

MAYER–VIETORIS SQUARES IN ALGEBRAIC GEOMETRY

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ABSTRACT. We consider various notions of Mayer–Vietoris squares in algebraic geometry. We use these to generalize a number of gluing and pushout results of Moret-Bailly, Ferrand–Raynaud, Joyet and Bhatt. An important intermediate step is Gabber’s rigidity theorem for henselian pairs, which our methods give a new proof of.

1. INTRODUCTION

Let X be a topological space and let $U, V \subseteq X$ be open subsets such that $X = U \cup V$. Frequently, natural invariants of X can be determined by the restriction of these invariants to U, V and $U \cap V$. The prototypical examples are the Mayer–Vietoris exact sequences in algebraic topology. These results have proved to be very useful for inductive arguments.

Now let X be a variety, scheme, algebraic space, or algebraic stack. It is straightforward to adapt the topological results (e.g., Mayer–Vietoris with open coverings) to this situation. In algebraic geometry, however, open coverings are often too restrictive to use in inductive arguments. A consideration of the existing literature, motivated us to make the following definition.

Definition 1.1. Consider a cartesian diagram of algebraic stacks

$$(1.1) \quad \begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ f_U \downarrow & \square & \downarrow f \\ U & \xrightarrow{j} & X, \end{array}$$

where j is an open immersion. If $W \rightarrow X$ is a morphism of algebraic stacks, then we let $f_W: W' = X' \times_X W \rightarrow W$ be the induced morphism. It will be convenient to let $i: Z \hookrightarrow X$ denote a closed immersion with complement U .

A *weak Mayer–Vietoris square* is a cartesian diagram of algebraic stacks as in (1.1) such that for every morphism of algebraic stacks $W \rightarrow X$ with image disjoint from U , the induced morphism $f_W: W' \rightarrow W$ is an isomorphism.

The condition of being a weak Mayer–Vietoris square is trying to capture that X' contains all neighborhoods of the complement of U in X . In particular, if X and X' are locally noetherian, then being a weak Mayer–Vietoris square is equivalent to f_Z being an isomorphism and f being flat at all points over Z (Lemma 3.3).

Fix a weak Mayer–Vietoris square as in (1.1). If f is étale, then it is also known as an *étale neighborhood*, or upper distinguished square, or Nisnevich square. These were recently treated in depth in [Ryd11]. Some highlights of the theory are that

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étale neighborhoods are pushouts in the 2-category of algebraic stacks and that quasi-coherent sheaves (and many more things) can be glued along étale neighborhoods. Étale neighborhoods feature prominently in the interactions between algebraic geometry and topology.

There is another class of weak Mayer–Vietoris squares, which generalize étale neighborhoods, that have been considered and applied to great effect in the past [FR70, Joy96]. These are our main object of interest.

Definition 1.2. A *flat Mayer–Vietoris square* is a weak Mayer–Vietoris square as in (1.1) such that f is flat.

Our first main result is the following.

Theorem A. *Fix a flat Mayer–Vietoris square as in (1.1). If X is locally excellent, then the square (1.1) is a pushout in the 2-category of algebraic stacks.*

An algebraic stack is locally excellent if it admits a smooth cover by an excellent scheme; in particular, algebraic stacks that are locally of finite type over the spectrum of a field, \mathbb{Z} , or a complete local noetherian ring are locally excellent. Theorem A is the key technical result used to establish Tannaka duality for algebraic stacks with non-separated diagonals [HR14a]—if the diagonals are separated, then [MB96, Cor. 6.5.1(g)] is sufficient for the Tannakian application.

In general, Mayer–Vietoris squares are interesting since objects can be glued along them. To formalize this, consider a 2-presheaf $F: (\mathbf{Stacks}/X)^\circ \rightarrow \mathbf{Cat}$ [Ryd11, App. D], e.g., the 2-presheaf $F(-) = \mathbf{QCoh}(-)$ of quasi-coherent sheaves of modules. By pull-back, we obtain a functor

$$\Phi_F: F(X) \rightarrow F(X') \times_{F(U')} F(U)$$

where the right-hand side denotes triples (W', θ, W_U) where $W' \in F(X')$, $W_U \in F(U)$ and $\theta: j'^*W' \rightarrow f_U^*W_U$ is an isomorphism. When the functor Φ_F is an equivalence, we say that we can glue F along the square.

We do not have any general gluing results for 2-*sheaves* as for étale neighborhoods [Ryd11, Thm. A]—nor do we expect such—but we will give gluing results for the following 2-presheaves, where the values over an algebraic stack Y are as follows:

- $\mathbf{QCoh}(Y)$ the category of quasi-coherent sheaves of \mathcal{O}_Y -modules.
- $\mathbf{Aff}(Y)$ the category of affine morphisms to Y .
- $\mathbf{Qaff}(Y)$ the category of quasi-affine morphisms to Y .
- $\mathbf{AlgSp}(Y)$ the category of representable morphisms $Y' \rightarrow Y$.
- $\mathbf{AlgSp}_{\text{lfp}}(Y)$ the category of representable morphisms $Y' \rightarrow Y$, locally of finite presentation.
- $\mathbf{Hom}(Y, W)$ the category of morphisms to a fixed algebraic stack W .
- $\mathbf{Et}(Y)$ the category of étale representable morphisms $Y' \rightarrow Y$, or equivalently, the category of cartesian sheaves of sets on the lisse-étale site of Y .
- $\mathbf{Et}_c(Y)$ the category of finitely presented étale representable morphisms $Y' \rightarrow Y$, or equivalently, the category of constructible sheaves of sets.

We prove the following gluing results.

Theorem B. *Fix a flat Mayer–Vietoris square as in (1.1). If j is quasi-compact, then*

- (1) $\Phi_{\mathbf{QCoh}}$, $\Phi_{\mathbf{Aff}}$ and $\Phi_{\mathbf{Qaff}}$ are equivalences of categories;
- (2) $\Phi_{\mathbf{AlgSp}}$ is fully faithful;
- (3) $\Phi_{\mathbf{Hom}(-, W)}$ is fully faithful for every algebraic stack W and an equivalence if W has quasi-affine diagonal; and
- (4) $\Phi_{\mathbf{AlgSp}_{\text{lfp}}}$ is an equivalence of categories if X is locally excellent.

Theorem C. *Fix a weak Mayer–Vietoris square as in (1.1). If j is quasi-compact, then Φ_{Et} and Φ_{Et_c} are equivalences of categories.*

Theorem A essentially follows from Theorem B(4). Theorem B(2)–(4) relies upon Theorem C and general Néron–Popescu desingularization [Pop85]. Gabber’s rigidity theorem (Theorem 6.4) features in the proof of Theorem C. Both results also depend upon some further gluing results for quasi-coherent sheaves.

To prove Theorem B, we approximately follow the approach of [MB96]. The main idea is pass to a square as in (1.1) where $f: X' \rightarrow X$ is replaced by its diagonal $\Delta_f: X' \rightarrow X' \times_X X'$ and $j: U \rightarrow X$ is replaced by $j' \times j': U' \times_X U' \rightarrow X' \times_X X'$. In particular, unless f is unramified the resulting square will not be a flat Mayer–Vietoris square. Moreover, even if X is a locally noetherian algebraic stack, then unless f is locally of type, $X' \times_X X'$ has no reason to be locally noetherian. For example, in applications one often takes $X = \text{Spec } A$ and $X' = \text{Spec } \hat{A}$, where \hat{A} denotes the I -adic completion with respect to some ideal I of A ; in this situation, $X' \times_X X'$ is only noetherian when $X' = X$.

To manage such squares, we have the following natural variant of what Moret-Bailly considered.

Definition 1.3. A *tor-independent Mayer–Vietoris square* is a weak Mayer–Vietoris square as in (1.1) such that every morphism of algebraic stacks $W \rightarrow X$ with image disjoint from U is tor-independent of f (Definition 2.1).

If f is affine and U is the complement of a finitely presented closed immersion $i: Z \hookrightarrow X$, then a tor-independent Mayer–Vietoris square is the same as a triple (X, Z, X') satisfying the (TI) condition in the terminology of [MB96, 0.2, 0.6] (Lemma 3.2(2)). If X' and X are locally noetherian, then tor-independent Mayer–Vietoris squares are very similar to flat Mayer–Vietoris squares (Lemma 3.3). We now state our gluing result for tor-independent Mayer–Vietoris squares, which we can prove for f -flat objects (see Definition 2.1).

Theorem D. *Fix a tor-independent Mayer–Vietoris square as in (1.1). If j is quasi-compact, then*

- (1) $\Phi_{\text{QCoh}_{f-\text{fl}}}$ is an equivalence and
- (2) $\Phi_{\text{AlgSp}_{f-\text{fl}}}$ is fully faithful.

For tor-independent Mayer–Vietoris squares, we prove the following non-noetherian variant of Theorem A.

Theorem E. *Fix a tor-independent Mayer–Vietoris square as in (1.1). If j is quasi-compact, then it is a pushout in the 2-category of Deligne–Mumford stacks.*

Since we make no separation assumptions on our algebraic stacks, Theorem E generalizes recent work of Bhatt [Bha14]. Note, however, that while Bhatt uses (derived) Tannaka duality to prove a version of Theorem E for quasi-compact and quasi-separated algebraic spaces, we work in the opposite direction (i.e., we use pushouts to prove Tannaka duality in [HR14a]).

