Lecture #1

I. Introduction
   Challenges
   History
   Applications

II. Singularities
   Examples of tangent cones
   Invariants of singularities: Hilbert–Samuel functionality and multiplicity
Introduction

X singular (variety/k or ...)

- Weak resolution of singularities: \( \Pi: \tilde{X} \to X \) proper surjective w/ \( \tilde{X} \) regular
- Resolution of sing: if in addition \( \Pi \) birational.
- Strong res of sing: \( \xrightarrow{\eta} \quad \Pi|_{X\setminus X_{\text{sing}}} : \Pi^{-1}(X\setminus X_{\text{sing}}) \to X\setminus X_{\text{sing}} \) isomorphism
  \( + \quad \Pi^{-1}(X_{\text{sing}}) \) snc
- Functorial: If \( \exists \text{ Res } : X \xrightarrow{R(x)} (\eta \xrightarrow{\pi_x} X) \) that commutes w/ smooth morphisms:
  \( \xrightarrow{\eta} \quad R(x') \xrightarrow{\eta} x' \\
  \quad \downarrow \quad \bigcirc \quad \downarrow \text{ smooth} \\
  R(x) \xrightarrow{\eta} X \\

- by blow-ups (in smooth centers): \( \Pi \) is a sequence of blow-ups in smooth centers
  \( \xrightarrow{\Pi} \quad \tilde{X} = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_1} X_0 \xrightarrow{\pi_0} X \)

- Local uniformization: "Resolve singularities locally on \( \tilde{X} \)"
  For every valuation ring \( V \) and \( \text{Spec}(V) \to X \), find \( \tilde{V} : \tilde{X} \to X \text{ proper birational} \)
  such that \( \tilde{X} \) regular in a nbhd of the image of the unique lift \( \text{Spec}(V) \to \tilde{X} \).

- Embedded resolution: Given \( X \hookrightarrow Y \) w/ \( Y \) regular, \( \exists \) resolution \( \tilde{X} \xrightarrow{\pi_X} X \) sitting in:
  \( \xrightarrow{\Pi_X} \quad \tilde{Y} \xrightarrow{\Pi_Y} Y \)
  where \( \tilde{Y} \) regular, \( \Pi_Y \) proper birational.

  Variant: Make \( \Pi_Y^{-1}(X) \) into a snc divisor.
Challenges

- **Patching**: local algo $\rightarrow$ global algo. "surprisingly serious obstacle" (in dim $\geq 4$)
  
  Optimal solution: show that choices don't matter (Kaloderosz 05)

- Writing down an algorithm is difficult - depends on history etc. (in dim $\geq 3$)
  
  Does not exist an algorithm that works on smooth blow-up at a line.

Ex 3.6.2: $X = \{ x^2 + y^2 + z^2 t^2 = 0 \} \subset \mathbb{A}^4$

$X_{sing} = \{ x = y = z = 0 \} \cup \{ x = y = t = 0 \}$

Any sensible (factorial) algorithm has to blow-up $x = y = z = t = 0$:

$X_{\mathbb{C}^4}$

Both these challenges are wide open in dim $\geq 4$ resp. dim $\geq 3$. 

- easy dim 2
  
  possible in dim 3
History

Newton n/1650, ...

resolution of curves

1899, 1935: Jung, Walker, Hirzebruch

resolution of surfaces (char 0)

Levi 1899, Chisini, Albanese 1924

Zariski 1939

Zariski 1940

local uniformization, char 0

res. of sing in dim ≤ 3 (using local unif.), char 0

Abhyankar 1956

local uni. + res. of syng in dim 2, char p > 0.

→ Hironaka 1964

strong, emb, res. by blow-ups in char 0 (arb. dim.)

218 pages!

emb. res. of surfaces, char p > 0.

res. of syng in dim 3, char > 31 = c. (not emb.)

Lipman 1978

res. of exc. surfaces (indeed char) (not emb.)

'30 '40 '44

Bennett, Giraud 30's

Villamayor 89-96

→ Bierstone-Milman 1997

simplifications of Hironaka's proof (maximal (simplifications)

functionally strong emb

controlled transform (simplifications)

"all choices are equivalent" (simplifications)

Kollár 2007

"no invariant"

Enchas - Hauser 2002

Wlodarczyk 2005

Kollár 2007

de Jong 1996

algorithms (weak res. of syng.) arb. char (incl. mixed)

Bogomolov-pankov 1996

(res. of syng in char 0: simple proof.

