathrm{id}_X\times f}{\nearrow} & \\ & & X \times_{Z_X} W & & \end{array} \tag{3.8.1}$$

There is a one-to-one correspondence between on one hand quotients $W \to Z_W$ such that $Z_W \to Z_X$ is étale and the diagram (3.6.1) *is cartesian and on the other hand effective descent data (f, φ) such that φ is an isomorphism fitting into the diagram* (3.8.1)*. Let Γ be the set-theoretic image of the morphism*

$$\gamma : R_W \xrightarrow{\big((s,f\circ t),(f\circ s,t)\big)} (W \times_{Z_X} X) \times_{X\times_{Z_X} X} (X \times_{Z_X} W). \tag{3.8.2}$$

An isomorphism φ as above exists if and only if Γ is open. If this is the case, the graph of φ coincides with the subspace induced by Γ. In particular, there is at most one isomorphism φ as in diagram (3.8.1) *and at most one quotient Z_W in the descent condition* (3.6)*.*

Moreover, the subset Γ is open if and only if the second projection

$$(W \times_{Z_X} X) \times_{X\times_{Z_X} X} (X \times_{Z_X} W) \to X \times_{Z_X} W$$

is universally submersive over Γ.

Proof. There is a correspondence between étale morphisms $Z_W \to Z_X$ such that $W = Z_W \times_{Z_X} X$ and effective descent data (f, φ) where $\varphi : W \times_{Z_X} X \to X \times_{Z_X} W$ is an isomorphism over $X \times_{Z_X} X$. This is because q is universally submersive and hence a morphism of descent for étale morphisms by Theorem (A.10). Given a quotient Z_W of $R_W \rightrightarrows W$ as in Definition (3.6) we have that the corresponding isomorphism φ is the composition of the canonical isomorphisms $W \times_{Z_X} X \cong W \times_{Z_W} W \cong X \times_{Z_X} W$ and that φ fits into the commutative diagram (3.8.1).

Further, since $W \to X$ is étale there is a one-to-one correspondence between morphisms $\varphi : W \times_{Z_X} X \to X \times_{Z_X} W$ and open subspaces

$$U \subseteq (W \times_{Z_X} X) \times_{X \times_{Z_X} X} (X \times_{Z_X} W)$$

such that $U \to W \times_{Z_X} X$ is universally bijective. This correspondence is given by $\varphi \mapsto \Gamma_\varphi$ [SGA$_1$, Exp. IX, Cor. 1.6]. If φ is an isomorphism fitting into diagram (3.8.1) then set-theoretically its graph Γ_φ coincides with Γ. As Γ_φ is open so is Γ.

Conversely, assume that Γ is open and let $(x, y) \in W \times_{Z_X} X$ be a point. Then there is a point $r \in R_W$, unique up to an element in the stabilizer $S(y)$, such that $(x, y) = (\pi_1 \circ \gamma)(r)$. The image of r by $\pi_2 \circ \gamma$ is $(f(x), x')$ where x' is independent upon the choice of r as $S(x') = S(y)$. Thus $\Gamma \to W \times_{Z_X} X$ is universally bijective. Similarly it follows that $\Gamma \to X \times_{Z_X} W$ is universally bijective. This shows that Γ corresponds to an isomorphism as in diagram (3.8.1).

The last statement follows from the following lemma:

Lemma (3.9). *Let $f : X \to Y$ be an étale and separated morphism of algebraic spaces. Let $Z \subseteq X$ be a subset such that $f|_Z$ is a universal homeomorphism. Then $Z \subseteq X$ is open.*

Proof. As the question is local in the étale topology on Y, we can assume that Y is an affine scheme. As f is locally quasi-finite and separated, it follows that X is a scheme [LMB00, Thm. A.2]. Let $x \in Z$ be a point with image $y = f(x) \in Y$. Let y' be the closed point in the strict henselization $Y' = \operatorname{Spec}({}^{\mathrm{sh}}\mathcal{O}_{Y,y})$ and choose a lifting x' in $Z' = g^{-1}(Z)$ where $g : X \times_Y Y' \to X$ is the pull-back of $Y' \to Y$. As $\mathcal{O}_{X',x'} = \mathcal{O}_{Y',y'}$ it follows that $Z' = \operatorname{Spec}(\mathcal{O}_{X',x'})$. Hence we have that $W \cap X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}) = \operatorname{Spec}(\mathcal{O}_{X,x})$ and that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an isomorphism. There is thus an open neighborhood $U \subseteq Y$ of y and a section s of $f^{-1}(U) \to U$ with $s(y) = x$. Since f is étale we have that $s(U)$ is open. As $Z \cap s(U)$ is open in Z, and both $f|_Z$ and $f|_{s(U)}$ are homeomorphisms onto their images, it follows that $Z \cap s(U)$ is open in X. Thus Z is open. □

Corollary (3.10). *Let $R \rightrightarrows X$ be a groupoid and let $q : X \to Z$ be a topological quotient. Further, let $g : Z' \to Z$ be a flat morphism and let $q' : X' \to Z'$ be the pull-back of q along g. Then q' is a topological quotient by Proposition* (2.8)*.*

(i) *Assume that g is étale and separated. If q satisfies the descent condition then so does q'.*
(ii) *Assume that g is fppf. Then q satisfies the descent condition if q' satisfies the descent condition.*

Proof. To prove (i), let $f' : W' \to X'$ be an étale, separated, square and fpr morphism of groupoids. As q satisfies the descent condition there is a topological quotient $W' \to Z_{W'}$ and an étale morphism $Z_{W'} \to Z_X$. Set-theoretically it is clear that $f' \times f' : W' \times_{Z_{W'}} W' \to X' \times_{Z_X} X'$ factors through the subset $X' \times_{Z_{X'}} X'$. As $X' \to X$ is étale this subset is open and thus we have a morphism $f' \times f' : W' \times_{Z_{W'}} W' \to X' \times_{Z_{X'}} X'$. As q is a morphism of descent for étale morphisms, the morphism f' descends to an étale morphism $Z_{W'} \to Z_{X'}$ such that $W' = Z_{W'} \times_{Z_{X'}} X'$.

(ii) follows from an easy application of fppf descent taking into account that by Proposition (3.8) the quotient Z_W figuring in the descent condition is unique. □

Theorem (3.11). *Let $R \rightrightarrows X$ be a groupoid and $q : X \to Z$ a topological quotient (resp. a strongly topological quotient) such that q satisfies effective descent for étale and separated morphisms. Then q satisfies the weak descent condition (resp. the descent condition).*

Proof. Let $f : W \to X$ be an étale, separated, square and fpr morphism. As q satisfies effective descent for f it is by Proposition (3.8) enough to show that the image Γ of the morphism γ in equation (3.8.2) is open, or equivalently, that $\pi_2|_\Gamma$ is universally submersive. If q is a strongly topological quotient, then $\pi_2 \circ \gamma = f^* j_Z$ is universally submersive. Thus $\pi_2|_\Gamma$ is universally submersive. This shows that q satisfies the descent condition.

Let $q : X \to Z$ be a topological quotient and let $r : X \to Q$ be a quotient such that Q is a locally separated algebraic space. Then $X \times_Q X \hookrightarrow X \times_S X$ is an *immersion*. Thus

$$\pi_1 : (X \times_Z X) \times_{X \times_S X} (X \times_Q X) \to X \times_Z X$$

is an immersion. Moreover, as $R_X \to X \times_Z X$ is surjective we have that π_1 is surjective. Hence we obtain a monomorphism $(X \times_Z X)_{\mathrm{red}} \to X \times_Q X$ over $X \times_S X$. Let $Q' \to Q$ be an étale and separated morphism and $W = X \times_Q Q'$. Then the image of

$$\gamma_Q : R_W \to (W \times_Q X) \times_{X \times_Q X} (X \times_Q W)$$

is open. As $(\gamma_Q)_{\mathrm{red}}$ factors through γ_{red} and the monomorphism

$$\big((W \times_Z X) \times_{X \times_Z X} (X \times_Z W)\big)_{\mathrm{red}} \to \big((W \times_Q X) \times_{X \times_Q X} (X \times_Q W)\big)_{\mathrm{red}}$$

it follows that the image of γ is open. □

Remark **(3.12).** The morphism q satisfies effective descent for étale and separated morphisms in the following two cases

(i) q is integral and universally open.
(ii) q is universally open and locally of finite presentation and Z is locally noetherian.

This follows from Theorems (A.13) and (A.14) respectively. Theorem (3.11) is proven in [Kol97, Thm. 2.14] under the additional assumptions that (ii) holds and that $j : R \to X \times_S X$ is proper.

In [Ryd07c] it is shown that q satisfies effective descent for, not necessarily separated, étale morphisms if q is *any* universally open morphism, cf. Theorem (A.15).

Remark **(3.13).** In [KM97] all algebraic spaces are locally noetherian and the quotients are locally of finite type and universally open. Hence they satisfy (ii). Moreover every quotient $q : X \to Y$ is such that $j_Y : R \to X \times_Y X$ is proper and thus strongly topological. All quotients in [KM97] thus satisfy the descent condition.