Remark 1.4. While it may appear that our results are weaker than the corresponding étale gluing results because we require j to be quasi-compact, this turns out to not be the case. Indeed, for étale neighborhoods, at least smooth-locally on X there is always an étale neighborhood X'' of Z' in X' and an open $U_0 \subseteq U$ such that $U_0 \rightarrow X$ is quasi-compact and the resulting square with $X'' \rightarrow X$ is an étale neighborhood of $X \setminus U_0$.

Overview. In Section 2 we give some preliminaries on tor-independence. In Section 3 we compare the different notions of Mayer–Vietoris squares and give several examples. In Section 4 we glue quasi-coherent sheaves in tor-independent Mayer–Vietoris squares (Theorem D(1) and Theorem B(1)).

In Section 5 we prove some fundamental theorems for étale sheaves of sets on algebraic stacks. In particular, we prove that every sheaf on a quasi-compact and quasi-separated algebraic stack is a filtered colimit of constructible sheaves. We also discuss henselian pairs of stacks.

In Section 6 we prove Gabber’s rigidity theorem and glue étale sheaves in weak Mayer–Vietoris squares (Theorem C). In the noetherian case, Gabber’s rigidity theorem follows immediately from Ferrand–Raynaud [FR70, App.]. In the non-noetherian case, which is essential for the applications in this paper, the previous proof [ILO14, Exp. 20] was much more involved. Using our results on gluing of sheaves, we provide a self-contained proof (for H^0 but the methods can be extended to H^1).

In Section 7 we glue algebraic spaces and prove that Mayer–Vietoris squares are pushouts (Theorems A, B, D(2), and E).

2. PRELIMINARIES

Here we record some preliminary results that will be of use in subsequent sections. Most of these are globalizations of the affine results proved in [MB96, §2]. We begin with the following definition.

Definition 2.1. Let $f: X' \rightarrow X$ and $g: W \rightarrow X$ be morphisms of algebraic stacks. Let $N \in \mathrm{QCoh}(X')$ and $M \in \mathrm{QCoh}(W)$.

- (1) We say that M and N are *tor-independent* if $\mathcal{T}or_i^{X,f,g}(N, M) = 0$ for all $i > 0$ [Hal14, App. C]. Equivalently, for all smooth morphisms $\mathrm{Spec} A \rightarrow X$, $\mathrm{Spec} A' \rightarrow \mathrm{Spec} A \times_X X'$ and $\mathrm{Spec} B \rightarrow \mathrm{Spec} A \times_X W$ we have

$$\mathrm{Tor}_i^A(N(\mathrm{Spec} A' \rightarrow X'), M(\mathrm{Spec} B \rightarrow W)) = 0$$

for all $i > 0$.

- (2) We say that M is *f-flat* if it is tor-independent of $\mathcal{O}_{X'}$. We let $\mathrm{QCoh}_{f\text{-fl}}(W) \subseteq \mathrm{QCoh}(W)$ denote the subcategory of *f-flat* quasi-coherent sheaves on W .
- (3) We say that g is *f-flat* if \mathcal{O}_W is *f-flat*. Note that g is *f-flat* if and only if f is *g-flat*. In particular, we may also say that f and g are tor-independent.

The following lemma is immediate from the definitions (we employ the notational conventions from [HR14c]).

Lemma 2.2. *Let $f: X' \rightarrow X$ be a morphism of algebraic stacks and let $M \in \mathrm{QCoh}(X)$. Then M is *f-flat* if and only if the natural map $\mathbb{L}f_{\mathrm{qc}}^* M \rightarrow f^* M$ is a quasi-isomorphism in $\mathrm{D}_{\mathrm{qc}}(X')$.*

The following notation will also be useful.

Notation 2.3. Let $i: Z \hookrightarrow X$ be a closed immersion of algebraic stacks, which is defined by quasi-coherent ideal I . For each integer $n \geq 0$, let $i^{[n]}: Z^{[n]} \hookrightarrow X$ be the closed immersion defined by the quasi-coherent ideal I^{n+1} . Note that if i is a finitely presented closed immersion, then so too is $i^{[n]}$ for all $n \geq 0$.

The following lemma will eventually be improved (see Corollary 4.8), but is for the meantime sufficient for our purposes.

Lemma 2.4. *Consider a cartesian diagram of algebraic stacks:*

$$\begin{array}{ccc} V' & \xrightarrow{v'} & W' \\ g_V \downarrow & & \downarrow g \\ V & \xrightarrow{v} & W. \end{array}$$

Assume that v is g -flat and a closed immersion.

- (1) *If $M \in \mathrm{QCoh}_{g_V\text{-fl}}(V)$, then $v_*M \in \mathrm{QCoh}_{g\text{-fl}}(W)$.*
- (2) *If g_V is an isomorphism, then $V^{[n]}$ is g -flat and $g_{V^{[n]}}: W' \times_W V^{[n]} \rightarrow V^{[n]}$ is an isomorphism for every $n \geq 0$.*

Proof. Both claims are smooth local on W , so we may assume that $W = \mathrm{Spec} A$ is an affine scheme. Claim (1) follows from tor-independent base change [HR14c, Cor. 4.13]. Indeed, this provides quasi-isomorphisms:

$$\mathrm{L}g_{\mathrm{qc}}^*(v_*M) \simeq \mathrm{L}g_{\mathrm{qc}}^*Rv_*M \simeq Rv'_*\mathrm{L}(g_V)_{\mathrm{qc}}^*M \simeq v'_*g_V^*M \simeq g^*v_*M.$$

Claim (2) is essentially the local criterion for flatness [EGA, 0_{III}.10], but we will spell out the details. Assume that $V = \mathrm{Spec}(A/I)$. Fix an integer $n \geq 1$. By induction we may also assume that $g_{V^{[n-1]}}: W' \times_W \mathrm{Spec}(A/I^n) \rightarrow \mathrm{Spec}(A/I^n)$ is an isomorphism and A/I^n is g -flat. Now (1) implies that every A/I^n -module is g -flat. In particular, I^n/I^{n+1} is g -flat so the distinguished triangle

$$\mathrm{L}g_{\mathrm{qc}}^*(I^n/I^{n+1}) \rightarrow \mathrm{L}g_{\mathrm{qc}}^*(A/I^{n+1}) \rightarrow \mathrm{L}g_{\mathrm{qc}}^*(A/I^n) \rightarrow \mathrm{L}g_{\mathrm{qc}}^*(I^n/I^{n+1})[1]$$

now implies that A/I^{n+1} is g -flat. By (1), we see that every A/I^{n+1} -module is g -flat. Clearly $g_{V^{[n]}}$ is a nil-immersion, so to prove that it is an isomorphism it is sufficient to prove that it is also flat. This is smooth local on W' , so we may assume that $W' = \mathrm{Spec} A'$. Let N be an A/I^{n+1} -module; then N is g -flat as seen above. It follows that there are quasi-isomorphisms:

$$\begin{aligned} (A'/I^{n+1}A') \otimes_{A/I^{n+1}}^{\mathrm{L}} N &\simeq (A' \otimes_A [A/I^{n+1}]) \otimes_{A/I^{n+1}}^{\mathrm{L}} N \\ &\simeq (A' \otimes_A^{\mathrm{L}} [A/I^{n+1}]) \otimes_{A/I^{n+1}}^{\mathrm{L}} N \\ &\simeq A' \otimes_A^{\mathrm{L}} N \\ &\simeq (A' \otimes_A N)[0]. \end{aligned}$$

Hence, $g_{V^{[n]}}$ is flat and we have the claim. \square

Let X be an algebraic stack and let $i: Z \hookrightarrow X$ be a closed immersion with complement $j: U \rightarrow X$. Define

$$\mathrm{QCoh}_Z(X) = \{M \in \mathrm{QCoh}(X) : j^*M \cong 0\}.$$

Note that $\mathrm{QCoh}_Z(X)$ only depends on the closed subset $|Z| \subseteq |X|$.

Lemma 2.5. *Let X be a quasi-compact algebraic stack. Let $i: Z \hookrightarrow X$ be a finitely presented closed immersion.*

- (1) *Let $M \in \mathrm{QCoh}_Z(X)$. If M is of finite type, then there exists an $n \gg 0$ such that the natural map $M \rightarrow i_*^{[n]}(i^{[n]})^*M$ is an isomorphism.*
- (2) *If $W \subseteq X$ is a closed substack with $|W| \subseteq |Z|$, then $W \subseteq Z^{[n]}$ for some $n \gg 0$.*

Proof. For (1): we may assume that $X = \mathrm{Spec} A$ is an affine scheme and $Z = \mathrm{Spec}(A/I)$, where $I = (f_1, \dots, f_r)$ is a finitely generated ideal of A . By assumption, $M_{f_i} = 0$ for each $i = 1, \dots, r$. As M is finitely generated, it follows that there exists $n \gg 0$ such that $f_i^n M = 0$ for all $i = 1, \dots, r$. The claim follows.

