PERFECT COMPLEXES ON ALGEBRAIC STACKS

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Abstract. Let X be a quasi-compact algebraic stack with quasi-compact and separated diagonal. We show that the unbounded derived category $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated if (i) X has quasi-finite and separated diagonal or (ii) X is a $\mathbb{Q}$-stack of s-global type. These are both simple consequences of our main result: compact generation of $\mathsf{D}_{\mathrm{qc}}(X)$ is quasi-finite flat local on X.

Introduction

Our first main result is the following.

Theorem A. *Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. Then the unbounded derived category $\mathsf{D}_{\mathrm{qc}}(X)$, of $\mathcal{O}_X$-modules with quasi-coherent cohomology, is compactly generated by a single perfect complex.*

This generalizes the results of A. Bondal and M. Van den Bergh [BB03, Thm. 3.1.1] for schemes and B. Toën [Toë12, Cor. 5.2] for Deligne–Mumford stacks admitting coarse moduli spaces (i.e., X has finite inertia). Every compact object of $\mathsf{D}_{\mathrm{qc}}(X)$ is a perfect complex. A subtlety in Theorem A is that the converse does not always hold in positive characteristic: if X is not tame, then there are perfect complexes that are not compact.

Extending Theorem A to stacks with infinite stabilizer groups is our second main result. For this we recall a notion from [Ryd09, §2]. We say that an algebraic stack X is of *s-global type* if there exists a finitely presented, separated and representable étale surjection $X' \to X$ such that X' is a quotient of a quasi-affine scheme by GL_N for some N. If X is of s-global type, then X is quasi-compact with quasi-affine diagonal. Stacks as in Theorem A are of s-global type [Ryd09, Cor. 2.7]. Replacing étale with quasi-finite flat gives the same class of stacks [Ryd09, Prop. 2.7.1].

Theorem B. *Let X be a $\mathbb{Q}$-stack of s-global type. Then the unbounded derived category $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by a countable set of perfect complexes.*

In general, it is not possible to generate $\mathsf{D}_{\mathrm{qc}}(X)$ by a single perfect complex (e.g., $X = B\mathbb{G}_m$); thus Theorem B could be viewed as sharp. For a variant in positive characteristic, see [HR14, Thm. D].

Theorems A and B are both consequences of a more general result, which essentially asserts that compact generation of $\mathsf{D}_{\mathrm{qc}}(X)$ by perfect complexes is local for the quasi-finite flat topology. To be precise, we need a slightly stronger notion. If β is cardinal, then we say that X is *β-crisp* if $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by a

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set of cardinality $\leq \beta$ and for every quasi-compact open subset $U \subseteq X$, there is a perfect complex supported on the complement. Furthermore, we require that this also holds étale-locally on X, cf. Definition 8.1. If X is β-crisp and β is finite, then X is compactly generated by a single perfect complex. Hence Theorems A and B are essentially implied by

Theorem C. *Let $p : X' \to X$ be a morphism of quasi-compact and quasi-separated algebraic stacks that is representable, separated, quasi-finite, locally of finite presentation, and faithfully flat. If X' is β-crisp, then X is β-crisp.*

Theorem C is proved using the technique of quasi-finite flat dévissage for algebraic stacks, due to the second author [Ryd11], together with some descent results for compact generation. In sections 5–6 these descent results are stated in great generality—for presheaves of triangulated categories—without requiring monoidal or linear structures. We also establish compact generation for many other categories such as twisted derived categories of sheaves on stacks and derived stacks (see Theorem 6.9, §9 and further below).

Perfect and compact objects. As we already have mentioned, some care has to be taken since perfect objects are not necessarily compact. The perfect objects are the *locally compact* objects or, equivalently, the dualizable objects. If X is a quasi-compact and quasi-separated algebraic stack, then the following conditions are equivalent (Remark 4.12):

- every perfect object of $\mathsf{D}_{\mathrm{qc}}(X)$ is compact;
- the structure sheaf $\mathcal{O}_X$ is compact;
- there exists an integer d_0 such that for all quasi-coherent sheaves M on X, the cohomology groups $H^d(X, M)$ vanish for all $d > d_0$; and
- the derived global section functor $\mathsf{R}\Gamma\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(\mathsf{Ab})$ commutes with small coproducts.

We say that a stack is *concentrated* when it satisfies the conditions above.

In [HR14, Thm. B] we give a complete list of the group schemes G/k such that BG is concentrated: every linear group and certain non-affine groups in characteristic zero but only the linearly reductive groups in positive characteristic. Note that [HR14, Thm. A] also gives many examples of classifying stacks that are not concentrated, yet compactly generated.

Drinfeld and Gaitsgory have proved that noetherian algebraic stacks with affine stabilizer groups in characteristic zero are concentrated [DG13, Thm. 1.4.2]. This is generalized in [HR14, Thm. C] to positive characteristic. In particular, a stack with finite stabilizers is concentrated if and only if it is tame.

Perfect stacks. Ben–Zvi, Francis and Nadler introduced the notion of a *perfect* (derived) stack in [BZFN10]. In our context, an algebraic stack X is perfect if and only if it has affine diagonal, it is concentrated and its derived category $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated [BZFN10, Prop. 3.9]. A direct consequence of our main theorems and [HR14, Thm. C] is that the following classes of algebraic stacks are perfect:

(1) quasi-compact tame Deligne–Mumford stacks with affine diagonal; and
(2) $\mathbb{Q}$-stacks of s-global type with affine diagonal.

The affine diagonal assumption is needed only because it is required in the definition of a perfect stack. It is useful though: if X is perfect, then $\mathsf{D}(\mathsf{QCoh}(X)) = \mathsf{D}_{\mathrm{qc}}(X)$ by [HNR14].

In the terminology of Lurie [Lur11a, Def. 8.14], an algebraic stack is perfect if it has quasi-affine diagonal, is concentrated and $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated. Thus in Lurie's terminology, we have shown that

(1) quasi-compact tame Deligne–Mumford stacks with quasi-compact and separated diagonals; and
(2) $\mathbb{Q}$-stacks of s-global type

are perfect.

Applications. Compact generation is extremely useful and we will illustrate this with a simple application—also the origin of this paper. Let A be a ring. Then an A-linear functor F is *coherent* if there exists a homomorphism of A-modules $M_1 \to M_2$ together with a natural isomorphism for every A-module N:

$$F(N) \cong \operatorname{coker}(\operatorname{Hom}_A(M_2, N) \to \operatorname{Hom}_A(M_1, N)).$$

This definition is due to Auslander [Aus66] who initiated the study of coherent functors. Hartshorne studied in detail [Har98] coherent functors when A is noetherian and M_1 and M_2 are coherent and obtained some very nice applications to classical algebraic geometry. For background material on coherent functors, we refer the reader to Hartshorne's article. Recently, the first author has used coherent functors to prove Cohomology and Base Change for algebraic stacks [Hal14a] and to give a new criterion for algebraicity of a stack [Hal14b].

Using the compact generation results of this article, we can give a straightforward proof of the following Theorem (combine Theorem A with Corollary 4.19).

Theorem D. *Let A be a noetherian ring and let $\pi\colon X \to \operatorname{Spec} A$ be a proper morphism of algebraic stacks with finite diagonal. If $\mathcal{F} \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{G} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$, then the functor*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}}(-))\colon \mathsf{Mod}(A) \to \mathsf{Mod}(A)$$

is coherent.

Theorem D generalizes a result of the first author for algebraic spaces [Hal14a, Thm. E], which was proved using a completely different argument. The first author has also proved a non-noetherian and infinite stabilizer variant of Theorem D at the expense of assuming that $\mathcal{G}$ has flat cohomology sheaves over S [Hal14a, Thm. C].

Azumaya algebras and the cohomological Brauer group. Our work is strongly influenced by Toën's excellent paper [Toë12] on derived Azumaya algebras and generators of twisted derived categories. In fact, after reading this paper a few years ago, the authors realized that it should extend from schemes to algebraic spaces and stacks using the étale dévissage of Raynaud–Gruson, cf. [Ryd11].

In [Toë12], Toën shows that compact generation of certain linear categories on derived schemes is an fppf-local question. The salient example is the *derived category of twisted sheaves* $\mathsf{D}(\mathsf{QCoh}^{\alpha}(X))$, where the twisting is given by a Brauer class α of $H^2(X, \mathbb{G}_m)$. A compact generator of $\mathsf{D}(\mathsf{QCoh}^{\alpha}(X))$ gives rise to a *derived Azumaya algebra*—the endomorphism algebra of the generator [Toë12, Prop. 4.6]. More classically, a twisted vector bundle that is generating gives rise to an ordinary Azumaya algebra $\mathcal{A}$.

The Brauer group $\mathrm{Br}(X)$ classifies Azumaya algebras $\mathcal{A}$ up to Morita equivalence, that is, up to equivalence of the category of modules $\mathsf{Mod}(\mathcal{A})$. Moreover, $\mathsf{Mod}(\mathcal{A}) \simeq \mathsf{QCoh}^{\alpha}(X)$ for a unique element α in the cohomological Brauer group $\mathrm{Br}'(X) := H^2(X, \mathbb{G}_m)_{\mathrm{tors}}$. Existence of twisted vector bundles thus answers the question whether $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ is surjective.

Constructing twisted vector bundles is difficult: the question is not local as vector bundles do not extend well over open immersions. The state of the art is due to Gabber: if X is quasi-projective, then a twisted vector bundle exists and hence $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ is surjective, see the proof by de Jong [Jon03]. Compact objects of the derived category, on the other hand, can be glued using Thomason's localization theorem (see below).

Our methods also apply to compact generation of twisted derived categories. In particular, we prove that on a quasi-compact algebraic stack with quasi-finite and separated diagonal every twisted derived category has a compact generator (Example 9.1). We thus establish a derived analogue of $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for such stacks.

Although we work with non-derived schemes and stacks, our methods are strong enough to deduce similar results for derived (and spectral) Deligne–Mumford stacks (Example 9.2). Indeed, if X is a derived Deligne–Mumford stack, then the small étale site of X is equivalent to the small étale site of the non-derived 0-truncation $\pi_0 X$. Thus, (local) compact generation of a presheaf of triangulated categories on X can be studied on $\pi_0 X$. If the triangulated categories are tensored over $\mathsf{QCoh}(X)$, then the existence of perfect complexes with prescribed supports follows using Koszul complexes—see [Toë12, Lem. 4.5(2)] for the derived modifications of the classical arguments.

Sometimes results for stacks can be deduced from schemes using a similar approach: if $\pi\colon X \to X_{\mathrm{cms}}$ is a coarse moduli space, then a presheaf $\mathcal{T}$ of triangulated categories on X induces a presheaf $\pi_*\mathcal{T}$ of triangulated categories on X_{cms}. If $\pi_*\mathcal{T}$ is locally compactly generated, then it is enough to show that compact generation is local on X_{cms} to deduce compact generation of $\mathcal{T}(X)$. This is how Toën extends his result to Deligne–Mumford stacks admitting a coarse moduli scheme [Toë12, Cor. 5.2].

Gluing compact objects: local-global principle. Thomason's localization theorem is readily applied to show that compact generation (with supports) is *Zariski-local*. This leads to the proofs that $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated when X is a quasi-compact and quasi-separated scheme (or derived scheme).

Another construction of compact objects is via *finite flat* maps: if $f\colon X \to Y$ is finite and flat of finite presentation, then $\mathsf{R}f_*$ takes compact objects to compact objects. This is because $\mathsf{R}f_*$ has a right adjoint $f^! = f^{\times}$ which preserves coproducts (Proposition 3.3). Combining Thomason localization and this fact, one sees that compact generation is fppf-local for schemes. Indeed, the fppf topology on a scheme is generated by Zariski coverings and finite flat coverings [Stacks, 05WM]. A similar approach is used by Toën in [Toë12, Prop. 4.13] for derived schemes.

For algebraic spaces and algebraic stacks, it is not possible to reduce from fppf coverings to Zariski coverings and finite flat coverings. At our disposal we instead have the following reductions:

- quasi-finite flat coverings are dominated by finite flat coverings and étale coverings;

- étale coverings are dominated by Nisnevich coverings and finite étale coverings; and
- Nisnevich coverings are described using étale neighborhoods (also known as elementary neighborhoods, distinguished squares and excision squares).

The first reduction is standard for schemes and extended to stacks in [Ryd11, Thm. 6.3]. The last two reductions is what is called étale dévissage in [Ryd11] and goes back to Raynaud–Gruson and Morel–Voevodsky.

In [Lur11a, Thm. 6.1] and [Lur11b, Thm. 1.5.10] Lurie proves that compact generation is étale local on $\mathbb{E}_\infty$-algebras and on spectral algebraic spaces for quasi-coherent stacks (sheaves of linear ∞-categories). Lurie uses *scallop decompositions*—a special form of étale neighborhoods where the covering space is affine. Due to this requirement, scallop decompositions do not exist for algebraic stacks. Antieau and Gepner [AG14, Thm. 6.1] expands on Lurie's proof for spectral schemes.

Combining Toën's and Lurie's results, it follows that compact generation is local for Deligne–Mumford stacks with coarse moduli spaces [Toë12, p. 587, Cor. 5.2].

In our local-global principle, Theorem C, we must use general étale neighborhoods and do not have access to affine schemes. Therefore, we need a stronger inductive assumption—β-crispness—that includes the existence of perfect complexes with prescribed supports.

Historical remarks. The first proof that $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated when X is a quasi-compact separated scheme appears to be due to Neeman [Nee96, Prop. 2.5] although he attributes the ideas to Thomason [TT90]. Bondal–van der Bergh [BB03, Thm. 3.1.1] adapted the proof to deal with quasi-separated schemes and noted that there is a single compact generator. Lipman and Neeman further refined the result by giving an effective bound on the existence of maps from the compact generator [LN07, Thm. 4.2]. As noted by Ben-Zvi, Francis and Nadler, the proof of Bondal and van der Bergh readily extends to derived schemes [BZFN10, Prop. 3.19].

To apply Thomason's localization theorem, it is necessary to establish the existence of compact objects with prescribed support. On affine schemes this is done using Koszul complexes, cf. Bökstedt and Neeman [BN93, Prop. 6.1]. This is the basis for our induction and also used in all previous proofs, e.g., Toën [Toë12, Lem. 4.10] and [AG14, Prop. 6.9].

Related results. Drinfeld and Gaitsgory [DG13, Thm. 8.1.1] prove that on an algebraic stack of finite type over a field of characteristic zero with affine stabilizers, the derived category of D-modules is compactly generated. They remark that compact generation of $\mathsf{D}_{\mathrm{qc}}(X)$ is much subtler and open in general [DG13, 0.3.3].

Antieau [Ant14] has considered local-global results for the telescope conjecture.

Krishna [Kri09] has considered the K-theory and G-theory for tame Deligne–Mumford stacks with the resolution property admitting projective coarse moduli schemes (i.e., projective stacks).

Future extensions. In [LN07, Thm. 4.1], Lipman and Neeman proves that pseudo-coherent complexes can be approximated arbitrarily well by perfect complexes on a quasi-compact and quasi-separated scheme. Local approximability by perfect complexes is essentially the definition of pseudo-coherence so this is a local-global result in the style of Theorem C. This result has been extended to algebraic spaces in [Stacks, 08HH] and we expect that it can be extended to stacks with quasi-finite diagonal using the methods of this paper amplified with t-structures. Similarly, we expect that there is an effective bound on the compact generator in Theorem A as in [LN07, Thm. 4.2].

Contents of this paper. In §§1–2 we recall and develop some generalities on unbounded derived categories of quasi-coherent sheaves on stacks and concentrated morphisms—working in the unbounded derived category is absolutely essential for this range of mathematics. Unfortunately some important foundational results, such as concentrated morphisms, had not been considered in the literature before.

In §3, we prove finite flat duality for stacks and unbounded derived categories.

In §4, we recall the concept of compact objects and Thomason's localization theorem for triangulated categories. We also take the opportunity to establish a general projection formula for stacks and a tor-independent base change result. We also prove Theorem D assuming Theorem A.

In §§5–6 we introduce presheaves of triangulated categories and Mayer–Vietoris triangles. We also prove our main result on descent of compact generation (Theorem 6.9).

In §7 we introduce the β-resolution property, which gives a convenient method to keep track of the number of vector bundles needed for generating the derived category of a stack with the resolution property.

In §8, we introduce compact generation with supports and β-crispness and relate these to Koszul complexes.

In §9, we prove the main theorems.

In Appendix A, we apply the Mayer–Vietoris results obtained in §5 to recover a descent result of Moret–Bailly [MB96].

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Notations and assumptions. Let **TCat** denote the 2-category of triangulated categories.

For an abelian category $\mathcal{A}$, denote by $\mathsf{D}(\mathcal{A})$ its unbounded derived category. For a complex $M \in \mathsf{D}(\mathcal{A})$, denote its ith cohomology group by $\mathcal{H}^i(M)$. For a sheaf of rings A on a site E, denote by $\mathsf{Mod}(A)$ (resp. $\mathsf{QCoh}(A)$) the category of A-modules (resp. the category of quasi-coherent A-modules). If the sheaf of rings A on the site E is implicit, it will be convenient to denote $\mathsf{Mod}(A)$ as $\mathsf{Mod}(E)$ and $\mathsf{D}(\mathsf{Mod}(A))$ as $\mathsf{D}(E)$.

For algebraic stacks, we adopt the conventions of the *Stacks Project* [Stacks]. This means that algebraic stacks are stacks over the big fppf site of some scheme, admitting a smooth, representable, surjective morphism from a scheme (note that there are no separation hypotheses here). A morphism of algebraic stacks is *quasi-separated* if its diagonal and double diagonal are represented by quasi-compact morphisms of algebraic spaces.

For a scheme X denote its underlying topological space by $|X|$. For a scheme X and a point $x \in |X|$, denote by $\kappa(x)$ the residue field at x.

Let $f\colon X \to Y$ be a 1-morphism of algebraic stacks. Then for any other 1-morphism of algebraic stacks $g\colon Z \to Y$, we denote by $f_Z\colon X_Z \to Z$ the pullback of f by g.

1. Quasi-coherent sheaves on algebraic stacks

In this section we review derived categories of quasi-coherent sheaves on algebraic stacks. For generalities on unbounded derived categories on ringed sites we refer the reader to [KS06, §18.6]. In [KS06, §18.6], a morphism of ringed sites is assumed to have a left exact inverse image—we will not make this assumption, but instead indicate explicitly when it does and does not hold.

Let X be an algebraic stack. Let $\mathsf{Mod}(X)$ (resp. $\mathsf{QCoh}(X)$) denote the abelian category of $\mathcal{O}_X$-modules (resp. quasi-coherent $\mathcal{O}_X$-modules) on the lisse-étale site of X [LMB, 12.1]. Let $\mathsf{D}(X)$ (resp. $\mathsf{D}_{\mathrm{qc}}(X)$) denote the unbounded derived category of $\mathsf{Mod}(X)$ (resp. the full subcategory of $\mathsf{D}(X)$ with cohomology in $\mathsf{QCoh}(X)$). Superscripts such as $+$, $-$, $\geq n$, and b decorating $\mathsf{D}(X)$ and $\mathsf{D}_{\mathrm{qc}}(X)$ are to be interpreted as usual.

If X is a Deligne–Mumford stack (e.g., a scheme or an algebraic space), then there is an associated small étale site which we denote as $X_{\text{ét}}$. There is a natural morphism of ringed sites $\mathrm{res}_X\colon X_{\text{lis-ét}} \to X_{\text{ét}}$. Let $\mathsf{Mod}(X_{\text{ét}})$ (resp. $\mathsf{QCoh}(X_{\text{ét}})$) denote the abelian category of $\mathcal{O}_{X_{\text{ét}}}$-modules (resp. quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-modules). The restriction of $(\mathrm{res}_X)_*\colon \mathsf{Mod}(X) \to \mathsf{Mod}(X_{\text{ét}})$ to $\mathsf{QCoh}(X)$ is fully faithful with essential image $\mathsf{QCoh}(X_{\text{ét}})$. Let $\mathsf{D}_{\mathrm{qc}}(X_{\text{ét}})$ denote the triangulated category $\mathsf{D}_{\mathsf{QCoh}(X_{\text{ét}})}(\mathsf{Mod}(X_{\text{ét}}))$. Then the natural functor $\mathsf{R}(\mathrm{res}_X)_*\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X_{\text{ét}})$ is an equivalence of categories. If X is a scheme, then the corresponding statement for the Zariski site is also true.