Proposition (3.14) ([Kol97, Cor. 2.15])**.** *Let $R \rightrightarrows X$ be a groupoid and let $q : X \to Z$ be a geometric quotient satisfying the descent condition (resp. the weak descent condition). Then q is a categorical quotient (resp. a categorical quotient among locally separated algebraic spaces).*

Proof. Let $r : X \to T$ be any quotient (resp. a quotient with T locally separated). We have to prove that there is a unique morphism $f : Z \to T$ such that $r = f \circ q$. As geometric quotients commute with open immersions, we can assume that T is quasi-compact. Let $T' \to T$ be an étale covering with T' an affine scheme. Let $X' = X \times_T T'$. As q satisfies the descent condition (resp. the weak descent condition), there is a geometric quotient $q' : X' \to Z'$ such that $X' = X \times_Z Z'$. As T' is affine, the morphism $X' \to T'$ is determined by $\Gamma(T') \to \Gamma(X')$. Further as q' is geometric, we have that the image of $\Gamma(T')$ lies in $\Gamma(Z') = \Gamma(X')^{R'}$. The homomorphism $\Gamma(T') \to \Gamma(Z')$ gives a morphism $f' : Z' \to T'$ such that $r' = f' \circ q'$ and this is the only f' with this property. By étale descent the morphism f' descends to a unique morphism $f : Z \to T$ such that $r = f \circ q$. □

The following theorem answers the conjecture in [Kol97, Rmk. 2.20] and generalizes [Kol97, Cor. 2.15].

Theorem (3.15). *Let $R \rightrightarrows X$ be a groupoid and let $q : X \to Z$ be a geometric (resp. strongly geometric) quotient. If q is universally open, then q satisfies the weak descent (resp. descent) condition. In particular q is then a categorical quotient among locally separated algebraic spaces (resp. a categorical quotient).*

Proof. See Remark (3.12) which refers to the result of [Ryd07c]. □

Definition (3.16). A GC *quotient* is a strongly geometric quotient satisfying the descent condition. As a GC quotient is categorical by Proposition (3.14) we will speak about *the* GC quotient when it exists.

Remark **(3.17).** The definition of GC quotient given by Keel and Mori differs slightly from ours. In [KM97] it simply means a geometric and uniform categorical quotient. However, all GC quotients figuring in [KM97] are strongly geometric and satisfies the descent condition by Remark (3.13).

Theorem (3.18) ([Kol97, Cor. 2.17])**.** *Let $f : (R_W, W) \to (R_X, X)$ be a surjective, étale, separated, square and fpr morphism of groupoids and let $Q = W \times_X W$. Assume that $j : R_X \to X \times_S X$ is quasi-compact and assume that a* GC *quotient $W \to Z_W$ of (R_W, W) exists. Then* GC *quotients Z_Q and Z_X exist such that Z_X is the quotient of the étale equivalence relation*

$Z_Q \rightrightarrows Z_W$. *Furthermore the natural squares of the diagram*

$$\begin{array}{ccccc} Q & \rightrightarrows & W & \xrightarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ Z_Q & \rightrightarrows & Z_W & \longrightarrow & Z_X \end{array} \tag{3.18.1}$$

are cartesian.

Proof. Since $f : W \to X$ is separated, square, étale and fix-point reflecting so are the two projections $\pi_1, \pi_2 : Q = W \times_X W \to W$. As $W \to Z_W$ satisfies the descent condition, there exists quotients $Q \to (Z_Q)_1$ and $Q \to (Z_Q)_2$ induced by the two projections π_1 and π_2. Further by Corollary (3.10) the quotients $Q \to (Z_Q)_1$ and $Q \to (Z_Q)_2$ satisfy the descent condition and it follows that $(Z_Q)_1 \cong (Z_Q)_2$ is the unique GC quotient. The two canonical morphisms $Z_Q \to Z_W$ are étale and the corresponding squares cartesian.

We have that $Z_Q \rightrightarrows Z_W$ is an equivalence relation. Indeed, as $Z_Q \to Z_W \times_S Z_W$ is unramified, it is enough to check that it is an equivalence relation set-theoretically and this is clear. Let Z_X be the quotient sheaf of the equivalence relation in the étale topology. This is an "algebraic space" except that we have not verified that it is quasi-separated. There is a canonical morphism $X \to Z_X$ and this makes the diagram (3.18.1) cartesian.

We will now prove that Z_X is quasi-separated by showing that $Z_Q \hookrightarrow Z_W \times_S Z_W$ is quasi-compact. As Z_Q and X are quasi-separated it follows that $X \times_{Z_X} X$ is an (quasi-separated) algebraic space. Further, it is easily seen that $R_X \to X \times_S X$ factors through $X \times_{Z_X} X \hookrightarrow X \times_S X$ and that $R_X \to X \times_{Z_X} X$ is surjective. As $R_X \to X \times_S X$ is quasi-compact by assumption, so is $X \times_{Z_X} X \to X \times_S X$ and $W \times_{Z_X} W \to W \times_S W$. As $W \to Z_W$ is universally submersive in the constructible topology it follows that $Z_Q = Z_W \times_{Z_X} Z_W \to Z_W \times_S Z_W$ is quasi-compact. Thus Z_X is quasi-separated and an algebraic space.

As strongly geometric quotients and the descent condition are descended by étale base change by Proposition (2.8) and Corollary (3.10), it follows that $X \to Z_X$ is a GC quotient. □

Remark **(3.19).** We can extend the notion of algebraic spaces to also include spaces which are not quasi-separated. This is possible as any monomorphism locally of finite type between schemes satisfies effective descent with respect to the fppf topology. In fact, such a monomorphism is locally quasi-finite and separated and thus quasi-affine over any quasi-compact open subset. This is also valid for monomorphisms locally of finite type between algebraic spaces [LMB00, Thm. A.2]. With this extended notion, we obtain a much neater proof of the quasi-separatedness of Z_X in Theorem (3.18).

4. Finite quotients of affine and AF-schemes

In this section we give a resume of the known results on quotients of finite locally free groupoids of affine schemes. These are then easily extended to groupoids of schemes such that every orbit is contained in an affine open subscheme. The general existence results were announced in [FGA, Exp.

212] by Grothendieck and proven in [Gab63] by Gabriel. An exposition of these results with full proofs can also be found in [DG70, Ch. III, §2].

Besides the existence results, a list of properties of the quotient when it exists is given in Proposition (4.5). This proposition is also valid for algebraic spaces.

Theorem (4.1) ([FGA, Exp. 212, Thm. 5.1])**.** *Let $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$ and $R = \operatorname{Spec}(C)$ be affine schemes and $R \rightrightarrows X$ a finite locally free S-groupoid. Let $p_1, p_2 : B \to C$ be the homomorphisms corresponding to s and t. Let $Y = \operatorname{Spec}\left(B^R\right)$ where B^R is the equalizer of the homomorphisms p_1 and p_2. Let $q : X \to Y$ be the morphism corresponding to the inclusion $B^R \hookrightarrow B$. Then q is integral and a geometric quotient.*

Proof. The theorem is proven in [Gab63, Thm. 4.1]. Another proof, for noetherian rings is given by Keel and Mori [KM97, Prop. 5.1]. In [Con05, §3] the general case is proven and it is also shown that the quotient is categorical. □

Remark **(4.2).** Theorem (4.1) has a long history and there are several proofs. See [SGA_1, Exp. V] for a good exposition for the quotient by an abstract group, i.e. for groupoids induced by a finite group G/S with the trivial scheme structure. For general group schemes a proof is given in [GM07, §4]. For algebraic group schemes the result can be found in [Mum70, Thm 1, p. 111].

Definition (4.3). We say that X/S is AF if it satisfies the following condition.

(AF) Every finite set of points $x_1, x_2, \ldots, x_n \in X$ over the same point $s \in S$ is contained in an open subset $U \subseteq X$ such that $U \to S$ is quasi-affine.

Remark **(4.4).** The condition (AF) is stable under base change and local on the base in the Zariski topology. It is not clear whether the (AF) condition is local on the base in the étale topology. If S is a *scheme* and X/S is AF then X is a scheme and the subset U in (AF) can be chosen such that U is an affine scheme. If $X \to S$ admits an ample sheaf then it is AF [$\mathrm{EGA}_{\mathrm{II}}$, Cor. 4.5.4]. This is the case if $X \to S$ is (quasi-)affine or (quasi-)projective. If $X \to S$ is locally quasi-finite and separated, then $X \to S$ is AF [LMB00, Thm. A.2].

Proposition (4.5). *Let $R \rightrightarrows X$ be a finite locally free groupoid of algebraic spaces and assume that a geometric quotient $q : X \to Y$ exists such that q is* affine*. Then q is an integral and universally open* GC *quotient and j_Y is proper. Consider the following properties of a morphism of algebraic spaces:*

- (A) *quasi-compact, universally closed, universally open, separated, affine, quasi-affine*
- (A′) AF
- (B) *finite type, locally of finite type, proper*
- (B′) *projective, quasi-projective*

If $X \to S$ has one of the properties in (A) *then $Y \to S$ has the same property. The same holds for the properties in* (A′) *if S is a scheme, for those in* (B) *if S is locally noetherian and for those in* (B′) *if S is a noetherian scheme.*

Proof. As s is universally open and proper, it follows by Proposition (2.9) that q is universally open and universally closed. In particular q is integral [$\mathrm{EGA_{IV}}$, Prop. 18.2.8]. It follows that j_Y is proper and thus we have that q is strongly topological. Furthermore q satisfies the descent condition by Remark (3.12) and is thus a GC quotient.

The statement about the properties in group (A), (A′) and (B) is local on the base so we can assume that S is an affine scheme. The statement about the first three properties in (A) follows immediately as q is surjective. The property "separated" follows from Propositions (2.11) and (2.13) and "affine" from Theorem (4.1).