For (2): let $W_0 = Z \times_X W$. Then $W_0 \hookrightarrow W$ is a surjective and finitely presented closed immersion. From (1), it follows that $W \subseteq W_0^{[n]}$ for some $n \gg 0$. But $W_0^{[n]} \subseteq Z^{[n]}$ and we have the claim. \square

3. MAYER–VIETORIS SQUARES

In this section, we compare various notions of Mayer–Vietoris squares.

Lemma 3.1. *Fix a cartesian diagram as in (1.1).*

- (1) *If the square is a weak (resp. flat) Mayer–Vietoris square, then it remains so after arbitrary base change on X .*
- (2) *If the square is a tor-independent Mayer–Vietoris square, then it remains so after f -flat base change on X .*
- (3) *The properties of being a flat, tor-independent, or weak Mayer–Vietoris square are flat local on X .*

Proof. Claim (1) is trivial. For (2): it is sufficient to prove that if $w: W \rightarrow X$ is f -flat and $v: V \rightarrow X$ is f -flat, then $W \times_X V \rightarrow W$ is f_V -flat, where $f_V: X' \times_X V \rightarrow V$. This is local on W , V , X and X' , so we may assume that they are all affine. Let $X = \text{Spec } A$, $X' = \text{Spec } A'$, $V = \text{Spec } C$ and $W = \text{Spec } B$. It is thus sufficient to prove that $(B \otimes_A C) \otimes_C^{\mathbb{L}} (C \otimes_A A') \simeq (B \otimes_A C) \otimes_C (C \otimes_A A')$. To see this, we observe that

$$(B \otimes_A C) \otimes_C^{\mathbb{L}} (C \otimes_A A') \simeq (B \otimes_A C) \otimes_C^{\mathbb{L}} (C \otimes_A^{\mathbb{L}} A') \simeq B \otimes_A (C \otimes_A^{\mathbb{L}} A') \simeq B \otimes_A (C \otimes_A A'),$$

which gives the claim. The claim (3) is immediate from flat descent. \square

As the following Lemma shows, the conditions for Mayer–Vietoris squares are much easier to check when a description of the complement is given.

Lemma 3.2. *Fix a cartesian diagram as in (1.1). Suppose that U is the complement of a finitely presented closed immersion $i: Z \hookrightarrow X$.*

- (1) *If $f_{Z^{[n]}}: X' \times_X Z^{[n]} \rightarrow Z^{[n]}$ is an isomorphism for all n , then the square is a weak Mayer–Vietoris square.*
- (2) *If $f_Z: X' \times_X Z \rightarrow Z$ is an isomorphism and Z and f are tor-independent, then the square is a tor-independent Mayer–Vietoris square.*

Proof. We may assume that X is affine (Lemma 3.1(3)). Let $g: W \rightarrow X$ be a morphism of algebraic stacks with image disjoint from U . We must prove that $f_W: X' \times_X W \rightarrow W$ is an isomorphism. This is smooth local on W , so we may also assume that W is affine. The morphism g is now affine, so its schematic image V exists and is disjoint from U . In particular, $|V| \subseteq |Z|$. By Lemma 2.5(2), $V \subseteq Z^{[n]}$ for some $n \gg 0$. Hence, $W \rightarrow X$ factors through $Z^{[n]}$ for some $n \gg 0$. The claim (1) is now immediate. For (2), the result follows from Lemma 2.4(2). \square

Note that if X is quasi-compact and quasi-separated and j is quasi-compact, then $i: Z \hookrightarrow X$ as in Lemma 3.2 always exist [Ryd16, Prop. 8.2].

The following lemma connects the various types of Mayer–Vietoris squares to each other.

Lemma 3.3. *Fix a square as in (1.1). Consider the following conditions.*

- (1) *The square is a flat Mayer–Vietoris square.*
- (2) *The square is a weak Mayer–Vietoris square and f is flat at every point of Z' .*
- (3) *The square is a tor-independent Mayer–Vietoris square.*
- (4) *The square is a weak Mayer–Vietoris square.*

Then (1) \implies (2) \implies (3) \implies (4). If X and X' are locally noetherian, then (4) \implies (2). If there exists a Cartier divisor $i: Z \hookrightarrow X$ with complement U such that $f^{-1}(Z) \rightarrow Z$ is an isomorphism and $f^{-1}(Z) \hookrightarrow X'$ is also a Cartier divisor, then (3) holds.

Proof. That (1) \implies (2) \implies (3) \implies (4) is obvious. If X and X' are locally noetherian, then (2) follows from the local criterion of flatness [EGA, 0_{III}.10.2.1–2] (the conditions are flat-local on X and X' so reduces to schemes).

For the last claim, it is sufficient to prove that \mathcal{O}_Z is f -flat (Lemma 3.2(2)), which is local on X . So we may assume that $Z = V(d)$ and obtain an exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{d} \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0.$$

Applying $\mathbf{L}f_{\text{qc}}^*$ to this, we obtain a distinguished triangle in $\mathbf{D}_{\text{qc}}(X')$:

$$\mathcal{O}_{X'} \rightarrow \mathcal{O}_{X'} \rightarrow \mathbf{L}f_{\text{qc}}^* \mathcal{O}_Z \rightarrow \mathcal{O}_{X'}[1].$$

The resulting long exact exact of cohomology yields:

$$0 \rightarrow \mathcal{H}^{-1}(\mathbf{L}f_{\text{qc}}^* \mathcal{O}_Z) \rightarrow \mathcal{O}_{X'} \xrightarrow{d} \mathcal{O}_{X'} \rightarrow f^* \mathcal{O}_Z \rightarrow 0,$$

with all other terms 0. Since $Z \hookrightarrow X'$ is a Cartier divisor, d is regular on $\mathcal{O}_{X'}$ and so $\mathcal{H}^{-1}(\mathbf{L}f_{\text{qc}}^* \mathcal{O}_Z) = 0$. Hence, $\mathbf{L}f_{\text{qc}}^* \mathcal{O}_Z \rightarrow f^* \mathcal{O}_Z$ is a quasi-isomorphism and the square is a tor-independent Mayer–Vietoris square. \square

As the following lemma shows, blowing up provides a natural way to move from the weak Mayer–Vietoris setting to the tor-independent setting.

Lemma 3.4. *Fix a weak Mayer–Vietoris square as in (1.1). If there is a finitely presented closed immersion $i: Z \hookrightarrow X$ with complement U , then*

$$\begin{array}{ccc} U' & \xrightarrow{j'} & \text{Bl}_{Z'} X' \\ f_U \downarrow & \square & \downarrow \tilde{f} \\ U & \xrightarrow{j} & \text{Bl}_Z X \end{array}$$

is a tor-independent Mayer–Vietoris square.

Proof. Since the exceptional divisors $E \hookrightarrow \text{Bl}_Z X$ and $E' \hookrightarrow \text{Bl}_{Z'} X'$ are Cartier divisors it is enough to verify that $E' \rightarrow E$ is an isomorphism (Lemma 3.3). Let I be the ideal defining $Z \hookrightarrow X$ and I' the ideal defining $Z \hookrightarrow X'$. Then the inverse images of Z in the two blow-ups are

$$E = \text{Proj}_X(\oplus_{k \geq 0} I^k / I^{k+1}) \quad \text{and} \quad E' = \text{Proj}_{X'}(\oplus_{k \geq 0} I'^k / I'^{k+1}).$$

Since $I' = I\mathcal{O}_{X'}$ and $\mathcal{O}_X / I^m \rightarrow \mathcal{O}_{X'} / I'^m$ is an isomorphism for every m , these two graded rings are isomorphic $\mathcal{O}_X / I = \mathcal{O}_{X'} / I'$ -algebras. The result follows. \square

The following lemma is a key observation of Moret-Bailly and will be essential to the article.

Lemma 3.5 ([MB96, Cor. 2.5.1]). *Fix a weak Mayer–Vietoris square as in (1.1). If f admits a section $s: X \rightarrow X'$, then*

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ s_U \downarrow & \square & \downarrow s \\ U' & \xrightarrow{j'} & X' \end{array}$$

is a weak Mayer–Vietoris square. Moreover, if the square (1.1) is a tor-independent Mayer–Vietoris square, then so too is the one above.

Proof. Let $w' : W' \rightarrow X'$ be a morphism with image disjoint from U' . It follows that the composition $W' \xrightarrow{w'} X' \xrightarrow{f} X$ has image disjoint from U and so $W' \times_X X' \rightarrow W'$ is an isomorphism. In particular, the following diagram is cartesian:

$$\begin{array}{ccc} X & \xleftarrow{f \circ w'} & W' \\ s \downarrow & & \parallel \\ X' & \xleftarrow{w'} & W' \\ f \downarrow & & \parallel \\ X & \xleftarrow{f \circ w'} & W' \end{array}$$

and so $W' \times_X X \rightarrow W'$ is an isomorphism as required. For the latter claim, it suffices to prove that w' and s are tor-independent. But $f \circ w'$ and f are tor-independent and $f \circ w'$ and $f \circ s = \text{id}$ are tor-independent, so w' and s are tor-independent. The claim now follows from the cartesian diagram above. \square

Example 3.6 ([MB96, Prop. 2.5.2]). Fix a weak Mayer–Vietoris square as in (1.1). Then

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ \Delta_{f_U} \downarrow & \square & \downarrow \Delta_f \\ U' \times_U U' & \xrightarrow{j' \times j'} & X' \times_X X' \end{array}$$

is a weak Mayer–Vietoris square. Indeed, we can base change the square (1.1) by $X' \rightarrow X$ and the resulting square is still weak (Lemma 3.1). Taking the diagonal section to the projection $X' \times_X X' \rightarrow X'$ and using Lemma 3.5 gives the claim. If the square is a tor-independent Mayer–Vietoris square and f is f -flat (e.g., flat), then the square above is a tor-independent Mayer–Vietoris square. This claim follows from the same argument.

