Mini-course: Resolutions of singularities

2013-02-22

Lecture #1

I. Introduction
   - Challenges
   - History
   - Applications

II. Singularities
   - Examples of tangent cones
   - Invariants of singularities: Hilbert-Samuel fcn and multiplicity

Introduction

X singular (variety/k or ...)

- Weak resolution of singularities: $\pi : \tilde{X} \to X$ proper surjective w/ $\tilde{X}$ regular
- Resolution of sing: if in addition π birational.
- Strong res of sing: —— " —— $\pi|_{X \setminus X_{sing}} : \pi^{-1}(X \setminus X_{sing}) \to X \setminus X_{sing}$ isomorphism

  $+ \ \pi^{-1}(X_{sing})$ snc

- Functorial: If $\exists$ Res: $X \mapsto \left(R(X) = \tilde{X} \xrightarrow{\pi_X} X\right)$ that commutes w/ Smooth morphisms:

$$\begin{tikzcd} R(X') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d, "\text{smooth}"] \\ R(X) \arrow[r] & X \end{tikzcd}$$

- by blow-ups (in smooth centers): π is a sequence of blow-ups in smooth centers

$$\pi : \tilde{X} = X_n \to X_{n-1} \to X_{n-2} \to \dots \to X_1 \to X$$

- Local uniformization: "Resolve singularities locally on $\tilde{X}$".

  For every valuation ring $\underset{K(X)}{\overset{V}{\cap}}$ and $\operatorname{Spec}(V) \to X$, find $\pi : \tilde{X} \to X$ proper birational such that $\tilde{X}$ regular in a nbhd of the image of the unique lift $\operatorname{Spec} V \to \tilde{X}$.

- Embedded resolution: Given $X \overset{\text{closed}}{\hookrightarrow} Y$ w/ Y regular, $\exists$ resolution $\tilde{X} \xrightarrow{\pi_X} X$ sitting in:

$$\begin{tikzcd} \tilde{X} \arrow[r, hook] \arrow[d, "\pi_X"'] & \tilde{Y} \arrow[d, "\pi_Y"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

where $\tilde{Y}$ regular, π_Y proper birational.

Variant: Make $\pi_Y^{-1}(X)$ into a snc divisor.

Challenges

easy dim 2
possible in dim 3

- Patching: local algo ⇝ global algo. "surprisingly serious obstacle" (in dim ≥ 4)

  Optimal solution: show that choices don't matter (Włodarczyk '05)

- Writing down an algorithm is difficult – depends on history etc. (in dim ≥ 3)

  Does not exists an algorithm that works one smooth blow-up at a time.

  Ex 3.6.2: $X = \{ x^2 + y^2 + z^2 t^2 = 0 \} \subset \mathbb{A}^4$

  $$X_{sing} = \{ x = y = z = 0 \} \cup \{ x = y = t = 0 \}$$

  Any sensible (functorial) algorithm has to blow-up $x = y = z = t = 0$:

  $$\tilde{X}_{\tilde{t} \neq 0} = \{ x_1^2 + y_1^2 + z_1^2 t_1^2 = 0 \} \subset \mathbb{A}^4$$

Both these challenges are wide open in dim ≥ 4 resp. dim ≥ 3.

History

Newton ~1650, ...	resolution of curves
Jung^{1908}, Walker^{1935}, Hirzebruch Levi 1899, Chisini, Albanese 1924 Zariski 1939	resolution of surfaces (char 0)
Zariski 1940	local uniformization, char 0
Zariski 1944	res. of sing in dim ≤ 3 (using local unif.), char 0
Abhyankar 1956	local unif + res. of sings in dim 2, char $p > 0$.
→ Hironaka 1964	strong, emb, res. by blow-ups in char 0 (arb. dim.) 218 pages!
Abhyankar 1966	emb res. of surfaces, char $p > 0$. res. of sings in dim 3, char $> 3! = 6$. (not emb)
Lipman 1978	res of exc. surfaces (mixed char) (not emb)
Bennett'70, Giraud'74 70's	simplifications of Hironaka's proof (maximal
Villamayor 89-96	(simplifications)
→ Bierstone-Milman 1997	functorial strong emb res. of sings by blow-ups, char 0
Encinas-Hauser 2002	controlled transform (simplification)
Włodarczyk 2005	"all choices are equivalent" (simplification)
Kollár 2007	"no invariants"
de Jong 1996	alterations (weak res. of sing.) arb char (incl. mixed)
Bogomolov-Pantev 1996	(non-strong) res of sings in char 0 : simple proof.
Cossart Piltant 2009	res. of sings in dim 3, arb. char (not emb, uses loc unif.)
Temkin 2008	insep local uniformization, arb. dim.
Cossart-Jannsen-Saito 2009	emb res of surfaces, mixed char.