We now recall the relationship between the unbounded derived categories of quasi-coherent sheaves on an algebraic stack and those on a smooth hypercovering, following [LO08, Ex. 2.2.5] (i.e., cohomological descent). Let X be an algebraic stack and let $p_\bullet\colon U_\bullet \to X$ be a smooth hypercovering by algebraic spaces. Let $U^+_{\bullet,\text{lis-ét}}$ and $U^+_{\bullet,\text{ét}}$ denote the two strictly simplicial sites naturally induced by the simplicial algebraic space $U_\bullet$. There are natural morphisms of ringed sites:

$$X_{\text{lis-ét}} \xleftarrow{p^+_{\bullet,\text{lis-ét}}} U^+_{\bullet,\text{lis-ét}} \xrightarrow{\mathrm{res}_{U_\bullet}} U^+_{\bullet,\text{ét}}, \tag{1.1}$$

which induce equivalences of triangulated categories:

$$\mathsf{D}_{\mathrm{qc}}(X) \xleftarrow{\mathsf{R}(p^+_{\bullet,\text{lis-ét}})_*} \mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{lis-ét}}) \xrightarrow{\mathsf{R}(\mathrm{res}_{U_\bullet})_*} \mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}}). \tag{1.2}$$

The morphisms $p^+_{\bullet,\text{lis-ét}}$ and $\mathrm{res}_{U_\bullet}$ have left exact inverse image functors.

We now record for future reference some useful formulae. If $\mathcal{M}$ and $\mathcal{N} \in \mathsf{D}(X)$, then there is

$$\begin{aligned} \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N} &\in \mathsf{D}(X) \quad \text{(the derived tensor product)}\\ \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) &\in \mathsf{D}(X) \quad \text{(the derived sheaf Hom functor)}\\ \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) &\in \mathsf{D}(\mathsf{Ab}) \quad \text{(the derived global Hom functor)} \end{aligned}$$

If in addition $\mathcal{P} \in \mathsf{D}(X)$, then we have a functorial isomorphism:

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}, \mathcal{P}) \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{M}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{P})), \tag{1.3}$$

as well as a functorial quasi-isomorphism:

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}, \mathcal{P}) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{P})). \tag{1.4}$$

Letting $\mathsf{R}\Gamma(X,-) = \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,-)$, there is also a natural quasi-isomorphism:

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \simeq \mathsf{R}\Gamma\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}). \tag{1.5}$$

If $\mathcal{M}$ and $\mathcal{N}$ belong to $\mathsf{D}_{\mathrm{qc}}(X)$, then $\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$. These results are all consequences of [Ols07, §6] and [LO08, §§2.1–2.2, Ex. 2.2.4].

For a morphism of algebraic stacks $f\colon X \to Y$, the induced morphism of ringed sites $f_{\text{lis-ét}}\colon X_{\text{lis-ét}} \to Y_{\text{lis-ét}}$ does not necessarily have a left exact inverse image functor [Beh03, 5.3.12]. Thus the construction of the correct derived functors of $f^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is somewhat subtle. There are currently two approaches to constructing these functors. The first, due to M. Olsson [Ols07] and Y. Laszlo and M. Olsson [LO08], uses cohomological descent. The other approach appears in the Stacks Project [Stacks]. In this article, we will employ the approach of Olsson and Laszlo–Olsson, which we now briefly recall.

Let $f\colon X \to Y$ be a morphism of algebraic stacks. Let $q\colon V \to Y$ be a smooth surjection from an algebraic space. Let $U \to X \times_Y V$ be another smooth surjection from an algebraic space. Let $\tilde{f}\colon U \to V$ be the resulting morphism of algebraic spaces and let $p\colon U \to X$ be the resulting smooth covering. By (1.1), there is an induced 2-commutative diagram of ringed sites:

(1.6)
$$\begin{array}{ccccc} X & \xleftarrow{p^+_{\bullet,\text{lis-ét}}} & U^+_{\bullet,\text{lis-ét}} & \xrightarrow{\mathrm{res}_{U_\bullet}} & U^+_{\bullet,\text{ét}} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{f}^+_{\bullet,\text{lis-ét}}} & & \downarrow{\scriptstyle \tilde{f}^+_{\bullet,\text{ét}}} \\ Y & \xleftarrow[q^+_{\bullet,\text{lis-ét}}]{} & V^+_{\bullet,\text{lis-ét}} & \xrightarrow[\mathrm{res}_{V_\bullet}]{} & V^+_{\bullet,\text{ét}}. \end{array}$$

The 2-commutativity of the diagram above induces natural transformations:

$$\mathsf{R}(f_{\text{lis-ét}})_* \Rightarrow \mathsf{R}(q^+_{\bullet,\text{lis-ét}})_*\mathsf{R}(\tilde{f}^+_{\bullet,\text{lis-ét}})_*\mathsf{L}(p^+_{\bullet,\text{lis-ét}})^* \quad \text{and} \tag{1.7}$$

$$\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_* \Rightarrow \mathsf{R}(\mathrm{res}_{V_\bullet})_*\mathsf{R}(\tilde{f}^+_{\bullet,\text{lis-ét}})_*\mathsf{L}(\mathrm{res}_{U_\bullet})^*, \tag{1.8}$$

which are natural isomorphisms for those complexes with quasi-coherent cohomology that are sent to complexes with quasi-coherent cohomology by $\mathsf{R}(\tilde{f}^+_{\bullet,\text{lis-ét}})_*$ or $\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*$.

Remark 1.1. Note, however, that if $f\colon X \to Y$ is not representable, then $\mathsf{R}(f_{\text{lis-ét}})_*$ does not, in general, send $\mathsf{D}_{\mathrm{qc}}(X)$ to $\mathsf{D}_{\mathrm{qc}}(Y)$—even if f is proper and étale and X and Y are smooth Deligne–Mumford stacks [Stacks, 07DC]. The problem is that quasi-compact and quasi-separated morphisms of algebraic stacks can have unbounded cohomological dimension, which is in contrast to the case of schemes and algebraic spaces [Stacks, 073G]. In the next section we will clarify this with the concept of a *concentrated* morphism.

Another crucial observation here is that the morphism of sites $\tilde{f}^+_{\bullet,\text{ét}}$ has a left exact inverse image functor. The general theory now gives rise to an unbounded derived functor $\mathsf{L}(\tilde{f}^+_{\bullet,\text{ét}})^*\colon \mathsf{D}(V^+_{\bullet,\text{ét}}) \to \mathsf{D}(U^+_{\bullet,\text{ét}})$, which is right adjoint to $\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*\colon \mathsf{D}(U^+_{\bullet,\text{ét}}) \to \mathsf{D}(V^+_{\bullet,\text{ét}})$. The functor $\mathsf{L}(\tilde{f}^+_{\bullet,\text{ét}})^*$ is easily verified to preserve small coproducts and complexes with quasi-coherent cohomology. Using the equivalences of (1.2), we may now define a functor $\mathsf{L}f^*_{\mathrm{qc}}\colon \mathsf{D}_{\mathrm{qc}}(Y) \to \mathsf{D}_{\mathrm{qc}}(X)$ such that $\mathcal{H}^0(\mathsf{L}f^*_{\mathrm{qc}}\mathcal{M}[0]) \cong f^*\mathcal{M}$, whenever $\mathcal{M} \in \mathsf{QCoh}(Y)$. If $f\colon X \to Y$ is flat, then for all integers q and all $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ there is a natural isomorphism:

$$f^*\mathcal{H}^q(\mathcal{M}) \cong \mathcal{H}^q(\mathsf{L}f^*_{\mathrm{qc}}\mathcal{M}) \tag{1.9}$$

Since the category $\mathsf{D}_{\mathrm{qc}}(Y)$ is well generated [HNR14, Thm. B.1] and the functor $\mathsf{L}f^*_{\mathrm{qc}}$ preserves small coproducts, it admits a right adjoint [Nee01, Thm. 8.4.4]

$$\mathsf{R}(f_{\mathrm{qc}})_*\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(Y).$$

The functor above is closely related to the functors we have already seen. Indeed, since $\mathsf{L}f^*_{\mathrm{qc}}\mathcal{O}_Y[0] \simeq \mathcal{O}_X[0]$, it follows that if $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$, then

$$\mathsf{R}\Gamma(Y, \mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M}) \simeq \mathsf{R}\Gamma(X, \mathcal{M}). \tag{1.10}$$

We now describe $\mathsf{R}(f_{\mathrm{qc}})_*$ locally. Let

$$\mathcal{Q}_{V^+_{\bullet,\text{ét}}}\colon \mathsf{D}(V^+_{\bullet,\text{ét}}) \to \mathsf{D}_{\mathrm{qc}}(V^+_{\bullet,\text{ét}})$$

be a right adjoint to the natural inclusion functor $\mathsf{D}_{\mathrm{qc}}(V^+_{\bullet,\text{ét}}) \to \mathsf{D}(V^+_{\bullet,\text{ét}})$, which exists by [Nee01, Thm. 8.4.4]. A straightforward calculation, utilizing the equivalences (1.2), induces a natural isomorphism of functors:

$$\mathsf{R}(f_{\mathrm{qc}})_* \simeq \mathsf{R}(q^+_{\bullet,\text{lis-ét}})_*\mathsf{L}(\mathrm{res}_{V_\bullet})^*\mathcal{Q}_{V^+_{\bullet,\text{ét}}}\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*\mathsf{R}(\mathrm{res}_{U_\bullet})_*\mathsf{L}(p^+_{\bullet,\text{lis-ét}})^*. \tag{1.11}$$

It is our hope that the following Lemma clarifies the situation somewhat.

Lemma 1.2. *If $f\colon X \to Y$ be a morphism of algebraic stacks that is quasi-compact and quasi-separated, then the following hold.*

1. *The restriction of $\mathsf{R}(f_{\text{lis-ét}})_*$ to $\mathsf{D}^+_{\mathrm{qc}}(X)$ factors through $\mathsf{D}^+_{\mathrm{qc}}(Y)$.*
2. *The restriction of $\mathsf{R}(f_{\text{lis-ét}})_*$ and $\mathsf{R}(f_{\mathrm{qc}})_*$ to $\mathsf{D}^+_{\mathrm{qc}}(X)$ are isomorphic.*
3. *For each integer d, the restriction of the functor $\mathsf{R}(f_{\mathrm{qc}})_*$ to $\mathsf{D}^{[d,\infty)}_{\mathrm{qc}}(X)$ preserves direct limits (in particular, small coproducts).*
4. *Consider a 2-cartesian diagram of algebraic stacks:*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

If g is flat, then the base change transformation

$$\mathsf{L}g^*_{\mathrm{qc}}\mathsf{R}(f_{\mathrm{qc}})_* \Rightarrow \mathsf{R}(f'_{\mathrm{qc}})_*\mathsf{L}(g')^*_{\mathrm{qc}}$$

is an isomorphism upon restriction to $\mathsf{D}^+_{\mathrm{qc}}(X)$.

Proof. Claim (1) is [Ols07, Lem. 6.20]. Claim (2) follows from cohomological descent (1.2), claim (1) and equations (1.7), (1.8), and (1.11).

For (3), by (1), we may replace $\mathsf{R}(f_{\mathrm{qc}})_*$ by $\mathsf{R}(f_{\text{lis-ét}})_*$. The hypercohomology spectral sequence:

$$\mathsf{R}^r(f_{\text{lis-ét}})_*\mathcal{H}^s(\mathcal{M}) \Rightarrow \mathsf{R}^{r+s}(f_{\text{lis-ét}})_*\mathcal{M} \tag{1.12}$$

now applies and it is thus sufficient to prove that the higher pushforwards

$$\mathsf{R}^r(f_{\text{lis-ét}})_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$$

preserve direct limits for every integer $r \geq 0$. This is local on Y for the smooth topology, so we may assume that Y is an affine scheme. Thus it suffices to prove that the cohomology functors $H^r(X_{\text{lis-ét}}, -)\colon \mathsf{QCoh}(X) \to \mathsf{Ab}$ preserve direct limits for every integer $r \geq 0$. Since X is quasi-compact and quasi-separated, this is well-known (e.g., [Stacks, 0739]).

The base change transformation of (4) exists by functoriality of the adjoints. Applying (2) we may replace $(f_{\mathrm{qc}})_*$ and $(f'_{\mathrm{qc}})_*$ by $(f_{\text{lis-ét}})_*$ and $(f'_{\text{lis-ét}})_*$ respectively. The statement is now local on Y and Y' for the smooth topology, so we may assume that both Y and Y' are affine schemes. Small modifications to the argument of [Stacks, 073K] complete the proof. □

In §2 we describe a class of morphisms for which the conclusions of Lemma 1.2 remain valid in the unbounded derived category.

2. Concentrated morphisms of algebraic stacks

A morphism of schemes $f\colon X \to Y$ is concentrated if it is quasi-compact and quasi-separated [Lip09, §3.9]. Concentrated morphisms of schemes are natural to consider when working with unbounded derived categories of quasi-coherent sheaves. Indeed, if f is concentrated, then

(1) $\mathsf{R}(f_{\mathrm{qc}})_*$ coincides with the restriction of $\mathsf{R}(f_{\mathrm{Zar}})_*$ to $\mathsf{D}_{\mathrm{qc}}(X)$,
(2) $\mathsf{R}(f_{\mathrm{qc}})_*$ preserves small coproducts, and
(3) $\mathsf{R}(f_{\mathrm{qc}})_*$ is compatible with flat base change on Y.

Here, as before, $\mathsf{R}(f_{\mathrm{qc}})_*$ denotes the right adjoint to the unbounded derived functor $\mathsf{L}f^*_{\mathrm{qc}}\colon \mathsf{D}_{\mathrm{qc}}(Y) \to \mathsf{D}_{\mathrm{qc}}(X)$. In this section we isolate a class of morphisms of algebraic stacks, which we will also call concentrated, that enjoy the same properties.

Definition 2.1. Let $n \geq 0$ be an integer. A quasi-compact and quasi-separated morphism of algebraic stacks $f\colon X \to Y$ has *cohomological dimension* $\leq n$ if for all $i > n$ and all $\mathcal{M} \in \mathsf{QCoh}(X)$ we have that $\mathsf{R}^i(f_{\text{lis-ét}})_*\mathcal{M} = 0$ (by Lemma 1.2(1), this is equivalent to $\mathsf{R}^i(f_{\mathrm{qc}})_*\mathcal{M} = 0$).

The next result is inspired by [Alp08, Prop. 3.9], where similar results are proven in the context of cohomologically affine morphisms. Note, however, that cohomologically affine morphisms are not the same as morphisms of cohomological dimension ≤ 0 [Alp08, Rem. 3.5].

Lemma 2.2. *Let $f\colon X \to Y$ be a 1-morphism of algebraic stacks that is quasi-compact and quasi-separated. Let $n \geq 0$ be an integer.*

(1) *Let $\alpha\colon f \Rightarrow f'$ be a 2-morphism. If f has cohomological dimension $\leq n$, then so does f'.*
(2) *Let $g\colon Z \to Y$ be a 1-morphism of algebraic stacks that is faithfully flat. If $f_Z\colon X \times_Y Z \to Z$ has cohomological dimension $\leq n$, then so does f.*
(3) *If f is affine, then it has cohomological dimension ≤ 0.*
(4) *Let $h\colon W \to X$ be a 1-morphism of algebraic stacks that is quasi-compact and quasi-separated and let $m \geq 0$ be an integer. If f (resp. h) has cohomological dimension $\leq n$ (resp. $\leq m$) , then the composition $f \circ h\colon W \to Y$ has cohomological dimension $\leq m + n$.*
(5) *Let $g\colon Z \to Y$ be a 1-morphism of algebraic stacks that is* quasi-affine. *If f has cohomological dimension $\leq n$, then so has the 1-morphism $f_Z\colon X \times_Y Z \to Z$.*
(6) *Let $g\colon Z \to Y$ be a 1-morphism of algebraic stacks. If f has cohomological dimension $\leq n$ and Y has quasi-affine diagonal, then the 1-morphism $f_Z\colon X \times_Y Z \to Z$ has cohomological dimension $\leq n$.*

Proof. The claim (1) is trivial. To address the claim (2) we note that higher pushforwards commute with flat base change (Lemma 1.2(4)). As faithfully flat morphisms are conservative, the morphism f has cohomological dimension $\leq n$. The claim (3) follows trivially from (2). The claim (4) follows from the Leray spectral sequence.

We now address the claim (5). Denote the pullback of g by f as $g_X : Z_X \to X$ and throughout we fix $M \in \mathsf{QCoh}(Z_X)$. We first assume that the morphism g is a quasi-compact open immersion. In this situation the adjunction $(g_X)^*_{\mathrm{qc}}((g_X)_{\text{lis-ét}})_*M \to M$ is an isomorphism. For $i \geq 0$ we deduce that there are isomorphisms in $\mathsf{QCoh}(Z)$:

$$\mathsf{R}^i((f_Z)_{\text{lis-ét}})_*\big((g_X)^*_{\mathrm{qc}}((g_X)_{\text{lis-ét}})_*M\big) \to \mathsf{R}^i((f_Z)_{\text{lis-ét}})_*M.$$

Since higher pushforward commute with flat base change, we deduce that for all $i \geq 0$ there are isomorphisms:

$$g^*\mathsf{R}^i(f_{\text{lis-ét}})_*\big(((g_X)_{\text{lis-ét}})_*M\big) \to \mathsf{R}^i((f_Z)_{\text{lis-ét}})_*M.$$

Since $((g_X)_{\text{lis-ét}})_*M \in \mathsf{QCoh}(X)$, it follows that $f_Z \colon X_Z \to Z$ has cohomological dimension $\leq n$. Next assume that the morphism g is affine. Then the morphism g_X is also affine and so, by (3), both morphisms are of cohomological dimension ≤ 0. By (4) we conclude that the composition $f \circ g_X : Z_X \to Y$ has cohomological dimension $\leq n$. But we have a 2-isomorphism $f \circ g_X \Rightarrow g \circ f_Z$ and so by (1) the morphism $g \circ f_Z$ is of cohomological dimension $\leq n$. By the Leray spectral sequence, however, we see that there is an isomorphism for all $i \geq 0$:

$$(g_{\text{lis-ét}})_*\mathsf{R}^i((f_Z)_{\text{lis-ét}})_*M \to \mathsf{R}^i((g \circ f_Z)_{\text{lis-ét}})_*M.$$

Since the morphism g is affine, the functor g_* is faithful; thus we conclude that the morphism f_Z has cohomological dimension $\leq n$. In general, quasi-affine morphisms $g \colon Z \to Y$ factor as $Z \xrightarrow{j} \overline{Z} \xrightarrow{\overline{g}} Y$, where the morphism j is a quasi-compact open immersion and the morphism $\overline{g}$ is affine. Combining the above completes the proof of (5).

To prove the claim (6) we observe that by (2) the statement is smooth local on Z—thus we are free to assume that Z is an affine scheme. Since the diagonal of the stack Y is quasi-affine, the morphism $g \colon Z \to Y$ is quasi-affine. An application of (5) now gives the claim. □

We wish to point out that Lemma 2.2(6) is false if Y does not have affine stabilizers [HR14, Rem. 2.6].

Definition 2.3. A quasi-compact and quasi-separated morphism $f \colon X \to Y$ of algebraic stacks is said to be of *finite cohomological dimension* if there exists an integer $n \geq 0$ such that the morphism f is of cohomological dimension $\leq n$.

Morphisms of quasi-compact and quasi-separated algebraic spaces have finite cohomological dimension [Stacks, 073G]. V. Drinfeld and D. Gaitsgory [DG13, §2] have shown that a morphism of noetherian algebraic stacks $f \colon X \to Y$ has finite cohomological dimension if Y is a $\mathbb{Q}$-stack and f has affine stabilizers. This result will be refined and generalized in [HR14].

Definition 2.4. A morphism of algebraic stacks $f \colon X \to Y$ is *concentrated* if it is quasi-compact, quasi-separated, and for any quasi-compact and quasi-separated algebraic stack Z and any morphism $g \colon Z \to Y$, the pulled back morphism $f_Z : X_Z \to Z$ is of finite cohomological dimension.