Assume that X/S is quasi-affine. To show that Y/S is quasi-affine it is enough to show that there is an *affine* cover of the form $\{Y_f\}$ with $f \in \Gamma(Y/S)$. Let $y \in Y$ be a point and $q^{-1}(y)$ the corresponding orbit in X. Then as X/S is quasi-affine there is a global section $g \in \Gamma(X/S)$ such that X_g is an affine neighborhood of $q^{-1}(y)$, cf. [$\mathrm{EGA_{II}}$, Cor. 4.5.4]. Let $f = \mathrm{N}_t(s^*g) \in \Gamma(X/S)$ be the norm of s^*f along t [$\mathrm{EGA_{II}}$, 6.4.8]. Then f is invariant, i.e. $s^*f = t^*f$, and $X_f \subseteq X_g$ is an affine neighborhood of $q^{-1}(y)$, cf. the proof of [$\mathrm{SGA_1}$, Exp. VIII, Cor. 7.6]. Thus $f \in \Gamma(Y/S)$ and $q(X_f) = Y_f$ is a geometric quotient of the groupoid $R_f \rightrightarrows X_f$. As X_f is affine so is Y_f by (A) for "affine". This shows that Y/S is quasi-affine.

Finally assume that X/S is AF and let $Z \subseteq Y_s$ be a finite subset in the fiber of Y over $s \in S$. Then $q^{-1}(Z)$ is a finite subset of X_s and thus contained in an affine open subset $U \subseteq X$. Proceeding as in the proof of [$\mathrm{SGA_1}$, Exp. VIII, Cor. 7.6] we obtain an affine open invariant subset $V \subseteq U \subseteq X$ containing $q^{-1}(Z)$. The open subset $q(V)$ is affine, by the "affine" part of (A), and contains Z. This shows that Y/S is AF.

Now assume that S is noetherian. As we have already shown the statement for quasi-compact, universally closed and separated, it is enough to show the statement for the property "locally of finite type" in group (B). Assume that $X \to S$ is locally of finite type. Then q is finite. As the quotient is uniform we can, in order to prove that $Y \to S$ is locally of finite type, assume that Y is affine and hence also X. It is then easily seen that $Y \to S$ is of finite type. For details, see the argument in [$\mathrm{SGA_1}$, Exp. V, Cor. 1.5].

For the properties in (B′) we cannot assume that S is affine as projectivity and quasi-projectivity is not local on the base. The statement about (quasi-)projectivity is probably well-known but I do not know of a full proof. A sketch is given in [Knu71, Ch. IV, Prop 1.5] and also discussed in [Ryd07b]. Both these, however, are for quotients of an abstract group G, i.e. a group with trivial scheme structure. The general case is proven as follows:

Let $\mathcal{L}$ be an ample sheaf of X and let $\mathcal{L}' = \mathrm{N}_t(s^*\mathcal{L}) = \mathrm{N}_{t_*\mathcal{O}_R/\mathcal{O}_X}(t_*s^*\mathcal{L})$. This is an ample invertible sheaf by [$\mathrm{EGA_{II}}$, Cor. 6.6.2]. Further, it comes with a canonical R-linearization [MFK94, Ch. 1, §3], i.e. a canonical isomorphism $\phi : s^*\mathcal{L}' \to t^*\mathcal{L}'$ satisfying a co-cycle condition. This is obvious from

the description $\mathcal{L}' = p^*\mathrm{N}_{X/\mathscr{X}}(\mathcal{L})$ where $p : X \to \mathscr{X}$ is the stack quotient of $R \rightrightarrows X$. Consider the graded $\mathcal{O}_X$-algebra $\mathcal{A} = \oplus_{n\geq 0}\mathcal{L}'^n$. As $\mathcal{L}'$ is ample, we have a canonical (closed) immersion $X \hookrightarrow \mathrm{Proj}(f_*\mathcal{A})$ where $f : X \to S$ is the structure morphism. Let $\mathcal{A}^R$ be the invariant ring, where $(\mathcal{L}'^n)^R$ is the equalizer of $s^*, t^* : \mathcal{L}'^n \to s^*\mathcal{L}'^n \cong t^*\mathcal{L}'$. It can then be shown, as in the case with an abstract group, that the quotient Y is a subscheme of $\mathrm{Proj}_S(f_*\mathcal{A}^R)$. As S is locally noetherian it follows that $f_*\mathcal{A}^R$ is a finitely generated $\mathcal{O}_S$-algebra, but it is not necessarily generated by elements degree 1. As S is noetherian there is an integer m such that $(f_*\mathcal{A}^R)^{(m)} = \oplus_{n\geq 0} f_*(\mathcal{L}'^{mn})^R$ is generated in degree 1. Hence Y is (quasi-)projective. □

Remark **(4.6).** If S is of characteristic zero, i.e. a $\mathbb{Q}$-space, then the GC quotient $q : X \to Y$ of the proposition is *universal*, i.e. it commutes with any base change. In fact, there exists a Reynolds operator, i.e. an $\mathcal{O}_Y$-module retraction $\mathcal{R} : q_*\mathcal{O}_X \to \mathcal{O}_Y$ of the inclusion $\mathcal{O}_Y = (q_*\mathcal{O}_X)^R \hookrightarrow q_*\mathcal{O}_X$. The sequence (2.1.1) is thus split exact which shows that q is a universal geometric quotient. We construct the Reynolds operator as follows: The rank r of s is constant on each connected component and constant on R-orbits. Locally the Reynolds operator $\mathcal{R}$ is defined by $\frac{1}{r}\mathrm{Tr}_{s^*} \circ t^*$ where $s^*, t^* : q_*\mathcal{O}_X \to (q \circ s)_*\mathcal{O}_R$ are the $\mathcal{O}_Y$-homomorphisms induced by s and t.

More general, in any characteristic, the quotient is universal if the stack $[R \rightrightarrows X]$ is *tame* [AOV07].

Theorem (4.7) ([FGA, Exp. 212, Thm. 5.3])**.** *Let $R \rightrightarrows X$ be a finite locally free groupoid of* schemes*. Then a* GC *quotient $q : X \to Y$ with q affine and Y a* scheme *exists if and only if every R-orbit of X is contained in an affine open subset.*

Proof. The result follows from Theorem (4.1) if we find an open covering of X with R-invariant affines. Let $x \in X$ and let $U \subseteq X$ be an affine open subset containing the R-orbit of x. Using the procedure of [Gab63, 5b)] from the proof of [SGA_1, Exp. VIII, Cor. 7.6], we obtain an R-invariant affine $U' \subseteq U$ containing the R-orbit of x. □

Remark **(4.8).** When S is a *scheme* and $X \to S$ is AF then any R-orbit of X is contained in an affine open subset and the conclusion of Theorem (4.7) holds. In particular this is true for $X \to S$ (quasi-)affine or (quasi-)projective. Furthermore Proposition (4.5) shows that geometric quotients exist in the following categories:

(1) Affine schemes over S.
(2) Quasi-affine schemes over S.
(3) Schemes over S satisfying AF.
(4) Projective schemes over a noetherian base scheme S.
(5) Quasi-projective schemes over a noetherian base scheme S.

Remark **(4.9).** In Theorem (5.3) we will show that if X/S is a *separated* algebraic space then a GC quotient $q : X \to Y$ exists and is *affine*. Thus, if X/S is a separated scheme then it follows from Theorem (4.7) that a geometric quotient $Y = X/R$ exists as a *scheme* if and only if every R-orbit of X is contained in an affine open subset.

Remark **(4.10).** When we replace the groupoid with a finite group scheme or a finite abstract group, then Theorem (4.7) is a classic result. It can be traced back to Serre [Ser59, Ch. III, §12, Prop. 19] when X is an algebraic variety. Also see [Mum70, Thm 1, p. 111] for the case when X is an algebraic scheme and [SGA$_1$, Exp. V §1] or [GM07, Thm. 4.16] for arbitrary schemes.

5. Finite quotients of algebraic spaces

Let $R_X \rightrightarrows X$ be a groupoid. For any étale morphism $U \to X$ we will construct a groupoid $R_W \rightrightarrows W$ with a square étale morphism $h : (R_W, W) \to (R_X, X)$. The construction requires that $R_X \rightrightarrows X$ is *proper*, flat and locally of finite presentation. If $R_X \rightrightarrows X$ is *finite* and $U \to X$ is surjective, then $h|_{\mathrm{fpr}}$ will be surjective.

Proposition (5.1). *Let $R_X \rightrightarrows X$ be a groupoid which is proper, flat and of finite presentation. Let further $f : U \to X$ be an étale and separated morphism. There is then a groupoid (R_W, W) with a square separated étale morphism $h : (R_W, W) \to (R_X, X)$ together with an étale and separated morphism $W \to U$. If U is a disjoint union of quasi-affine schemes then so is W. If in addition $R_X \rightrightarrows X$ is* finite *and f surjective then $h|_{\mathrm{fpr}(h)}$ is surjective.*

Proof. Set-theoretically the points of W in the fiber over $x \in X$ will correspond to a choice of a point of U in the fiber over every point in the orbit of x. More precisely, given an X-scheme T a morphism $T \to W$ should correspond to a section of $\pi_1 : (R_X \times_{t,X} T) \times_{s,X} U \to R_X \times_{t,X} T$. This is the functor $\mathrm{Hom}_{R_X/_sX}(R_X, R_X \times_{s,X} U)$ of Definition (A.7). As R_X is proper, flat and of finite presentation over X, this functor is representable by the algebraic space $W = \Pi(R_X \times_{t,X} U/_{\pi_1} R_X/_s X)$, cf. Definition (A.7), which is separated and locally of finite presentation over X.

An easier description is given using the stack $\mathscr{X} = [R_X \rightrightarrows X]$. Then $W = \mathscr{W} \times_{\mathscr{X}} X$ where $\mathscr{W} = \Pi(U/X/\mathscr{X})$. This induces a groupoid (R_W, W) with $R_W = W \times_{\mathscr{W}} W$ and the morphism $\mathscr{W} \to \mathscr{X}$ induces a square morphism $(R_W, W) \to (R_X, X)$.