Applications

1) Existence of smooth compactifications: X/k $\overset{\text{regular}}{\text{variety}}$.

Nagata gives $X \subset \bar{X}$ complete variety but $\bar{X}$ singular at boundary.

Hironaka (~~emb.~~ strong res) gives $X \subset \tilde{X}$ complete regular.

2) Study of singularities via exc. fiber of a strong resolution.

3) Resolving indeterminancy locus: $V \subset H^0(X, \mathcal{L})$

$$\begin{tikzcd} & \overset{\text{regular}}{X} \arrow[r, dashed, "|V|"] & \mathbb{P}^N \\ \underset{\text{res.}}{\tilde{X}} \arrow[ur, "\text{seq. blow-up}"'] \arrow[urr, bend right] & & \end{tikzcd}$$

(strong emb res via blow-ups)

4) Multiplier ideals: X regular, D bad $\mathbb{Q}$-divisor. $\mathcal{J}(X, D) = \pi_*(K_{\tilde{X}/X} \otimes \mathcal{I}(\lfloor \pi^* D \rfloor))$

for any $\pi : \tilde{X} \to X$, $\pi^{-1}(D)$ snc. Kawamata-Viehweg-Nadel vanishing.

5) Mixed Hodge structures: X sing variety. Simplicial resolution $X_\bullet \subset \underset{\substack{\text{regular}\\\text{complete}}}{\bar{X}_\bullet} \supset \underset{\text{snc}}{D.}$

Singularities

Def: A scheme X is regular in $x \in X$ if $\mathcal{O}_{X,x}$ is regular.

A local ring (A, m) is regular if one of the following equiv cond's holds:

(i) $m = (f_1, f_2, \dots, f_n)$ where $f_1, f_2, \dots, f_n$ is a reg seq.

(ii) $\dim_k m/m^2 = \dim A$

(iii) $\bigoplus_{k \geq 0} m^k/m^{k+1} =: \mathrm{Gr}_m A$ is a polynomial ring

Rmk: Always
- $\dim_k m/m^2 \geq \dim A$
- $\mathrm{Sym}_k\, m/m^2 = \bigoplus_{d \geq 0} S^d(m/m^2) \overset{\varphi}{\twoheadrightarrow} \bigoplus_{d \geq 0} m^d/m^{d+1}$
- $\dim \mathrm{Gr}_m A = \dim A$.
- $(m/m^2)^\vee$ is the (Zariski) tangent space. (a vector space $/k$)

  Corresponding tangent space scheme $\mathrm{Spec}(\mathrm{Sym}\, m/m^2)$
- Surjection φ corresponds to

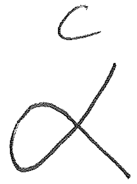
$$\mathrm{Spec}\left(\bigoplus m^d/m^{d+1}\right) \overset{j}{\hookrightarrow} \mathrm{Spec}\left(\mathrm{Sym}_k\, m/m^2\right) \cong \mathbb{A}^r_k$$

<u>tangent cone</u>, a scheme of $\dim = \dim A$ <u>tangent space</u>, of dimension $r = \dim_k m/m^2$

- (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ j is an isomorphism.