In the next Proposition, we show that general Mayer–Vietoris squares can smoothly locally be dominated by much simpler ones.

Proposition 3.7. *Fix a weak Mayer–Vietoris square as in (1.1). Then smoothly locally on X , there is an étale neighborhood $p : X'' \rightarrow X'$ of Z' such that the composition $f \circ p : X'' \rightarrow X$ is quasi-affine.*

Proof. By Lemma 3.1, we may assume that X is an affine scheme. Observe that the Deligne–Mumford locus of X' is an open substack containing Z' . In particular, there exists an affine scheme V and an étale morphism $V \rightarrow X'$ whose image contains Z' . Let $Z_V = V \times_{X'} Z'$; then the composition $z : Z_V \rightarrow Z' \simeq Z = X \setminus U$ is affine and étale. After passing to an étale cover of X , we may assume that the morphism $Z_V \rightarrow Z' \simeq Z$ has a section $s : Z \rightarrow Z''$. Since z is étale and separated, s is an open and closed immersion; it follows that $X'' = V \setminus (Z_V \setminus s(Z))$ is an open subscheme of V . After replacing X'' with a quasi-compact open neighborhood of $s(Z)$, we can assume that X'' is quasi-compact. Thus, $X'' \rightarrow X' \rightarrow X$ is a quasi-affine morphism and $X'' \rightarrow X'$ is an isomorphism over Z' . \square

The following is the last lemma of the section.

Lemma 3.8. *Fix a tor-independent Mayer–Vietoris square as in (1.1). If j is quasi-compact, then $\text{QCoh}_Z(X) \subseteq \text{QCoh}_{f-\text{fl}}(X)$.*

Proof. We may assume that X is affine and that $i : Z \rightarrow X$ is finitely presented. In this case if $N \in \text{QCoh}_Z(X)$, then we may write it as a limit of quasi-coherent subsheaves of finite type that also belong to $\text{QCoh}_Z(X)$. Thus, it is sufficient to

prove the result for such sheaves. By Lemma 2.5(1), there is an $n \gg 0$ such that $N \rightarrow i_*^{[n]}(i^{[n]})^*N$ is an isomorphism. The result now follows from Lemma 2.4(1). \square

Flat Mayer–Vietoris squares and weak Mayer–Vietoris squares are stable under arbitrary base change but tor-independent Mayer–Vietoris squares are not. We now give six examples:

- Two examples of weak Mayer–Vietoris squares that are not tor-independent Mayer–Vietoris squares.
- Two examples of closed immersions that give rise to non-flat tor-independent Mayer–Vietoris squares.
- A tor-independent Mayer–Vietoris square that is not universally a tor-independent Mayer–Vietoris square.
- A flat Mayer–Vietoris as in (1.1) with j not quasi-compact that is not a pushout in the category of affine schemes.

As we will see later, tor-independent Mayer–Vietoris squares satisfy gluing of quasi-coherent sheaves. In particular, $\Gamma(X, \mathcal{O}_X) = \Gamma(X', \mathcal{O}_{X'}) \times_{\Gamma(U', \mathcal{O}_{U'})} \Gamma(U, \mathcal{O}_U)$, which does not always hold for weak Mayer–Vietoris squares.

Example 3.9. Let $A = k[x]$, $B = A[z_1, z_2, \dots]/(xz_1, \{z_k - xz_{k+1}\}_{k \geq 1})$ and $C = B/(z_1)$. Then $A/(x^n) = B/(x^n) = C/(x^n) = k[x]/(x^n)$ and $A_x = B_x = C_x = k[x]_x$. Let $X = \text{Spec } A$, $Z = \text{Spec } A/(x)$, $U = X \setminus Z$, $X' = \text{Spec } B$, $U' = X' \setminus Z$ and $X'' = \text{Spec } C$. Then the squares

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & \square & \downarrow \\ U & \longrightarrow & X \end{array} \quad \begin{array}{ccc} U'' & \longrightarrow & X'' \\ \downarrow & \square & \downarrow \\ U' & \longrightarrow & X' \end{array}$$

are weak Mayer–Vietoris squares but not tor-independent Mayer–Vietoris squares. Indeed $A \rightarrow B \times_{B_x} A_x = B$ and $B \rightarrow C \times_{C_x} B_x = C$ are not isomorphisms.

Note that $Z \hookrightarrow X$ is a Cartier divisor but $Z \hookrightarrow X'$ is not a Cartier divisor.

Example 3.10. A diagonal Mayer–Vietoris square (Example 3.6) is typically not flat, e.g., let A be a noetherian ring, $I \subseteq A$ an ideal and consider the I -adic completion \hat{A}_I . Let $f: X' = \text{Spec } \hat{A}_I \rightarrow X = \text{Spec } A$ and $j: U = X \setminus V(I) \rightarrow X$. This gives rise to a flat Mayer–Vietoris square as in (1.1) and the diagonal Mayer–Vietoris square is tor-independent. However, Δ_f is a non-flat closed immersion.

Example 3.11. Let V be a valuation ring with valuation $\nu: K(V)^\times \rightarrow \Gamma$ and let $x \in V$. Then V_x is also a valuation ring and $V_x = V_P$ where $P \subseteq V$ is the maximal prime ideal properly contained in the prime ideal $Q = \sqrt{(x)}$. Explicitly:

$$\begin{aligned} P &= \{a \in V : \forall n \in \mathbb{N} : \nu(a) \geq n\nu(x)\} \\ Q &= \{a \in V : \exists n \in \mathbb{N} : n\nu(a) > \nu(x)\}. \end{aligned}$$

Let $X = \text{Spec}(V)$, $U = \text{Spec}(V_P)$, $Z = \text{Spec}(V/x)$ and $X' = \text{Spec}(V/P)$. The resulting square as in (1.1) is tor-independent. Indeed, it is a weak Mayer–Vietoris square since $P \subseteq (x^n)$ for all n . It remains to verify that $\text{Tor}_i^A(V/x, V/P) = 0$ for all $i > 0$. But $x \notin P$ so x is V/P -regular, hence the Tors vanish.

Example 3.12. Consider the valuation $\nu: k(x, y) \rightarrow \mathbb{Z}^2$ with $\nu(x) = (0, 1)$ and $\nu(y) = (1, 0)$ where \mathbb{Z}^2 is lexicographically ordered. Then $Q = (x)$ is the maximal ideal and $P = (y, y/x, y/x^2, \dots)$. Then $X' = \text{Spec } V/P \rightarrow X = \text{Spec } V$, $U = \text{Spec } V_P = \text{Spec } V_x$ is a tor-independent Mayer–Vietoris square as in the previous example.

Let $A = V/y$ and let $z_n = y/x^n$ denote the image of y/x^n . Then $A = k[x, z_1, \dots]/(xz_1, z_k - xz_{k+1})_{(x, z_1, z_2, \dots)}$ and $B = A/PA = k[x]_{(x)}$. Let $Y' = \text{Spec } B$, $Y = \text{Spec } A$ and $U = \text{Spec } A_x$. As in Example 3.9, $A/(x^n) = k[x]/(x^n)$ and $B/(x^n) = k[x]/(x^n)$ but $A \rightarrow B \times_{B_x} A_x = B$ is not an isomorphism.

Example 3.13. Let $X = \text{Spec } A$ be the spectrum of an absolutely flat ring such that there exists a non-discrete point $x \in |X|$. Let $\mathfrak{m} \subseteq A$ be the corresponding maximal ideal. For a concrete example, let \mathbb{P} be the set of primes of \mathbb{Z} , let $A = \prod_{p \in \mathbb{P}} \mathbb{F}_p$ and let \mathfrak{m} be a maximal ideal containing the ideal $\bigoplus_{p \in \mathbb{P}} \mathbb{F}_p$. Let $X' = \text{Spec } A/\mathfrak{m}$ and let $f: X' \rightarrow X$ be the induced closed immersion. Let $j: U = X \setminus \{x\} \rightarrow X$ be the open immersion of the complement. Since A is an absolutely flat ring, f is also flat. Let $U' = X' \times_X U = \emptyset$; then the resulting square is a flat Mayer–Vietoris square but j is not quasi-compact.

Note that the natural map $|X'| \amalg |U| = |X'| \amalg_{|U'|} |U| \rightarrow |X|$ is not a homeomorphism since $|X'| \subseteq |X|$ is not open. In particular, the functor Φ_{Et} is not an equivalence, cf. Corollary 6.6.