The next result is immediate from Lemma 2.2.

Lemma 2.5. *Let $f\colon X \to Y$ be a 1-morphism of algebraic stacks that is quasi-compact and quasi-separated.*

(1) *If f is concentrated, then it remains so after base change.*
(2) *Let $g\colon Z \to Y$ be a 1-morphism that is faithfully flat. If $f_Z\colon X \times_Y Z \to Z$ is concentrated, then so is f.*
(3) *If f is representable, then it is concentrated.*
(4) *Let $h\colon Y \to W$ be a 1-morphism that is concentrated. Then the composition $h \circ f\colon X \to W$ is concentrated if and only if f is concentrated.*
(5) *Assume that Y is quasi-compact with quasi-affine diagonal. Then f is concentrated if and only if it is of finite cohomological dimension.*

Thus, by [DG13, §2] and Lemma 2.5(2), for a quasi-compact morphism of locally noetherian algebraic $\mathbb{Q}$-stacks $f\colon X \to Y$, if f has affine stabilizers, then f is concentrated.

The main result of this section is the following Theorem that refines Lemma 1.2.

Theorem 2.6. *Let $f\colon X \to Y$ be a concentrated 1-morphism of algebraic stacks*

(1) *If Y is quasi-compact and quasi-separated, then there is an integer n such that the natural morphism:*

$$\tau^{\geq j}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \to \tau^{\geq j}\mathsf{R}(f_{\mathrm{qc}})_*(\tau^{\geq j-n}\mathcal{M})$$

for every integer j and $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$.
(2) *The restriction of $\mathsf{R}(f_{\text{lis-ét}})_*$ to $\mathsf{D}_{\mathrm{qc}}(X)$ coincides with $\mathsf{R}(f_{\mathrm{qc}})_*$.*
(3) *The functor $\mathsf{R}(f_{\mathrm{qc}})_*$ preserves small coproducts.*
(4) *If $g\colon Y' \to Y$ is a flat morphism of algebraic stacks, then the 2-cartesian square:*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

induces a natural quasi-isomorphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$:

$$\mathsf{L}g^*_{\mathrm{qc}}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \simeq \mathsf{R}(f'_{\mathrm{qc}})_*\mathsf{L}g'^*_{\mathrm{qc}}\mathcal{M}.$$

Proof. For (2), choose a diagram as in (1.6). By the natural transformations (1.7), (1.8), and (1.11), it is sufficient to prove that the restriction of $\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*$ to $\mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}})$ factors through $\mathsf{D}_{\mathrm{qc}}(V^+_{\bullet,\text{ét}})$. This can be verified smooth locally on Y, so we may assume that Y is quasi-compact and quasi-separated and f has cohomological dimension $\leq n$ for some integer n. In particular, $\mathsf{R}^i(\tilde{f}^+_{\bullet,\text{ét}})_*\mathcal{M} = 0$ for every $i > n$ and $\mathcal{M} \in \mathsf{QCoh}(U^+_{\bullet,\text{ét}})$. By [LO08, Lem. 2.1.10], for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}})$ and integer j the natural morphism:

$$\tau^{\geq j}\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*\mathcal{M} \to \tau^{\geq j}\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*(\tau^{\geq j-n}\mathcal{M}) \qquad (2.1)$$

is an isomorphism. By Lemma 1.2(2) the result follows. Note that the equation (2.1) now proves (1). Finally, the claims (3) and (4) follow from (1) and the corresponding results for the bounded below category in Lemma 1.2. □

3. Perfect complexes and finite duality

Let A be a ring. A complex $P \in \mathsf{D}(\mathsf{Mod}(A))$ is *perfect* if it is quasi-isomorphic to a bounded complex of projective A-modules of finite presentation.

Let X be an algebraic stack. A complex $P \in \mathsf{D}_{\mathrm{qc}}(X)$ is *perfect* if for any smooth morphism $\operatorname{Spec} A \to X$, where A is a ring, the complex of A-modules $\mathsf{R}\Gamma(\operatorname{Spec} A, P|_{\operatorname{Spec} A})$ is perfect.

Example 3.1. Let X be an algebraic stack. Then any $\mathcal{O}_X$-module of finite presentation that is flat defines a perfect complex in $\mathsf{D}_{\mathrm{qc}}(X)$. In particular, if $f\colon X \to Y$ is a morphism of algebraic stacks that is finite, flat, and of finite presentation, then $f_*\mathcal{O}_X$ is perfect in $\mathsf{D}_{\mathrm{qc}}(Y)$.

The following Lemma is straightforward but crucial.

Lemma 3.2. *Let X be an algebraic stack and let $P \in \mathsf{D}_{\mathrm{qc}}(X)$ be a perfect complex. Then the restriction of the functor $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(P,-) : \mathsf{D}(X) \to \mathsf{D}(X)$ to $\mathsf{D}_{\mathrm{qc}}(X)$ factors through $\mathsf{D}_{\mathrm{qc}}(X)$ and preserves small coproducts in $\mathsf{D}_{\mathrm{qc}}(X)$.*

Proof. Both statements are local on X for the smooth topology. Thus we may assume that $X = \operatorname{Spec} A$, where A is a ring. The collection of all P that satisfy the conclusions of the Lemma are closed under finite coproducts, direct summands, shifts, and the taking of cones. Since $\mathcal{O}_X$ satisfies the conclusions of the Lemma and X is affine, the result follows. □

The main result of this section is the following.

Proposition 3.3. *Let $f\colon X \to Y$ be a finite morphism of algebraic stacks. Suppose that $\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{O}_X \in \mathsf{D}_{\mathrm{qc}}(Y)$ is perfect. Then the functor $\mathsf{R}(f_{\mathrm{qc}})_*\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(Y)$ admits a right adjoint $f^\times\colon \mathsf{D}_{\mathrm{qc}}(Y) \to \mathsf{D}_{\mathrm{qc}}(X)$ that preserves small direct sums. If, in addition, f is faithfully flat, then $f^\times$ is conservative.*

Proof. Pick a diagram as in (1.6). Since f is an affine morphism, we may assume that $U_i = X \times_Y V_i$. By Theorem 2.6, it is sufficient to prove that the restriction of $\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*$ to $\mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}})$ admits a right adjoint with the desired properties. The morphism of ringed sites $\tilde{f}^+_{\bullet,\text{ét}}\colon U^+_{\bullet,\text{ét}} \to V^+_{\bullet,\text{ét}}$ factors as:

$$(U^+_{\bullet,\text{ét}}, \mathcal{O}_{U^+_{\bullet,\text{ét}}}) \xrightarrow{g} (V^+_{\bullet,\text{ét}}, (\tilde{f}^+_{\bullet,\text{ét}})_*\mathcal{O}_{U^+_{\bullet,\text{ét}}}) \xrightarrow{k} (V^+_{\bullet,\text{ét}}, \mathcal{O}_{V^+_{\bullet,\text{ét}}}).$$

Since f is affine, the functor $\mathsf{R}g_*$ induces an equivalence of categories

$$\mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}}) \to \mathsf{D}_{\mathrm{qc}}(V^+_{\bullet,\text{ét}}, (\tilde{f}^+_{\bullet,\text{ét}})_*\mathcal{O}_{U^+_{\bullet,\text{ét}}})$$

with quasi-inverse $\mathsf{L}g^*$. Next observe that Lemma 3.2 combined with the definitions implies that the functor

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_{V^+_{\bullet,\text{ét}}}}(\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*\mathcal{O}_{U^+_{\bullet,\text{ét}}}, -)\colon \mathsf{D}_{\mathrm{qc}}(V^+_{\bullet,\text{ét}}) \to \mathsf{D}(V^+_{\bullet,\text{ét}})$$

factors canonically through $\mathsf{D}_{\mathrm{qc}}(V^+_{\bullet,\text{ét}}, (\tilde{f}^+_{\bullet,\text{ét}})_*\mathcal{O}_{U^+_{\bullet,\text{ét}}})$. Thus we define:

$$f^\times(-) = \mathsf{L}g^*\mathsf{R}\mathcal{H}om_{\mathcal{O}_{V^+_{\bullet,\text{ét}}}}(\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*\mathcal{O}_{U^+_{\bullet,\text{ét}}}, -)\colon \mathsf{D}_{\mathrm{qc}}(V^+_{\bullet,\text{ét}}) \to \mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}}).$$

It remains to verify the adjointness. Let $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(V^+_{\bullet,\mathrm{\acute{e}t}})$ and $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\mathrm{\acute{e}t}})$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_{U^+_{\bullet,\mathrm{\acute{e}t}}}}(\mathcal{N}, f^{\times}\mathcal{M}) &= \mathrm{Hom}_{\mathcal{O}_{U^+_{\bullet,\mathrm{\acute{e}t}}}}\big(\mathcal{N}, \mathsf{L}g^*\mathsf{R}\mathcal{H}om_{\mathcal{O}_{V^+_{\bullet,\mathrm{\acute{e}t}}}}(\mathsf{R}(\tilde{f}^+_{\bullet,\mathrm{\acute{e}t}})_*\mathcal{O}_{X^+_\bullet}, \mathcal{M})\big)\\ &\cong \mathrm{Hom}_{(\tilde{f}^+_{\bullet,\mathrm{\acute{e}t}})_*\mathcal{O}_{U^+_{\bullet,\mathrm{\acute{e}t}}}}\big(\mathsf{R}(\tilde{f}^+_{\bullet,\mathrm{\acute{e}t}})_*\mathcal{N}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_{V^+_{\bullet,\mathrm{\acute{e}t}}}}(\mathsf{R}(\tilde{f}^+_{\bullet,\mathrm{\acute{e}t}})_*\mathcal{O}_{X^+_\bullet}, \mathcal{M})\big)\\ &\cong \mathrm{Hom}_{\mathcal{O}_{V^+_{\bullet,\mathrm{\acute{e}t}}}}(\mathsf{R}(\tilde{f}^+_{\bullet,\mathrm{\acute{e}t}})_*\mathcal{N}, \mathcal{M}).\end{aligned}$$

That $f^{\times}$ has the asserted properties follows from its definition and Lemma 3.2. □

4. Triangulated categories

In this section we will recall some results on triangulated categories that may not be familiar to everyone. For excellent and comprehensive treatments of these topics see [Nee92b] and [Tho97, §2]. In particular, we will recall *thick* and *localizing* triangulated subcategories. This leads to the concept of *compact* objects and Thomason's localization theorem.

For this section, we fix a triangulated category $\mathcal{S}$ and we denote the shift operator by Σ.

A functor $f\colon \mathcal{S} \to \mathcal{S}'$ between triangulated categories is *triangulated* if f sends triangles to triangles and is compatible with shifts. We say that a full subcategory $\mathcal{R}$ of $\mathcal{S}$ is *triangulated* if the category $\mathcal{R}$ is triangulated and the inclusion functor $i\colon \mathcal{R} \to \mathcal{S}$ is triangulated. A subcategory $\mathcal{R} \subseteq \mathcal{S}$ is *thick* (also known as épaisse or saturated) if it is full, triangulated, and any $\mathcal{S}$-direct summand of any $r \in \mathcal{R}$ belongs to $\mathcal{R}$.

Example 4.1. For a triangulated functor $f\colon \mathcal{S} \to \mathcal{S}'$ we denote by $\ker f$ the full triangulated subcategory consisting of those $x \in \mathcal{S}$ such that $f(x) \simeq 0$. The subcategory $\ker f \subseteq \mathcal{S}$ is thick.

Example 4.2. Given triangulated subcategories $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{S}$ such that $\mathcal{R}_1$ is a thick subcategory of $\mathcal{S}$, then $\mathcal{R}_1$ is a thick subcategory $\mathcal{R}_2$.

Kernels of triangulated functors produce essentially all thick subcategories [Tho97, §1.3]. Indeed, for any thick subcategory $\mathcal{R} \subseteq \mathcal{S}$ there is a *quotient* functor $q\colon \mathcal{S} \to \mathcal{S}/\mathcal{R}$ such that $\mathcal{R} \cong \ker q$ and q is essentially surjective.

For a class $R \subseteq \mathcal{S}$ the *thick closure* of R is the smallest thick subcategory $\mathcal{R} \subseteq \mathcal{S}$ containing R. A subcategory $\mathcal{R} \subseteq \mathcal{S}$ is *dense* if it is full, triangulated, and its thick closure coincides with $\mathcal{S}$.

If the triangulated category $\mathcal{S}$ is essentially small, then there is a notion of $\mathsf{K}_0(\mathcal{S})$: it is the free abelian group on the *set* of isomorphism classes of objects in $\mathcal{S}$ modulo the relation that given an $\mathcal{S}$-triangle $s_1 \to s_2 \to s_3$, then $[s_2] = [s_1] + [s_3]$. It is easy to see that for s, $t \in \mathcal{S}$ then $[s \oplus t] = [s] + [t]$ and $[s] = -[\Sigma s]$. Also, for any $\sigma \in \mathsf{K}_0(\mathcal{S})$, there exists $s \in \mathcal{S}$ such that $\sigma = [s]$. Given a triangulated functor $f : \mathcal{S} \to \mathcal{S}'$ between essentially small triangulated categories, there is an induced group homomorphism $\mathsf{K}_0(f) : \mathsf{K}_0(\mathcal{S}) \to \mathsf{K}_0(\mathcal{S}')$. The following is a nice result of A. Neeman [Nee92b, Cor. 0.10] (also see [Tho97, Lem. 2.2]).

Lemma 4.3. *Let $\mathcal{S}$ be an essentially small triangulated category and let $\mathcal{R}$ be a* dense *subcategory. If $s \in \mathcal{S}$, then $s \in \mathcal{R}$ if and only if s belongs to the image of $\mathsf{K}_0(\mathcal{R})$ in $\mathsf{K}_0(\mathcal{S})$. In particular, if $s \in \mathcal{S}$, then $s \oplus \Sigma s \in \mathcal{R}$.*

A pair of triangulated functors $\mathcal{R} \xrightarrow{i} \mathcal{S} \xrightarrow{p} \mathcal{T}$ is *left-exact* (resp. *almost exact*, resp. *exact*) if the functor i makes $\mathcal{R}$ a thick subcategory of $\mathcal{S}$ and the functor $p\colon \mathcal{S} \to \mathcal{T}$ factors through the quotient $\overline{p}\colon \mathcal{S}/\mathcal{R} \to \mathcal{T}$ and this functor is fully faithful (resp. dense, resp. an equivalence). The following (well-known) Lemma will be useful.

Lemma 4.4. *Let $f\colon \mathcal{S} \to \mathcal{T}$ be a triangulated functor. Suppose that f has a triangulated right adjoint $g\colon \mathcal{T} \to \mathcal{S}$ such that the adjunction $fg \to \mathrm{Id}_{\mathcal{T}}$ is an isomorphism. Then the sequence $\ker f \to \mathcal{S} \xrightarrow{f} \mathcal{T}$ is exact.*

Proof. Set $\overline{\mathcal{S}} = \mathcal{S}/\ker f$. There are induced functors $\overline{f}\colon \overline{\mathcal{S}} \to \mathcal{T}$ and $\overline{g}\colon \mathcal{T} \to \overline{\mathcal{S}}$, which we claim are quasi-inverse. Since f and g are adjoints, there are natural transformations $\eta\colon \mathrm{Id}_{\mathcal{S}} \to gf$ and $\epsilon\colon fg \to \mathrm{Id}_{\mathcal{T}}$. It remains to show that the induced natural transformations $\overline{\eta} : \mathrm{Id}_{\overline{\mathcal{S}}} \to \overline{g}\overline{f}$ and $\overline{\epsilon} = \epsilon\colon \overline{f}\overline{g} \to \mathrm{Id}_{\mathcal{T}}$ are isomorphisms. By hypothesis, $\epsilon = \overline{\epsilon}$ is an isomorphism, so it remains to address $\overline{\eta}$. If $s \in \mathcal{S}$, then we have a distinguished triangle $s' \to s \xrightarrow{\eta(s)} gf(s)$. Applying f we obtain a triangle $f(s') \to f(s) \xrightarrow{f_*\eta(s)} fgf(s)$ in $\mathcal{T}$. In particular, since $fg \simeq \mathrm{Id}_{\mathcal{T}}$, it follows that $f(s') \simeq 0$ and so $s' \in \ker f$. Thus it follows that the map $\eta(s)\colon s \to gf(s)$ is an isomorphism in $\overline{\mathcal{S}}$, hence $\overline{\eta}$ is an isomorphism. □

A triangulated category is said to be *closed under small coproducts* if it admits small categorical coproducts and small coproducts of triangles remain triangles. If the triangulated category $\mathcal{S}$ is closed under small coproducts, then we say that a subcategory $\mathcal{R} \subseteq \mathcal{S}$ is *localizing* if it is a full triangulated subcategory, closed under small coproducts, and the functor $\mathcal{R} \to \mathcal{S}$ preserves small coproducts. For a class $R \subseteq \mathcal{S}$, where $\mathcal{S}$ is closed under small coproducts, there is a smallest subcategory $\mathcal{R} \subseteq \mathcal{S}$ that is localizing and contains R. We refer to $\mathcal{R}$ as the *localizing envelope* of R.

Example 4.5. If a subcategory $\mathcal{R} \subseteq \mathcal{S}$ is localizing, then it is thick. Indeed, given $r \in \mathcal{R}$ and $r \simeq r' \oplus r''$ in $\mathcal{S}$, then the Eilenberg swindle produces an $\mathcal{S}$-isomorphism:

$$r'' \oplus r \oplus r \oplus \cdots \simeq r \oplus r \oplus r \oplus \cdots .$$

Since $\mathcal{R}$ is localizing, then $r'' \oplus r \oplus r \oplus \cdots \in \mathcal{R}$. The cone of the natural morphism $r \oplus r \oplus \cdots \to r'' \oplus r \oplus r \oplus \cdots$ is r''; thus $r'' \in \mathcal{R}$.

A result of A. Neeman [Nee92b, Prop. 1.9] says that if a subcategory $\mathcal{R} \subseteq \mathcal{S}$ is localizing, then the quotient $q : \mathcal{S} \to \mathcal{S}/\mathcal{R}$ preserves small coproducts. In particular, since the quotient is essentially surjective, the category $\mathcal{S}/\mathcal{R}$ is closed under small coproducts.

Example 4.6. For any triangulated functor $f\colon \mathcal{S} \to \mathcal{S}'$ that preserves small coproducts, the subcategory $\ker f$ is localizing.

An object $s \in \mathcal{S}$ is *compact* if the functor $\mathrm{Hom}_{\mathcal{S}}(s, -)$ preserves small coproducts. Denote by $\mathcal{S}^c$ the full subcategory of compact objects of $\mathcal{S}$.

Example 4.7. Let A be a ring. A complex of A-modules is compact in $\mathsf{D}(A)$ if and only if it is quasi-isomorphic to a bounded complex of projective A-modules [Stacks, 07LT]. That is, the compact objects of $\mathsf{D}(A)$ are the perfect complexes of A-modules.

Example 4.8. Let $f\colon \mathcal{S} \to \mathcal{S}'$ be a triangulated functor that admits a right adjoint $g\colon \mathcal{S}' \to \mathcal{S}$. If g preserves small coproducts, then f sends $\mathcal{S}^c$ to $\mathcal{S}'^c$ [Nee96, Thm. 5.1 "⇒"].

Example 4.9. Let X be a quasi-compact and quasi-separated algebraic stack. If $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)^c$, then $\mathcal{P}$ is perfect. This follows by combining Examples 4.7 and 4.8. If X is concentrated and $\mathcal{P}$ is perfect on X, then $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)^c$. This follows from Lemma 3.2, Theorem 2.6(3), and the quasi-isomorphism (1.5).

In the following Lemma we refine Example 4.9. We have not seen this characterization in the literature before.

Lemma 4.10. *Let X be a quasi-compact and quasi-separated algebraic stack and let $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)$ be a perfect complex. Then the following conditions are equivalent.*

(1) *$\mathcal{P}$ is a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$.*
(2) *There exists an integer $r \geq 0$ such that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{N}[i]) = 0$ for all $\mathcal{N} \in \mathsf{QCoh}(X)$ and $i > r$.*
(3) *There exists an integer $r \geq 0$ such that the natural map*
$$\tau^{\geq j}\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}) \to \tau^{\geq j}\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \tau^{\geq j-r}\mathcal{M})$$
is a quasi-isomorphism for all $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ and integers j.

