TANNAKA DUALITY FOR ALGEBRAIC STACKS WITH QUASI-AFFINE DIAGONAL

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Abstract. We extend Tannaka duality of algebraic stacks with quasi-affine diagonals to the non-noetherian setting. **Preliminary draft!**

1. Introduction

Let T and X be algebraic stacks. In [HR14], we studied the Tannaka duality functor

$$\omega_X(T)\colon \operatorname{Hom}(T,X) \to \operatorname{Hom}_{c\otimes}(\mathsf{QCoh}(X), \mathsf{QCoh}(T)),$$

that takes a morphism $f\colon T \to X$ to the functor $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$, as well as the variants:

$$\omega_X^{\mathrm{ft}}(T)\colon \operatorname{Hom}(T,X) \to \operatorname{Hom}_{c\otimes}^{\mathrm{ft}}(\mathsf{QCoh}(X), \mathsf{QCoh}(T)),$$
$$\omega_{X,\simeq}^{\mathrm{ft}}(T)\colon \operatorname{Hom}(T,X) \to \operatorname{Hom}_{c\otimes,\simeq}^{\mathrm{ft}}(\mathsf{QCoh}(X), \mathsf{QCoh}(T)).$$

The right hand sides denotes the categories of monoidal functors that are cocontinuous (i.e., preserves colimits and thus in particular are right exact). In $\operatorname{Hom}_{c\otimes}^{\mathrm{ft}}$ and $\operatorname{Hom}_{c\otimes,\simeq}^{\mathrm{ft}}$ we also require the functors to preserve finite type objects. In the categories $\operatorname{Hom}_{c\otimes}$ and $\operatorname{Hom}_{c\otimes}^{\mathrm{ft}}$, the morphisms are all natural transformations of such functors. In the category $\operatorname{Hom}_{c\otimes,\simeq}^{\mathrm{ft}}$, we only include natural isomorphisms. We showed that [HR14, Thm. 5.10, Lem. 6.1, Thm. 8.4]

(i) $\omega_X(T)$ is an equivalence if X is quasi-compact and quasi-separated with affine stabilizer groups and there exists a nilpotent closed immersion $X_0 \hookrightarrow X$ such that X_0 has the resolution property (or merely has it Zariski-locally, e.g., if X is a scheme);
(ii) $\omega_{X,\simeq}^{\mathrm{ft}}(T)$ is an equivalence if T is locally excellent and X is quasi-compact and quasi-separated with affine stabilizer groups; and
(iii) $\omega_X^{\mathrm{ft}}(T)$ is an equivalence if T is locally noetherian and X is quasi-compact with quasi-affine diagonal.

The right hand side of ω_X is a stack for the fpqc topology whereas the left hand side is only known to be a stack for the fppf topology unless X has quasi-affine diagonal. This indicates that the restriction on T may be necessary when Δ_X is not quasi-affine.

The main result of this article is that for stacks with quasi-affine diagonals we can relax some of the finiteness assumptions on T.

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Theorem 1.1. *Let T and X be algebraic stacks and assume that X is quasi-compact with quasi-affine diagonal. Then $\omega_X^{\mathrm{ft}}(T)$ is fully faithful. If either*

(i) *T is noetherian; or*

(ii) *T has no embedded weakly associated points, e.g., T is reduced;*

then $\omega_X^{\mathrm{ft}}(T)$ is an equivalence of categories.

It is possible that the theorem holds for every T. Also see the more precise Theorem 6.2. It is also tempting to believe that $\omega_X(T)$ is an equivalence if it is so étale-locally. It would then follow that $\omega_X(T)$ is an equivalence if X has quasi-finite diagonal or more generally is of (s-)global type [Ryd15, Def. 2.1].

When X is a stack with quasi-affine diagonal, then the situation is simpler than for general X: if $f^*\colon \mathsf{QCoh}(T) \to \mathsf{QCoh}(X)$ is a tensor functor that preserves finitely generated objects, then a smooth presentation $p\colon X' \to X$ can be "pulled back" to a morphism of finite type $q\colon T' \to T$. That q is surjective is easy to establish. If in addition q is flat, then the equivalence of $\omega_X(T)$ follows immediately from fpqc descent. One way to assure that q is flat is to require that f^* is *tame*, that is, preserves flat objects. This approach is taken in [Lur04]. We prove that flatness of q often is automatic using the stratification methods of [HR14], thus obtaining Theorem 1.1.