Let $B = \Gamma(U, \mathcal{O}_U)$. If the square was a pushout in the category of affine schemes, then corresponding to the maps $X' \rightarrow X$ and $U \rightarrow \text{Spec } B$, there would be a unique map $g: X \rightarrow \text{Spec } B \amalg X' = \text{Spec}(B \times A/\mathfrak{m})$. Then $g^{-1}(X') = X'$ which is a contradiction since $X' \subseteq X$ is not open.

This example also shows that

$$\Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_U) \times \Gamma(X', \mathcal{O}_{X'})$$

is not an isomorphism. In particular, the functor Φ_{QCoh} is not even fully faithful.

4. GLUING OF MODULES IN MAYER–VIETORIS SQUARES

In this section, we show that quasi-coherent sheaves of modules, and related objects such as quasi-coherent sheaves of algebras, can be glued in tor-independent Mayer–Vietoris squares. This generalizes previous results of Ferrand–Raynaud [FR70, App.] and Moret-Bailly [MB96]. We will prove this using some ideas from the theory developed in [HR14c, §5] for triangulated categories that are perfectly suited to simultaneously deal with the non-flatness of f and the non-affineness of j . For quasi-compact and quasi-separated algebraic spaces and in the context of stable ∞ -categories, this was recently accomplished (independently) by Bhatt [Bha14, Prop. 5.6]. Since we work with morphisms of algebraic stacks that may not have finite cohomological dimension, we do not expect gluing results to hold in this generality in the unbounded derived category. Before we get to gluing, we characterize the tor-independent squares in terms of derived categories.

Notation 4.1. Let $i: Z \hookrightarrow X$ be a closed immersion of algebraic stacks with complement $j: U \rightarrow X$. Define

$$\mathbf{D}_{\text{qc}, Z}(X) = \{M \in \mathbf{D}_{\text{qc}}(X) : \mathbf{L}j_{\text{qc}}^* M \simeq 0\}.$$

Proposition 4.2. *Fix a cartesian diagram as in (1.1) with f concentrated, j quasi-compact and X quasi-compact and quasi-separated. Consider the following conditions:*

- (1) *the square is a tor-independent Mayer–Vietoris square;*
- (2) *$\mathbf{R}f_*$ and $\mathbf{L}f_{\text{qc}}^*$ induce equivalences $\mathbf{D}_{\text{qc}, Z}(X) \simeq \mathbf{D}_{\text{qc}, Z'}(X')$;*
- (3) *$\mathbf{R}f_*$ and $\mathbf{L}f_{\text{qc}}^*$ induce t -exact equivalences $\mathbf{D}_{\text{qc}, Z}(X) \simeq \mathbf{D}_{\text{qc}, Z'}(X')$; and*
- (4) *$\mathbf{R}f_*$ and $\mathbf{L}f_{\text{qc}}^*$ induce t -exact equivalences $\mathbf{D}_{\text{qc}, Z}^b(X) \simeq \mathbf{D}_{\text{qc}, Z'}^b(X')$.*

Then (1) \implies (2) \iff (3) \iff (4). If either f_Z is affine or f_Z is representable and Z has quasi-affine diagonal, then all conditions are equivalent.

In the application of Proposition 4.2 to the main result of this section (Theorem 4.4), we will only need (1) \implies (2) when X is an affine scheme and X' is a quasi-affine scheme. We have included the general situation for independent interest. Note that condition (2) is the definition of a Mayer–Vietoris D_{qc} -square [HR14c, Def. 5.4] and Proposition 4.2 gives another proof of [HR14c, Ex. 5.5].

Remark 4.3. Taking $f: \text{Spec } k \rightarrow BG$ as in [HR15, Rem. 1.6] and $U = \emptyset$ provides an example where (4) is satisfied, but (1) is not satisfied (f is representable but not affine and its target does not have affine stabilizers).

Proof of Proposition 4.2. Trivially, (3) implies (2) and (4). Since X is quasi-compact and quasi-separated, we may assume that there is a finitely presented complement $i: Z \hookrightarrow X$ of U . If (3) is satisfied, then $\mathbf{L}f_{\text{qc}}^* \mathcal{O}_Z = f^* \mathcal{O}_Z$ and the adjunction maps $\mathcal{O}_Z \rightarrow f_* f^* \mathcal{O}_Z$ and $f^* f_* \mathcal{O}_{Z'} \rightarrow \mathcal{O}_{Z'}$ are isomorphisms. If f_Z is affine, then (1) holds (Lemma 3.2(2)). Otherwise, if Z has quasi-affine diagonal, then we start by noting that the t -exactness of $\mathbf{R}f_*$ also shows that $\mathbf{R}(f_Z)_*$ is t -exact. By [HR14c, Lem. 2.2 (6)], it follows that if $\tilde{Z} \rightarrow Z$ is a smooth morphism, where \tilde{Z} is an affine scheme, then the pullback \tilde{f}_Z of f_Z to \tilde{Z} is such that $\mathbf{R}(\tilde{f}_Z)_*$ is t -exact. Since \tilde{f}_Z is representable, we conclude that \tilde{f}_Z is affine from Serre’s Criterion [Ryd15, Thm. 8.7]. By smooth descent, f_Z is affine, and we again see that (1) holds.

Let us show that (2) \implies (3). Since f is concentrated and X is quasi-compact and quasi-separated, there exists an integer n such that

$$\tau^{\geq k} \mathbf{R}f_* N \rightarrow \tau^{\geq k} \mathbf{R}f_{\text{qc}}^* (\tau^{\geq k-n} N)$$

for every integer k and $N \in D_{\text{qc}}(X')$ [HR14c, Thm. 2.6(1)]. In particular, if $N \in D_{\text{qc}, Z'}^{\leq 0}(X')$, then $\mathbf{R}f_* N$ is bounded above. Let m denote the largest integer such that $\mathcal{H}^m(\mathbf{R}f_* N) \neq 0$. Then $\mathcal{H}^m(N) \cong \mathcal{H}^m(\mathbf{L}f_{\text{qc}}^* \mathbf{R}f_* N) \neq 0$ so $m \leq 0$. Thus, $\mathbf{R}f_*$ is right t -exact and hence t -exact on $D_{\text{qc}, Z'}(X')$. Similarly, if $M \in D_{\text{qc}, Z}^{\geq 0}(X)$, then we have isomorphisms $M \simeq \mathbf{R}f_* \tau^{\geq -m} \mathbf{L}f_{\text{qc}}^* M$ for every $m \geq n$. It follows that $\mathbf{L}f_{\text{qc}}^* M$ is supported in degrees ≥ 0 so $\mathbf{L}f_{\text{qc}}^*$ is t -exact on $D_{\text{qc}, Z}(X)$.

We will finish the proof by showing that (1) \implies (4) \implies (3). For every $M \in D_{\text{qc}, Z}(X)$ and $N \in D_{\text{qc}, Z'}(X')$ we have adjunction maps

$$\eta_M: M \rightarrow \mathbf{R}f_* \mathbf{L}f_{\text{qc}}^* M \quad \text{and} \quad \epsilon_N: \mathbf{L}f_{\text{qc}}^* \mathbf{R}f_* N \rightarrow N.$$

We will show that these are quasi-isomorphisms.

For (1) \implies (4) it is enough—by standard truncation arguments—to prove:

- $\eta_{M[0]}$ and $\epsilon_{N[0]}$ are quasi-isomorphisms; and
- $\mathbf{L}f^* M[0] \rightarrow (f^* M)[0]$ and $(f_* N)[0] \rightarrow \mathbf{R}f_* N[0]$ are quasi-isomorphisms,

where M is a quasi-coherent \mathcal{O}_X -module such that $j^* M \cong 0$ and N is a quasi-coherent $\mathcal{O}_{X'}$ such that $j'^* N \cong 0$. If M and N are of finite type, then there exists an integer $n \gg 0$ such that $M \rightarrow i_*^{[n]}(i^{[n]})^* M$ and $N \rightarrow i_*'^{[n]}(i'^{[n]})^* N$ are isomorphisms (Lemma 2.5(1)). Now Lemma 2.4 informs us that $f_{Z^{[n]}}: Z'^{[n]} \rightarrow Z^{[n]}$ is an isomorphism, $i^{[n]}$ and f are tor-independent, and M is f -flat. This immediately proves the claims when M and N are of finite type. But every quasi-coherent sheaf on X or X' is a directed limit of its quasi-coherent subsheaves of finite type [Ryd16], so we have the claim in general.