Proof. Assume that (2) does not hold. Then there is an infinite sequence of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{M}_1, \mathcal{M}_2, \dots$ and strictly increasing sequence of integers $d_1 < d_2 < \dots$ such that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}_i[d_i]) \neq 0$ for every i. Since $\mathsf{D}_{\mathrm{qc}}(X)$ is left-complete [HNR14, Thm. B.1], there is a quasi-isomorphism in $\mathsf{D}_{\mathrm{qc}}(X)$:
$$\bigoplus_{i=1}^{\infty} \mathcal{M}_i[d_i] \simeq \prod_{i=1}^{\infty} \mathcal{M}_i[d_i].$$
This implies that the natural morphism
$$\bigoplus_{i=1}^{\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}_i[d_i]) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \bigoplus_{i=1}^{\infty} \mathcal{M}_i[d_i]) \simeq \prod_{i=1}^{\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}_i[d_i])$$
is not an isomorphism. In particular, $\mathcal{P}$ is not compact. Thus, by the contrapositive, we have proved the implication (1) ⇒ (2).

For (2)⇒(3), first choose a diagram as in (1.6) with $V = Y = \mathrm{Spec}(\mathbb{Z})$. Let $\mathcal{P}^+_{\text{ét}} = \mathsf{R}(\mathrm{res}_{U_\bullet})_*\mathsf{L}(p^+_{\bullet,\text{lis-ét}})^*\mathcal{P}$ and $\mathcal{M}^+_{\text{ét}} = \mathsf{R}(\mathrm{res}_{U_\bullet})_*\mathsf{L}(p^+_{\bullet,\text{lis-ét}})^*\mathcal{M}$. Note that
$$\mathsf{R}(\mathrm{res}_{U_\bullet})_*\mathsf{L}(p^+_{\bullet,\text{lis-ét}})^*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_{U^+_{\text{ét}}}}(\mathcal{P}^+_{\text{ét}}, \mathcal{M}^+_{\text{ét}})$$
has quasi-coherent cohomology (Lemma 3.2) and that $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_{U^+_{\text{ét}}}}(\mathcal{P}^+_{\text{ét}}, \mathcal{N}[i]) = 0$ for all $i > r$ and $\mathcal{N} \in \mathsf{QCoh}(U^+_{\text{ét}})$. It is enough to prove that
$$\tau^{\geq j}\mathsf{R}\mathrm{Hom}_{\mathcal{O}_{U^+}}(\mathcal{P}^+_{\text{ét}}, \mathcal{M}) \to \tau^{\geq j}\mathsf{R}\mathrm{Hom}_{\mathcal{O}_{U^+}}(\mathcal{P}^+_{\text{ét}}, \tau^{\geq j-r}\mathcal{M})$$
for every integer n and $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(U^+_{\text{ét}})$. This follows as in the proof of [LO08, Lem. 2.1.10] with ϵ_* replaced by $\mathrm{Hom}_{\mathcal{O}_{U^+_{\text{ét}}}}(\mathcal{P}^+_{\text{ét}}, -)$.

Finally, for (3)⇒(1), we note that because $\mathcal{P}$ is perfect, it is dualizable. Thus for all $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ and integers j there is a natural quasi-isomorphism
$$\begin{aligned}\tau^{\geq j}\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}) &\simeq \tau^{\geq j}\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \tau^{\geq j-r}\mathcal{M}) \\ &\simeq \tau^{\geq j}\mathsf{R}\Gamma(X_{\text{lis-ét}}, \mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \tau^{\geq j-r}\mathcal{M}).\end{aligned}$$

Also since $\mathcal{P}$ is perfect, it follows that the restriction of the functor $\mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} (-)$ to $\mathsf{D}^{\geq j-r}_{\mathrm{qc}}(X)$ factors through $\mathsf{D}^{\geq j-r'}_{\mathrm{qc}}(X)$ for some fixed integer r'. The result now follows from Lemma 1.2(3). □

Example 4.11. Let X be an algebraic stack. If $\mathcal{Q} \in \mathsf{D}_{\mathrm{qc}}(X)^c$ and $\mathcal{P}$ is perfect, then $\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)^c$. Indeed, Hom-$\otimes$ adjunction (1.3) shows that there is a natural isomorphism of functors $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}, -) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, -))$. Since $\mathcal{P}$ is perfect, $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, -)$ preserves small coproducts (Lemma 3.2), and the claim follows.

Remark 4.12. Let X be a quasi-compact and quasi-separated algebraic stack. Then the following are equivalent:

(1) every perfect object of $\mathsf{D}_{\mathrm{qc}}(X)$ is compact;
(2) the structure sheaf $\mathcal{O}_X$ is compact;
(3) X has finite cohomological dimension; and
(4) the derived global section functor $\mathsf{R}\Gamma\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(\mathsf{Ab})$ commutes with small coproducts.

The equivalence of the first two conditions follows from Example 4.11. The structure sheaf is compact, if and only if X has finite cohomological dimension (Lemma 4.10). The last condition is equivalent to the definition of $\mathcal{O}_X$ being compact.

A class $S \subseteq \mathcal{S}$ is *generating* if given $x \in \mathcal{S}$ such that $\forall s \in S$ and all $n \in \mathbb{Z}$ we have $\mathrm{Hom}_{\mathcal{S}}(\Sigma^n s, x) = 0$, then $x \simeq 0$. The triangulated category $\mathcal{S}$ is *compactly generated* if it admits a *set* of generators consisting of compact objects.

Example 4.13. Let A be a ring. Denote the unbounded derived category of A-modules by $\mathsf{D}(A)$. Then the set $\{A\}$ compactly generates $\mathsf{D}(A)$. Hence $\mathsf{D}(A)$ is compactly generated.

We now recall Thomason's Localization Theorem, which was proved in this generality by A. Neeman [Nee92b, Nee96, Thm. 2.1].

Theorem 4.14 (Thomason's Localization)**.** *Consider an exact sequence* $\mathcal{R} \xrightarrow{i} \mathcal{S} \xrightarrow{q} \mathcal{T}$ *of triangulated categories that are closed under small coproducts. If the triangulated category* $\mathcal{S}$ *is compactly generated and* $\mathcal{R}$ *is the localizing envelope of a subset* $R \subseteq \mathcal{S}^c$*, then there is an induced sequence* $\mathcal{R}^c \to \mathcal{S}^c \to \mathcal{T}^c$ *which is almost exact. In particular,* $\mathcal{R}^c = \mathcal{S}^c \cap \mathcal{R}$ *and* $\mathcal{R}^c$ *is the thick closure of* R*.*

Combining Theorem 4.14 with the elementary Lemma 4.3 produces something very surprising, which was observed by A. Neeman [Nee92b, Cor. 0.9].

Corollary 4.15. *In Theorem 4.14 assume that the category* $\mathcal{T}^c$ *is essentially small. Then for every* $t \in \mathcal{T}^c$*, there exists an* $s \in \mathcal{S}^c$ *and an isomorphism* $t \oplus \Sigma t \simeq q(s)$*.*

Another useful Corollary is the following [Nee96, Thm. 2.1.2].

Corollary 4.16. *In Theorem 4.14 suppose that* R *is a generating set for* $\mathcal{S}$*, then* $\mathcal{R} = \mathcal{S}$*.*

A typical application of Corollary 4.16 is given by the following Proposition. The given argument is a variant of [Nee96, Prop. 5.3], though we have not seen this Proposition in the literature before.

Proposition 4.17 (Strong projection formula)**.** *Let A be a ring and let $\pi\colon X \to \operatorname{Spec} A$ be a morphism of algebraic stacks. Let $\mathcal{Q}$ be a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$ and let $\mathcal{G} \in \mathsf{D}_{\mathrm{qc}}(X)$. Then for every $I \in \mathsf{D}(A)$, there is a natural quasi-isomorphism:*

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q},\mathcal{G}) \otimes^{\mathsf{L}}_A I \simeq \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q},\mathcal{G}\otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I).$$

Proof. First we describe the morphism: by adjunction, there is a natural morphism

$$\mathsf{L}\pi^*_{\mathrm{qc}}\mathsf{R}(\pi_{\mathrm{qc}})_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q},\mathcal{G}) \to \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q},\mathcal{G})$$

and so by (1.3) there is a natural morphism

$$(\mathsf{L}\pi^*_{\mathrm{qc}}\mathsf{R}(\pi_{\mathrm{qc}})_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q},\mathcal{G})) \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I \to \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I.$$

By (1.3) again, there is a natural morphism:

$$\mathsf{L}\pi^*_{\mathrm{qc}}\left[\mathsf{R}(\pi_{\mathrm{qc}})_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q},\mathcal{G}) \otimes^{\mathsf{L}}_A I\right] \to \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q},\mathcal{G}\otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I).$$

By adjunction and (1.5) and (1.10) we deduce the existence of the required natural morphism

$$\phi_I\colon \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q},\mathcal{G}) \otimes^{\mathsf{L}}_A I \to \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q},\mathcal{G}\otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I).$$

Let $\mathcal{K} \subseteq \mathsf{D}(A)$ be the full subcategory with objects those I such that ϕ_I is a quasi-isomorphism. It remains to show that $\mathcal{K} = \mathsf{D}(A)$. Clearly, $\mathcal{K}$ is a triangulated subcategory that contains $A[k]$ for every integer k. Moreover, since $\mathcal{Q}$ is a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$, $\mathcal{K}$ is closed under small coproducts. The result now follows from Corollary 4.16. □

A straightforward implication is the usual projection formula for concentrated morphisms of algebraic stacks.

Corollary 4.18 (Projection formula)**.** *Let $f\colon X \to Y$ be a concentrated 1-morphism of algebraic stacks. The natural map*

$$(\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M}) \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{N} \to \mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{N})$$

is a quasi-isomorphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(Y)$.

Proof. By adjunction, there is a natural morphism

$$\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{N} \to \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{N}.$$

By adjunction again, we deduce the existence of a natural morphism

$$\psi_{\mathcal{N}}\colon (\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M}) \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{N} \to \mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{N}).$$

It remains to show that $\psi_{\mathcal{N}}$ is a quasi-isomorphism for every $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(Y)$. Note that the verification of this is smooth local on Y, so by Theorem 2.6(4), we may reduce to the situation where Y is an affine scheme. By Example 4.9, $\mathcal{O}_X$ is compact and the result now follows from Proposition 4.17. □

Combining the strong projection formula of Proposition 4.17 with the characterization of compact objects in Lemma 4.10, we can prove a lot of Theorem D.

Corollary 4.19. *Let A be a noetherian ring and let $\pi\colon X \to \operatorname{Spec} A$ be a proper morphism of algebraic stacks. If $\mathcal{F} \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{G} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$ and $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated, then the functor*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}\otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}}(-))\colon \mathsf{Mod}(A) \to \mathsf{Mod}(A)$$

is coherent.

Proof. We begin by observing that the coherent functors $\mathsf{Mod}(A) \to \mathsf{Mod}(A)$ constitute a full abelian subcategory of the category of A-linear functors that is closed under products (where everything is computed "pointwise") [Hal14a, Ex. 3.9]. Let $\mathcal{T} \subseteq \mathsf{D}_{\mathrm{qc}}(X)$ denote the full subcategory with objects those $\mathcal{F} \in \mathsf{D}_{\mathrm{qc}}(X)$ where the functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}}(-))$ is coherent for every $\mathcal{G} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. In particular, $\mathcal{T}$ is closed under small coproducts, shifts, and triangles. By Corollary 4.16, it is enough to prove that $\mathcal{T}$ contains the compact objects of $\mathsf{D}_{\mathrm{qc}}(X)$. If $\mathcal{Q} \in \mathsf{D}_{\mathrm{qc}}(X)$ is compact, then the strong projection formula (Proposition 4.17) implies that there is a natural quasi-isomorphism:

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \otimes^{\mathsf{L}}_A I \simeq \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I).$$

Since $\mathcal{Q}$ is compact, it is perfect (Example 4.9) and so $\mathcal{Q} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. The morphism π is proper, thus $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \in \mathsf{D}^+_{\mathsf{Coh}}(A)$ [Ols07, Prop. 6.4.3 & Thm. 10.13]. By Lemma 4.10 we also have $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \in \mathsf{D}^b(A)$. Thus the functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}}(-))$ is coherent [Hal14a, Ex. 3.13] and we deduce the result. ☐

Let $f\colon X \to Y$ and $g\colon Y' \to Y$ be morphism of algebraic stacks. We say that f and g are *tor-independent* if for every smooth morphism $\operatorname{Spec} A \to Y$ and any pair of smooth morphisms $\operatorname{Spec} B \to X \times_Y \operatorname{Spec} A$ and $\operatorname{Spec} A' \to Y' \times_Y \operatorname{Spec} A$ we have that $\mathrm{Tor}^A_i(B, A') = 0$ for all $i > 0$. Equivalently, $\mathcal{T}or^{Y,f,g}_i(\mathcal{O}_X, \mathcal{O}_{Y'}) = 0$ for every integer $i > 0$ (see [Hal14b, App. C] for details). Note that if g is flat, then it is tor-independent of every f. The projection formula of Corollary 4.18 is powerful enough to prove a very general tor-independent base change result which extends Theorem 2.6(4).

Corollary 4.20. *Fix a 2-cartesian square of algebraic stacks*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

If f and g are tor-independent and f is concentrated, then there is a natural quasi-isomorphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$:

$$\mathsf{L}g^*_{\mathrm{qc}}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \simeq \mathsf{R}(f'_{\mathrm{qc}})_*\mathsf{L}g'^*_{\mathrm{qc}}\mathcal{M}.$$

Proof. By Theorem 2.6(4), the result can be verified smooth-locally on Y and Y'. Thus, we may assume that $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$. In particular, g and g' are affine. Since g is affine, $\mathsf{R}(g_{\mathrm{qc}})_*$ is conservative. Hence, it is sufficient to verify that the morphism in question is a quasi-isomorphism after application of the functor $\mathsf{R}(g_{\mathrm{qc}})_*$. By the projection formula applied to g and then f, there are natural quasi-isomorphisms

$$\begin{aligned} \mathsf{R}(g_{\mathrm{qc}})_*\mathsf{L}g^*_{\mathrm{qc}}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} &\simeq \mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \otimes_{\mathcal{O}_Y} \mathsf{R}(g_{\mathrm{qc}})_*\mathcal{O}_{Y'} \\ &\simeq \mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(g_{\mathrm{qc}})_*\mathcal{O}_{Y'}). \end{aligned}$$

Note, however, that because f and g are tor-independent and g is affine, the natural map

$$\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(g_{\mathrm{qc}})_*\mathcal{O}_{Y'} \to \mathsf{R}(g'_{\mathrm{qc}})_*\mathcal{O}_{X'}$$

is a quasi-isomorphism. Indeed, this may be verified smooth-locally on X, so we may assume that $X = \operatorname{Spec} C$. The morphism in question corresponds to the map

$C \otimes^{\mathsf{L}}_A A' \to (C \otimes_A A')[0]$ in $\mathsf{D}(C)$, which is a quasi-isomorphism because f and g are tor-independent. With the projection formula and functoriality, we now obtain the following natural sequence of quasi-isomorphisms:

$$\begin{aligned}\mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(g_{\mathrm{qc}})_*\mathcal{O}_{Y'}) &\simeq \mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}(g'_{\mathrm{qc}})_*\mathcal{O}_{X'}) \\ &\simeq \mathsf{R}(f_{\mathrm{qc}})_*\mathsf{R}(g'_{\mathrm{qc}})_*\mathsf{L}g'^*_{\mathrm{qc}}\mathcal{M} \\ &\simeq \mathsf{R}(g_{\mathrm{qc}})_*\mathsf{R}(f'_{\mathrm{qc}})_*\mathsf{L}g'^*_{\mathrm{qc}}\mathcal{M}.\end{aligned}$$

The result follows. □

5. Presheaves of triangulated categories

Throughout this section we fix a small category $\mathcal{D}$ that is closed under finite limits. A *$\mathcal{D}$-presheaf of triangulated categories* is a 2-functor $\mathcal{T} \colon \mathcal{D}^\circ \to \mathbf{TCat}$.

Given a morphism $i \colon U \to V$ in $\mathcal{D}$, there is an induced pullback functor $i^*_{\mathcal{T}} \colon \mathcal{T}(U) \to \mathcal{T}(V)$. When there is no cause for confusion, we will suppress the subscript $\mathcal{T}$ from $i^*_{\mathcal{T}}$. Let:

$$\mathcal{T}_{V\setminus U}(V) = \ker(i^* \colon \mathcal{T}(V) \to \mathcal{T}(U)).$$

For the moment, the following condition on a $\mathcal{D}$-presheaf of triangulated categories $\mathcal{T}$ will be of interest.

- (**A**) For any morphism $i \colon U \to V$ in $\mathcal{D}$, the pullback functor $i^* \colon \mathcal{T}(V) \to \mathcal{T}(U)$ admits a right adjoint $i_* \colon \mathcal{T}(U) \to \mathcal{T}(V)$.

Suppose that $\mathcal{T}$ is $\mathcal{D}$-presheaf of triangulated categories satisfying (**A**). Let $i \colon U \to V$ be a morphism in $\mathcal{D}$ and let $N \in \mathcal{T}(V)$. We denote by $\eta^i_N \colon N \to i_*i^*N$ the unit of the adjunction. A morphism $j \colon W \to V$ in $\mathcal{D}$ is *$\mathcal{T}$-preflat* if for any cartesian square in $\mathcal{D}$:

$$\begin{array}{ccc} U_W & \xrightarrow{j_U} & U \\ {\scriptstyle i_W}\downarrow & & \downarrow{\scriptstyle i} \\ W & \xrightarrow{j} & V, \end{array}$$

the natural transformation $j^*i_* \to (i_W)_*(j_U)^*$ is an isomorphism. A morphism $j \colon W \to V$ in $\mathcal{D}$ is *$\mathcal{T}$-flat* if for any morphism $V' \to V$, the pullback $j' \colon W' \to V'$ of j is $\mathcal{T}$-preflat.

Note that because monomorphisms are stable under base change, $\mathcal{T}$-flat monomorphisms are stable under base change.

Example 5.1. Let Y be an algebraic stack that is quasi-compact and quasi-separated. Let $\mathbf{Rep}^{\mathrm{fp}}/Y$ denote the category of 1-morphisms $X \to Y$ that are representable and of finite presentation. The category $\mathbf{Rep}^{\mathrm{fp}}/Y$ is small. We have a $\mathbf{Rep}^{\mathrm{fp}}/Y$-presheaf of triangulated categories $\mathsf{D}_{\mathrm{qc}} \colon (\mathbf{Rep}^{\mathrm{fp}}/Y)^\circ \to \mathbf{TCat}$, that sends $X \to Y$ to $\mathsf{D}_{\mathrm{qc}}(X)$ and a 1-morphism $f \colon X' \to X$ in $\mathbf{Rep}^{\mathrm{fp}}/Y$ to $\mathsf{L}f^*_{\mathrm{qc}} \colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X')$. The functor $\mathsf{L}f^*_{\mathrm{qc}}$ admits a right adjoint $\mathsf{R}(f_{\mathrm{qc}})_*$ that preserves small coproducts (Theorem 2.6(3)) so D_{qc} satisfies (**A**).

By Theorem 2.6(4), if f is a flat morphism, then it is D_{qc}-flat. Conversely, if $f \colon X' \to X$ is D_{qc}-flat, then f is flat. Indeed, this is local on the source and target of f, so it is sufficient to show that if $f \colon \operatorname{Spec} B \to \operatorname{Spec} A$ is D_{qc}-preflat, then B is a flat A-algebra. For this, we note that if I is an ideal of A, then corresponding to $i \colon \operatorname{Spec}(A/I) \to \operatorname{Spec} A$ we see that there is a quasi-isomorphism $(A/I) \otimes^{\mathsf{L}}_A B \simeq$

$(B/IB)[0]$. That is, for all $i > 0$ and ideals I of A we have that $\operatorname{Tor}^i_A(B, A/I) = 0$—hence B is flat over A. It follows that the D_{qc}-flat monomorphisms are the quasi-compact open immersions [EGA, IV.17.9.1].