Disclaimer. It is not clear how useful Theorem 1.1 is; all applications of Tannaka duality that we are aware of uses tensor functors that are constructed using completeness in one way or another. If A is a noetherian I-adically complete ring, then the functor $\mathsf{Coh}(A) \to \varprojlim_n \mathsf{Coh}(A/I^n)$ is an equivalence. Thus, if $f_n^*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(A/I^n)$ is a compatible system of tensor functors, then there is an induced tensor functor $f^*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(A)$ defined by $f^*(\mathcal{F}) = \varprojlim_n f_n^*(\mathcal{F})$. If A is not noetherian and (M_n) is an adic system of finitely generated A/I^n-modules, then $M := \varprojlim_n M_n$ is a finitely generated A-module, but in general we cannot define f^* in this way. The problem is that this forces M to be complete (at least if I is finitely generated) whereas not every finitely generated A-module is complete. In fact, there are examples of A with I principal such that there exists finitely presented A-modules that are not complete [Stacks, 05JD].

If X has the resolution property, then it is possible to define f^* using vector bundles: every vector bundle on $\mathrm{Spec}(A)$ is automatically complete. In [Bha14], Bhatt takes a similar approach and defines f^* using perfect complexes. He obtains a derived version of Tannaka duality for algebraic spaces that is strong enough to be applied to complete non-noetherian rings.

2. Monoidal and tensor categories

Let $\mathsf{CAlg}(\mathbf{C})$ be the category of commutative $\mathbf{C}$-algebras, which inherits a symmetric monoidal structure from $\mathbf{C}$. If the underlying category of $\mathbf{C}$ is cocomplete and $\otimes_{\mathbf{C}}$ is cocontinuous in each argument, then the underlying category of $\mathsf{CAlg}(\mathbf{C})$ is cocomplete and the forgetful functor $\mathsf{CAlg}(\mathbf{C}) \to \mathbf{C}$ preserves limits and directed colimits [Joh02, Lem. 1.1.8].

If $F\colon \mathbf{C} \to \mathbf{D}$ is a monoidal functor, then F induces a functor $\mathsf{CAlg}(F)\colon \mathsf{CAlg}(\mathbf{C}) \to \mathsf{CAlg}(\mathbf{D})$. If $G\colon \mathbf{D} \to \mathbf{C}$ is a (lax monoidal) right adjoint to F, then there is also an induced (lax monoidal) right adjoint $\mathsf{CAlg}(G)\colon \mathsf{CAlg}(\mathbf{D}) \to \mathsf{CAlg}(\mathbf{C})$.

The functors $\mathsf{CAlg}(F)$ and $\mathsf{CAlg}(G)$ are compatible with the forgetful functors $\mathsf{CAlg}(\mathbf{C}) \to \mathbf{C}$ and $\mathsf{CAlg}(\mathbf{D}) \to \mathbf{D}$.

3. Locally finitely generated and presentable categories

In this section we recall some concepts from [AR94, Ch. 1]. Let $\mathbf{C}$ be a cocomplete category. An object c of $\mathbf{C}$ is *finitely presentable* if the natural map:

$$\varinjlim_{\lambda} \mathrm{Hom}_{\mathbf{C}}(c, d_\lambda) \to \mathrm{Hom}_{\mathbf{C}}(c, \varinjlim_{\lambda} d_\lambda)$$

is bijective for every direct system $\{d_\lambda\}_\lambda$ in $\mathbf{C}$. An object c of $\mathbf{C}$ is *finitely generated* if the map above is bijective for those direct systems $\{d_\lambda\}_\lambda$ with monomorphic bonding maps. We denote the collection of all finitely presentable (resp. finitely generated) objects of $\mathbf{C}$ by $\mathbf{C}^{\mathrm{fp}}$ (resp. $\mathbf{C}^{\mathrm{fg}}$).

A category $\mathbf{C}$ is *locally finitely presentable* if it is cocomplete and has a set $\mathcal{A}$ of finitely presentable objects such that every object c of $\mathbf{C}$ is a directed colimit of objects from $\mathcal{A}$. Similarly, $\mathbf{C}$ is *locally finitely generated* if it is cocomplete and has a set $\mathcal{A}$ of finitely generated objects such that every object c of $\mathbf{C}$ is a directed colimit of objects from $\mathcal{A}$.

Example 3.1. If $\mathbf{C}$ is a locally finitely presentable category, then an object of $\mathbf{C}$ is finitely generated if and only if it is the epimorphic image of a finitely presentable object. In particular, every object of $\mathbf{C}$ is the directed colimit of its finitely generated subobjects [AR94, Ch. 1.E]. Also if $\mathbf{C}^{\mathrm{fp}}$ has a small skeleton, then [AR94, Thm. 1.58] shows that $\mathbf{C}^{\mathrm{fg}}$ also has a small skeleton.