By Proposition (A.9) the morphism $W \to X$ is étale and separated. Furthermore the unit section of $R_X \to X$ gives a factorization of $W \to X$ through f and an étale and separated morphism $W \to U$ by the same proposition. Replacing $\mathscr{W}$ with an open covering of $\mathscr{W}$ consisting of quasi-compact substacks we can further assume that W is a disjoint union of quasi-compact algebraic spaces. If U is a disjoint union of quasi-affine schemes, then as an étale, separated and quasi-compact morphism is quasi-affine, the space W is a disjoint union of quasi-affine schemes.

Finally, we show the surjectivity of $h|_{\mathrm{fpr}(h)}$. Let $x : \mathrm{Spec}(\overline{k}) \to X$ be a geometric point of X and let $w : \mathrm{Spec}(\overline{k}) \to W$ be a lifting. Then w corresponds to a morphism $\varphi : s^*(x) \to U$ such that $t = f \circ \varphi$. Let $R_x = s^*(x)_{\mathrm{red}}$ which we consider as an X-scheme using t. As $U \to X$ is étale any X-morphism $R_x \to U$ induces a unique morphism φ as above. If $R_X \to X$ is finite, then R_x is a finite set of points. We may then choose a morphism $R_x \to U$ such that if the image of two points $r_1, r_2 \in R_x$ is in the

same fiber of $f : U \to X$ then $r_1 = r_2$. This will correspond to a point w in the fix-point reflecting locus of $W \to X$. □

Remark **(5.2).** Let $G = \{g_1, g_2, \dots, g_n\}$ be a finite group with the trivial scheme structure acting on the algebraic space X and $R_X = G \times_S X$ the induced groupoid. Let $f : U \to X$ be an étale and separated morphism. Then the étale cover $W \to X$ of Proposition (5.1) is the fiber product of $g_1 \circ f, g_2 \circ f, \dots, g_n \circ f$. The morphism $W \to U$ is the projection on the factor corresponding to the identity element $g_i = e \in G$.

Theorem (5.3). *Let $R \rightrightarrows X$ be a finite locally free groupoid with finite stabilizer $S(X) = j^{-1}\big(\Delta(X)\big) \to X$. Then a* GC *quotient $q : X \to X/R$ exists and q is affine. Hence it has the properties of Proposition* (4.5).

Proof. The question is étale-local over S so we can assume that S is affine. Let $\varphi : U \to X$ be an étale covering such that U is a disjoint union of affine schemes. Let $W \to X$ be the étale and separated cover constructed in Proposition (5.1). As the stabilizer is proper, the subset $W|_{\mathrm{fpr}}$ is open by Proposition (3.5). Further $W|_{\mathrm{fpr}} \to X$ is surjective by Proposition (5.1). Thus $W|_{\mathrm{fpr}} \to X$ is an étale separated square fpr cover such that $W|_{\mathrm{fpr}}$ is a disjoint union of quasi-affine schemes. By Theorem (4.7) and Remark (4.8) a GC quotient of $W|_{\mathrm{fpr}}$ exists. Hence a geometric quotient X/R exists by Theorem (3.18). □

Theorem (5.4) (Deligne)**.** *Let G/S be a finite locally free group scheme acting on a separated algebraic space X/S. Then a* GC *quotient $q : X \to X/G$ exists such that q is affine and X/G is a separated algebraic space. Furthermore, there is an étale G-equivariant surjective morphism $f : U \to X$ with U a disjoint union of quasi-affine schemes such that $U/G \to X/G$ is a surjective étale morphism.*

Proof. As X is separated, the finite locally free groupoid $G \times_S X \rightrightarrows X \times_S X$ has a finite diagonal. In particular, its stabilizer is finite. The existence of a GC quotient q thus follows from Theorem (5.3). The morphism $f : U \to X$ can be taken as the morphism $W|_{\mathrm{fpr}} \to X$ in the proof of Theorem (5.3). □

In the case of the action of symmetric products we can find a more explicit étale cover:

Theorem (5.5). *Let S be an algebraic space and X/S a separated algebraic space. Then a uniform geometric and categorical quotient* $\mathrm{Sym}^d(X/S) := (X/S)^d/\mathfrak{S}_d$ *exists as a separated algebraic space. Let $\{S_\alpha \to S\}_\alpha$ and $\{U_\alpha \to X \times_S S_\alpha\}_\alpha$ be sets of étale morphisms of separated algebraic spaces. Then the diagram*

$$\begin{array}{ccc} \coprod_\alpha (U_\alpha/S_\alpha)^d|_{\mathrm{fpr}} & \longrightarrow & (X/S)^d \\ \downarrow & \square & \downarrow \\ \coprod_\alpha \mathrm{Sym}^d(U_\alpha/S_\alpha)|_{\mathrm{fpr}} & \longrightarrow & \mathrm{Sym}^d(X/S) \end{array} \tag{5.5.1}$$

is cartesian and the horizontal morphisms étale. If the U_α's are such that for every $s \in S$, any set of d points in X_s lies in the image of some U_α then the

horizontal morphisms are surjective. In particular, there is an étale cover of $\operatorname{Sym}^d(X/S)$ *of the form* $\left\{\operatorname{Sym}^d(U_\alpha/S_\alpha)|_{V_\alpha}\right\}$ *with* U_α *and* S_α *affine and* V_α *an open subset.*

Proof. Theorem (5.4) shows the existence of the GC quotient $\operatorname{Sym}^d(X/S)$. As $(U_\alpha/S_\alpha)^d \to (X/S)^d$ is étale, the diagram (5.5.1) is cartesian by the descent condition. Let $x \in (X/S)^d$ be a point and let $x_1, x_2, \dots, x_n \in X$ be its projections, which are all over the same point $s \in S$. If $x_1, x_2, \dots, x_n$ lies in the image of some U_α we can choose liftings $u_1, u_2, \dots, u_n \in U_\alpha$ such that $u_i = u_j$ if and only if $x_i = x_j$. The u_i's then determine a point $u \in (U_\alpha/S_\alpha)^d$ in the fix-point reflecting locus of $(U_\alpha/S_\alpha)^d \to (X/S)^d$. This shows the surjectivity of the horizontal morphisms. □

Remark **(5.6).** Note that the action of $\mathfrak{S}_d$ on $(X/S)^d$ by permutations has a proper stabilizer exactly when X/S is separated. Thus Theorem (5.5) is the most general possible with these methods.

6. Algebraic stacks with quasi-finite diagonal

In this section we will establish some basic facts about algebraic stacks with quasi-finite diagonal. These are well-known for such stacks locally of finite type over a noetherian base scheme S but we will extend the results to all stacks with quasi-finite diagonal. The main result is that every algebraic stack with quasi-finite (resp. unramified) diagonal has a locally quasi-finite fppf (resp. étale) presentation. In the noetherian case this goes back to Grothendieck-Gabriel [Gab63, Lem. 7.2] and is also shown in [KM97, Lem. 3.3].

Remark **(6.1).** If $\mathscr{X}$ is an algebraic stack with quasi-finite diagonal then the diagonal is strongly representable, i.e. schematic. In fact, any separated and quasi-finite morphism of algebraic spaces is schematic [LMB00, Thm. A.2]. In particular if $U \to \mathscr{X}$ is a presentation with a *scheme* U and $R = U \times_{\mathscr{X}} U$ then R is also a scheme.

Remark **(6.2).** If $\mathscr{X}$ is an algebraic S-stack with quasi-finite diagonal and S is of characteristic 0 then $\mathscr{X}$ is Deligne-Mumford. In fact, as any group scheme over a field of characteristic 0 is smooth, it follows that the diagonal $\Delta_{\mathscr{X}/S}$ is unramified, i.e. formally unramified and locally of finite type.

Definition (6.3). Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks. We say that f is quasi-finite (resp. étale) if there is an epimorphism $U \to \mathscr{Y}$ in the fppf topology and a representable quasi-finite fppf (resp. étale and surjective) morphism $V \to \mathscr{X} \times_{\mathscr{Y}} U$ such that $V \to \mathscr{X} \times_{\mathscr{Y}} U \to U$ is representable and quasi-finite (resp. étale).

These definitions are stable under base change, descend under fppf base change and agree with the usual definitions when f is representable.

Definition (6.4) ([LMB00, 9.7])**.** A morphism of stacks $f : \mathscr{X} \to \mathscr{Y}$ is an fppf-gerbe if f and the diagonal $\Delta_{\mathscr{X}/\mathscr{Y}}$ are epimorphisms in the fppf topology.

Proposition (6.5). *Let $\mathscr{X}$ be an algebraic stack such that the inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ is fppf. Then the coarse fppf-sheaf X associated to $\mathscr{X}$ is an algebraic space and the structure morphism $\mathscr{X} \to X$ as well as the diagonal $\Delta_{\mathscr{X}/X}$ are fppf. Furthermore, if $I_{\mathscr{X}} \to \mathscr{X}$ is quasi-finite (resp. étale) then so are $\Delta_{\mathscr{X}/X}$ and $\mathscr{X} \to X$. If $I_{\mathscr{X}} \to \mathscr{X}$ is finite, then $\Delta_{\mathscr{X}/X}$ is finite and $\mathscr{X} \to X$ is separated.*

Proof. By [LMB00, Cor. 10.8] $\mathscr{X} \to X$ is an fppf-gerbe and $\mathscr{X} \to X$ is fppf. Let P be one of the properties fppf, quasi-finite, finite and étale. Then P is local on the target in the fppf topology. As $\mathscr{X} \to X$ is a gerbe $\Delta_{\mathscr{X}/X} : \mathscr{X} \to \mathscr{X} \times_X \mathscr{X}$ is an epimorphism and hence $\Delta_{\mathscr{X}/X}$ has property P if and only if $\Delta^*_{\mathscr{X}/X}\Delta_{\mathscr{X}/X} : I_{\mathscr{X}} \to \mathscr{X}$ has property P. If $\Delta_{\mathscr{X}/X}$ is finite then $\mathscr{X} \to X$ is separated.