Example 5.2. Our notion of $\mathcal{T}$-flatness is not always optimal. In particular, it is weaker than expected in the derived setting. If $\mathcal{T}$ is a presheaf of triangulated categories *with t-structures*, then a better definition is that f is $\mathcal{T}$-flat if f^* is t-exact.

To illustrate this, suppose $\mathcal{D} = \mathsf{SCR}^\circ$ is the category of affine derived schemes, that is, the opposite category to the category of simplicial commutative rings. Further, let $\mathcal{T} = \mathsf{Mod}(-)$ be the presheaf that takes a simplicial commutative ring A to the homotopy category $\mathsf{Mod}(A)$ of (not necessarily connective) A-modules. Then every morphism in $\mathcal{D}$ is $\mathcal{T}$-flat whereas $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat exactly when the pullback $B \overset{\mathsf{L}}{\otimes}_A - \colon \mathsf{Mod}(A) \to \mathsf{Mod}(B)$ is t-exact. Nevertheless, just as in the non-derived case, the finitely presented $\mathcal{T}$-flat monomorphisms are exactly the quasi-compact open immersions since every monomorphism of derived schemes is formally étale [HAGII, 2.2.2.5 (2)].

Lemma 5.3. *Let $\mathcal{T}$ be a $\mathcal{D}$-presheaf of triangulated categories satisfying* (**A**)*.*

(1) *If $i\colon V \to X$ is a $\mathcal{T}$-preflat monomorphism in $\mathcal{D}$, then the adjunction $i^*i_* \to \mathrm{Id}_{\mathcal{T}(V)}$ is an isomorphism.*

(2) *Fix a commutative diagram in $\mathcal{D}$:*

$$\begin{array}{ccc} W_V & \xrightarrow{i_W} & W \\ {\scriptstyle j_V}\downarrow & & \downarrow{\scriptstyle j} \\ V & \xrightarrow{i} & X. \end{array}$$

(a) *Then, $j^* : \mathcal{T}_{X\setminus V}(X) \to \mathcal{T}(W)$ factors through $\mathcal{T}_{W\setminus W_V}(W)$.*

(b) *If the diagram is cartesian and i is $\mathcal{T}$-preflat, then the functor $j_* : \mathcal{T}_{W\setminus W_V}(W) \to \mathcal{T}(X)$ factors through $\mathcal{T}_{X\setminus V}(X)$ and is right adjoint to $j^*\colon \mathcal{T}_{X\setminus V}(X) \to \mathcal{T}_{W\setminus W_V}(W)$.*

Proof. If $i\colon V \to X$ is a monomorphism, then the commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{Id}_V} & V \\ {\scriptstyle \mathrm{Id}_V}\downarrow & & \downarrow{\scriptstyle i} \\ V & \xrightarrow{i} & X \end{array}$$

is cartesian, whence $i^*i_* \simeq (\mathrm{Id}_V)_*(\mathrm{Id}_V)^* \simeq \mathrm{Id}_{\mathcal{T}(V)}$.

By functoriality (2a) is trivial. For (2b), given $M \in \mathcal{T}_{W\setminus W_V}(W)$, then $i^*j_*M \simeq (j_V)_*(i_W)^*M \simeq 0$, hence $j_*M \in \mathcal{T}_{X\setminus V}(X)$. □

Definition 5.4. Fix a $\mathcal{D}$-presheaf of triangulated categories $\mathcal{T}$ satisfying (**A**). A *Mayer–Vietoris $\mathcal{T}$-square* is a cartesian diagram in $\mathcal{D}$:

$$\begin{array}{ccc} U' & \xrightarrow{i'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{i} & X, \end{array}$$

satisfying the following three conditions:

(1) i is a $\mathcal{T}$-flat monomorphism,

(2) the natural transformation $f^*i_* \to i'_* f_U^*$ is an isomorphism, and
(3) the induced functor $f^*\colon \mathcal{T}_{X\setminus U}(X) \to \mathcal{T}_{X'\setminus U'}(X')$ is an equivalence of categories.

Condition (2) for a Mayer–Vietoris $\mathcal{T}$-square is satisfied if f is a $\mathcal{T}$-(pre)flat morphism. By tor-independent base change (Corollary 4.20), if $\mathcal{T} = \mathsf{D}_{\mathrm{qc}}$, then condition (2) is satisfied by every f. In Appendix A, we consider applications of these Mayer–Vietoris triangles to a result of Moret-Bailly [MB96]. For this intended application, it is essential that we permit f to be non-flat.

Example 5.5. We continue with Example 5.1. Let $f\colon X' \to X$ be a representable, quasi-compact, and quasi-separated étale neighborhood of a closed subset $|Z| \subseteq |X|$ with quasi-compact complement $|U|$. Let $i\colon U \hookrightarrow X$ be the resulting quasi-compact open immersion. Then the cartesian square:

$$\begin{array}{ccc} U' & \xrightarrow{i'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{i} & X, \end{array}$$

is a Mayer–Vietoris $\mathbf{Rep}^{\mathrm{fp}}/Y$-square. To see this, it remains to prove that the functor $\mathsf{L}f^*_{\mathrm{qc}}$ induces the desired equivalence. Now the exact functor $f^*\colon \mathsf{Mod}(X) \to \mathsf{Mod}(X')$ admits an exact left adjoint $f_!\colon \mathsf{Mod}(X') \to \mathsf{Mod}(X)$ [Stacks, 03DI]. Explicitly, for $M \in \mathsf{Mod}(X')$ we have that $f_!M$ is the sheafification of the presheaf

$$U \mapsto \bigoplus_{\phi \in \mathrm{Hom}_X(U,X')} M(U \xrightarrow{\phi} X').$$

Note that the natural map $M \to f^*f_!M$ is an isomorphism for all $M \in \mathsf{Mod}(X')$ such that $i'^*M = 0$. Also, if $N \in \mathsf{Mod}(X)$ and $i^*N = 0$, then the natural map $f_!f^*N \to N$ is an isomorphism. The exactness of the adjoint pair $(f_!, f^*)$ now gives an adjoint pair on the level of derived categories $(f_!, f^*)\colon \mathsf{D}(X) \leftrightarrows \mathsf{D}(X')$ and that the relations just given also hold on the derived category. Next, we observe that the restriction of f^* to $\mathsf{D}_{\mathrm{qc}}(X)$ coincides with $\mathsf{L}f^*_{\mathrm{qc}}$. Thus it remains to prove that if $M \in \mathsf{D}_{\mathrm{qc}}(X')$ and $\mathsf{L}(i')^*_{\mathrm{qc}}M \simeq 0$, then $f_!M \in \mathsf{D}_{\mathrm{qc}}(X)$. The exactness of $f_!$ and $\mathsf{L}(i')^*_{\mathrm{qc}}$ show that is sufficient to prove this result when M is a quasi-coherent sheaf such that $(i')^*M = 0$. Note that $(U \xrightarrow{i} X, X' \xrightarrow{f} X)$ is an étale cover of X, $i^*f_!M = (f_U)_!(i')^*M = 0$ and $f^*f_!M \cong M$. We deduce that étale locally $f_!M$ is quasi-coherent. By descent $f_!M$ is quasi-coherent and the result is proved. A different proof is given in Lemma A.1.

Mayer–Vietoris $\mathcal{T}$-squares give rise to many nice properties. In particular, we obtain a familiar distinguished triangle.

Lemma 5.6. *Let $\mathcal{T}$ be a $\mathcal{D}$-presheaf of triangulated categories satisfying* ($\mathbf{A}$)*. Consider a Mayer–Vietoris $\mathcal{T}$-square in $\mathcal{D}$:*

$$\begin{array}{ccc} U' & \xrightarrow{i'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{i} & X. \end{array}$$

(1) *If $N \in \mathcal{T}(X)$, then there is a unique map d that makes the triangle:*

$$N \xrightarrow{\begin{pmatrix} \eta^i_N \\ \eta^f_N \end{pmatrix}} i_*i^*N \oplus f_*f^*N \xrightarrow{\left(\eta^f_{i_*i^*N}\ -f_*f^*\eta^i_N\right)} f_*f^*i_*i^*N \xrightarrow{d} N[1].$$

distinguished. Moreover, this d is functorial in N.

(2) *Let $M \in \mathcal{T}_{X' \setminus U'}(X')$ and let $N \in \mathcal{T}(X)$. Then there is a natural bijection:*

$$\mathrm{Hom}_{\mathcal{T}(X)}(f_*M, N) \cong \mathrm{Hom}_{\mathcal{T}(V)}(M, f^*N).$$

(3) *Given $N_U \in \mathcal{T}(U)$, $N' \in \mathcal{T}(X')$, and an isomorphism $\delta \colon i'^*N' \to (f_U)^*N_U$, define N by a distinguished triangle in $\mathcal{T}(X)$:*

$$N \longrightarrow i_*N_U \oplus f_*N' \xrightarrow{\left(\eta^f_{i_*N_U}\ -\alpha\right)} f_*f^*i_*N_U \longrightarrow N[1],$$

where $\alpha \colon f_*N' \to f_*f^*i_*N_U$ is the composition:

$$f_*N' \xrightarrow{f_*\eta^{i'}_{N'}} f_*i'_*i'^*N' \xrightarrow{f_*i'_*\delta} f_*i'_*f^*_U N_U \cong f_*f^*i_*N_U.$$

Then the induced maps $i^*N \to N_U$ and $f^*N \to N'$ are isomorphisms.

(4) *If $N \in \mathcal{T}(X)$ satisfies $i^*N \in \mathcal{T}(U)^c$, $f^*N \in \mathcal{T}(X')^c$, and $f^*_U i^*N \in \mathcal{T}(U')^c$, then $N \in \mathcal{T}(X)^c$.*

Proof. An equivalent formulation of (1) is that

$$\begin{array}{ccc} N & \xrightarrow{\eta^i_N} & i_*i^*N \\ \Big\downarrow{\scriptstyle \eta^f_N} & & \Big\downarrow{\scriptstyle \eta^f_{i_*i^*N}} \\ f_*f^*N & \xrightarrow{f_*f^*\eta^i_N} & f_*f^*i_*i^*N \end{array}$$

is a homotopy cartesian square [Nee01, Def. 1.4.1] whose differential d is unique and is functorial in N. To see that the square is cartesian, first choose C such that we have a distinguished triangle:

$$C \xrightarrow{l} N \xrightarrow{\eta^i_N} i_*i^*N \xrightarrow{m} C[1].$$

Since i is a $\mathcal{T}$-flat monomorphism, $i^*\eta^i_N$ is an isomorphism (Lemma 5.31). It follows that $i^*C \cong 0$, so $C \in \mathcal{T}_{X \setminus U}(X)$, and η^f_C is an isomorphism. We thus obtain a morphism of distinguished triangles:

$$\begin{array}{ccccccc} C & \xrightarrow{l} & N & \xrightarrow{\eta^i_N} & i_*i^*N & \xrightarrow{m} & C[1] \\ \Big\| & & \Big\downarrow{\scriptstyle \eta^f_N} & & \Big\downarrow{\scriptstyle u} & & \Big\| \\ C & \xrightarrow{\eta^f_N \circ l} & f_*f^*N & \xrightarrow{f_*f^*\eta^i_N} & f_*f^*i_*i^*N & \xrightarrow{(\eta^f_{C[1]})^{-1} \circ f_*f^*m} & C[1] \end{array}$$

for some morphism u. We can certainly let $u = \eta^f_{i_*i^*N}$ and we will soon see that this is actually the only possible u. On the other hand, we can choose u such that the middle square is a homotopy cartesian square by the Octahedral Axiom [Nee01, Lem. 1.4.3]. After applying i_*i^* to the middle square and adjoining to it the natural square relating u and i_*i^*u we obtain the commutative diagram:

$$\begin{array}{ccccc} i_*i^*N & \xrightarrow{i_*i^*\eta^i_N} & i_*i^*i_*i^*N & \xleftarrow{\eta^i_{i_*i^*N}} & i_*i^*N \\ \downarrow{\scriptstyle i_*i^*\eta^f_N} & & \downarrow{\scriptstyle i_*i^*u} & & \downarrow{\scriptstyle u} \\ i_*i^*f_*f^*N & \xrightarrow{i_*i^*f_*f^*\eta^i_N} & i_*i^*f_*f^*i_*i^*N & \xleftarrow{\eta^i_{f_*f^*i_*i^*N}} & f_*f^*i_*i^*N. \end{array}$$

Since i is a $\mathcal{T}$-flat monomorphism, the horizontal maps are all isomorphisms and it follows that $u = \eta^f_{i_*i^*N}$. Moreover, it is readily verified that the induced differential

$$d := l[1] \circ (\eta^f_C)^{-1}[1] \circ f_*f^*m \colon f_*f^*i_*i^*N \to N[1]$$

is independent of the choice of the triangle $C \xrightarrow{l} N \xrightarrow{\eta^i_N} i_*i^*N \xrightarrow{m} C[1]$. The functoriality of the Mayer–Vietoris triangle now follows from the construction. Finally, to show that d is unique, if $d' \colon f_*f^*i_*i^*N \to N[1]$ is another morphism that makes a distinguished triangle, then there is an induced morphism of distinguished triangles:

$$\begin{array}{ccccccc} N & \longrightarrow & i_*i^*N \oplus f_*f^*N & \longrightarrow & f_*f^*i_*i^*N & \xrightarrow{d} & N[1] \\ \| & & \| & & \downarrow{\scriptstyle \theta} & & \| \\ N & \longrightarrow & i_*i^*N \oplus f_*f^*N & \longrightarrow & f_*f^*i_*i^*N & \xrightarrow{d'} & N[1]. \end{array}$$

It remains to show that θ is the identity morphism. Starting from the middle square, we obtain the commutative diagram

$$\begin{array}{ccccc} i_*i^*f_*f^*N & \xrightarrow{i_*i^*f_*f^*\eta^i_N} & i_*i^*f_*f^*i_*i^*N & \xleftarrow{\eta^i_{f_*f^*i_*i^*N}} & f_*f^*i_*i^*N \\ \| & & \downarrow{\scriptstyle i_*i^*\theta} & & \downarrow{\scriptstyle \theta} \\ i_*i^*f_*f^*N & \xrightarrow{i_*i^*f_*f^*\eta^i_N} & i_*i^*f_*f^*i_*i^*N & \xleftarrow{\eta^i_{f_*f^*i_*i^*N}} & f_*f^*i_*i^*N, \end{array}$$

where the horizontal arrows are isomorphisms. It follows that θ is the identity.

To obtain the isomorphism in (2), apply the homological functor $\mathrm{Hom}_{\mathcal{T}(X)}(f_*M, -)$ to the triangle $N \to i_*i^*N \oplus f_*f^*N \to f_*f_*i_*i^*N$ from (1). This readily produces an isomorphism $\mathrm{Hom}_{\mathcal{T}(X)}(f_*M, N) \cong \mathrm{Hom}_{\mathcal{T}(X)}(f_*M, f_*f^*N)$. There is also a natural isomorphism $\mathrm{Hom}_{\mathcal{T}(X)}(f_*M, f_*f^*N) \cong \mathrm{Hom}_{\mathcal{T}(X')}(f^*f_*M, f^*N)$. But $M \in \mathcal{T}_{X' \setminus U'}(X')$, so $f^*f_*M \to M$ is an isomorphism. The result follows.

For (3), the natural maps $v_i \colon i^*N \to N_U$ and $v_f \colon f^*N \to N'$ are obtained by adjunction from the maps $v_i^\vee \colon N \to i_*N_U$ and $v_f^\vee \colon N \to f_*N'$ in the defining triangle of N. The defining triangle exhibits N as a homotopy pullback. We may thus find a morphism of distinguished triangles [Nee01, Lem. 1.4.4] (Octahedral axiom):

$$\begin{array}{ccccccc} C & \longrightarrow & N & \xrightarrow{v_i^\vee} & i_*N_U & \longrightarrow & C[1] \\ \| & & \downarrow{\scriptstyle v_f^\vee} & & \downarrow{\scriptstyle \eta^f_{i_*N_U}} & & \| \\ C & \longrightarrow & f_*N' & \xrightarrow{\alpha} & f_*f^*i_*N_U & \longrightarrow & C[1] \end{array}$$

Since i is a $\mathcal{T}$-flat monomorphism, we have that $i^*\alpha$ is an isomorphism, so $i^*C \cong 0$. It follows that $i^*v_i^\vee\colon i^*N \to i^*i_*N_U$ is an isomorphism and hence that $v_i\colon i^*N \to N_U$ is an isomorphism (Lemma 5.3(1)).

Now, by (1), we have a morphism of distinguished triangles:

$$\begin{array}{ccccccc} N & \longrightarrow & i_*i^*N \oplus f_*f^*N & \longrightarrow & f_*f^*i_*i^*N & \xrightarrow{d} & N[1] \\ \| & & \downarrow{\scriptstyle i_*v_i \oplus f_*v_f} & & \downarrow{\scriptstyle \theta} & & \| \\ N & \longrightarrow & i_*N_U \oplus f_*N' & \longrightarrow & f_*f^*i_*N_U & \longrightarrow & N[1]. \end{array}$$

As before, it follows that $\theta = f_*f^*i_*v_i$ by considering the application of i_*i^* to the middle square. Since v_i is an isomorphism, it follows that f_*v_f is an isomorphism. Now, if we let W be a cone of v_f, then $f_*W \simeq 0$. We will be done if we can show that i'^*v_f is an isomorphism. Indeed, it would then follow that $i'^*W \simeq 0$ and so $W \simeq f^*f_*W \simeq f^*0 \simeq 0$. To this end, since the following diagram commutes:

$$\begin{array}{ccc} i'^*f^*N & \xrightarrow{i'^*v_f} & i'^*N' \\ \downarrow & & \downarrow{\scriptstyle \delta} \\ f_U^*i^*N & \xrightarrow{f_U^*v_i} & f_U^*N_U, \end{array}$$

and all appearing morphisms except i'^*v_f are known to be isomorphisms, it follows that i'^*v_f is an isomorphism.

For (4), let $h \in \{i, f, i \circ f_U\}$. Since h^* admits a right adjoint, it commutes with small coproducts. Thus if $\{Q_\lambda\}$ is a set of objects of $\mathcal{T}(X)$, then

$$\begin{aligned} \oplus_\lambda \operatorname{Hom}(N, h_*h^*Q_\lambda) &\cong \oplus_\lambda \operatorname{Hom}(h^*N, h^*Q_\lambda) \\ &\cong \operatorname{Hom}(h^*N, \oplus_\lambda h^*Q_\lambda) \\ &\cong \operatorname{Hom}(h^*N, h^*(\oplus_\lambda Q_\lambda)) \\ &\cong \operatorname{Hom}(N, h_*h^*(\oplus_\lambda Q_\lambda)). \end{aligned}$$

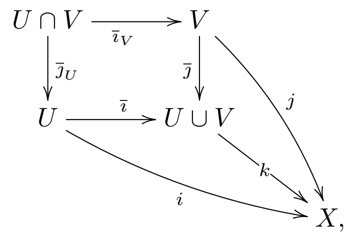
The result now follows by consideration of the Mayer–Vietoris triangles associated to Q_λ and $\oplus_\lambda Q_\lambda$, together with the long exact sequence given by the homological functor $\operatorname{Hom}(N, -)$. $\square$

In the following definition, we axiomatize the required properties of open immersions of algebraic stacks.

Definition 5.7. Let $\mathcal{T}$ be a $\mathcal{D}$-presheaf of triangulated categories satisfying (**A**). Let $\mathcal{L}$ be a collection of morphisms in $\mathcal{D}$. We say that $\mathcal{L}$ is $\mathcal{T}$-*localizing* if it satisfies the following five conditions:

(1) if $i\colon U \to X$ belongs to $\mathcal{L}$, then i is a $\mathcal{T}$-flat monomorphism;
(2) if $a\colon X \to Y$ is an isomorphism, then a belongs to $\mathcal{L}$;
(3) if $a\colon X \to Y$ and $b\colon Y \to Z$ belong to $\mathcal{L}$, then $b \circ a$ belongs to $\mathcal{L}$;
(4) if $i\colon U \to X$ belongs to $\mathcal{L}$ and $f\colon X' \to X$ is a morphism in $\mathcal{D}$, then the induced morphism $i'\colon U \times_X X' \to X'$ belongs to $\mathcal{L}$; and

(5) if $i\colon U \hookrightarrow X$ and $j\colon V \hookrightarrow X$ belong to $\mathcal{L}$, then there is a commutative diagram:



where k belongs to $\mathcal{L}$, and the square is a Mayer–Vietoris $\mathcal{T}$-square.