Example 3.2. If $\mathbf{C}$ is a locally finitely presentable symmetric monoidal category with $\otimes_{\mathbf{C}}$ cocontinuous in each variable (e.g., a Grothendieck abelian tensor category), then $\mathsf{CAlg}(\mathbf{C})$ is locally finitely presentable.

Example 3.3. Let X be a quasi-compact and quasi-separated algebraic stack. The finitely presentable (resp. finitely generated) objects of $\mathsf{QCoh}(X)$ are the quasi-coherent $\mathcal{O}_X$-modules of finite presentation (resp. finite type). Similarly, in $\mathsf{CAlg}(\mathsf{QCoh}(X))$, the finitely presentable (resp. finitely generated) objects are the quasi-coherent $\mathcal{O}_X$-algebras of finite presentation (resp. finite type).

An algebraic stack X has the *completeness property* if every quasi-coherent $\mathcal{O}_X$-module is a directed colimit of quasi-coherent $\mathcal{O}_X$-modules of finite presentation. In particular, if X is quasi-compact, quasi-separated, and has the completeness property, then $\mathsf{QCoh}(X)$ is locally finitely presentable. The completeness property for algebraic stacks was investigated in detail by the second author [Ryd15].

Example 3.4. Let X be an algebraic stack. If X is noetherian or of global type, then X has the completeness property [Ryd15, §4].

An algebraic stack X has the *partial completeness property* if every quasi-coherent $\mathcal{O}_X$-module is a directed colimit of quasi-coherent $\mathcal{O}_X$-submodules of finite type. If X is quasi-compact and quasi-separated, then X has the partial completeness property [Ryd16]. In particular, $\mathsf{QCoh}(X)$ is then locally finitely generated.

4. Bounded tensor functors

A functor $q^*\colon \mathbf{C} \to \mathbf{D}$ between cocomplete categories is *bounded* if it admits a right adjoint $q_*\colon \mathbf{D} \to \mathbf{C}$ that preserves directed colimits. It is *weakly bounded* if q_* preserves the directed colimits with monomorphic bonding maps.

Proposition 4.1. *Let $q^*\colon \mathbf{C} \to \mathbf{D}$ be a functor between cocomplete categories.*

(i) *If q^* is bounded, then $q^*(\mathbf{C}^{\mathrm{fp}}) \subseteq \mathbf{D}^{\mathrm{fp}}$. Conversely, if $\mathbf{C}$ is locally finitely presentable, q^* is cocontinuous, and $q^*(\mathbf{C}^{\mathrm{fp}}) \subseteq \mathbf{D}^{\mathrm{fp}}$, then q^* is bounded.*

(ii) *If q^* is weakly bounded, then $q^*(\mathbf{C}^{\mathrm{fg}}) \subseteq \mathbf{D}^{\mathrm{fg}}$. Conversely, if $\mathbf{C}$ is locally finitely generated, q^* is cocontinuous, and $q^*(\mathbf{C}^{\mathrm{fg}}) \subseteq \mathbf{D}^{\mathrm{fg}}$, then q^* is weakly bounded.*

Proof. We only prove (i) as (ii) has an almost identical proof. Let $c \in \mathbf{C}$ be finitely presentable and let $\{d_\lambda\}_\lambda$ be a direct system of objects in $\mathbf{D}$ with colimit d. There are natural bijections:

$$\varinjlim_\lambda \mathrm{Hom}_{\mathbf{C}}(q^*c, d_\lambda) \cong \varinjlim_\lambda \mathrm{Hom}_{\mathbf{D}}(c, q_*d_\lambda) \cong \mathrm{Hom}_{\mathbf{D}}(c, \varinjlim_\lambda q_*d_\lambda)$$
$$\cong \mathrm{Hom}_{\mathbf{D}}(c, q_*d) \cong \mathrm{Hom}_{\mathbf{C}}(q^*c, d).$$

Hence, $q^*(\mathbf{C}^{\mathrm{fp}}) \subseteq \mathbf{D}^{\mathrm{fp}}$. Conversely, if q^* is cocontinuous and $q^*(\mathbf{C}^{\mathrm{fp}}) \subseteq \mathbf{D}^{\mathrm{fp}}$, then there is an isomorphism $\mathrm{Hom}_{\mathbf{D}}(c, \varinjlim_\lambda q_*d_\lambda) \cong \mathrm{Hom}_{\mathbf{D}}(c, q_*d)$ for every finitely presentable $c \in \mathbf{C}$ and direct system as before. Thus, if in addition $\mathbf{C}$ is locally finitely presentable, then q^* is bounded. □

Corollary 4.2. *Let $q^*\colon \mathbf{C} \to \mathbf{D}$ be a cocontinuous tensor functor between Grothendieck abelian tensor categories.*