To see that (4) \implies (3) it is enough to prove that $\mathbf{L}f_{\text{qc}}^*$ is left t -exact on $D_{\text{qc}, Z}(X)$ and that $\mathbf{R}f_*$ is right t -exact on $D_{\text{qc}, Z'}(X')$. For the first claim, let M be a complex in $D_{\text{qc}, Z}^{\geq 0}(X)$. We may write M as a homotopy colimit of its truncations $\tau^{\leq n} M$. Since $\mathbf{L}f_{\text{qc}}^*$ commutes with coproducts and is t -exact on $D_{\text{qc}, Z}^b(X)$, it follows that $\mathbf{L}f_{\text{qc}}^* M \in D_{\text{qc}, Z'}^{\geq 0}(X')$ so $\mathbf{L}f_{\text{qc}}^*$ is t -exact. Also, if N is a complex in $D_{\text{qc}, Z'}^{\leq 0}(X')$, then

since f is concentrated and X is quasi-compact and quasi-separated, there exists an integer n such that $\tau^{>0}\mathbf{R}f_*N \rightarrow \tau^{>0}\mathbf{R}f_*(\tau^{\geq -n}N) \simeq 0$. Hence, $\mathbf{R}f_*$ is t -exact. \square

The following theorem generalizes [MB96, Thm. 3.1] (f affine) and [FR70, App.] (f affine and flat) and is Theorem D(1).

Theorem 4.4. *Fix a tor-independent Mayer–Vietoris square as in (1.1) with j quasi-compact. The functors*

$$\Phi_{\text{Mod}}: \text{Mod}(X) \rightleftarrows \text{Mod}(X') \times_{\text{Mod}(U')} \text{Mod}(U): \Psi$$

where

$$\Phi_{\text{Mod}}(N) = (f^*N, j^*N, \delta) \quad \text{and} \quad \Psi(N', N_U, \alpha) = f_*N' \times_{\alpha} j_*N_U$$

and δ is the canonical isomorphism $j'^*f^*N \cong f_U^*j^*N$, are adjoint. Also, Φ_{Mod} preserves tensor products and the restriction of Φ_{Mod} to $\text{QCoh}_{f\text{-fl}}(X)$ induces an equivalence of categories

$$(4.1) \quad \Phi_{\text{QCoh}, f\text{-fl}}: \text{QCoh}_{f\text{-fl}}(X) \rightarrow \text{QCoh}(X') \times_{\text{QCoh}(U')} \text{QCoh}_{f_U\text{-fl}}(U)$$

that preserves short exact sequences. Moreover,

- (1) $f^*: \text{QCoh}_Z(X) \rightarrow \text{QCoh}_{Z'}(X')$ is an equivalence;
- (2) Φ_{QCoh} preserves and reflects
 - (a) zero objects,
 - (b) surjective homomorphisms and
 - (c) modules of finite type; and
- (3) $\Phi_{\text{QCoh}, f\text{-fl}}$ preserves and reflects
 - (a) modules of finite presentation and
 - (b) flat modules.

Proof. Let $k = j \circ f_U$. That Φ_{Mod} and Ψ are adjoints is clear. Hence, to prove the equivalence (4.1) it is enough to show that the unit $N \rightarrow \Psi(\Phi_{\text{Mod}}(N))$ and the counit $\Phi_{\text{Mod}}(\Psi(N', N_U, \delta)) \rightarrow (N', N_U, \delta)$ of the adjunction are isomorphisms when restricted to the relevant subcategories. This is smooth local on X , so we may assume that X is an affine scheme.

Until further notice, we will assume that X' is a quasi-compact and quasi-separated algebraic space (even quasi-affine scheme is sufficient). Now the functor

$$\mathbf{L}f^*: \text{D}_{\text{qc}, Z}(X) \rightarrow \text{D}_{\text{qc}, Z'}(X')$$

is an equivalence of categories (Proposition 4.2). Thus, we have a Mayer–Vietoris D_{qc} -square in the sense of [HR14c, Def. 5.4], which provides some natural distinguished triangles [HR14c, Lem. 5.6] that we now describe.

- (i) For every $N \in \text{D}_{\text{qc}}(X)$, there is a distinguished triangle:

$$N \longrightarrow \mathbf{R}j_*\mathbf{L}j^*N \oplus \mathbf{R}f_*\mathbf{L}f^*N \longrightarrow \mathbf{R}f_*\mathbf{L}f^*\mathbf{R}j_*\mathbf{L}j^*N \longrightarrow N[1].$$

- (ii) Conversely, given $N_U \in \text{D}_{\text{qc}}(U)$, $N' \in \text{D}_{\text{qc}}(X')$ and an isomorphism $\delta: \mathbf{L}j'^*N' \rightarrow \mathbf{L}f_U^*N_U$, we define N by the following distinguished triangle in $\text{D}_{\text{qc}}(X)$:

$$N \longrightarrow \mathbf{R}j_*N_U \oplus \mathbf{R}f_*N' \xrightarrow{(\eta_{\mathbf{R}j_*N_U}^f \quad -\alpha)} \mathbf{R}f_*\mathbf{L}f^*\mathbf{R}j_*N_U \longrightarrow N[1],$$

where $\alpha: \mathbf{R}f_*N' \rightarrow \mathbf{R}f_*\mathbf{L}f^*\mathbf{R}j_*N_U$ is the composition:

$$\begin{aligned} \mathbf{R}f_*N' &\xrightarrow{\mathbf{R}f_*\eta_{N'}^j} \mathbf{R}f_*\mathbf{R}j'_*\mathbf{L}j'^*N' \\ &\xrightarrow{\mathbf{R}f_*\mathbf{R}j'_*\delta} \mathbf{R}f_*\mathbf{R}j'_*\mathbf{L}f_U^*N_U \cong \mathbf{R}f_*\mathbf{L}f^*\mathbf{R}j_*N_U. \end{aligned}$$

Then the induced maps $\mathbf{L}j^*N \rightarrow N_U$ and $\mathbf{L}f^*N \rightarrow N'$ are isomorphisms.

Now let $N \in \mathbf{QCoh}_{f\text{-fl}}(X)$; then the distinguished triangle from (i) reduces to the following distinguished triangle:

$$N \longrightarrow \mathbf{R}j_*j^*N \oplus \mathbf{R}f_*f^*N \longrightarrow \mathbf{R}f_*\mathbf{L}f^*\mathbf{R}j_*j^*N \longrightarrow N[1].$$

Observe that tor-independent base change [HR14c, Cor. 4.13] implies that:

$$\mathbf{R}f_*\mathbf{L}f^*\mathbf{R}j_*j^*N \simeq \mathbf{R}f_*\mathbf{R}j'_*\mathbf{L}f'_U\mathbf{L}j^*N \simeq \mathbf{R}k_*\mathbf{L}j'^*\mathbf{L}f^*N \simeq \mathbf{R}k_*k^*N.$$

Hence, taking the long exact cohomology sequence of the distinguished triangle above, we obtain the following exact sequence:

$$0 \rightarrow N \rightarrow j_*j^*N \oplus f_*f^*N \rightarrow k_*k^*N \rightarrow 0.$$

So the natural map $N \rightarrow \Psi(\Phi_{\text{Mod}}(N))$ is an isomorphism when N is f -flat.

Conversely, given a triple (N', N_U, δ) , where $N' \in \mathbf{QCoh}(X')$ and $N_U \in \mathbf{QCoh}_{f_U\text{-fl}}$, (ii) provides a distinguished triangle:

$$N \longrightarrow \mathbf{R}j_*N_U \oplus \mathbf{R}f_*N' \longrightarrow \mathbf{R}k_*N'_U \longrightarrow N[1],$$

such that the induced maps $\mathbf{L}j^*N \rightarrow N_U$ and $\mathbf{L}f^*N \rightarrow N'$ are isomorphisms. Since $\Phi(N', N_U, \delta) = \mathcal{H}^0(N)$, it is enough to show that N is concentrated in degree 0. To see this, we have a distinguished triangle:

$$\mathcal{H}^0(N)[0] \longrightarrow N \longrightarrow \tau^{\geq 1}(N) \longrightarrow \mathcal{H}^0(N)[1].$$

If we apply the t -exact functor $\mathbf{L}j^*$ to this triangle, then the third term vanishes so $\tau^{\geq 1}(N) \in \mathbf{D}_{\text{qc}, Z}(X)$. If we instead apply the right t -exact functor $\mathbf{L}f^*$ to this triangle, we obtain the triangle:

$$\mathbf{L}f^*\mathcal{H}^0(N)[0] \longrightarrow N'[0] \longrightarrow \mathbf{L}f^*\tau^{\geq 1}(N) \longrightarrow \mathbf{L}f^*\mathcal{H}^0(N)[1].$$

The first two terms are concentrated in degrees ≤ 0 and the third is concentrated in degrees ≥ 1 since $\mathbf{D}_{\text{qc}, Z}(X) \rightarrow \mathbf{D}_{\text{qc}, Z'}(X')$ is a t -exact equivalence. It follows that $\tau^{\geq 1}(N) \simeq 0$.