Example 5.8. We continue the Example 5.5. Let $\mathcal{I}$ be the collection of morphisms in $\mathbf{Rep}^{\mathrm{fp}}/Y$ that are open immersions. By Example 5.5 and standard arguments, $\mathcal{I}$ is D_{qc}-localizing.

We now have a straightforward lemma.

Lemma 5.9. *Let $\mathcal{T}$ be a $\mathcal{D}$-presheaf of triangulated categories satisfying* (**A**). *Let $\mathcal{L}$ be a $\mathcal{T}$-localizing collection of morphisms in $\mathcal{D}$. If $i\colon U \hookrightarrow X$ and $j\colon V \hookrightarrow X$ belong to $\mathcal{L}$, then there is an exact sequence of triangulated categories:*

$$\mathcal{T}_{X\setminus U\cup V}(X) \to \mathcal{T}_{X\setminus V}(X) \xrightarrow{i^*} \mathcal{T}_{U\setminus U\cap V}(U).$$

Proof. Lemma 5.3(2a) shows that the functor $i^*\colon \mathcal{T}_{X\setminus V}(X) \to \mathcal{T}(U)$ factors through $\mathcal{T}_{U\setminus U\cap V}(U)$. Also, Lemma 5.3(2b) shows that $i_*\colon \mathcal{T}_{U\setminus U\cap V}(U) \to \mathcal{T}_{X\setminus V}(X)$ is a right adjoint to i^* and Lemma 5.3(1) shows that the natural transformation $i^*i_* \to \mathrm{Id}$ is an isomorphism. Thus, by Lemma 4.4, it suffices to prove that $(\ker i^*) \cap (\ker j^*) = \mathcal{T}_{X\setminus U\cup V}(X)$. Certainly, we have $\mathcal{T}_{X\setminus U\cup V}(X) \subseteq (\ker i^*) \cap (\ker j^*)$. For the other inclusion let $M \in (\ker i^*) \cap (\ker j^*)$, then $k^*M \in \mathcal{T}(U \cup V)$. Let $\overline{k}\colon U \cap V \to U \cup V$ denote the induced morphism, then we obtain a triangle in $\mathcal{T}(U \cup V)$:

$$k^*M \to \bar{\imath}_*\bar{\imath}^*k^*M \oplus \bar{\jmath}_*\bar{\jmath}^*k^*M \to \overline{k}_*\overline{k}^*k^*M.$$

Functoriality induces isomorphisms $\bar{\imath}^*k^* \simeq i^*$, $\bar{\jmath}^*k^* \simeq j^*$, and $\overline{k}^*k^* \simeq \bar{\jmath}_U^*i^*$. By hypothesis i^*M and j^*M vanish, so the triangle gives $k^*M \simeq 0$. Hence $M \in \mathcal{T}_{X\setminus U\cup V}(X)$. □

6. Descent of compact generation

For this section we fix a small category $\mathcal{D}$ with an initial object $\emptyset$ that admits all finite limits. In particular, for any $X \in \mathcal{D}$ there is a unique monomorphism $\emptyset \hookrightarrow X$. We also fix a collection $\mathcal{L}$ of morphisms in $\mathcal{D}$ and a $\mathcal{D}$-presheaf of triangulated categories $\mathcal{T}$.

Definition 6.1. We say that $\mathcal{T}$ is an *admissible* $(\mathcal{L}, \mathcal{D})$*-presheaf* of triangulated categories if the following conditions are satisfied:

(1) $\mathcal{T}$ satisfies (**A**);
(2) $\mathcal{L}$ is $\mathcal{T}$-localizing;
(3) for all $X \in \mathcal{D}$, the triangulated category $\mathcal{T}(X)$ is closed under small coproducts;
(4) for all $(f\colon X \to Y) \in \mathcal{D}$, the push-forward $f_*\colon \mathcal{T}(X) \to \mathcal{T}(Y)$ preserves small coproducts; and
(5) $\mathcal{T}(\emptyset) \simeq 0$.

Example 6.2. We continue the Example 5.8: D_{qc} is an admissible $(\mathcal{I}, \mathbf{Rep}^{\mathrm{fp}}/Y)$-presheaf of triangulated categories. Indeed, the only non-trivial condition is that f_* preserves small coproducts, which follows from Theorem 2.6 since representable morphisms are concentrated.

Definition 6.3. Let β be a cardinal and let $X \in \mathcal{D}$. We say that an admissible $(\mathcal{L}, \mathcal{D})$-presheaf of triangulated categories $\mathcal{T}$ is *compactly generated with $\mathcal{L}$-supports by β objects at X* if for every $j: V \to X$ in $\mathcal{L}$ the triangulated category $\mathcal{T}_{X \setminus V}(X)$ is generated by a set of cardinality $\leq \beta$ whose elements have compact image in $\mathcal{T}(X)$.

Since $\emptyset \hookrightarrow X$ is a $\mathcal{T}$-flat monomorphism and $\mathcal{T}(\emptyset) \simeq 0$, it follows that $\mathcal{T}(X)$ is also compactly generated by a set of cardinality $\leq \beta$. Also observe that if β is a finite cardinal, then $\mathcal{T}$ can always be compactly generated with supports by *one* object at X. In this section we will give conditions on $\mathcal{T}$ that guarantee that the condition of compact generation with $\mathcal{L}$-supports by β objects descends along certain morphisms and diagrams in $\mathcal{D}$.

Our first result is of an elementary nature and is similar to the arguments of Töen [Toë12, Lem. 4.11]. First we require a definition:

Definition 6.4. Let $\mathcal{T}$ be an admissible $(\mathcal{L}, \mathcal{D})$-presheaf of triangulated categories. A morphism $f: X' \to X$ in $\mathcal{D}$ is *$\mathcal{T}$-quasiperfect with respect to $\mathcal{L}$* if the following three conditions are satisfied:

(1) f is $\mathcal{T}$-flat;
(2) if $P \in \mathcal{T}(X')^c$, then $f_*P \in \mathcal{T}(X)^c$;
(3) f_* admits a right adjoint $f^\times$ such that for every $j: V \to X$ in $\mathcal{L}$, the restriction of $f^\times$ to $\mathcal{T}_{X \setminus V}(X)$ factors through $\mathcal{T}_{X' \setminus V \times_X X'}(X')$.

By Example 4.8, a potentially easy way to verify condition (2) above is for $f^\times$ to preserve small coproducts. To verify condition (3) above, it is sufficient prove the following: for every $j: V \to X$ in $\mathcal{L}$, if $j': V' \to X'$ is the pullback of j along f and $f_V: V' \to V$ is the projection to V, then f_* and $(f_V)_*$ both admit right adjoints and the natural transformation $j'^* f^\times \to f_V^\times j^*$ is an isomorphism of functors.

Example 6.5. We continue with Example 6.2. If $q: W' \to W$ is a finite and faithfully flat morphism of finite presentation, then q is D_{qc}-quasiperfect with respect to $\mathcal{I}$ (Proposition 3.3). In [HR14, App. A], we prove that if $q: W' \to W$ is a proper smooth and representable morphism of noetherian algebraic stacks, then q is D_{qc}-quasiperfect with respect to $\mathcal{I}$.

We now have the first important result of this section.

Proposition 6.6. *Let β be a cardinal. Let $\mathcal{T}$ be an admissible $(\mathcal{L}, \mathcal{D})$-presheaf of triangulated categories. Let $f: X' \to X$ be a morphism in $\mathcal{D}$ that is $\mathcal{T}$-quasiperfect with respect to $\mathcal{L}$. If the functor $f^\times$ (which exists because f is $\mathcal{T}$-quasiperfect) is conservative and $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at X', then $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at X.*

Proof. Let $j: V \to X$ belong to $\mathcal{L}$ and set $V' = X' \times_X V$. By hypothesis, there exists a subset $\mathcal{B}' \subseteq \mathcal{T}(X')^c \cap \mathcal{T}_{X' \setminus V'}(X')$ of cardinality $\leq \beta$ generating $\mathcal{T}_{X' \setminus V'}(X')$. Set $\mathcal{B} = \{f_*P : P \in \mathcal{B}'\}$. Then $\mathcal{B}$ has cardinality $\leq \beta$, and $\mathcal{B} \subseteq \mathcal{T}_{X \setminus V}(X)$ by Lemma 5.3(2b). Since f is $\mathcal{T}$-quasiperfect with respect to $\mathcal{L}$, $\mathcal{B} \subseteq \mathcal{T}(X)^c$ too. It remains to show that $\mathcal{B}$ generates $\mathcal{T}_{X \setminus V}(X)$. Let $N \in \mathcal{T}_{X \setminus V}(X)$ satisfy $\mathrm{Hom}_{\mathcal{T}(X)}(f_*P[n], N) = 0$ for all $P \in \mathcal{B}'$ and all $n \in \mathbb{Z}$. Since f is $\mathcal{T}$-quasiperfect with respect to $\mathcal{L}$, it follows that $f^{\times}N \in \mathcal{T}_{X'\setminus V'}(X')$. As $\mathcal{B}'$ is generating for $\mathcal{T}_{X'\setminus V'}(X')$, we may conclude that $f^{\times}N \simeq 0$. By assumption, $f^{\times}$ is conservative, thus $M \simeq 0$, and $\mathcal{B}$ generates $\mathcal{T}_{X\setminus V}(X)$. □

Our next descent result is deeper, relying on Thomason's Localization Theorem 4.14. First, however, we require a lemma.

Lemma 6.7. *Let β be a cardinal. Let $\mathcal{T}$ be an admissible $(\mathcal{L}, \mathcal{D})$-presheaf of triangulated categories. Suppose that $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at $X \in \mathcal{D}$. Let $W \to V$ and $V \to X$ belong to $\mathcal{L}$. Then the following holds.*

1. *$\mathcal{T}_{X\setminus W}(X)$ is closed under small coproducts and the subcategory $\mathcal{T}_{X\setminus V}(X) \subseteq \mathcal{T}_{X\setminus W}(X)$ is localizing;*
2. *$\mathcal{T}_{X\setminus V}(X)$ is the localizing envelope of a set of compact objects of $\mathcal{T}_{X\setminus W}(X)$; and*
3. *$\mathcal{T}_{X\setminus V}(X)^c = \mathcal{T}_{X\setminus W}(X)^c \cap \mathcal{T}_{X\setminus V}(X)$.*

Proof. First, observe that $\mathcal{T}(X)$ is closed under small coproducts. Also, if $f\colon X' \to X$ is a morphism in $\mathcal{D}$, then f^* admits a right adjoint, so f^* preserves small coproducts. Hence, we see that $\mathcal{T}_{X\setminus X'}(X)$ is a localizing subcategory of $\mathcal{T}(X)$. In particular, $\mathcal{T}_{X\setminus X'}(X)$ is closed under small coproducts. The claim (1) is now immediate.

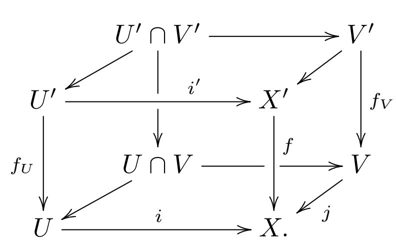
By hypothesis, $\mathcal{T}_{X\setminus V}(X)$ is generated by a subset R that has compact image in $\mathcal{T}(X)$, hence also in $\mathcal{T}_{X\setminus W}(X)$. Let $\mathcal{R} \subseteq \mathcal{T}_{X\setminus W}(X)$ denote the localizing envelope of R, then $\mathcal{R} \subseteq \mathcal{T}_{X\setminus V}(X)$. Applying Corollary 4.16 to $R \subseteq \mathcal{T}_{X\setminus V}(X)$, we find $\mathcal{R} = \mathcal{T}_{X\setminus V}(X)$, proving (2). The claim (3) is now an immediate consequence of (2) and Thomason's Theorem 4.14. □

Proposition 6.8. *Let β be a cardinal. Let $\mathcal{T}$ be an admissible $(\mathcal{L}, \mathcal{D})$-presheaf of triangulated categories. Consider a Mayer–Vietoris $\mathcal{T}$-square:*

$$\begin{array}{ccc} U' & \xrightarrow{i'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{i} & X. \end{array}$$

If $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at U and X', then $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at X.

Proof. Let $j\colon V \to X$ belong to $\mathcal{L}$. Form the cartesian cube:



By Lemma 5.9, we have an exact sequence

$$\mathcal{T}_{X'\setminus U'\cup V'}(X') \to \mathcal{T}_{X'\setminus V'}(X') \to \mathcal{T}_{U'\setminus U'\cap V'}(U'). \tag{6.1}$$

The category $\mathcal{T}_{X'\setminus V'}(X')$ is compactly generated and by Lemma 6.7(1) it is also closed under small coproducts. By Lemma 6.7(2) the subcategory $\mathcal{T}_{X'\setminus U'\cup V'}(X') \subseteq \mathcal{T}_{X'\setminus V'}(X')$ is the localizing envelope of a set of compact objects of $\mathcal{T}_{X'\setminus V'}(X')$.

Now let $P \in \mathcal{T}(U)^c \cap \mathcal{T}_{U\setminus U\cap V}(U)$. Then $(f_U)^*P \in \mathcal{T}(U')^c$ since $(f_U)_*$ preserves coproducts (Example 4.8) and $(f_U)^*P \in \mathcal{T}_{U'\setminus U'\cap V'}(U')$. Thus $(f_U)^*P \in \mathcal{T}_{U'\setminus U'\cap V'}(U')^c$ by Lemma 6.7(3). We now apply Thomason's localization Theorem, in the form of Corollary 4.15, to the exact sequence (6.1). This gives us $P' \in \mathcal{T}_{X'\setminus V'}(X')^c = \mathcal{T}(X')^c \cap \mathcal{T}_{X'\setminus V'}(X')$ and an isomorphism $i'^*P' \simeq (f_U)^*(P \oplus P[1])$. As in Lemma 5.6(3), form the following triangle in $\mathcal{T}(X)$:

$$\tilde{P} \to i_*(P \oplus P[1]) \oplus f_*P' \to f_*f^*i_*(P \oplus P[1]).$$

By Lemma 5.6(3,4) we have that $i^*\tilde{P} = P \oplus P[1]$ and $f^*\tilde{P} = P'$ and that $\tilde{P} \in \mathcal{T}(X)^c$. Since $i_*i^*\tilde{P}$, $f_*f^*\tilde{P}$ and $f_*f^*i_*i^*\tilde{P} \in \mathcal{T}_{X\setminus V}(X)$, it follows that $\tilde{P} \in \mathcal{T}_{X\setminus V}(X)$.

Now let $Q \in \mathcal{T}_{X'\setminus U'\cup V'}(X')$, and note that $f_*Q \in \mathcal{T}_{X\setminus U\cup V}(X)$. Moreover, $f^*f_*Q \to Q$ is an isomorphism, because $f^* : \mathcal{T}_{X\setminus U}(X) \to \mathcal{T}_{X'\setminus U'}(X')$ is an equivalence of categories. We also have that $i^*f_*Q \simeq 0$ and $i'^*f^*f_*Q \simeq 0$. Thus, if in addition $Q \in \mathcal{T}(X')^c$, then $f_*Q \in \mathcal{T}(X)^c$ by Lemma 5.6(4).

By hypothesis, there is a subset $\mathcal{B}_0 \subseteq \mathcal{T}(U)^c \cap \mathcal{T}_{U\setminus U\cap V}(U)$ (resp. $\mathcal{B}_1 \subseteq \mathcal{T}(X')^c \cap \mathcal{T}_{X'\setminus U'\cup V'}(X')$) of cardinality $\leq \beta$ generating $\mathcal{T}_{U\setminus U\cap V}(U)$ (resp. $\mathcal{T}_{X'\setminus U'\cup V'}(X')$). Define:

$$\mathcal{B} = \{\tilde{P} \,:\, P \in \mathcal{B}_0\} \cup \{f_*Q \,:\, Q \in \mathcal{B}_1\}.$$

If β is infinite, then the cardinality of $\mathcal{B}$ is $\leq \beta$, and if β is finite then the same is true of $\mathcal{B}$. By the above considerations, $\mathcal{B} \subseteq \mathcal{T}(X)^c \cap \mathcal{T}_{X\setminus V}(X)$ and it remains to show that $\mathcal{B}$ generates $\mathcal{T}_{X\setminus V}(X)$.

Let $M \in \mathcal{T}_{X\setminus V}(X)$ so that $f^*M \in \mathcal{T}_{X'\setminus V'}(X')$ and $i^*M \in \mathcal{T}_{U\setminus U\cap V}(U)$ (Lemma 5.3(2a)). Suppose that $\mathrm{Hom}_{\mathcal{T}(X)}(f_*Q[n], M) = 0$ for all $Q \in \mathcal{B}_1$ and all $n \in \mathbb{Z}$. By Lemma 5.6(2), we see that $\mathrm{Hom}_{\mathcal{T}(X')}(Q[n], f^*M) = 0$ for all $Q \in \mathcal{B}_1$ and all $n \in \mathbb{Z}$. Let K be a cone of $f^*M \to i'_*i'^*f^*M$. Note that

$$\mathrm{Hom}_{\mathcal{T}(X')}(Q[n], i'_*i'^*f^*M) = \mathrm{Hom}_{\mathcal{T}(X')}(i'^*Q[n], i'^*f^*M) = 0$$

so $\mathrm{Hom}_{\mathcal{T}(X')}(Q[n], K) = 0$. Since $K \in \mathcal{T}_{X'\setminus U'\cup V'}(X')$ and $\mathcal{B}_1$ is generating, we see that $K \simeq 0$, so $f_*f^*M \to f_*i'_*i'^*f^*M \simeq f_*f^*i_*i^*M$ is an isomorphism. From the Mayer-Vietoris triangle $M \to i_*i^*M \oplus f_*f^*M \to f_*f^*i_*i^*M$, we deduce that that natural map $M \to i_*i^*M$ is an isomorphism for all such M.

Now suppose that M also satisfies $\mathrm{Hom}_{\mathcal{T}(X)}(\tilde{P}[n], M) = 0$ for all $P \in \mathcal{B}_0$ and $n \in \mathbb{Z}$. Since the natural map $M \to i_*i^*M$ is an isomorphism, it follows that $\mathrm{Hom}_{\mathcal{T}(X)}(i^*\tilde{P}[n], i^*M) = 0$ for all $P \in \mathcal{B}_0$ and $n \in \mathbb{Z}$. By Lemma 5.6(3), $i^*\tilde{P} \simeq P \oplus P[1]$ and so $\mathrm{Hom}_{\mathcal{T}(U)}(P[n], i^*M) = 0$ for all $P \in \mathcal{B}_0$ and all $n \in \mathbb{Z}$. By assumption, $\mathcal{B}_0$ generates $\mathcal{T}_{U\setminus U\cap V}(U)$ and thus $i^*M \simeq 0$. Since $M \simeq i_*i^*M \simeq 0$, we deduce that $\mathcal{B}$ generates $\mathcal{T}_{X\setminus V}(X)$. □

We are now in a position to prove the main technical result of the article.