(strong) res. of syng in char 0: simple proof.

Cossart Pillant 2009

res. of syng in dim 3, arb. char (not emb, wos loc. uni.)

Temkin 2008

"isop. local uniformization", arb. dim.

Cossart-Jannsen-Saito 2009

emb. res. of surfaces, mixed char.
Applications

1) Existence of smooth compactifications: $X/\mathfrak{m}$ regular
   Nagata gives $X \subset \tilde{X}$ complete variety but $\tilde{X}$ singular at boundary,
   Hironaka (good res) gives $X \subset \tilde{X}$ complete regular.

2) Study of singularities via exc. fiber of a strong resolution.

3) Resolving indeterminacy locus: $X \longrightarrow \mathbb{P}^N$
   $V \subset H^0(X, L)$
   (strong and res) via blow-ups
   $X$ res.

4) Multiplier ideals: $X$ regular, $D$ bad divisor.
   $J(X, D) = \prod (K_X^i D) \otimes L(D)$
   for any $\Pi: \tilde{X} \longrightarrow X$, $\Pi^*(D)$ snc.
   Kawamata-Viehweg-Nadel vanishing.

5) Mixed Hodge structures: $X$ sing variety. Simplicial resolution $X_0 \subset \tilde{X}$ regular snc
**Singularities**

**Def:** A scheme $X$ is regular in $x \in X$ if $O_{X,x}$ is regular.

A local ring $(A, \mathfrak{m})$ is regular if one of the following equiv cond holds:

1. $\mathfrak{m} = (f_1, f_2, \ldots, f_n)$ where $f_1, f_2, \ldots, f_n$ is a reg seq.
2. $\dim \frac{\mathfrak{m}}{\mathfrak{m}^2} = \dim A$
3. $\mathfrak{m}/\mathfrak{m}^2 =: \text{Gr}_\mathfrak{m} A$ is a polynomial ring

**Rank:** Always $\dim \frac{\mathfrak{m}}{\mathfrak{m}^2} \geq \dim A$

- $\text{Sym}_\mathfrak{m} \frac{\mathfrak{m}}{\mathfrak{m}^2} = \bigoplus_{d \geq 0} S^d \left( \frac{\mathfrak{m}}{\mathfrak{m}^2} \right)$
- $\dim \text{Gr}_\mathfrak{m} A = \dim A$
- $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^\ast$ is the (Zariski) tangent space, (a vector space $V_A$)

   Correspondingly, tangent space scheme $\text{Spec} \left( \text{Sym} \frac{\mathfrak{m}}{\mathfrak{m}^2} \right)$

- Surjection $\mathbb{A}^r$ corresponds to

$$\text{Spec} \left( \bigoplus \mathfrak{m}^{d+1} \right) \setminus \text{Spec} \left( \text{Sym} \frac{\mathfrak{m}}{\mathfrak{m}^2} \right) \hookrightarrow \mathbb{A}^r$$

- Tangent cone, a scheme of $\dim = \dim A$
- Tangent space, of dimension $r = \dim \frac{\mathfrak{m}}{\mathfrak{m}^2}$

(iv) $\Leftrightarrow$ (iii) $\Leftrightarrow j$ is an isomorphism.
Examples of tangent cones

Ex 1: \[ h[x,y,z]/y^2-x-z \]

tangent cone at origin
\[ h[x,y,z]/y^2-x^2 \]

Ex 2: \[ h[x,y,z]/y^2-x^{n+1} \]

Fix \( n \geq 2 \)
tangent cone at origin
\[ h[x,y,z]/y^2 \]

Ex 3: \[ h[x,y,z]/y^3-3x^2y+x+yz \]

tangent cone at origin
\[ h[x,y,z]/y^2-3x^2y \]
\[ y(y-\sqrt[3]{a})(y+\sqrt[3]{a}) \]

Ex 4: (not planar, not l.c.i., not Gorenstein)
\[ h[x,y,z]/xy, yz, zx \]
= tangent cone at origin