Similarly, let P be one of the properties quasi-finite and étale and assume that $I_{\mathscr{X}} \to \mathscr{X}$ has property P. Then, as $\mathscr{X} \to X$ is an epimorphism, $\mathscr{X} \to X$ has property P if and only if $\pi_1 : \mathscr{X} \times_X \mathscr{X} \to \mathscr{X}$ has property P. As $\Delta_{\mathscr{X}/X}$ is a fppf morphism with property P and the composition $\pi_1 \circ \Delta_{\mathscr{X}/X} : \mathscr{X} \to \mathscr{X} \times_X \mathscr{X} \to \mathscr{X}$ has property P, by definition π_1 has property P. □

Corollary (6.6). *Let $\mathscr{X}$ be a quasi-compact algebraic stack with quasi-finite diagonal. Then there is a stratification $\mathscr{X} = \bigcup_{n=1}^N \mathscr{X}_n$ of locally closed substacks such that $I_{\mathscr{X}_n} \to \mathscr{X}_n$ is quasi-finite and fppf. In particular, for every n there is an algebraic space X_n and an fppf morphism $\mathscr{X}_n \to X_n$ making $\mathscr{X}_n$ into an fppf-gerbe over X_n.*

Proof. Let $p : U \to \mathscr{X}$ be an fppf presentation with a quasi-compact scheme U and let $R = U \times_{\mathscr{X}} U$ which is a scheme by Remark (6.1). Let $S(U) \to U$ be the pull-back of $j : R \to U \times_S U$ along the diagonal $\Delta_{U/S}$. Then $S(U) \to U$ is quasi-finite and separated and coincides with the pull-back of the inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ along $U \to \mathscr{X}$.

As U is quasi-compact (and quasi-separated as always) there is by Zariski's Main Theorem [EGA$_{\text{IV}}$, Cor. 18.12.13] a factorization

$$S(U) \xrightarrow{f} \overline{S(U)} \xrightarrow{g} U$$

where f is open and g is finite. Using Fitting ideals [Eis95, §20.2] we obtain a finite stratification $U = \bigcup_{n=1}^N U_n$ where U_n is such that $\overline{S(U)}|_{U_n} \to U_n$ is locally free of rank n. If we let $\mathscr{X}_n = p(U_n) \setminus \bigcup_{m=1}^{n-1} \mathscr{X}_m$ then $\mathscr{X} = \bigcup_{n=1}^N \mathscr{X}_n$ is a stratification of $\mathscr{X}$ such that $I_{\mathscr{X}_n} = I_{\mathscr{X}}|_{\mathscr{X}_n} \to \mathscr{X}_n$ is fppf with fibers of rank at most n. When $\mathscr{X}$ has unramified diagonal, then $I_{\mathscr{X}_n} \to \mathscr{X}_n$ is fppf and unramified, hence étale.

Now Proposition (6.5) shows that the coarse fppf-sheaf X_n associated to $\mathscr{X}_n$ is an algebraic space and that $\mathscr{X}_n \to X_n$ is an fppf-gerbe. □

Remark **(6.7).** If $\mathscr{X}$ is an algebraic stack with finite inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ then there is a canonical stratification $\mathscr{X} = \bigcup_{n=1}^\infty \mathscr{X}_n$ in locally closed, not necessarily reduced, substacks such that $I_{\mathscr{X}_n} \to \mathscr{X}_n$ is locally free of rank n. The substack $\mathscr{X}_n$ is supported on the subset of $\mathscr{X}$ where the automorphism groups have order n.

Let ξ be a point of $\mathscr{X}$ and let $x : \operatorname{Spec}(k) \to \mathscr{X}$ be a representation of ξ. There is then a canonical factorization

$$\operatorname{Spec}(k) \twoheadrightarrow \mathscr{X}_\xi \hookrightarrow \mathscr{X}$$

into an epimorphism followed by a monomorphism where $\mathscr{X}_\xi$ is independent of the choice of x. Recall the following definition from [LMB00, Def. 11.2]:

Definition (6.8). A point ξ of an algebraic stack $\mathscr{X}$ is called *algebraic* if

(i) The coarse fppf-sheaf associated to $\mathscr{X}_\xi$ is an algebraic space, necessarily of the form $\operatorname{Spec}(k(\xi))$.
(ii) The canonical monomorphism $\mathscr{X}_\xi \hookrightarrow \mathscr{X}$ is representable and hence $\mathscr{X}_\xi$ is an algebraic $k(\xi)$-stack.
(iii) The algebraic $k(\xi)$-stack $\mathscr{X}_\xi$ is of finite type.

Proposition (6.9). *If $\mathscr{X}$ is an algebraic stack with quasi-finite diagonal then every point is algebraic. Furthermore $\mathscr{X}_\xi \to \operatorname{Spec}(k(\xi))$ is quasi-finite and separated. If $\mathscr{X}$ has unramified diagonal, then $\mathscr{X}_\xi \to \operatorname{Spec}(k(\xi))$ is étale.*

Proof. The first statement follows immediately from Corollary (6.6). We also note that $I_{\mathscr{X}_\xi} \to \mathscr{X}_\xi$ is fppf, quasi-finite and that every fiber has the same rank, the rank of the automorphism group of ξ. Thus $I_{\mathscr{X}_\xi} \to \mathscr{X}_\xi$ is fppf and *finite*. Further, if $\mathscr{X}$ has unramified diagonal then $I_{\mathscr{X}_\xi} \to \mathscr{X}_\xi$ is étale. The last two statements of the corollary now follows from Proposition (6.5). □

We may now prove the following theorem almost exactly as for morphisms of schemes, cf. [$\mathrm{EGA}_{\mathrm{IV}}$, §17.16].

Theorem (6.10). *Let $\mathscr{X}$ be an algebraic stack with quasi-finite (resp. unramified) diagonal. Then there is a locally quasi-finite fppf (resp. étale) presentation $U \to \mathscr{X}$ with a scheme U.*

Proof. Let $U \to \mathscr{X}$ be a flat (resp. smooth) presentation with U a scheme. Let $\xi \in |\mathscr{X}|$ be a point and let $U_\xi = U \times_{\mathscr{X}} \mathscr{X}_\xi$. Then by Proposition (6.9) the point ξ is algebraic and $\mathscr{X}_\xi \to \operatorname{Spec}\big(k(\xi)\big)$ is quasi-finite fppf (resp. étale) as well as $\Delta_{\mathscr{X}_\xi/k(\xi)}$. Thus U_ξ is a scheme and $U_\xi \to \mathscr{X}_\xi \to \operatorname{Spec}\big(k(\xi)\big)$ is fppf (resp. smooth and surjective). We can then find a closed point $u \in U_\xi$ and a regular sequence $f_1, f_2, \dots, f_n$ such that $\mathcal{O}_{U_\xi,u}/(f_1, f_2, \dots, f_n)$ is artinian (resp. a separable extension of $k(\xi)$). As $\Delta_{\mathscr{X}_\xi/k(\xi)}$ is quasi-finite fppf (resp. étale) it follows that $\operatorname{Spec}\big(\mathcal{O}_{U_\xi,u}/(f_1, f_2, \dots, f_n)\big) \to \mathscr{X}_\xi$ is quasi-finite fppf (resp. étale).

There exists an open subset $V \subseteq U$ and sections $g_1, g_2, \dots, g_n$ of $\mathcal{O}_U$ over V which lifts the regular sequence $f_1, f_2, \dots, f_n$. Let $Z \hookrightarrow V$ be the closed subscheme of V defined by the ideal generated by the f_i:s. Replacing V by a smaller open subset, we can assume that $Z \hookrightarrow V \to \mathscr{X}$ is *flat* and of finite presentation [$\mathrm{EGA}_{\mathrm{IV}}$, Thm. 11.3.8]. Further as u is isolated in $Z_\xi = Z \times_{\mathscr{X}} \mathscr{X}_\xi$ it follows that after replacing V with a smaller open subset that $Z \to \mathscr{X}$ is quasi-finite and flat of finite presentation (resp. étale).

Repeating the procedure for every ξ and taking the disjoint union, we obtain a locally quasi-finite (resp. étale) cover $Z \to \mathscr{X}$. □

Remark **(6.11).** Theorem (6.10) gives a proof of the fact that a stack with unramified diagonal is Deligne-Mumford which is independent of [LMB00, Thm. 8.1]. We also note that $\mathscr{X}_\xi \to k(\xi)$ is *proper* as $U_\xi \to \mathscr{X}_\xi \to k(\xi)$ is finite and hence universally closed.

7. Coarse moduli spaces of stacks

We begin by rephrasing the results of §3 in stack language. If $U \to \mathscr{X}$ is an fppf presentation and $R = U \times_{\mathscr{X}} U$ then there is a one-to-one correspondence between quotients $U \to Z$ of the groupoid $R \rightrightarrows U$ and morphisms $\mathscr{X} \to Z$. We say that a morphism $\mathscr{X} \to Z$ is a topological (resp. geometric, resp. categorical) quotient if $U \to \mathscr{X} \to Z$ is such a quotient. This definition does not depend on the choice of presentation $U \to \mathscr{X}$ and can be rephrased as follows:

Definition (7.1). Let $\mathscr{X}$ be an algebraic stack, Z an algebraic space and $q : \mathscr{X} \to Z$ a morphism. Then q is

(i) *categorical* if q is initial among morphisms from $\mathscr{X}$ to algebraic spaces.
(ii) *topological* if q is a universal homeomorphism.
(iii) *strongly topological* if q is a strong homeomorphism, cf. §A.2
(iv) *geometric* if q is topological and $\mathcal{O}_Z \to q_*\mathcal{O}_{\mathscr{X}}$ is an isomorphism.
(v) *strongly geometric* if q is strongly topological and $\mathcal{O}_Z \to q_*\mathcal{O}_{\mathscr{X}}$ is an isomorphism.