Hence, we have proven the equivalence (4.1) when $f: X' \rightarrow X$ is quasi-compact, quasi-separated and representable. We now address the general case. By Proposition 3.7, smooth-locally on X there is an étale neighborhood X'' of Z' in X' such that the induced composition $w: X'' \rightarrow X$ is quasi-affine. Let $U'' = X'' \times_X U$. It now follows from the case considered already, as well as [Ryd11, Ex. 1.2], that we have equivalences:

$$\begin{aligned} \mathbf{QCoh}_{w\text{-fl}}(X) &\simeq \mathbf{QCoh}(X'') \times_{\mathbf{QCoh}(U'')} \mathbf{QCoh}_{w_U\text{-fl}}(U) \\ &\simeq (\mathbf{QCoh}(X'') \times_{\mathbf{QCoh}(U'')} \mathbf{QCoh}(U')) \times_{\mathbf{QCoh}(U')} \mathbf{QCoh}_{w_U\text{-fl}}(U) \\ &\simeq \mathbf{QCoh}(X') \times_{\mathbf{QCoh}(U')} \mathbf{QCoh}_{w_U\text{-fl}}(U). \end{aligned}$$

Note that $\mathbf{QCoh}_{f\text{-fl}}(X) \subseteq \mathbf{QCoh}_{w\text{-fl}}(X)$ is a full subcategory and that we have an equivalence

$$\mathbf{QCoh}_{f\text{-fl}}(X) \rightarrow \mathbf{QCoh}_{w\text{-fl}}(X) \times_{\mathbf{QCoh}_{w_U\text{-fl}}(U)} \mathbf{QCoh}_{f_U\text{-fl}}(U).$$

It follows that $\Phi_{\mathbf{QCoh}, f\text{-fl}}$ is an equivalence and it preserves short exact sequences.

Now for (1): if $N \in \mathbf{QCoh}_Z(X)$, then N is f -flat (Lemma 3.8). Hence,

$$N \rightarrow \Psi(f^*N, 0, 0) = f_*f^*N$$

is an isomorphism. Also if $N' \in \mathbf{QCoh}_{Z'}(X')$, then

$$(f^*f_*N', 0, 0) = \Phi_{\text{Mod}}(f_*N') \rightarrow (N', 0, 0)$$

is an isomorphism and the claim follows.

For (2a): the preservation is obvious. For the reflection: if $N \in \mathbf{QCoh}(X)$ and $j^*N \cong 0$, then $N \in \mathbf{QCoh}_Z(X)$. But if $f^*N \cong 0$ too, then $N \cong 0$ by (1).

For (2b): the preservation is because Φ_{Mod} admits a right adjoint Ψ and so is right exact. For the reflection: if $u: N \rightarrow M$ is a morphism in $\text{QCoh}(X)$ and j^*u and f^*u are surjective, then $j^* \text{coker}(u) = 0$ and $f^* \text{coker}(u) = \text{coker}(f^*u) = 0$. It follows from (2a) that $\text{coker}(u) = 0$ and u is surjective.

For (2c): the preservation is clear. For the reflection: we may assume that X is affine. Write M as a filtered union of quasi-coherent subsheaves M_λ of finite type. For sufficiently large λ we see that $\Phi_{\text{Mod}}(M_\lambda) \rightarrow \Phi(M)$ is surjective. By (2b), we see that $M_\lambda = M$ and so M is of finite type.

For (3a): the preservation is clear. For the reflection: Let M be a f -flat quasi-coherent \mathcal{O}_X -module such that $\Phi(M)$ is of finite presentation. By (2c) we know that M is of finite type. Since we are free to assume that X is affine, there is an exact sequence $0 \rightarrow K \rightarrow \mathcal{O}_X^{\oplus n} \rightarrow M \rightarrow 0$. But M is f -flat, so the sequence remains exact after applying f^* . Since $\Phi(K)$ is of finite type, so is K and hence M is of finite presentation.

For (3b): the preservation is clear. For the reflection: as before, we may assume that X is affine and f is quasi-affine. Let $N \in \text{QCoh}_{f\text{-fl}}(X)$ and let $M \in \text{QCoh}(X)$. It is sufficient to prove that $\tau^{<0}(M \otimes_{\mathcal{O}_X}^{\mathbb{L}} N) \simeq 0$. We begin with the following distinguished triangle:

$$C \longrightarrow M \longrightarrow \text{R}j_*j^*M \longrightarrow C[1].$$

Observe that the derived projection formula [HR14c, Cor. 4.12] implies that

$$(\text{R}j_*j^*M) \otimes_{\mathcal{O}_X}^{\mathbb{L}} N \simeq \text{R}j_*((j^*M) \otimes_{\mathcal{O}_U}^{\mathbb{L}} j^*N).$$

But j^*N is flat and so we conclude immediately that $\tau^{<0}((\text{R}j_*j^*M) \otimes_{\mathcal{O}_X}^{\mathbb{L}} N) \simeq 0$. It remains to prove that $\tau^{<0}(C \otimes_{\mathcal{O}_X}^{\mathbb{L}} N) \simeq 0$. To this end, we first note that $\tau^{<0}C \simeq 0$ and $j^*C \simeq 0$. Moreover,

$$\text{L}f^*(C \otimes_{\mathcal{O}_X}^{\mathbb{L}} N) \simeq (\text{L}f^*C) \otimes_{\mathcal{O}_{X'}}^{\mathbb{L}} \text{L}f^*N \simeq (\text{L}f^*C) \otimes_{\mathcal{O}_{X'}}^{\mathbb{L}} f^*N.$$

By assumption, f^*N is flat and so for all integers k there are isomorphisms:

$$\mathcal{H}^k((\text{L}f^*C) \otimes_{\mathcal{O}_{X'}}^{\mathbb{L}} f^*N) \cong \mathcal{H}^k(\text{L}f^*C) \otimes_{\mathcal{O}_{X'}} f^*N.$$

But $C \in \text{D}_{\text{qc},Z}(X)$, so $\tau^{<0}C \simeq 0$ implies $\tau^{<0}(\text{L}f^*C) \simeq 0$ (Proposition 4.2). Putting this all together, we see that $\tau^{<0}(\text{L}f^*(C \otimes_{\mathcal{O}_X}^{\mathbb{L}} N)) \simeq 0$. But $j^*(C \otimes_{\mathcal{O}_X}^{\mathbb{L}} N) \simeq 0$, which implies that $\tau^{<0}(C \otimes_{\mathcal{O}_X}^{\mathbb{L}} N) \simeq 0$ (Proposition 4.2 again). \square

Note that 4.4(3b) gives a vast generalization of [MB96, Prop. 4.1(iii)], where only the descent of étaleness is proved.

Remark 4.5. Assume that we are in the situation of Theorem 4.4. If f is concentrated, then the Mayer–Vietoris triangle shows that the functor $\text{D}_{\text{qc}}(X) \rightarrow \text{D}_{\text{qc}}(X') \times_{\text{D}_{\text{qc}}(U')} \text{D}_{\text{qc}}(U)$ is essentially surjective. It is, however, not fully faithful. The reason is a well-known fault of the derived category: whereas cones are unique up to isomorphism, morphisms between cones are not unique. One way to fix this problem is to work with ∞ -categories. Then one obtains the expected equivalence, cf. [Bha14, Prop. 5.6].

We now have a number of corollaries.

Corollary 4.6. *Assume that we are in the situation of Theorem 4.4. If $M \in \text{Mod}(X)$ and $N \in \text{QCoh}_{f\text{-fl}}(X)$, then the natural map:*

$$\text{Hom}(M, N) \rightarrow \text{Hom}(f^*M, f^*N) \times_{\text{Hom}(j'^*f^*M, j'^*f^*N)} \text{Hom}(j^*M, j^*N)$$

is bijective.

Proof. Follows from the unit $N \rightarrow \Psi(\Phi_{\text{Mod}}(N))$ being an isomorphism. \square

Corollary 4.7. *Assume that we are in the situation of Theorem 4.4. If f is flat, then Φ_{QCoh} is an equivalence of abelian categories and preserves and reflects flatness.*

Corollary 4.8 ([MB96, Cor. 3.4.3]). *Assume that we are in the situation of Theorem 4.4. Then $N \in \mathrm{QCoh}(X)$ is f -flat if and only if j^*N is f_U -flat.*

Proof. The necessity is clear. For the sufficiency: if j^*N is f_U -flat, then $\tilde{N} = \Psi(f^*N, j^*N, \delta)$ is an f -flat quasi-coherent sheaf and there is a natural map $\eta: N \rightarrow \tilde{N}$. Now $j^*\eta$ and $f^*\eta$ are isomorphisms, so $\ker(\eta)$ is f -flat (Lemma 3.8) and η is surjective (Theorem 4.4(2b)). So we have an exact sequence:

$$0 \rightarrow \ker(\eta) \rightarrow N \rightarrow \tilde{N} \rightarrow 0$$

and $\ker(\eta)$ and \tilde{N} are f -flat. It follows that N is f -flat, which gives the sufficiency. \square

We now consider Theorem 4.4 in the context of algebras, or equivalently, affine schemes.

Corollary 4.9. *Assume that we are in the situation of Theorem 4.4. The natural functor*

$$\Phi_{\mathrm{Aff}, f\text{-fl}}: \mathrm{Aff}_{f\text{-fl}}(X) \rightarrow \mathrm{Aff}(X') \times_{\mathrm{Aff}(U')} \mathrm{Aff}_{f_U\text{-fl}}(U),$$

is an equivalence of categories. Moreover, the functor Φ_{Aff} preserves and reflects

- (1) *closed immersions;*
- (2) *finite morphisms;*
- (3) *integral morphisms; and*
- (4) *morphisms of finite type.*

and $\Phi_{\mathrm{Aff}, f\text{-fl}}$ preserves and reflects

- (5) *morphisms of finite presentation.*

Proof. An \mathcal{O}_X -algebra structure on a \mathcal{O}_X -module M is given by homomorphisms $\mathcal{O}_X \rightarrow M$ and $M \otimes_{\mathcal{O}_X} M \rightarrow M$ satisfying various compatibility conditions. If M is f -flat, then an algebra structure on $\Phi_{\mathrm{Mod}}(M)$ descends to a unique algebra structure on M by Corollary 4.6.