(i) *If $\mathbf{C}$ is locally finitely presentable, q^* is bounded and A is a finitely presentable commutative $\mathbf{C}$-algebra, then q^*A is a finitely presentable commutative $\mathbf{D}$-algebra.*

(ii) *If $\mathbf{C}$ is locally finitely generated, q^* is weakly bounded and A is a finitely generated commutative $\mathbf{C}$-algebra, then q^*A is a finitely generated commutative $\mathbf{D}$-algebra.*

Proof. Again, we only prove (i). By definition, the right adjoint q_* preserves directed colimits. It follows that $\mathsf{CAlg}(q_*)\colon \mathsf{CAlg}(\mathbf{D}) \to \mathsf{CAlg}(\mathbf{C})$ preserves directed colimits because the forgetful functor $\mathsf{CAlg}(\mathbf{C}) \to \mathbf{C}$ (and similarly with $\mathbf{D}$) preserves and reflects directed colimits. Applying Proposition 4.1, we deduce the result, since $\mathsf{CAlg}(q^*)$ is left adjoint to $\mathsf{CAlg}(q_*)$. □

Corollary 4.3. *Let $\mathbf{C}$ be an abelian category and let $q^*\colon \mathbf{C} \to \mathbf{Q}$ be a localization that is bounded. Then $q^*(\mathbf{C}^{\mathrm{fp}}) = \mathbf{Q}^{\mathrm{fp}}$ and $q^*(\mathbf{C}^{\mathrm{fg}}) = \mathbf{Q}^{\mathrm{fg}}$. If $\mathbf{C}$ is locally finitely presentable (resp. locally finitely generated) then so is $\mathbf{Q}$.*

Proof. A simple consequence of Proposition 4.1. □

5. Bounded abelian tensor categories

In general, the question of whether a tensor localization of an abelian tensor category is supported [HR14, Def. 5.6] appears to be a subtle one, and is not something we wish to become involved with. Instead, we make the following definition, which simplifies this situation greatly.

Definition 5.1. A Grothendieck abelian tensor category $\mathbf{C}$ is *weakly bounded* if it satisfies the following conditions:

(i) $\mathbf{C}$ is locally finitely generated and
(ii) $\mathcal{O}_{\mathbf{C}}$ is finitely presentable.

A Grothendieck abelian tensor category $\mathbf{C}$ is *bounded* if it satisfies the following conditions:

(i) $\mathbf{C}$ is locally finitely presentable,
(ii) $\mathcal{O}_{\mathbf{C}}$ is finitely presentable,
(iii) $\mathbf{C}^{\mathrm{fp}}$ has small skeleton,
(iv) if M and N are finitely presentable objects of $\mathbf{C}$, then $M \otimes_{\mathbf{C}} N$ is finitely presentable.

Example 5.2. If X is an algebraic stack with the completeness property, then $\mathsf{QCoh}(X)$ is bounded. If X is a quasi-compact and quasi-separated algebraic stack, then $\mathsf{QCoh}(X)$ is weakly bounded.

Example 5.3. Let $\mathbf{C}$ be a weakly bounded Grothendieck abelian tensor category that is generated by dualizable objects. If $\mathbf{K}$ is a tensor ideal of $\mathbf{C}$, then it is supported. To see this, we note that it is sufficient to prove that if $M \in \mathbf{C}^{\mathrm{fg}}$ and $N \in \mathbf{K}$, then $\mathcal{H}om_{\mathbf{C}}(M, N) \in \mathbf{K}$. By hypothesis, if $M \in \mathbf{C}^{\mathrm{fg}}$, then there is a dualizable object E of $\mathbf{C}$ and an epimorphism $E \twoheadrightarrow M$. It follows that $\mathcal{H}om_{\mathbf{C}}(M, N) \hookrightarrow \mathcal{H}om_{\mathbf{C}}(E, N)$. But E is dualizable, so $\mathcal{H}om_{\mathbf{C}}(E, N) \cong E^{\vee} \otimes_{\mathbf{C}} N \in \mathbf{K}$, and the result follows.

Example 5.4. Let $\mathbf{C}$ be a weakly bounded Grothendieck abelian tensor category. If $q^* \colon \mathbf{C} \to \mathbf{Q}$ is a bounded tensor localization, then $\mathbf{Q}$ is weakly bounded. This is an immediate consequence of Corollary 4.3.

Presumably the following Lemma is known, though we were unable to locate a reference in the literature.