Remark **(7.2).** If $q : \mathscr{X} \to Z$ is a strongly topological quotient and $q' : \mathscr{X} \to Z'$ is a topological quotient, then any morphism $Z' \to Z$ is separated by Corollary (A.6). Thus, a strong topological quotient $q : \mathscr{X} \to Z$ is "maximally non-separated" among topological quotients.

Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of stacks. There is then an induced morphism $\varphi : I_{\mathscr{X}} \to I_{\mathscr{Y}} \times_{\mathscr{Y}} \mathscr{X}$. If $x : \operatorname{Spec}(k) \to \mathscr{X}$ is a point and $y = f \circ x$, then φ_x is the natural morphism of k-groups $\mathcal{I}som_{\mathscr{X}}(x,x) \to \mathcal{I}som_{\mathscr{Y}}(y,y)$. It can be shown that f is representable if and only if φ is a monomorphism.

Definition (7.3). Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of stacks. We say that f is *fix-point reflecting* or fpr if φ is an isomorphism. We let $\operatorname{fpr}(f) \subseteq |\mathscr{X}|$ be the subset over which φ is an isomorphism.

Remark **(7.4).** Let $\mathscr{X}$ and $\mathscr{Y}$ be stacks with presentations $U \to \mathscr{X}$ and $V \to \mathscr{Y}$. Then there is a one-to-one correspondence between square morphisms $(R_U, U) \to (R_V, V)$ and morphisms $\mathscr{X} \to \mathscr{Y}$ such that $U = V \times_{\mathscr{Y}} \mathscr{X}$. Further there is a one-to-one correspondence between square fpr morphisms $(R_U, U) \to (R_V, V)$ and fpr morphisms $\mathscr{X} \to \mathscr{Y}$ such that $U = V \times_{\mathscr{Y}} \mathscr{X}$.

The following is a reformulation of Proposition (3.5) for stacks.

Proposition (7.5). *Let $f : \mathscr{X} \to \mathscr{Y}$ be a representable and unramified morphism of stacks. If the inertia stack $I_{\mathscr{Y}} \to \mathscr{Y}$ is proper, then the subset $\operatorname{fpr}(f) \subseteq \mathscr{X}$ is open.*

Definition (7.6). Let $\mathscr{X}$ be an algebraic stack and $q : \mathscr{X} \to X$ a topological quotient. We say that q satisfies the *descent condition* if for any étale

and separated fpr morphism $f : \mathscr{W} \to \mathscr{X}$ there exists a quotient $\mathscr{W} \to W$ and a 2-cartesian square

$$\begin{array}{ccc} \mathscr{W} & \xrightarrow{f} & \mathscr{X} \\ \downarrow & \square & \downarrow q \\ W & \longrightarrow & X \end{array}$$

where $W \to X$ is *étale*.

Proposition (7.7). *Let $\mathscr{X} \to X$ be a geometrical quotient satisfying the descent condition. Then q is a categorical quotient, i.e. a coarse moduli space.*

Proof. See Proposition (3.14). □

Definition (7.8). We say that a strongly geometric quotient $\mathscr{X} \to X$ is a GC quotient if it satisfies the descent condition. As a GC quotient is categorical by Proposition (7.7) we will speak about *the* GC quotient when it exists.

Remark **(7.9).** Let $\mathscr{X}$ be an algebraic stack with an fppf presentation $U \to \mathscr{X}$ and let $q : \mathscr{X} \to X$ be a topological quotient. Then q satisfies the descent condition if and only if $U \to \mathscr{X} \to X$ satisfies the descent condition of Definition (3.6). Similarly q is a GC quotient if and only if $U \to X$ is a GC quotient.

For completeness, we mention the following theorem which we will not need:

Theorem (7.10). *If $q : \mathscr{X} \to X$ is a strongly geometric quotient, then q satisfies the descent condition. In particular q is the* GC *quotient.*

Proof. As q is universally open this follows from Theorem (3.15). □

Theorem (7.11). *Let $f : \mathscr{W} \to \mathscr{X}$ be a surjective, étale, separated and fpr morphism of algebraic stacks. Let $\mathscr{Q} = \mathscr{W} \times_{\mathscr{X}} \mathscr{W}$. If $\mathscr{W}$ has a* GC *quotient W then* GC *quotients $\mathscr{Q} \to Q$ and $\mathscr{X} \to X$ exists. Further, the diagram*

$$\begin{array}{ccccc} \mathscr{Q} & \rightrightarrows & \mathscr{W} & \xrightarrow{f} & \mathscr{X} \\ \downarrow & & \downarrow & & \downarrow \\ Q & \rightrightarrows & W & \longrightarrow & X \end{array}$$

is cartesian.

Proof. Follows immediately from Theorem (3.18) and Remark (7.9). □

Proposition (7.12) ([KM97, §4],[Con05, Lemma 2.2])**.** *Let $\mathscr{X}$ be a stack with quasi-finite diagonal. Then there is a representable étale separated morphism $\mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ has a finite fppf presentation $U \to \mathscr{W}$ with U a disjoint union of quasi-affine schemes. Furthermore $\mathscr{W}_{\mathrm{fpr}} \to \mathscr{X}$ is surjective.*

Proof. By Theorem (6.10) there is a locally quasi-finite fppf cover $V \to \mathscr{X}$ with V a scheme. Taking an open cover, we can assume that V is a disjoint union of affine schemes. Let $\mathscr{H} = \mathrm{Hilb}(V/\mathscr{X})$ be the Hilbert "stack",

which exists by fppf-descent as the Hilbert functor commutes with base change and $V \to \mathscr{X}$ is locally of finite presentation and separated. The structure morphism $\mathscr{H} \to \mathscr{X}$ is representable, locally of finite presentation and separated. As $V \to \mathscr{X}$ is locally quasi-finite, a morphism $T \to \mathscr{H}$ corresponds to a finite and finitely presented subscheme $Z \hookrightarrow V \times_{\mathscr{X}} T$.

Let $\mathscr{W}$ be the substack parameterizing *open* and closed subschemes $Z \hookrightarrow V \times_{\mathscr{X}} T$. Let $f : Z \to T$ be a family of closed subschemes. Note that $g : Z \hookrightarrow V \times_{\mathscr{X}} T$ is open if and only if it is étale. As the family f is flat and f and $V \to \mathscr{X}$ are locally of finite presentation, we have that the morphism $g : Z \hookrightarrow V \times_{\mathscr{X}} T$ is étale at z if and only if $g_{f(z)}$ is étale at z by [EGA$_{\text{IV}}$, Rem. 17.8.3]. Let $Z_{\text{ét}}$ be the open subset of Z where g is étale. Then the open subset $T \setminus f(Z \setminus Z_{\text{ét}}) \subseteq T$ is the set of $t \in T$ such that f_t is open and closed. Thus $\mathscr{W}$ is an open substack of $\mathscr{H}$. It is immediately clear that $\mathscr{W} \to \mathscr{X}$ is formally étale and hence étale. Replacing $\mathscr{W}$ with an open cover, we can assume that $\mathscr{W}$ is quasi-compact.

We let U be the universal family over $\mathscr{W}$. Then U is a quasi-affine scheme. In fact, as $U \hookrightarrow V \times_{\mathscr{X}} \mathscr{W}$ is an open and closed immersion and $\mathscr{W} \to \mathscr{X}$ is étale, quasi-compact and separated, we have that $U \to V$ is quasi-affine.

Now let $x : \operatorname{Spec}(k) \to \mathscr{X}$ be a point. Then it lifts uniquely to the point $w : \operatorname{Spec}(k) \to \mathscr{W}$ corresponding to the family $V_x = V \times_{\mathscr{X}} \operatorname{Spec}(k) \to \operatorname{Spec}(k)$. It is further clear that $\mathcal{I}som(w, w) = \mathcal{I}som(x, x)$ and hence that $w \in \mathscr{W}_{\text{fpr}}$. □

We are now ready to prove the full generalization of Keel and Mori's theorem [KM97]:

Theorem (7.13). *Let $\mathscr{X}$ be a stack with finite inertia stack. Then $\mathscr{X}$ has a* GC *coarse moduli space $q : \mathscr{X} \to X$. In particular q is a universal homeomorphism and hence quasi-compact. Furthermore q is* separated *and if $\mathscr{X} \to S$ is locally of finite type, then q is proper and quasi-finite. Consider the properties:*

(A) *quasi-compact, universally closed, universally open, separated*
(B) *finite type, locally of finite type, proper*

If $\mathscr{X} \to S$ has one of the properties in (A) *then $X \to S$ has the same property. If S is locally noetherian, the same holds for the properties in* (B).

Proof. By Propositions (7.5) and (7.12) there is a representable, étale, separated, fix-point reflecting and surjective morphism $\mathscr{W} \to \mathscr{X}$ and a finite fppf presentation $U \to \mathscr{W}$ with U a disjoint union of quasi-affine schemes. A GC quotient $\mathscr{W} \to W$ exists by Theorem (4.7). Then Theorem (7.11) shows that a GC quotient $q : \mathscr{X} \to X$ exists and is such that $W \to X$ is étale and $\mathscr{W} = \mathscr{X} \times_X W$.

As $\mathscr{W} \to W$ is a separated, so is q. If $\mathscr{X} \to S$ is locally of finite type then $\mathscr{X} \to X$ is locally of finite type and hence proper. Also, for any locally quasi-finite fppf presentation $V \to \mathscr{X}$, the composition $V \to X$ is locally quasi-finite as $U \to \mathscr{W} \to W$ is finite. Thus q is quasi-finite.