Theorem 6.9. *Let X be a quasi-compact and quasi-separated algebraic stack. Let $\mathcal{D}$ be $\mathbf{Rep}^{\mathrm{fp}}/X$ or one of the full subcategories $\mathbf{Rep}^{\mathrm{fp,qff,sep}}/X$ or $\mathbf{Rep}^{\mathrm{et,sep}}/X$. Let $\mathcal{I}$ denote the set of open immersions in $\mathcal{D}$. Let $\mathcal{T}$ be a presheaf of triangulated categories on $\mathcal{D}$. Assume that*

(1) $\mathcal{T}(\emptyset) \simeq 0$,
(2) $\mathcal{T}(W)$ *is closed under small coproducts for all $W \in \mathcal{D}$,*

(3) *for every morphism* $f\colon W_1 \to W_2$ *in* $\mathcal{D}$*, the pullback* $f^*\colon \mathcal{T}(W_2) \to \mathcal{T}(W_1)$ *admits a right adjoint* f_* *that preserves small coproducts,*

(4) *for every cartesian square in* $\mathcal{D}$

$$\begin{array}{ccc} U_W & \xrightarrow{j_U} & U \\ {\scriptstyle i_W}\downarrow & & \downarrow{\scriptstyle i} \\ W & \xrightarrow{j} & V, \end{array}$$

such that j *is flat, the natural transformation* $j^*i_* \to (i_W)_*(j_U)^*$ *is an isomorphism,*

(5) *for every open immersion* $U \to W$ *and étale neighborhood* $f\colon W' \to W$ *of* $W \setminus U$*, the pullback* f^* *induces an equivalence* $\mathcal{T}_{W\setminus U}(W) \to \mathcal{T}_{W'\setminus U'}(W')$*,*

(6) *for every finite faithfully flat morphism* $W' \to W$ *of finite presentation, the functor* $f_*\colon \mathcal{T}(W') \to \mathcal{T}(W)$ *admits a right adjoint* $f^\times$ *that preserves small coproducts, is conservative, and commutes with pullback along open immersions.*

Let $\mathcal{C} \subseteq \mathcal{D}$ *be the collection of all objects* W *such that for every separated étale morphism* $q\colon W' \to W$ *in* $\mathcal{D}$ *and every open immersion* $V' \to W'$ *in* $\mathcal{I}$*, the triangulated category* $\mathcal{T}_{W'\setminus V'}(W')$ *is generated by a set of cardinality* $\leq \beta$ *whose elements have compact image in* $\mathcal{T}(W')$*.*

If $p\colon W \to X$ *is a separated, quasi-finite and faithfully flat morphism in* $\mathcal{D}$ *such that* $W \in \mathcal{C}$*, then* $X \in \mathcal{C}$*.*

Proof. Condition (3) says that $\mathcal{T}$ satisfies (**A**). Condition (4) says that flat morphisms are $\mathcal{T}$-flat. Conditions (4) and (5) imply that étale neighborhoods are Mayer–Vietoris squares. Thus conditions (3), (4) and (5) imply that $\mathcal{I}$ is $\mathcal{T}$-localizing. Combining this with condition (3), we conclude that $\mathcal{T}$ is an admissible $(\mathcal{I},\mathcal{D})$-presheaf of triangulated categories.

By assumption, there exists an object $W \in \mathcal{C}$ with $W \to X$ separated, quasi-finite and faithfully flat. We will apply [Ryd11, Thm. 6.1] to deduce that $X \in \mathcal{C}$. To do this, we need to verify the following three conditions for a flat morphism $q\colon W' \to W$ in $\mathcal{D}$.

(D1) If $W \in \mathcal{C}$ and q is étale and separated, then $W' \in \mathcal{C}$;
(D2) if $W' \in \mathcal{C}$ and q is finite and surjective, then $W \in \mathcal{C}$; and
(D3) if q is an étale neighborhood of $W \setminus U$, where $U \to W$ is an open immersion in $\mathcal{D}$, and U and W' belong to $\mathcal{C}$, then $W \in \mathcal{C}$.

Now (D1) tautologically follows from the definition of $\mathcal{C}$. For (D2), Condition (6) implies that p is $\mathcal{T}$-quasiperfect with respect to $\mathcal{I}$. By Proposition 6.6, we deduce that (D2) is satisfied. As noted previously, Conditions (4) and (5) imply that étale neighborhoods are Mayer–Vietoris squares. By Proposition 6.8, it follows that (D3) is satisfied. The result follows. □

7. Algebraic stacks with the β-resolution property

Let X be an algebraic stack. Recall that X is said to have the *resolution property* if every quasi-coherent $\mathcal{O}_X$-module M is a quotient of a direct sum of locally free $\mathcal{O}_X$-modules of finite type. The resolution property is a subtle and difficult property, although it is always satisfied for quasi-projective schemes. It has been studied systematically by several authors, with notable contributions due to Thomason [Tho87], Totaro [Tot04], and Gross [Gro10, Gro13].

The following simple refinement of the resolution property will be useful for us. Let $\mathbf{VB}(X) \subseteq \mathsf{QCoh}(X)$ denote the subcategory of locally free $\mathcal{O}_X$-modules of finite type. Let β be a cardinal. We say that X has the *β-resolution property* if there exists a set $\mathcal{B} \subseteq \mathbf{VB}(X)$ of cardinality $\leq \beta$, with the property that every quasi-coherent $\mathcal{O}_X$-module M is a quotient of a direct sum of elements of $\mathcal{B}$. Equivalently, every quasi-coherent $\mathcal{O}_X$-module M is the direct limit of its finitely generated submodules and every finitely generated submodule is a quotient of a finite direct sum of objects in $\mathcal{B}$.

The following simple Lemma will be important.

Lemma 7.1 ([Gro13, Prop. 1.8 (v)])**.** *Let $f\colon X \to Y$ be a quasi-affine morphism of algebraic stacks. Let β be a cardinal. If Y has the β-resolution property, then so does X.*

Proof. Let $\mathcal{B} \subseteq \mathbf{VB}(Y)$ be a resolving set of cardinality β. Then $f^*\mathcal{B} = \{f^*E : E \in \mathcal{B}\}$ is a resolving set of cardinality β. This follows immediately from the fact that f^* is right-exact and $f^*f_*M \to M$ is surjective for every $M \in \mathsf{QCoh}(X)$. □

Remark 7.2. Similarly, there is the following partial converse: if $f : X \to Y$ is finite and faithfully flat of finite presentation and X has the β-resolution property, then so does Y. In this case, one takes $f_*\mathcal{B}$ as the resolving set and uses that f_* is right-exact and that $f_*f^!M \to M$ is surjective [Gro13, Prop. 1.13].

Proposition 7.3. *Let X be an algebraic stack. If X is quasi-affine, then X has the 1-resolution property. If X is quasi-compact and quasi-separated with affine stabilizer groups, then the following are equivalent:*

1. *X has the $\aleph_0$-resolution property;*
2. *X has the resolution property; and*
3. *$X = [V/\mathrm{GL}_{n,\mathbb{Z}}]$ where V is a quasi-affine scheme.*

When these conditions hold, X has affine diagonal.

Proof. The first statement follows from the fact that $\mathcal{O}_X \in \mathsf{QCoh}(X)$ is a generator if and only if X is quasi-affine. Trivially, (1) $\implies$ (2). That (2) $\implies$ (3) is Totaro's theorem [Tot04, Gro13]. Finally, to see that (3) $\implies$ (1) it is enough to prove that $B\mathrm{GL}_{n,\mathbb{Z}}$ has the $\aleph_0$-resolution property since $X \to B\mathrm{GL}_{n,\mathbb{Z}}$ is quasi-affine. That $B\mathrm{GL}_{n,\mathbb{Z}}$ has the resolution property is a special case of [Tho87, Lem. 2.4]: every coherent sheaf on $B\mathrm{GL}_{n,\mathbb{Z}}$ is a quotient of a finite dimensional subrepresentation of a finite number of copies of the regular representation. Since there is a countable number of vector bundles on $B\mathrm{GL}_{n,\mathbb{Z}}$, the $\aleph_0$-resolution property holds.

The last statement follows since $[V/\mathrm{GL}_{n,\mathbb{Z}}]$ has affine diagonal. □

Question 7.4. Every algebraic stack that admits a finite flat cover $V \to X$ with V quasi-affine has the 1-resolution property by Remark 7.2. Are all stacks with the 1-resolution of this form? Is every algebraic space with the 1-resolution property quasi-affine?

Many quotient stacks have the resolution property:

Example 7.5. Let $S = \mathrm{Spec}(R)$ be a regular scheme of dimension at most 1 (e.g., $R = \mathbb{Z}$ or R is a field). Let $G \to S$ be a flat affine group scheme of finite type and let V be an algebraic space with an action of G. Then $X = [V/G]$ has the resolution property in the following cases:

(1) V is quasi-affine.
(2) V is normal, noetherian and has an ample family of line bundles (e.g. V quasi-projective) and G is an extension of a finite flat group scheme by a smooth group scheme with connected fibers (this is automatic if R is a field).

For (1), use that BG_S has the resolution property [Tho87, Lem. 2.4] and Lemma 7.1. For (2) use [Tho87, Lem. 2.10 and 2.14]. Note that in (2) it is crucial that V is normal to apply Sumihiro's theorem [Sum75, Thm. 1.6] and deduce that sufficiently high powers of the line bundles carry a G-action. In fact, this is false otherwise (cf. [Tot04, §9]) and whether X has the resolution property in this case is not known in general. Alternatively, we could assume that V is a quasi-projective scheme with a *linearizable action* of G, as in [BZFN10, Cor. 3.22].

8. Crisp stacks

In this section we define β-crisp stacks and show that the β-resolution property implies β-crispness.

Definition 8.1. Let β be a cardinal. We say that an algebraic stack X is *β-crisp* if

(1) $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by a set of cardinality $\leq \beta$.
(2) for every étale morphism $W \to X$ that is quasi-compact and separated, and every quasi-compact open immersion $i\colon U \hookrightarrow W$ there exists a perfect object P of $\mathsf{D}_{\mathrm{qc}}(W)$ with support $W \setminus U$.

By the following two Lemmas 8.2 and 8.4, an equivalent definition is that for every étale morphism $W \to X$ that is quasi-compact and separated, and every quasi-compact open immersion $i\colon U \hookrightarrow W$, the triangulated category

$$\mathsf{D}_{\mathrm{qc},|W|\setminus|U|}(W) = \{M \in \mathsf{D}_{\mathrm{qc}}(W) \,:\, i^*M \simeq 0\}$$

is generated by a set of cardinality $\leq \beta$ consisting of compact objects of $\mathsf{D}_{\mathrm{qc}}(W)$.

Lemma 8.2. *Let $f\colon X \to Y$ be a quasi-affine morphism of algebraic stacks. Let β be a cardinal. If $\mathsf{D}_{\mathrm{qc}}(Y)$ is compactly generated by β objects, then so is $\mathsf{D}_{\mathrm{qc}}(X)$.*

Proof. Since f is concentrated, f_* preserves small coproducts; hence f^* takes compact objects to compact objects. Since f_* is conservative, the result follows. □

Remark 8.3. We do not know if the analogue of Lemma 8.2 holds for β-crispness.

Lemma 8.4. *Let X be a quasi-compact and quasi-separated algebraic stack and let $i\colon U \to X$ be a quasi-compact open immersion with complement $|Z|$.*

(1) *If $\mathsf{D}_{\mathrm{qc},|Z|}(X)$ is generated by a set whose elements have compact image in $\mathsf{D}_{\mathrm{qc}}(X)$, then there exists a compact object Q of $\mathsf{D}_{\mathrm{qc}}(X)$ with support $|Z|$.*
(2) *If $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by a set of cardinality at most β and there exists a perfect complex P on X with support $|Z|$, then $\mathsf{D}_{\mathrm{qc},|Z|}(X)$ is generated by a set of cardinality at most β whose elements have compact image in $\mathsf{D}_{\mathrm{qc}}(X)$.*

Proof. For (1): let $\{Q_\lambda\}_{\lambda\in\Lambda}$ be a set of generators for $\mathsf{D}_{\mathrm{qc},|Z|}(X)$ whose elements have compact image in $\mathsf{D}_{\mathrm{qc}}(X)$. Let $z \in |Z|$ be a point and choose a representative, that is, a field k and a 1-morphism of algebraic stacks $\overline{z}\colon \operatorname{Spec} k \to X$ with image z. Since the diagonal of X is quasi-compact and quasi-separated, it follows that $\overline{z}$ is quasi-compact and quasi-separated. By Lemma 1.2(4), it follows that $\mathsf{L}i_{\mathrm{qc}}^*\mathsf{R}(\overline{z}_{\mathrm{qc}})_*\mathcal{O}_{\operatorname{Spec} k} \simeq 0$ and so there exists a $\lambda \in \Lambda$, an integer n, and a non-zero morphism $Q_\lambda[n] \to \mathsf{R}(\overline{z}_{\mathrm{qc}})_*\mathcal{O}_{\operatorname{Spec} k}$. By adjunction, $\mathsf{L}\overline{z}_{\mathrm{qc}}^* Q_\lambda \neq 0$ and we deduce that $\cup_{\lambda\in\Lambda}\operatorname{supp}(Q_\lambda) = |Z|$. It suffices to show that there is a finite subset $\Lambda' \subseteq \Lambda$ such that $\cup_{\lambda\in\Lambda'}\operatorname{supp}(Q_\lambda) = |Z|$. This is obvious if X is noetherian or, more generally, if Z has a finite number of irreducible components.

In complete generality, we note that $|Z|$ is constructible and that $\operatorname{supp}(Q_\lambda)$ is constructible for every λ. Indeed, both claims are local on X so we can assume that $X = \operatorname{Spec}(A)$ is affine. Since U is quasi-compact and open, it is constructible; hence, so is its complement $|Z|$. That $\operatorname{supp}(Q_\lambda)$ is constructible follows from an easy approximation argument [Nee92a, Lem. A.2] (note that in [*loc. cit.*], $D^b(R)$ denotes the triangulated category of perfect complexes of R-modules). We conclude that $|Z| = \cup_{\lambda\in\Lambda}\operatorname{supp}(Q_\lambda)$ has a finite subcovering since the constructible topology is quasi-compact.

For (2): let P be a perfect complex with support $|Z|$ and let $\{Q_b\}_{b\in B}$ be a set of compact generators for $\mathsf{D}_{\mathrm{qc}}(X)$, where the cardinality of B is at most β. By Example 4.11, for each $b \in B$ the complex $Q_b \otimes^{\mathsf{L}}_{\mathcal{O}_X} P$, which belongs to $\mathsf{D}_{\mathrm{qc},|Z|}(X)$, is a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$. Thus it suffices to prove that the set $\{Q_b \otimes^{\mathsf{L}}_{\mathcal{O}_X} P\}_{b\in B}$ generates $\mathsf{D}_{\mathrm{qc},|Z|}(X)$.

Let $M \in \mathsf{D}_{\mathrm{qc},|Z|}(X)$ and suppose that $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(Q_b \otimes^{\mathsf{L}}_{\mathcal{O}_X} P, M) \simeq 0$. By adjunction (1.3), $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(Q_b, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(P,M)) \simeq 0$. Since the set $\{Q_b\}_{b\in B}$ is generating, it follows that $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(P,M) \simeq 0$. Verifying that this condition implies that $M \simeq 0$ is local on X for the smooth topology, so we may assume that X is an affine scheme.

By [BN93, Prop. 6.1], there is a perfect generating complex $K \in \mathsf{D}_{\mathrm{qc},|Z|}(X)^c$ (a Koszul complex). As P is perfect and the support of P is $|Z|$, it follows from [Nee92a, Lem. A.3] that K is in the thick closure of P, so P is also a generator of $\mathsf{D}_{\mathrm{qc},|Z|}(X)$. By assumption, $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(P,M) \simeq 0$, so $M \simeq 0$. □

The main result of this section is the following

Proposition 8.5. *Let X be a concentrated algebraic stack with affine diagonal. Let β be a cardinal. If X has the β-resolution property, then it is β-crisp.*

Proof. By Lemma 7.1 the β-resolution property is preserved under quasi-affine morphisms. By Zariski's Main Theorem [LMB, Thm. 16.5] étale morphisms that are quasi-compact, separated, and representable are quasi-affine. Thus it is enough to prove the following statement: if $i\colon V \hookrightarrow X$ is a quasi-compact open immersion with complement $|Z|$, then there exists a subset $\mathcal{B}_{|Z|} \subseteq \mathsf{D}_{\mathrm{qc},|Z|}(X)$, of cardinality $\leq \beta$, with compact image in $\mathsf{D}_{\mathrm{qc}}(X)$.

Let $\mathcal{B} \subseteq \mathbf{VB}(X)$ be a resolving set of cardinality $\leq \beta$. Since X is concentrated, if $E \in \mathcal{B}$, then $E[0] \in \mathsf{D}_{\mathrm{qc}}(X)^c$ (Example 4.9). Let $M \in \mathsf{D}_{\mathrm{qc}}(X)$. We claim that if $n \in \mathbb{Z}$ is such that $\mathcal{H}^{-n}(M) \neq 0$, then there exists an $E \in \mathcal{B}$ and a non-zero morphism $E[n] \to M$ in $\mathsf{D}_{\mathrm{qc}}(X)$. We prove this claim by a small modification (which is likely well-known) to the argument of A. Neeman [Nee96, Ex. 1.10].

Since $E[n]$ is compact, there exists (Lemma 4.10) an integer $r \geq n$ such that:

$$\mathrm{Hom}_{\mathcal{O}_X}(E[n], M) = \mathrm{Hom}_{\mathcal{O}_X}(E[n], \tau^{\geq -r} M).$$

We may consequently assume that $M \in \mathsf{D}^+_{\mathrm{qc}}(X)$. By [Lur04, Thm. 3.8], the natural functor $\mathsf{D}^+(\mathsf{QCoh}(X)) \to \mathsf{D}^+_{\mathrm{qc}}(X)$ is an equivalence of triangulated categories. Hence, we are free to assume that M is a complex $(\cdots \to M^k \xrightarrow{d^k} M^{k+1} \to \cdots)$ of quasi-coherent $\mathcal{O}_X$-modules. By assumption $\mathcal{H}^{-n}(M) \neq 0$, so there exists $E_0 \in \mathcal{B}$ and a non-zero morphism $E_0 \to \mathcal{H}^{-n}(M)$. Let W_0 denote the pullback of the map $E_0 \to \mathcal{H}^{-n}(M)$ along the surjection $\ker(d^{-n}) \to \mathcal{H}^{-n}(M)$. The morphism of quasi-coherent $\mathcal{O}_X$-modules $W_0 \to \ker(d^{-n}) \to \mathcal{H}^{-n}(M)$ is non-zero, thus there exists an $E \in \mathcal{B}$ and a morphism $E \to W_0$ such that the composition $E \to W_0 \to \mathcal{H}^{-n}(M)$ is non-zero. The composition $E \to \ker(d^{-n}) \to M^{-n}$ thus induces a non-zero morphism $E[n] \to M$ in $\mathsf{D}_{\mathrm{qc}}(X)$ and we deduce the claim.

We now return to the proof of the Proposition. The above considerations shows that the set $\mathcal{B}$ compactly generates $\mathsf{D}_{\mathrm{qc}}(X)$. Now let $j\colon Z \hookrightarrow X$ be a closed immersion with support $|Z|$. Since X has the resolution property and $i\colon V \to X$ is quasi-compact, we may choose j such that the quasi-coherent ideal sheaf I defining Z in X is of finite type. It follows that there is a surjection $F \to I$, where F is a finite direct sum of objects of $\mathcal{B}$. Corresponding to the morphism $s\colon F \to \mathcal{O}_X$, we obtain a section $s^\vee \in \Gamma(X, F^\vee)$ with vanishing locus $|Z|$. If $K(s^\vee)$ is the resulting Koszul complex [FL85, IV.2], then $K(s^\vee)$ is a perfect complex on X with support $|Z|$. By Lemma 8.4(2), we deduce the claim. □

We conclude this section with examples of algebraic stacks that are crisp.

Example 8.6. Let A be a ring. Then $\operatorname{Spec} A$ is 1-crisp. Indeed, $\operatorname{Spec} A$ has the 1-resolution property and is concentrated, thus the result follows from Proposition 8.5.

Example 8.7. Let X be a concentrated stack with affine stabilizers and the resolution property. Then X is $\aleph_0$-crisp. Examples are stacks of the form $X = [V/G]$ where V and G are as in Example 7.5 and either $S = \operatorname{Spec}(\mathbb{Q})$ or G is linearly reductive (e.g., a torus). Indeed, under these assumptions on G, the classifying stack BG is concentrated, so X is concentrated since $X \to BG$ is representable. More generally, we can take any stack $X = [V/G]$ as in Example 7.5 with linearly reductive stabilizers. Such stacks are concentrated by [HR14, Thm. C].

9. Quasi-finite flat locality of β-crispness and applications

We are now in a position to prove Theorems A, B, and C as well as addressing the applications mentioned in the Introduction.