Lemma 5.5. *Let $\mathbf{C}$ be a weakly bounded Grothendieck abelian tensor category. Let $q^* \colon \mathbf{C} \to \mathbf{Q}$ be a bounded tensor localization. If J is an $\mathcal{O}_{\mathbf{C}}$-ideal such that $q^*(\mathcal{O}_{\mathbf{C}}/J) \cong 0$, then there exists a $\mathcal{O}_{\mathbf{C}}$-ideal J' that is finitely generated, is contained in J, and $q^*(\mathcal{O}_{\mathbf{C}}/J') \cong 0$.*

Proof. We may express J as a directed colimit $\varinjlim_i J_i$, where each J_i is a finitely generated subobject. Let $C_i = \mathcal{O}_{\mathbf{C}}/J_i$ and $C = \mathcal{O}_{\mathbf{C}}/J$. Each C_i is finitely presentable and $\varinjlim_i C_i \cong C$. Since $q^*C \cong 0$, and $\mathcal{O}_{\mathbf{Q}}$ is finitely presented, it follows that $\varinjlim_i \mathrm{Hom}_{\mathbf{C}}(\mathcal{O}_{\mathbf{Q}}, q^*C_i) = 0$ and so eventually the surjections $\mathcal{O}_{\mathbf{Q}} \twoheadrightarrow q^*C_i$ are 0. Thus there is an $i_0 \in I$ such that $q^*(C_i) \cong 0$. □

Example 5.6. If $\mathbf{C}$ is a Grothendieck abelian tensor category and M and N are objects of $\mathbf{C}$, then $\mathrm{Ann}_{\mathbf{C}}(M) \subseteq \mathrm{Ann}_{\mathbf{C}}(M \otimes_{\mathbf{C}} N)$. This is a simple consequence of the existence of a natural map $\mathcal{H}om_{\mathbf{C}}(M, M) \to \mathcal{H}om_{\mathbf{C}}(M \otimes_{\mathbf{C}} N, M \otimes_{\mathbf{C}} N)$

Presumably the following Proposition is also known, though we were unable to locate a reference in the literature.

Proposition 5.7. *Let $q^*\colon \mathbf{C} \to \mathbf{Q}$ be a tensor localization of a Grothendieck abelian tensor category. Consider the following two assertions:*

(i) *the tensor localization q^* is supported and bounded; and*
(ii) *there is a set of finitely generated $\mathcal{O}_{\mathbf{C}}$-ideals $\{J_\lambda\}_{\lambda\in\Lambda}$ such that $\ker(q^*)$ is the smallest tensor ideal containing the set $\{\mathcal{O}_{\mathbf{C}}/J_\lambda\}_{\lambda\in\Lambda}$.*

If $\mathbf{C}$ is weakly bounded, then (i) $\Rightarrow$ (ii)*. If $\mathbf{C}$ is bounded, then* (ii) $\Rightarrow$ (i)*.*

Proof. Let $\mathbf{K} = \ker(q^*)$. For (i)$\Rightarrow$(ii), let Λ be a small and skeletal subset of $\mathbf{C}^{\mathrm{fg}} \cap \mathbf{K}$ (Example 3.1). For each $\lambda \in \Lambda$ let $J^0_\lambda = \mathrm{Ann}_{\mathbf{C}}(\lambda)$. Then $\mathcal{O}_{\mathbf{C}}/J^0_\lambda$ belongs to $\mathbf{K}$. By Lemma 5.5, there exists $J_\lambda \subseteq J^0_\lambda$ that is finitely generated such that $\mathcal{O}_{\mathbf{C}}/J_\lambda \in \mathbf{K}$. Clearly, the collection of $\mathbf{C}$-ideals $\{J_\lambda\}_{\lambda\in\Lambda}$ satisfies the conditions.

For (ii)$\Rightarrow$(i): let $\mathcal{V}$ be the set of $\mathcal{O}_{\mathbf{C}}$-ideals J such that $q^*(\mathcal{O}_{\mathbf{C}}/J) \cong 0$. Let $\mathbf{L}_0$ be the full subcategory of $\mathbf{C}^{\mathrm{fp}}$ with objects those M such that $\mathrm{Ann}(M) \in \mathcal{V}$. Clearly, $\mathbf{L}_0 \subseteq \mathbf{K}$ and for each $\lambda \in \Lambda$ we have that $\mathcal{O}_{\mathbf{C}}/J_\lambda$ belongs to $\mathbf{L}_0$. Observe that if $N \in \mathbf{C}^{\mathrm{fp}}$ and $L \in \mathbf{L}_0$, then $L \otimes_{\mathbf{C}} N \in \mathbf{L}_0$ (Example 5.6).