In (A) the property "separated" follows from Proposition (2.11) and the rest of the properties are obvious. In (B) we only need to prove that if S is locally noetherian and $\mathscr{X} \to S$ is locally of finite type then so is $X \to S$. As $\mathscr{W} \to S$ is locally of finite type then so is $W \to S$ by Proposition (4.5).

As $W \to X$ is étale and surjective it follows that $X \to S$ is locally of finite type. □

The following example shows that if $\mathscr{X}$ is a stack with quasi-finite but non-proper inertia, then a coarse moduli space does not always exist.

Example (7.14). Let k be a field and let $S = \operatorname{Spec}(k[x, y^2])$ be the affine plane. Let $U = \operatorname{Spec}(k[x, y])$ also be the affine plane, seen as a ramified double covering of S. Let $\tau : U \to U$ be the S-involution on U given by $y \mapsto -y$. We have a unique group structure on the scheme $G = S \amalg (S \setminus \{x = 0\})$ and we let G act on U by τ. Let $\mathscr{X} = [U/G]$ be the quotient stack.

Let $S' = S \setminus \{y = 0\}$ and let $\mathscr{X}' = \mathscr{X} \times_S S'$ and $U' = U \times_S S'$. We note that $\mathscr{X}' \to S'$ is an isomorphism outside $\{x = 0\}$. Moreover, the stack $\mathscr{X}'$ is represented by a non-separated algebraic space X'. Restricted to $x = 0$, the morphism $X' \to S'$ coincides with the étale double cover $U' \to S'$.

We will first show that $\mathscr{X}$ has *no topological quotient*. If there was a topological quotient $\mathscr{X} \to Z$ then we would have a factorization $R \twoheadrightarrow U \times_Z U \hookrightarrow U \times_S U$. As we will see, this is not possible. We have that $U \times_S U = \operatorname{Spec}\big(k[x, y_1, y_2]/(y_1^2 - y_2^2)\big)$, i.e. $U \times_S U$ is the union of two affine planes $U_i = \operatorname{Spec}(k[x, t_i])$ glued along the lines $t_i = 0$. In coordinates, we have that $t_1 = y_1 - y_2$ and $t_2 = y_1 + y_2$. The image of $R \to U \times_S U$ is the union of U_1 and $U_2 \setminus \{x = 0\}$. Restricted to U_2, the image of R is $U_2 \setminus \{x = 0, t_2 \neq 0\}$. The subfunctor of U_2 corresponding to this image is not representable, cf. [Art69b, Ex. 5.11]. This shows that there is no factorization $R \twoheadrightarrow T \hookrightarrow U \times_S U$.

In addition, the stack $\mathscr{X}$ has no *categorical quotient*. In fact, for any closed point $s \in S$ on the y^2-axis but not on the x-axis, let $Z_s \to S$ be the non-separated algebraic space which is isomorphic to S outside s but an étale extension of degree 2 at s. To be precise, over S' the space Z_s is the quotient of U' by the group $S' \amalg (S' \setminus \{s\})$ where the second component acts by τ. If k is algebraically closed then Z_s is even a scheme – the affine plane with a double point at s. It is clear that $\mathscr{X} \to S$ factors canonically through Z_s. If a categorical quotient $\mathscr{X} \to Z$ existed, then by definition we would have morphisms $Z \to Z_s$ for every s as above. This shows that $U \times_Z U \hookrightarrow U \times_S U$ would be contained in the union of U_1 and $U_2 \setminus Q$ where Q is the t_2-axis with all closed points except the origin removed. As for the case considered previously [Art69b, Ex. 5.11], it is clear that the existence of $U \times_Z U$ would violate criterion [4] of [Art69b, Thm 5.6].

Appendix A. Auxiliary results

A.1. **Topological results.** Recall that a morphism of topological spaces $f : X \to Y$ is *submersive* if it is surjective and has the quotient topology, i.e. a subset $Z \subseteq Y$ is open if and only if its inverse image $f^{-1}(Z)$ is open. Equivalently $Z \subseteq Y$ is closed if and only if $f^{-1}(Z)$ is closed.

Let $f : X \to Y$ be a morphism of algebraic spaces. We say that f is submersive (resp. universally submersive) if it is submersive in the Zariski topology (resp. submersive in the Zariski topology after any base change).

By slightly abuse of notation we say that f^{cons} is submersive (resp. universally submersive) if f is submersive in the constructible topology (resp. submersive in the constructible topology after any base change).

For details on the constructible topology on schemes see [$\mathrm{EGA}_{\mathrm{IV}}$, 1.9.13]. These results are easily extended from schemes to algebraic spaces as follows: An étale morphism is locally of finite presentation and hence open in the constructible topology. In particular, a surjective and étale morphism is submersive in the constructible topology. Let X be an algebraic space and $f : U \to X$ an étale presentation of X with a scheme U. Then we let X^{cons} be the unique topology on X such that f^{cons} is submersive. It is clear that this definition is independent on the choice of presentation.

Proposition (A.1). *Let $f : X \to Y$ and $g : Y' \to Y$ be morphism of algebraic spaces and $f' : X' \to Y'$ the pull-back of f along g. If f' is open (resp. closed, resp. submersive) and g is submersive then f is open (resp. closed, resp. submersive). If g is universally submersive, then f has one of the properties*

(i) *universally open*
(ii) *universally closed*
(iii) *universally submersive*
(iv) *separated*

if and only if f' has the same property.

Proof. Assume that f' is open (resp. closed) and let $Z \subseteq X$ be an open (resp. closed) subset. Then $g^{-1}(f(Z)) = f'(g'^{-1}(Z))$ is open (resp. closed) and thus so is $f(Z)$ as g is submersive. If f' is submersive then so is $g \circ f' = f \circ g'$ which shows that f is submersive. The first three properties of the second statement follows easily from the first. If f' is separated, then $\Delta_{X'/Y'}$ is universally closed and it follows that $\Delta_{X/Y}$ is universally closed and hence a closed immersion [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 18.12.6]. □

Proposition (A.2). *Let $f : X \to Y$ and $g : Y' \to Y$ be morphism of algebraic spaces and $f' : X' \to Y'$ the pull-back of f along g. If g^{cons} is submersive then f is quasi-compact if and only if f' is quasi-compact.*

Proof. If f is quasi-compact then f' is quasi-compact. Assume that f' is quasi-compact and g^{cons} is submersive. As f' is quasi-compact we have that f'^{cons} is closed [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 1.9.14 (iv)] and that the fibers of f' are quasi-compact. As g^{cons} is submersive it follows that f^{cons} is closed and that f has quasi-compact fibers. An easy argument using [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 1.9.15 (i)] shows that the fibers of f^{cons} are quasi-compact. Thus f^{cons} is proper and we have that f is quasi-compact by [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 1.9.15 (v)]. □

Proposition (A.3). *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is surjective and in addition has one of the following properties*

(i) *quasi-compact*
(ii) *locally of finite presentation*
(iii) *universally open*

then f^{cons} is universally submersive.

Proof. As properties (i)-(iii) are stable under base change it is enough to show that f^{cons} is submersive. We can furthermore assume that Y is quasi-compact [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 1.9.14 (vi)]. If f is open then there is a quasi-compact open subset $U \subseteq X$ such that $f|_U$ is surjective. As it is enough to show that $(f|_U)^{\mathrm{cons}}$ is submersive we can replace X with U and assume that f is quasi-compact.

If f is quasi-compact (resp. locally of finite presentation) then f^{cons} is open (resp. closed) by [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 1.9.14] and it follows that f^{cons} is submersive. □

A.2. **Strong homeomorphisms.** If $f : X \to Y$ is a homeomorphism of *topological spaces*, then the diagonal map is a homeomorphism. If $f : X \to Y$ is a universal homeomorphism of *schemes*, then the diagonal morphism is a universal homeomorphism. Indeed, it is a surjective immersion, i.e. a nil-immersion.

However, if $f : X \to Y$ is a universal homeomorphism of *algebraic spaces*, the diagonal is universally bijective but need not be a universal homeomorphism. A counterexample is given by Y as the affine line and X as a non-locally separated line. This motivates the following definition.

Definition (A.4). A morphism of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ is a *strong homeomorphism* if f is a universal homeomorphism and the diagonal morphism is universally submersive.

Proposition (A.5). *Let $f : \mathscr{X} \to \mathscr{Y}$ and $g : \mathscr{Y} \to \mathscr{Z}$ be morphisms of algebraic stacks.*

(i) *A separated universal homeomorphism is a strong homeomorphism.*
(ii) *If f is representable, then f is a universal homeomorphism if and only if f is locally separated or equivalently f is separated.*
(iii) *If f and g are strong homeomorphisms then so is $g \circ f$.*
(iv) *If $g \circ f$ is a strong homeomorphism and f is universally submersive then g is a strong homeomorphism.*
(v) *If $g \circ f$ is a strong homeomorphism and g is a representable universal homeomorphism then f is a strong homeomorphism.*

Corollary (A.6). *Let $\mathscr{X}$ be an algebraic stack and X and Y algebraic spaces together with morphisms $f : \mathscr{X} \to X$ and $g : X \to Y$. If $g \circ f$ is a strong homeomorphism and f is a universal homeomorphism, then g is separated.*

A.3. **Hilbert schemes.** Recall the following definition from [FGA, No. 195, §C 2].

Definition (A.7). Let X/S and Z/X be algebraic spaces. Consider the contravariant functor $\mathrm{Hom}_{X/S}(X, Z)$ which to an S-scheme T associates the set $\mathrm{Hom}_{X\times_S T}(X \times_S T, Z \times_S T)$, i.e. the set of sections of $Z \times_S T \to X \times_S T$. When $\mathrm{Hom}_{X/S}(X, Z)$ is representable, we denote the representing space with $\Pi(Z/X/S)$.