Examples of tangent cones

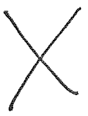
Ex 1: $k[x,y]/y^2-x^2-x^3$



$C^{sing} = \{(0,0)\}$

tgt cone at origin

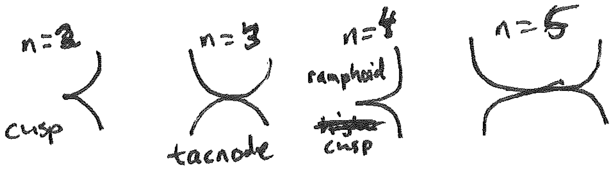
$k[x,y]/y^2-x^2$



mult 2

Ex 2: $k[x,y]/y^2-x^{n+1}$

Fix $n \geq 2$



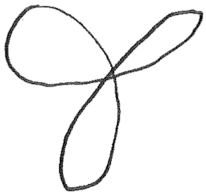
$C^{sing} = \{(0,0)\}$

tgt cone at origin

$k[x,y]/y^2$

mult 2

Ex 3: $k[x,y]/y^3-3x^2y+x^4+y^4$



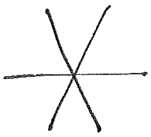
ordinary triple point

mult 3

tgt cone at origin

$k[x,y]/y^3-3x^2y$

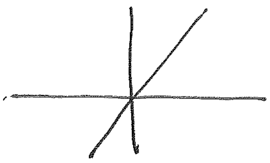
$y(y-\sqrt{3}x)(y+\sqrt{3}x)$



Ex 4: (not planar, not l.c.i.) not Gorenstein

$k[x,y,z]/xy,yz,zx$

= tgt cone at origin



"normal crossing" sing.

mult 3

Ex 5: $k[x,y,z]/x^2-f(y,z)$ isolated mult 2 sing, tgt cone: $k[x]/x^2$

$f(y,z) \in (y,z)^3$ curve plane sing of mult ≥ 3 "arbitrarily complicated"

Invariants of singularities $\quad \mathcal{O}_{X,x} = (A, m) \quad$ [K §2.8]

The easiest invariants come from the tangent cone $\quad Gr_m A = \bigoplus_{d \geq 0} m^d / m^{d+1}$

Hilbert function $\quad H(Gr_m A, d) = \dim_k m^d / m^{d+1}$

Hilbert-Samuel fcn $\quad HS(A, d) = \dim_k A / m^{d+1} = \sum_{s=0}^{d} H(Gr_m A, s)$

Standard fact [AM 11.2, 11.14] $\exists$ polynomials $HP(t), HSP(t) \in \mathbb{Q}[t]$ s.th.

$$H(d) = HP(t) \quad \forall t >> 0 \qquad \deg HP = \dim A - 1$$
$$HS(d) = HSP(d) \quad \forall d >> 0 \qquad \deg HSP = \dim A = \dim Gr_m A$$

Def: The multiplicity of A is $m = (\dim A)! \cdot (\text{coeff. of } t^{\dim A} \text{ in } HSP(t))$

So $\quad HSP(t) = \frac{m}{n!} t^n + \cdots, \quad HP(t) = \frac{m}{(n-1)!} t^{n-1} + \cdots \quad$ where $n = \dim A$

Rmk: A regular $\Leftrightarrow Gr_m A = k[x_1, \dots, x_n] \Leftrightarrow HS(d) = \binom{d+n}{n}$

$\Leftrightarrow H(1) = n$

Fact: A regular $\Leftrightarrow$ multiplicity $= 1$.

Ex: If A regular local ring and $f \in m^d \setminus m^{d+1}$ with leading term $f_d := \bar{f} \in m^d / m^{d+1}$ ("$f = f_d +$ higher order terms") then

$$Gr_m (A/f) = Gr_m A / f_d Gr_m A$$

$$HSP(A/f, t) = \binom{t+n}{n} - \binom{t-d+n}{n} = \frac{d}{(n-1)!} t^{n-1} + \cdots$$

So multiplicity $= d =$ order of vanishing of f.

Note that the multiplicity is the only invariant of the tangent cone for a hypersurface singularity.

More on Hilbert functions

$$HS(A) = H(A[x]_{(x,m)})$$

$H^n(A) := H(A[x_1,\dots,x_n]_{(x_1,\dots,x_n,m)})$ so that $HS = H^1$

The natural invariant of $x \in X$ is not $HS(\mathcal{O}_{X,x})$ but rather

$H^{d(x)+1}(\mathcal{O}_{X,x})$ where $d(x) = \dim \overline{\{x\}}$ (for X biequidim excellent)

The function $x \xmapsto{B} H^{d(x)+1}(\mathcal{O}_{X,x})$ is upper semi-continuous in the total order.
(Bennett)

Def: $Z \overset{\text{closed}}{\hookrightarrow} X$ is permissible if Z is regular and $Gr_I(\mathcal{O}_X)$ is a ~~locally~~ flat ~~free~~ $\mathcal{O}_Z$-module. Here I denotes the ideal sheaf defining Z.

($\Leftrightarrow$ X is normally flat along Z)

Rmk: $Z \hookrightarrow X$ permissible $\Rightarrow$ The exc div E of $Bl_Z X \longrightarrow X$ is flat over Z.

Thm (Bennett) $Z \hookrightarrow X$, Z regular. Then Z is permissible $\Leftrightarrow$ $B(x)$ constant along Z.