Proof of Theorem C. Take $\mathcal{D} = \mathbf{Rep}^{\mathrm{fp}}/X$. By Examples 5.1, 5.5, 6.2 and 6.5, the $\mathcal{D}$-presheaf of triangulated categories D_{qc} satisfies Conditions (1)-(6) of Theorem 6.9. The result now follows from Lemma 8.4 and Theorem 6.9. □

Proving Theorems A and B is now very simple.

Proof of Theorem A. By [Ryd11, Thm. 7.1] there exists a locally quasi-finite flat morphism $p\colon X' \to X$, where X' is a scheme. Since X is quasi-compact, we may further assume that X' is an affine scheme, and consequently the morphism p is also quasi-compact and separated. The result now follows by combining Example 8.6 with Theorem C. □

Proof of Theorem B. Since X is of s-global type, there exists an $N > 0$, a quasi-affine $\mathbb{Q}$-scheme V with an action of GL_N, together with an étale, representable, separated and finitely presented morphism $p\colon [V/\mathrm{GL}_N] \to X$. Now the result follows from Theorem C and Example 8.7. □

Example 9.1 (Brauer groups)**.** Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. Let $\alpha \in H^2(X, \mathbb{G}_m)$ be an element of the (bigger) cohomological Brauer group. Let $\mathcal{D} = \mathbf{Rep}^{\mathrm{fp}}/X$ and let $\mathcal{T} = \mathsf{D}(\mathsf{QCoh}^{\alpha}(-))$ be the presheaf of derived categories of α-twisted sheaves. Then the conditions of Theorem 6.9 are satisfied. Indeed, if $\mathcal{X}$ denotes the $\mathbb{G}_m$-gerbe corresponding to α, then there is a canonical decomposition $\mathsf{D}_{\mathrm{qc}}(\mathcal{X} \times_X T) = \mathsf{D}(\mathsf{QCoh}(\mathcal{X} \times_X T)) = \bigoplus_{m\in\mathbb{Z}} \mathsf{D}(\mathsf{QCoh}^{m\alpha}(T))$ which is respected by pullbacks[1]. Since $\mathsf{D}_{\mathrm{qc}}(\mathcal{X} \times_X -)$ satisfies the conditions of Theorem 6.9, so does $\mathcal{T}$.

Since X has quasi-finite diagonal, there exists a quasi-finite flat presentation $p\colon X' \to X$ such that X' is affine and $p^*\alpha = 0$. Thus, $\mathsf{D}(\mathsf{QCoh}^{\alpha}(X')) = \mathsf{D}(\mathsf{QCoh}(X')) = \mathsf{D}_{\mathrm{qc}}(X')$ is compactly generated by 1 object with supports. It follows that $\mathsf{D}(\mathsf{QCoh}^{\alpha}(X))$ is compactly generated by 1 object with supports (Theorem 6.9). The endomorphism algebra of this object is a derived Azumaya algebra [Toë12].

Example 9.2 (Sheaves of linear categories on derived stacks)**.** Let $(X, \mathcal{O}_X)$ be a derived (or spectral) Deligne–Mumford stack. The 0-truncation $(X, \pi_0\mathcal{O}_X)$ is an ordinary Deligne–Mumford stack with the same underlying topos X. In fact, even for a non-connective E_∞-algebra A, the category of étale A-algebras is equivalent to the category of étale $\pi_0 A$-algebras [Lur12, 8.5.0.6].

Let $F \in \mathrm{QStk}(X)$ be a quasi-coherent stack on X [Lur11a, §8], e.g., $F = \mathsf{QCoh}(X)$. For every object U in the small étale topos of X, this gives an $\mathcal{O}_X(U)$-linear ∞-category $F(U)$. Let $h^0(F)$ be the presheaf of triangulated categories that assigns to each étale $U \to X$ the homotopy category of $F(U)$. Compact generation of $F(U)$ is a statement about its homotopy category [Lur12, Rem. 1.4.4.3]. Moreover, since the conditions (1)-(6) of Theorem 6.9 can all be verified étale-locally (similarly to Töen's locally presentable dg-categories [Toë12]), it follows that Theorem 6.9 applies.

Appendix A. On Mayer–Vietoris squares

In this appendix, we consider some generalizations of a result of Moret-Bailly [MB96] using the Mayer–Vietoris squares developed in §5. For algebraic spaces and in the context of stable ∞-categories, this was recently accomplished (independently) by Bhatt [Bha14, §5].

The results of this appendix are not used in the body of the text, although Lemma A.1 can be used to give another proof of Example 5.5. The results of this appendix only depend upon §§1-5 of the article.

The main technical result of this appendix is the following description of Mayer–Vietoris D_{qc}-squares in the category of representable morphisms $\mathbf{Rep}/X$.

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[1]Here we have tacitly used [HNR14] to identify $\mathsf{D}_{\mathrm{qc}}(\mathcal{X} \times_X T) = \mathsf{D}(\mathsf{QCoh}(\mathcal{X} \times_X T))$. In general, one could *define* $\mathsf{D}^{\alpha}_{\mathrm{qc}}(X)$ as the degree one part of $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$.

Lemma A.1. *Fix a 2-cartesian square of algebraic stacks*

$$\begin{array}{ccc} U' & \xrightarrow{i'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{i} & X, \end{array}$$

where i is a quasi-compact open immersion and f is quasi-compact, quasi-separated, and representable. Let $j\colon Z \to X$ be a finitely presented closed immersion with support $X \setminus U$. Let $Z' = X' \times_X Z$ and let $f_Z\colon Z' \to Z$ be the induced morphism. Consider the following conditions:

(1) *The morphism f_Z is an isomorphism and the natural map $\mathsf{L}f^*_{\mathrm{qc}}\mathcal{O}_Z \to f^*\mathcal{O}_Z$ is a quasi-isomorphism;*
(2a) $\mathsf{R}(f_{\mathrm{qc}})_*$ *and* $\mathsf{L}f^*_{\mathrm{qc}}$ *induce triangulated equivalences* $\mathsf{D}_{\mathrm{qc},X\setminus U}(X) \simeq \mathsf{D}_{\mathrm{qc},X'\setminus U'}(X')$*;*
(2b) $\mathsf{R}(f_{\mathrm{qc}})_*$ *and* $\mathsf{L}f^*_{\mathrm{qc}}$ *induce t-exact equivalences* $\mathsf{D}_{\mathrm{qc},X\setminus U}(X) \simeq \mathsf{D}_{\mathrm{qc},X'\setminus U'}(X')$*; and*
(2c) $\mathsf{R}(f_{\mathrm{qc}})_*$ *and* $\mathsf{L}f^*_{\mathrm{qc}}$ *induce t-exact equivalences* $\mathsf{D}^b_{\mathrm{qc},X\setminus U}(X) \simeq \mathsf{D}^b_{\mathrm{qc},X'\setminus U'}(X')$.

Then (1) $\Longrightarrow$ (2a) $\Longleftrightarrow$ (2b) $\Longleftrightarrow$ (2c) and under any condition the square is a Mayer–Vietoris D_{qc}-square. If f_Z is affine or Z has quasi-affine diagonal, then all conditions are equivalent.

Note that if f is flat, then condition (1) simply says that f_Z is an isomorphism. Even if f is not flat, we always have that $\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(i_{\mathrm{qc}})_* \simeq \mathsf{R}(i'_{\mathrm{qc}})_*\mathsf{L}(f_U)^*_{\mathrm{qc}}$ since i is flat (Corollary 4.20). Also, taking f as in [HR14, Rem. 2.6] and $U = \emptyset$ provides an example where (2c) is satisfied, but (1) is not satisfied (f is not affine and its target does not have affine stabilizers).

Proof of Lemma A.1. Trivially, (2b) implies (2a) and (2c). If (2b) is satisfied, then $\mathsf{L}f^*_{\mathrm{qc}}\mathcal{O}_Z = f^*\mathcal{O}_Z$ and the adjunction maps $\mathcal{O}_Z \to f_*f^*\mathcal{O}_Z$ and $f^*f_*\mathcal{O}_{Z'} \to \mathcal{O}_{Z'}$ are isomorphisms. If f_Z is affine, then (1) holds. Otherwise, if Z has quasi-affine diagonal, then we start by noting that the t-exactness of $\mathsf{R}(f_{\mathrm{qc}})_*$ also shows that $\mathsf{R}((f_Z)_{\mathrm{qc}})_*$ is t-exact. By Lemma 2.2(6), it follows that if $\tilde{Z} \to Z$ is a smooth morphism, where $\tilde{Z}$ is an affine scheme, then the pullback $\tilde{f}_Z$ of f_Z to $\tilde{Z}$ is such that $\mathsf{R}((\tilde{f}_Z)_{\mathrm{qc}})_*$ is t-exact. By Serre's Criterion [Ryd09, Thm. 8.7], $\tilde{f}_Z$ is affine. By smooth descent, f_Z is affine, and we again see that (1) holds.

Let us show that (2a) $\Longrightarrow$ (2b). Since f is concentrated, this assertion is smooth local on X; thus, we may assume that X is an affine scheme. By Theorem 2.6(1), there exists an integer n such that

$$\tau^{\geq j}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N} \to \tau^{\geq j}\mathsf{R}f^*_{\mathrm{qc}}(\tau^{\geq j-n}\mathcal{N})$$

for every integer j and $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X')$. In particular, if $\mathcal{N} \in \mathsf{D}^{\leq 0}_{\mathrm{qc},X'\setminus U'}(X')$, then $\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N}$ is bounded above. Let m denote the largest integer such that $\mathcal{H}^m(\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N}) \neq 0$. Then $\mathcal{H}^m(\mathcal{N}) \cong \mathcal{H}^m(\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N}) \neq 0$ so $m \leq 0$. Thus, $\mathsf{R}(f_{\mathrm{qc}})_*$ is right t-exact and hence t-exact on $\mathsf{D}_{\mathrm{qc},X'\setminus U'}(X')$. Similarly, if $\mathcal{M} \in \mathsf{D}^{\geq 0}_{\mathrm{qc},X\setminus U}(X)$, then we have isomorphisms $\mathcal{M} \simeq \mathsf{R}(f_{\mathrm{qc}})_*\tau^{\geq -m}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{M}$ for every $m \geq n$. It follows that $\mathsf{L}f^*_{\mathrm{qc}}\mathcal{M}$ is supported in degrees ≥ 0 so $\mathsf{L}f^*_{\mathrm{qc}}$ is t-exact on $\mathsf{D}_{\mathrm{qc},X\setminus U}(X)$.

We will finish the proof by showing that (1) $\Longrightarrow$ (2c) $\Longrightarrow$ (2b). By assumption, the morphisms j and f are tor-independent and f_Z is an isomorphism. Let $j'\colon Z \to X'$ be the induced morphism such that $f \circ j' = j$. If $\mathcal{M} \in \mathsf{QCoh}(Z)$, then $f^*j_*\mathcal{M} =$

$j'_*\mathcal{M}$ since j is affine. By tor-independent base change (Corollary 4.20), there is an isomorphism of functors $\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(j_{\mathrm{qc}})_* \simeq \mathsf{R}(j'_{\mathrm{qc}})_*$. Thus $f_*j'_* = \mathsf{R}(f_{\mathrm{qc}})_*\mathsf{R}(j'_{\mathrm{qc}})_*$ and $f^*j_* = \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(j_{\mathrm{qc}})_*$.

For every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc},X\setminus U}(X)$ and $\mathcal{N} \in \mathsf{D}_{\mathrm{qc},X'\setminus U'}(X')$ we have adjunction maps

$$\eta_{\mathcal{M}} \colon \mathcal{M} \to \mathsf{R}(f_{\mathrm{qc}})_*\mathsf{L}f^*_{\mathrm{qc}}\mathcal{M} \quad \text{and} \quad \epsilon_{\mathcal{N}} \colon \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N} \to \mathcal{N}.$$

We will show that these are quasi-isomorphism. Since f is concentrated, these statements are smooth-local on X. We may henceforth assume that $X = \operatorname{Spec} A$, $Z = V(J)$ for a finitely generated ideal J of A, and X' is a quasi-compact and quasi-separated algebraic space over X.

To show that (1) $\implies$ (2c) it is enough, by standard truncation arguments, to prove that

- $\eta_{\mathcal{M}[0]}$ and $\epsilon_{\mathcal{N}[0]}$ are quasi-isomorphisms; and
- $\mathsf{L}f^*\mathcal{M}[0] \to (f^*\mathcal{M})[0]$ and $(f_*\mathcal{N})[0] \to \mathsf{R}f_*\mathcal{N}[0]$ are quasi-isomorphisms,

where $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module such that $i^*\mathcal{M} \cong 0$ and $\mathcal{N}$ is a quasi-coherent $\mathcal{O}_{X'}$ such that $i'^*\mathcal{N} \cong 0$.

Since every quasi-coherent sheaf on X or X' is a directed limit of its quasi-coherent subsheaves of finite type (for X this is trivial and for X' one applies [Ryd09, Cor. 4.12]), it follows that it is enough to prove these claims under the additional assumption that $\mathcal{M}$ and $\mathcal{N}$ are of finite type.

If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type such that $i^*\mathcal{M} \cong 0$, then there exists an integer n such that $J^n\mathcal{M} = 0$. Filtering $\mathcal{M}$ by the submodules $J^k\mathcal{M}$, we see that it is sufficient prove the claims under the further assumption that $\mathcal{M} = j_*\mathcal{M}_0$, where $\mathcal{M}_0$ is an $\mathcal{O}_Z$-module of finite type. We have already seen that $\mathsf{L}f^*_{\mathrm{qc}}j_* = f^*j_* = j'_*$ so $\mathsf{R}(f_{\mathrm{qc}})_*\mathsf{L}f^*_{\mathrm{qc}}j_* = f_*j'_* = j_*$. Thus $\eta_{\mathcal{M}[0]}$ is a quasi-isomorphism. Arguing as before, it only remains to prove that $\epsilon_{j'_*\mathcal{N}_0[0]}$ is a quasi-isomorphism for every $\mathcal{N}_0 \in \mathsf{QCoh}(Z)$ of finite type. Since f_Z is an isomorphism, this follows from the equality $\mathsf{L}f^*_{\mathrm{qc}}j_* = f^*j_* = j'_*$.

To see that (2c) $\implies$ (2b) it is enough to prove that $\mathsf{L}f^*_{\mathrm{qc}}$ is left t-exact on $\mathsf{D}_{\mathrm{qc},X\setminus U}(X)$ and that $\mathsf{R}(f_{\mathrm{qc}})_*$ is right t-exact on $\mathsf{D}_{\mathrm{qc},X'\setminus U'}(X')$. For the first claim, let $\mathcal{M}$ be a complex in $\mathsf{D}^{\geq 0}_{\mathrm{qc},X\setminus U}(X)$. We may write $\mathcal{M}$ as a homotopy colimit of its truncations $\tau^{\leq n}\mathcal{M}$. Since $\mathsf{L}f^*_{\mathrm{qc}}$ commutes with coproducts and is t-exact on $\mathsf{D}^b_{\mathrm{qc},X\setminus U}(X)$, it follows that $\mathsf{L}f^*_{\mathrm{qc}}\mathcal{M} \in \mathsf{D}^{\geq 0}_{\mathrm{qc},X'\setminus U'}(X')$ so $\mathsf{L}f^*_{\mathrm{qc}}$ is t-exact. Also, if $\mathcal{N}$ is a complex in $\mathsf{D}^{\leq 0}_{\mathrm{qc},X'\setminus U'}(X')$, then since f is concentrated and X is affine, there exists an integer n such that $\tau^{>0}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N} \to \tau^{>0}\mathsf{R}f^*_{\mathrm{qc}}(\tau^{\geq -n}\mathcal{N}) \simeq 0$. Hence, $\mathsf{R}(f_{\mathrm{qc}})_*$ is t-exact. □

Remark A.2. It can be shown that Lemma A.1 is also valid for non-representable concentrated morphisms, at least if f is of finite presentation or has quasi-affine diagonal and f_Z is representable.

Definition A.3. Let $f \colon X \to Y$ be a morphism. We say that $F \in \mathsf{QCoh}(Y)$ is f-flat if the natural map $\mathsf{L}f^*F \to f^*F$ is an isomorphism. We let $\mathsf{QCoh}_{f-\mathrm{fl}}(Y)$ denote the full subcategory of f-flat sheaves.

Theorem A.4. *Let*

$$\begin{array}{ccc} U' & \xrightarrow{i'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{i} & X, \end{array}$$

be a Mayer–Vietoris D_{qc}*-square of algebraic stacks, that is,*

(1) *the square is cartesian,*
(2) i *is a quasi-compact open immersion,*
(3) f *is a representable quasi-compact and quasi-separated morphism, and*
(4) $\mathsf{L}f^*_{\mathrm{qc}}$ *induces an equivalence of triangulated categories* $\mathsf{D}_{\mathrm{qc},X\setminus U}(X) \to \mathsf{D}_{\mathrm{qc},X'\setminus U'}(X')$.

Then there is an equivalence of categories

$$\Phi\colon \mathsf{QCoh}_{f-\mathrm{fl}}(X) \to \mathsf{QCoh}(X') \times_{\mathsf{QCoh}(U')} \mathsf{QCoh}_{f_U-\mathrm{fl}}(U),$$

which takes a sheaf N *to the triple* (f^*N, i^*N, δ)*, where* δ *is the canonical isomorphism* $i'^*f^*N \cong f_U^* i^*N$.

Proof. Let $k = i \circ f_U$. The right-adjoint of Φ is the functor Ψ which takes a triple (N', N_U, δ) to the sheaf $f_*N' \times_{k_*N'_U} i_*N_U$ where $N'_U := i'^*N' \overset{\delta}{\cong} f_U^*N_U$. It is enough to show that the unit $N \to \Psi(\Phi(N))$ and the counit $\Phi(\Psi(N', N_U, \delta)) \to (N', N_U, \delta)$ of the adjunction are isomorphisms. That the unit is an isomorphism follows immediately from Lemma 5.6(1) since all involved complexes are sheaves.

Conversely, given a triple (N', N_U, δ), we obtain by Lemma 5.6(3) a distinguished triangle

$$N \longrightarrow i_*N_U \oplus f_*N' \longrightarrow k_*N'_U \longrightarrow N[1],$$

such that the induced maps $i^*N \to N_U$, $f^*N \to N'$ and $k^*N \to N'_U$ are isomorphisms. Since $\Phi(N', N_U, \delta) = \mathcal{H}^0(N)$, it is enough to show that N is concentrated in degree 0. We note that a priori N is concentrated in degrees 0 and 1 so we have a distinguished triangle:

$$\mathcal{H}^0(N)[0] \longrightarrow N \longrightarrow \mathcal{H}^1(N)[-1] \longrightarrow \mathcal{H}^0(N)[1].$$

If we apply the t-exact functor i^* to this triangle, then the third term vanishes so $\mathcal{H}^1(N) \in \mathsf{D}_{\mathrm{qc},X\setminus U}(X)$. If we instead apply the right t-exact functor $\mathsf{L}f^*$ to this triangle, we obtain the triangle:

$$\mathsf{L}f^*\mathcal{H}^0(N)[0] \longrightarrow N'[0] \longrightarrow \mathsf{L}f^*\mathcal{H}^1(N)[-1] \longrightarrow \mathsf{L}f^*\mathcal{H}^0(N)[1].$$

The first two terms are concentrated in degrees ≤ 0 and the third is concentrated in degree 1 since $\mathsf{D}_{\mathrm{qc},X\setminus U}(X) \to \mathsf{D}_{\mathrm{qc},X'\setminus U'}(X')$ is a t-exact equivalence. It follows that $\mathcal{H}^1(N) = 0$. □

Remark A.5. Given a Mayer–Vietoris square as above, Lemma 5.6 shows that the functor $\mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X') \times_{\mathsf{D}_{\mathrm{qc}}(U')} \mathsf{D}_{\mathrm{qc}}(U)$ is essentially surjective. It is, however, not fully faithful. The reason is a well-known fault of the derived category: whereas cones are unique up to isomorphism, morphisms between cones are not unique. One way to fix this problem is to work with ∞-categories. Then one obtains the expected equivalence, cf. [Bha14, Prop. 5.6].

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