Let $\mathbf{L} = \mathrm{Ind}(\mathbf{L}_0)$. Since $\mathbf{K}$ is cocomplete and every object of $\mathbf{L}_0$ is finitely presentable in $\mathbf{K}$, there is an induced functor $\mathbf{L} \to \mathbf{K}$ and it is fully faithful [KS06, Prop. 6.3.4]. The result will be proved if we can show that $\mathbf{L} = \mathbf{K}$. To see this, it is sufficient to prove that $\mathbf{L}$ is a tensor ideal of $\mathbf{C}$.

Clearly, $\mathbf{L}$ is closed under directed colimits. Next we prove that if $L \in \mathbf{C}^{\mathrm{fg}}$ has $\mathrm{Ann}(L) \in \mathcal{V}$, then $L \in \mathbf{L}$. Since $\mathrm{Ann}(L) \in \mathcal{V}$, by Lemma 5.5, there is a finitely generated $\mathcal{O}_{\mathbf{C}}$-ideal J with $J \subseteq \mathrm{Ann}(L)$ and $q^*(\mathcal{O}_{\mathbf{C}}/J) \cong 0$. Since $\mathbf{C}$ is locally finitely presentable, we may express L as a directed colimit $\varinjlim_j L_j$, where each L_j is finitely presentable and the bonding maps $L_j \to L_k$ are epimorphisms. In particular, it follows that we have isomorphisms:

$$\varinjlim_j (L_j \otimes_{\mathbf{C}} [\mathcal{O}_{\mathbf{C}}/J]) \cong (\varinjlim_j L_j) \otimes_{\mathbf{C}} (\mathcal{O}_{\mathbf{C}}/J) \cong L.$$

For each j we have that $L_j \otimes_{\mathbf{C}} [\mathcal{O}_{\mathbf{C}}/J]$ is finitely presentable and $J \subseteq \mathrm{Ann}(L_j \otimes_{\mathbf{C}} [\mathcal{O}_{\mathbf{C}}/J])$, thus $L_j \otimes_{\mathbf{C}} [\mathcal{O}_{\mathbf{C}}/J] \in \mathbf{L}_0$. We deduce immediately that $L \in \mathbf{L}$. ☐

Proposition 5.8. *Let $q^*\colon \mathbf{C} \to \mathbf{Q}$ be a bounded and supported tensor localization. Let $f^*\colon \mathbf{C} \to \mathbf{C}'$ be a weakly bounded tensor functor. If $\mathbf{C}$ is weakly bounded and $\mathbf{C}'$ is bounded, then there is a 2-cocartesian diagram in* **AbTC***:*

$$\begin{array}{ccc} \mathbf{Q}' & \xleftarrow{\;f'^*\;} & \mathbf{Q} \\ {\scriptstyle q'^*}\big\uparrow & & \big\uparrow{\scriptstyle q^*} \\ \mathbf{C}' & \xleftarrow{\;f^*\;} & \mathbf{C}, \end{array}$$

where q'^* is a bounded and supported localization and f'^* is weakly bounded. If in addition f^* is bounded, then so is f'^*.

Proof. Let $\mathbf{K} = \ker q^*$. Since q^* is a bounded supported localization and $\mathbf{C}$ is weakly bounded, there is a set of finitely generated $\mathcal{O}_{\mathbf{C}}$-ideals $\{J_\lambda\}_{\lambda\in\Lambda}$ such that $\mathbf{K}$ is the smallest tensor ideal containing the set $\{\mathcal{O}_{\mathbf{C}}/J_\lambda\}_{\lambda\in\Lambda}$ (Proposition 5.7).

Let $\mathbf{K}'$ denote the smallest localizing tensor ideal of $\mathbf{C}'$ containing the set $\{f^*(\mathcal{O}_{\mathbf{C}}/J_\lambda)\}_{\lambda\in\Lambda}$. Denote by $q'^*\colon \mathbf{C}' \to \mathbf{Q}'$ the quotient of $\mathbf{C}'$ by $\mathbf{K}'$. Since $\mathbf{C}'$ is bounded, q'^* is a supported bounded localization (Proposition 5.7).

We now show that there exists a cocontinuous tensor functor $f'^*\colon \mathbf{Q} \to \mathbf{Q}'$ together with a natural isomorphism of functors $f'^*q^* \simeq q'^*f^*$. By [HR14, Thm. 5.8], it is sufficient to prove that if $K \in \mathbf{K}$, then $q'^*f^*(K) \simeq 0$. Since $\mathbf{K}$ is the smallest tensor ideal containing the set $\{\mathcal{O}_{\mathbf{C}}/J_\lambda\}_{\lambda\in\Lambda}$, it is sufficient to prove that for each $\lambda \in \Lambda$ we have $q'^*f^*(\mathcal{O}_{\mathbf{C}}/J_\lambda) \simeq 0$. By the definition of $\mathbf{K}'$, the claim follows.