Ex 5: \[ h[x,y,z]/x^2 - f(y,z) \]
isolated
mult 2 sing, tangent cone: \[ h[x]/x^2 \]
\[ f(y,z) \in (y,z)^3 \]
curve sing of mult \( \geq 3 \)
"arbitrarily complicated"
Invariants of singularities \( \mathcal{O}_{X,x} = (A, m) \) \[ K \S 2.8 \]

The easiest invariants come from the tangent cone \( \operatorname{Gr}_m A = \bigoplus_{d=0} m^d / m^{d+1} \)

**Hilbert function** \( H(\operatorname{Gr}_m A, s) = \dim_k m^s / m^{s+1} \)

**Hilbert-Samuel func** \( \operatorname{HS}(A, d) = \dim_k A / m^{d+1} = \sum_{s=0}^d H(\operatorname{Gr}_m A, s) \)

**Standard Fact** \[ AM \ II.12 \] \exists polynomials \( \operatorname{HP}(s), \operatorname{HSP}(s) \in k[s] \) s.t.

\[
\begin{align*}
H(d) &= \operatorname{HP}(d) \quad \forall d \gg 0 \quad \deg \operatorname{HP} = \dim A - 1 \\
\operatorname{HS}(d) &= \operatorname{HSP}(d) \quad \forall d \gg 0 \quad \deg \operatorname{HSP} = \dim A = \dim \operatorname{Gr}_m A
\end{align*}
\]

**Def:** The **multiplicity** of \( A \) is \( m = (\dim A)! \cdot \left( \text{coeff. of } t^{\dim A} \text{ in } \operatorname{HSP}(t) \right) \)

So \( \operatorname{HSP}(t) = \frac{m}{n!} t^n + \ldots, \quad \operatorname{HP}(t) = \frac{m}{(n-1)!} t^{n-1} + \ldots \) where \( n = \dim A \)

**Rmk:** A regular \( \iff \operatorname{Gr}_m A = k[x_1, \ldots, x_n] \iff \operatorname{HS}(d) = \binom{d+n}{n} \)

\( \iff H(1) = n \)

**Fact:** A regular \( \iff \) multiplicity = 1.

**Ex:** If \( A \) is a regular local ring and \( f \in m^d \setminus m^{d+1} \) with leading term \( f_d = \overline{f} \in m^d / m^{d+1} \) \( \left( \text{"} f = f_d + \text{higher order terms} \right) \) then

\[
\begin{align*}
\operatorname{Gr}_m (A/f) &= \operatorname{Gr}_m A / f_d \operatorname{Gr}_m A \\
\operatorname{HSP}(A/f, t) &= \frac{t^n}{n!} - \frac{t^{d+n}}{n} = \frac{d}{(n-1)!} t^{n-1} + \ldots
\end{align*}
\]

So **multiplicities** = \( d = \) order of vanishing of \( f \).

Note that the multiplicity is the **only** invariant of the tangent cone for a hypersurface singularity.
More on Hilbert functions

\[ HS(A) = H(A[x]_{(x,m)}) \]

\[ H^n(A) = H(A[x_1, \ldots, x_n]_{(x_1, \ldots, x_n, m)}) \quad \text{so that} \quad HS = H' \]

The natural invariant of \( x \in X \) is not \( HS(\mathcal{O}_x, x) \) but rather
\[ H^d(\mathcal{O}_x, x) \quad \text{where} \quad d = \dim S x^3 \quad \text{(for X biequidim)} \]

excellent

The function \( x \mapsto H^d(\mathcal{O}_x, x) \) is upper semi-continuous in the total order.

(Bennett)

Def: \( Z \rightarrow X \) is permissible if \( Z \) is regular and \( \text{Gr}_I(\mathcal{O}_x) \) is a locally flat free \( \mathcal{O}_Z \)-module. Here \( I \) denotes the ideal sheaf defining \( Z \).

(\( \Leftrightarrow \) \( X \) is normally flat along \( Z \))

Rmk: \( Z \rightarrow X \) permissible \( \Rightarrow \) The exceptional \( E \) of \( \mathcal{B}_Z \rightarrow X \)

is flat over \( Z \).

Thm (Bennett) \( Z \rightarrow X \), \( Z \) regular. Then \( Z \) is permissible \( \Leftrightarrow \) \( B(\mathcal{x}) \) constant along \( Z \).