Remark **(A.8).** Let X/S and Y/S be algebraic spaces. The functor of Definition (A.7) is a generalization of the functor $T \mapsto \mathrm{Hom}_T(X_T, Y_T)$ where

$X_T = X \times_S T$ and $Y_T = Y \times_S T$. In fact, if $Z = X \times_S Y$ and $Z_T = Z \times_S T$ then $\mathrm{Hom}_T(X_T, Y_T) = \mathrm{Hom}_{X_T}(X_T, Z_T)$.

It is easily seen that if X/S is flat, proper and of finite presentation, then $\mathrm{Hom}_{X/S}(X, Z)$ is an open subfunctor of the Hilbert functor $\mathrm{Hilb}(Z/S)$. If in addition Z/S is separated and locally of finite presentation $\mathrm{Hom}_{X/S}(X, Z)$ is represented by an algebraic space $\Pi(Z/X/S)$, locally of finite presentation and separated over S. In fact, Artin has shown the existence of $\mathrm{Hilb}(Z/S)$ under these hypotheses [Art69a, Cor. 6.2].

Proposition (A.9). *Let X/S be a flat and proper morphism of finite presentation between algebraic spaces and let $Z \to X$ be an étale and separated morphism. Then $\Pi(Z/X/S) \to S$ is étale and separated. If furthermore X/S has a section, then there is an étale and separated morphism $\Pi(Z/X/S) \to Z \times_X S$.*

Proof. As $\Pi(Z/X/S) \to S$ is locally of finite presentation, it is enough to show that the morphism is formally étale. Let $T_0 \hookrightarrow T$ be a closed immersion given by a nilpotent ideal of S-schemes. As $Z \to X$ is étale the natural map

$$\mathrm{Hom}_X(X \times_S T, Z) \to \mathrm{Hom}_X(X \times_S T_0, Z)$$

is bijective. By the functorial description of $\Pi(Z/X/S)$ this bijection is identified with

$$\mathrm{Hom}_S\big(T, \Pi(Z/X/S)\big) \to \mathrm{Hom}_S\big(T_0, \Pi(Z/X/S)\big)$$

and thus $\Pi(Z/X/S) \to S$ is étale.

If X/S has a section $s : S \to X$ and T is any S-scheme, then there is a natural map

$$\mathrm{Hom}_X(X \times_S T, Z) \to \mathrm{Hom}_X(T, Z) = \mathrm{Hom}_S(T, Z \times_X S)$$

which induces an S-morphism $\Pi(Z/X/S) \to Z \times_X S$. As $\Pi(Z/X/S)$ and $Z \times_X S$ are étale over S it follows that $\Pi(Z/X/S) \to Z \times_X S$ is étale. □

A.4. **Descent results.** Finally we need a few descent results for the category of étale morphisms. Note that a separated étale morphism of algebraic spaces is strongly representable, i.e. schematic [LMB00, Thm. A.2]. If $X \to Y$ is a possibly non-separated étale morphism of algebraic spaces any section is an open immersion. This is enough for the following results in this section to hold either in the category of schemes or in the category of algebraic spaces.

Theorem (A.10) ([SGA$_1$, Cor. 3.3, Exp. IX])**.** *Let $S' \to S$ be a universally submersive morphism. Then $S' \to S$ is a* morphism of descent *for étale morphisms over S, i.e. for every étale morphisms $X \to S$ and $Y \to S$ the sequence*

$$\mathrm{Hom}_S(X, Y) \longrightarrow \mathrm{Hom}_{S'}(X', Y') \rightrightarrows \mathrm{Hom}_{S''}(X'', Y'')$$

is exact, where X' and Y' are the pull-backs of X and Y by $S' \to S$, and X'' and Y'' are the pull-backs of X and Y by $S'' = S' \times_S S' \to S$.

Lemma (A.11). *Let $S' \to S$ be a universally open morphism. If $S' \to S$ is a morphism of effective descent for étale, separated and quasi-compact*

morphisms then it is a morphism of effective descent for étale and separated morphisms.

Proof. As étale morphisms are morphisms of effective descent for the fibered category of algebraic spaces, we can assume that S is an affine scheme. As $S' \to S$ is open we can replace S' by an open quasi-compact subset and assume that S' is quasi-compact. Let $X' \to S'$ be an étale and separated morphism with descent data, i.e. with an automorphism $X'' = X' \times_S S' \cong S' \times_S X'$ over $S' \times_S S'$. Then $\pi_1, \pi_2 : X'' \to X'$ is an equivalence relation.

If $U' \subseteq X'$ is a quasi-compact open subset then $V' = \pi_1(\pi_2^{-1}(U'))$ is a quasi-compact open subset containing U' and stable under the equivalence relation. By hypotheses $V' \to X'$ descends to an étale, quasi-compact and separated morphism $V \to X$. As the intersection of two quasi-compact open subsets is quasi-compact, we can then glue together the resulting V's to an étale and separated morphism $X \to S$ which descends $X' \to S'$. □

Theorem (A.12). *Let $S' \to S$ be an fppf morphism. Then $S' \to S$ is a morphism of* effective *descent for étale and separated morphisms over S.*

Proof. This follows immediately from the fact that $S' \to S$ is a morphism of effective descent for all morphisms in the category of algebraic spaces [LMB00, Cor. 10.4.2] and that étale and separated morphisms are a stable class of morphisms in the fppf topology.

It also follows easily from the fact that $S' \to S$ is a morphism of effective descent for quasi-affine morphisms. In fact, by Lemma (A.11) it is enough to show that $S' \to S$ is a morphism of effective descent for étale, separated and quasi-compact morphisms and these are quasi-affine [LMB00, Thm. A.2]. □

The following theorem is a slight generalization of Theorem 4.7 in [SGA_1, Exp. IX] mentioned in a footnote to the theorem.

Theorem (A.13). *Let $S' \to S$ be a surjective integral and universally open morphism. Then $S' \to S$ is a morphism of effective descent for étale and separated morphisms over S.*

Proof. By Lemma (A.11) it is enough to show that $S' \to S$ is a morphism of effective descent for étale, separated and quasi-compact morphisms.

Let $X' \to S'$ be an étale, quasi-compact and separated morphism with descent data, i.e. an isomorphism $\theta : \pi_1^* X' \to \pi_2^* X'$ over $S'' = S' \times_S S'$. By étale descent we can assume that $S = \operatorname{Spec}(A)$ is an affine scheme. Then $S' = \operatorname{Spec}(A')$ can be written as an inverse limit of S-schemes $S'_\alpha = \operatorname{Spec}(A'_\alpha)$ which are finite and of finite presentation. As $X' \to S'$ is of finite presentation standard limit results, cf. [$\mathrm{EGA}_{\mathrm{IV}}$, §8 and Cor. 17.7.9], show that the descent data (X', θ) over S' comes from descent data $(X'_\alpha, \theta_\alpha)$ for some α. We can thus assume that $S' \to S$ is finite and of finite presentation.

Similarly, writing now instead S as an inverse limit of affine schemes of finite type over $\operatorname{Spec}(\mathbb{Z})$ we can assume that S is noetherian. The theorem then follows from [SGA_1, Exp. IX, Thm. 4.7]. □

Theorem (A.14) ([SGA_1, Exp. IX, Cor. 4.9])**.** *Let S be locally noetherian. Let $S' \to S$ be a surjective universally open morphism, locally of finite*

presentation. Then $S' \to S$ is a morphism of effective descent for étale and separated morphisms over S. If $T \to S$ is any morphism, then $T' \to T$ is also a morphism of effective descent.

Proof. To show effectiveness of $T' \to T$, we can assume that S and T are affine schemes and that $S' \to S$ is quasi-compact. By Lemma (A.11) it is enough to prove effectiveness for étale, quasi-compact and separated morphisms. By a limit argument, we may then assume that $T \to S$ is of finite type, and in particular that T is noetherian. The theorem is now reduced to [SGA$_1$, Exp. IX, Cor. 4.9]. □

In Theorem (A.14) we cannot easily remove the noetherian hypothesis by a limit argument. The problem is that if $f : X \to S$ is universally open and an inverse limit of $f_\lambda : X_\lambda \to S_\lambda$ then we cannot deduce that f_λ is universally open for some λ. However, by other methods, we can show the following stronger result:

Theorem (A.15) ([Ryd07c])**.** *The following are morphisms of effective descent for the category of étale morphisms.*

(i) *Universally open and surjective morphisms.*
(ii) *Universally closed and surjective morphisms of finite presentation.*

Theorem (A.16) ([SGA$_1$, Exp. IX, Thm. 4.10])**.** *Let $S' \to S$ be a strongly representable universal homeomorphism. The functor*

$$\{\textit{étale schemes over } S\} \longrightarrow \{\textit{étale schemes over } S'\}$$
$$X \longmapsto X \times_S S'$$

is an equivalence of categories. The same hold with "schemes" replaced by "algebraic spaces".

Proof. It is enough to show the equivalence of categories for affine schemes étale over S and S'. As $S' \to S$ is a universal homeomorphism $S' \hookrightarrow S' \times_S S'$ is a surjective immersion. The functor from étale schemes over $S' \times_S S'$ to étale schemes over S' is therefore an equivalence by [SGA$_1$, Exp. IX, Prop. 1.7]. In particular, every étale scheme over S' comes with a unique descent datum. This shows that the functor in the theorem is fully faithful. Essential surjectivity follows from Theorem (A.13) as every universal homeomorphism is integral [EGA$_{\text{IV}}$, Cor. 18.12.11]. □

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