We now claim that $f'^*\colon \mathbf{Q} \to \mathbf{Q}'$ is weakly bounded. For this, we have to show that f'_* preserves directed colimits with monomorphic bonding maps. Since $f'_* \simeq q^*q_*f'_* \simeq q^*f_*q'_*$ and f^* and q'^* are weakly bounded, the claim follows. A similar argument shows that f'^* is bounded if f^* is bounded.

Finally, we show that the square is 2-cocartesian in **AbTC**. To see this, we consider cocontinuous tensor functors $h^*\colon \mathbf{C}' \to \mathbf{E}$ and $g^*\colon \mathbf{Q} \to \mathbf{E}$, together with an isomorphism of functors $h^*f^* \simeq g^*q^*$. It remains to produce an essentially unique functor $e^*\colon \mathbf{Q}' \to \mathbf{E}$ that is compatible with the data. As before, by [HR14, Thm. 5.8], it is sufficient to prove that $h^*f^*(\mathcal{O}_{\mathbf{C}}/J_\lambda) \cong 0$ for every $\lambda \in \Lambda$. Since $h^*f^* \simeq g^*q^*$, we deduce the result. □

Corollary 5.9. *Let $p\colon X' \to X$ be a quasi-affine morphism of quasi-compact and quasi-separated algebraic stacks. Let T be a quasi-compact and quasi-separated algebraic stack and let $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a weakly bounded tensor functor. If T has the completeness property, then there exists a quasi-affine morphism $p'\colon T' \to T$ and a 2-cocartesian diagram in the 2-category* **AbTC**

$$\begin{array}{ccc} \mathsf{QCoh}(T') & \xleftarrow{f'^*} & \mathsf{QCoh}(X') \\ {\scriptstyle p'^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ \mathsf{QCoh}(T) & \xleftarrow{f^*} & \mathsf{QCoh}(X), \end{array}$$

where f'^* is a weakly bounded tensor functor. Moreover,

(i) *if p is of finite type, then p' is of finite type;*
(ii) *if p is of finite presentation, f^* is bounded and X has the completeness property, then p' is of finite presentation and f'^* is bounded; and*
(iii) *if f^* arises from a morphism $f\colon T \to X$, then we can take $T' = T \times_X X'$.*

Proof. We can factor p as a quasi-compact open immersion $j\colon X' \to \overline{X'}$ followed by an affine morphism $\overline{p}\colon \overline{X'} \to X$. Let $\overline{T'} = \operatorname{Spec}_T(f^*p_*\mathcal{O}_{X'})$. By [HR14, Cor. 3.6], we have a 2-cocartesian diagram in the 2-category **AbTC**:

$$\begin{array}{ccc} \mathsf{QCoh}(\overline{T'}) & \xleftarrow{\overline{f'}^*} & \mathsf{QCoh}(\overline{X'}) \\ {\scriptstyle \overline{p'}^*}\uparrow & & \uparrow{\scriptstyle \overline{p}^*} \\ \mathsf{QCoh}(T) & \xleftarrow{f^*} & \mathsf{QCoh}(X), \end{array}$$

where $f^*\overline{p}_* = \overline{p'}_*\overline{f'}^*$. Since $\overline{p}_*$ and $\overline{p'}_*$ are exact, conservative and preserves colimits, and f^* is weakly bounded, it follows that $\overline{f'}^*$ is weakly bounded.

Since T has the completeness property, $\overline{T'} \to T$ is affine, and $\overline{X'}$ is quasi-compact and quasi-separated, it follows that $\overline{T'}$ has the completeness property and $\overline{X'}$ has the partial completeness property. Thus $\mathsf{QCoh}(\overline{X'})$ is weakly bounded and $\mathsf{QCoh}(\overline{T'})$ is bounded (Example 5.2). The tensor localization $j^*\colon \mathsf{QCoh}(\overline{X'}) \to \mathsf{QCoh}(X')$ is bounded and supported [HR14, Ex. 5.7]. The existence of the 2-cocartesian diagram with f'^* weakly bounded thus follows from Proposition 5.8.

The statements in (i) and (ii) follows from Corollary 4.2, noting that if X has the completeness property and $X' \to X$ is of finite presentation, then X' also has the completeness property. □

6. Proof of Theorem 1.1

Let T and X be algebraic stacks. When X has quasi-affine diagonal, then $\omega_X(T)$ is fully faithful [HR14, Prop. 4.8 (i)]. If $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ is a cocontinuous tensor functor, then we say that f^* is *algebraic* if it arises from a morphism of algebraic stacks $f\colon T \to X$. When X has quasi-affine diagonal, f is unique up to unique 2-isomorphism.

Proposition 6.1. *Let T and X be algebraic stacks and assume that X is quasi-compact with quasi-affine diagonal. Let $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor preserving objects of finite type. Then the functor* $\mathrm{Alg}(f^*)\colon \mathbf{Rep}/T \to \mathsf{Sets}$ *given by*

$$\mathrm{Alg}(f^*)(g\colon T' \to T) = \begin{cases} \{*\}, & \textit{if } g^*f^* \textit{ is algebraic} \\ \emptyset, & \textit{if } g^*f^* \textit{ is not algebraic} \end{cases}$$

is represented by a monomorphism of finite type. If X has the completeness property and f^* preserves objects of finite presentation, then $\mathrm{Alg}(f^*) \to T$ *is a monomorphism of finite presentation.*

Proof. Note that f^* is weakly bounded (Proposition 4.1). The question is local on T, so we may assume that T is an affine scheme.

Pick a presentation $p\colon X' \to X$ with X' affine. Then by Corollary 5.9, there exists a quasi-affine morphism $p'\colon T' \to T$ of finite type and a 2-cocartesian square

$$\begin{array}{ccc} \mathsf{QCoh}(T') & \xleftarrow{f'^*} & \mathsf{QCoh}(X') \\ {\scriptstyle p'^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ \mathsf{QCoh}(T) & \xleftarrow{f^*} & \mathsf{QCoh}(X). \end{array}$$

Since X' is affine, f'^* is algebraic. Let $X'' = X' \times_X X'$ and $T'' = T' \times_T T'$. Since $\pi_1, \pi_2\colon X'' \to X'$ are quasi-affine, we also have 2-cocartesian squares:

$$\begin{array}{ccc} \mathsf{QCoh}(T'') & \xleftarrow{f''^*} & \mathsf{QCoh}(X'') \\ {\scriptstyle \pi_i'^*}\uparrow & & \uparrow{\scriptstyle \pi_i^*} \\ \mathsf{QCoh}(T') & \xleftarrow{f'^*} & \mathsf{QCoh}(X'). \end{array}$$

Since f'^* is algebraic, we have that $\pi'_i\colon T'' \to T'$ are faithfully flat of finite presentation. We can thus form the quotient of the fppf equivalence relation $T'' \to T'$. This gives a monomorphism $j\colon T_0 \to T$ of finite type that is an isomorphism exactly when f^* is algebraic by fppf descent. It is clear that $\mathrm{Alg}(f^*) = T_0$.

If X has the completeness property and f^* preserves finitely presented objects, then $T' \to T$ is of finite presentation, hence $T_0 \to T$ is of finite presentation. □

That $\omega_X^{\mathrm{ft}}(T)$ is essentially surjective under the conditions of Theorem 1.1 follows from the slightly more precise result:

Theorem 6.2. *Let T and X be algebraic stacks and assume that X is quasi-compact with quasi-affine diagonal. Let $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor preserving objects of finite type. Then*

(i) $\mathrm{Alg}(f^*) \to T$ *is a closed immersion,*
(ii) *If T is noetherian or has no embedded weakly associated points, then* $\mathrm{Alg}(f^*) \to T$ *is an isomorphism,*
(iii) *If X has the completeness property and f^* preserves objects of finite presentation, then* $\mathrm{Alg}(f^*) \to T$ *is of finite presentation.*

Proof. We have seen (Proposition 6.1) that $j\colon \mathrm{Alg}(f^*) \to T$ is a monomorphism of finite type and that j is of finite presentation under the hypothesis (iii). We will now show that j is a closed immersion (resp. isomorphism) using [HR14, Prop. A.3].

The argument is similar to the proof of [HR14, Thm. 8.4]. Since X has affine stabilizers, there is a finitely presented filtration (X_k) with strata (Y_k) that have the resolution property [HR14, Prop. 8.2].

The filtration (X_k) induces a finitely presented filtration (T_k) on T. Indeed, if I_k denotes the ideal of $X_k \hookrightarrow X$, then $(f^*I_k)\mathcal{O}_T$ is a finitely generated ideal defining T_k. Let (W_k) denote the strata of (T_k). Then by [HR14, Thm. 5.8], we have cocontinuous tensor functors $f_k^{[n],*}\colon \mathsf{QCoh}(Y_k^{[n]}) \to \mathsf{QCoh}(W_k^{[n]})$. By [HR14, Lem. 6.1], $f_k^{[n],*}$ is algebraic for all k and n. We conclude that j is an isomorphism over $W_k^{[n]}$ for every k and n. It follows that j is a closed immersion (resp. an isomorphism) by [HR14, Lem. A.2 and Prop. A.3] □

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