That Φ_{Aff} preserves all the properties follows by definition. To see that Φ_{Aff} reflects the properties, we may work fppf-locally on X and assume that X is affine and work with the categories of algebras. We let $\Phi = \Phi_{\mathrm{Mod}}$ for the remainder of the proof.

(1) This follows from Theorem 4.4(2b).

(2) Let $A \rightarrow B$ be a homomorphism of \mathcal{O}_X -algebras. Write B as a filtered union of finitely generated A -submodules B_λ . If $\Phi(B)$ is a finite $\Phi(A)$ -algebra, then $\Phi(B_{\lambda_0}) \rightarrow \Phi(B)$ is surjective for sufficiently large λ . We conclude that $B = B_\lambda$ is a finite A -algebra from Theorem 4.4(2b).

(3) Let $A \rightarrow B$ be a homomorphism of \mathcal{O}_X -algebras. If $\Phi(A) \rightarrow \Phi(B)$ is integral, then $j^*A \rightarrow j^*B$ is integral. Thus, if B_0 is the integral closure of A in B , then $j^*B_0 = j^*B$. Write B as the filtered union of finitely generated B_0 -subalgebras $B_\lambda \subseteq B$. Since $j^*B_0 = j^*(B_\lambda) = j^*B$, we have that B/B_λ is f -flat; it follows that $\Phi(B_\lambda) \subseteq \Phi(B)$ is a $\Phi(B_0)$ -subalgebra of finite type. Thus $\Phi(B_\lambda)$ is a finite $\Phi(B_0)$ -algebra, so B_λ is a finite B_0 -algebra. It follows that $B = \bigcup_\lambda B_\lambda$ is integral over A .

(4) If $A \rightarrow B$ is a homomorphism of \mathcal{O}_X -algebras such that $\Phi(A) \rightarrow \Phi(B)$ is of finite type, then write B as a filtered union of finitely generated A -subalgebras B_λ . For sufficiently large λ , we have that $\Phi(B_\lambda) \rightarrow \Phi(B)$ is surjective, hence so is $B_\lambda \rightarrow B$ so $A \rightarrow B$ is of finite type.

(5) If $A \rightarrow B$ is a homomorphism of \mathcal{O}_X -algebras such that $\Phi(A) \rightarrow \Phi(B)$ is of finite presentation, then we have already seen that $A \rightarrow B$ is of finite type. There

is an exact sequence $0 \rightarrow I \rightarrow A[x_1, x_2, \dots, x_n] \rightarrow B \rightarrow 0$ and if B is f -flat, then this sequence remains exact after applying f^* . We conclude that I is a finitely generated ideal, hence that $A \rightarrow B$ is of finite presentation. \square

Corollary 4.10. *Assume that we are in the situation of Theorem 4.4. If f is flat, then Φ_{Aff} and Φ_{Qaff} are equivalences of categories.*

Proof. The equivalence of Φ_{Aff} follows immediately from Corollary 4.9. For Qaff , we must work a little more. Some notation will be useful: if $W \rightarrow Y$ is quasi-affine, then let $\overline{W} \rightarrow Y$ denote its affine hull. Note that the formation of $\overline{W} \rightarrow Y$ commutes with flat base change on Y . Similarly, for a morphism $\alpha: W_1 \rightarrow W_2$ of quasi-affine schemes over Y we let $\overline{\alpha}$ denote the induced morphism between the affine hulls.

Now for the faithfulness: let $\alpha, \beta: W_1 \rightarrow W_2$ be morphisms in $\text{Qaff}(X)$ such that $\Phi_{\text{Qaff}}(\alpha) = \Phi_{\text{Qaff}}(\beta)$. By the result for Aff , we see that $\overline{\alpha} = \overline{\beta}$ and the claim follows.

Next for the fullness: consider quasi-affine X -schemes W_1 and W_2 and a morphism $(\alpha', \alpha_U): \Phi_{\text{Qaff}}(W_1) \rightarrow \Phi_{\text{Qaff}}(W_2)$. The result for Aff implies that there is a morphism $\overline{\alpha}: \overline{W}_1 \rightarrow \overline{W}_2$ such that $\Phi_{\text{Aff}}(\overline{\alpha}) = (\alpha', \overline{\alpha_U})$. It is sufficient to prove that $W_1 \subseteq \overline{\alpha}^{-1}(W_2)$. But this may be checked on points and $X' \amalg U \rightarrow X$ is surjective. The claim follows.

Finally, for the essential surjectivity. Now fix a triple (W', W_U, θ) in the codomain for Φ_{Qaff} . This leads to a triple $(\overline{W}', \overline{W_U}, \overline{\theta})$ in the codomain of Φ_{Aff} that may be glued to an affine X -scheme \overline{W} . Since $U \subseteq X$ is quasi-compact and f is flat and an isomorphism over Z , it is easily verified that $X' \amalg U \rightarrow X$ is universally submersive. In particular, by base changing along $\overline{W} \rightarrow X$ we may glue the quasi-compact open subsets $W' \subseteq \overline{W}'$ and $W_U \subseteq \overline{W_U}$ to a quasi-compact open subset $W \subseteq \overline{W}$. This proves the claim. \square

We conclude this section with the following generalization of [FR70, Cor. 4.3].

Corollary 4.11. *Assume that we are in the situation of Theorem 4.4. Let $\eta: \mathcal{O}_X \rightarrow j_*\mathcal{O}_{U'}$ and let $\eta': \mathcal{O}_{X'} \rightarrow j'_*\mathcal{O}_{U'}$ denote the unit maps.*

- (1) η is injective if and only if η' is injective.
- (2) η is integrally closed if and only if η' is integrally closed.
- (3) If \overline{X} denotes the integral closure of X in U , i.e., $\text{Spec}_X(\mathcal{A})$ where \mathcal{A} is the integral closure of \mathcal{O}_X with respect to η , then $\overline{X}' := \overline{X} \times_X X'$ is the integral closure of X' in U' and the square of $U \rightarrow \overline{X}$ and $\overline{X}' \rightarrow \overline{X}$ is a tor-independent Mayer–Vietoris square.

Proof. By Corollary 4.9(1), there is a bijection of partially ordered sets

$$\Phi: \text{Cl}_{f\text{-fl}}(X) \rightarrow \text{Cl}(X') \times_{\text{Cl}(U')} \text{Cl}_{f_U\text{-fl}}(U),$$

where $\text{Cl}(X)$ denotes the set of closed substacks $V \hookrightarrow X$ and $\text{Cl}_{f\text{-fl}}$ denotes the subset of closed substacks such that \mathcal{O}_V is f -flat. If we let \overline{U} and \overline{U}' denote the schematic closures of U and U' in X and X' respectively, then \overline{U} is f -flat (Corollary 4.8) and \overline{U} corresponds to a triple $(\overline{U} \times_X X', U', U)$ on the right hand side. But \overline{U} is minimal among the closed substacks of X that contains U and \overline{U}' is minimal among the closed substacks of X' that contains U' . It follows that $\Phi(\overline{U}) = (\overline{U}', U', U)$. Thus, $X = \overline{U}$ if and only if $X' = \overline{U}'$. Equivalently, η is injective if and only if η' is injective.

Similarly, Corollary 4.9(3) induces an equivalence of categories of integral morphisms

$$\Phi: \text{Int}_{f\text{-fl}}(X) \rightarrow \text{Int}(X') \times_{\text{Int}(U')} \text{Int}_{f_U\text{-fl}}(U).$$

If we let $\text{Int}(X, U)$ denote the integral morphisms $W \rightarrow X$ such that $W|_U \rightarrow U$ is an isomorphism and U is schematically dense in W , then $\text{Int}(X, U)$ is equivalent to the bounded lattice of sub- \mathcal{O}_X -algebras of $j_*\mathcal{O}_U$ that are integral over \mathcal{O}_X . These extensions are automatically f -flat, since they are f_U -flat after restricting to \mathcal{O}_U (Corollary 4.8). We thus obtain a bijection of bounded lattices:

$$\Phi: \text{Int}(X, U) \rightarrow \text{Int}(X', U').$$

Indeed, the only non-obvious detail is that U' is schematically dense in $\Phi(W) = W \times_X X'$ and that U is schematically dense in $\Phi^{-1}(W', U', U)$. This follows from the previous part since the square

$$\begin{array}{ccc} W \times_X U' & \longrightarrow & W \times_X X' \\ \downarrow & \square & \downarrow \\ W \times_X U & \longrightarrow & W \end{array}$$

is a tor-independent Mayer–Vietoris square (Lemma 3.1(2)). Moreover, the minimal elements of these lattices are \overline{U} and \overline{U}' , and the maximal elements are \overline{X} and \overline{X}' . The result follows. \square

5. ÉTALE SHEAVES OF SETS ON STACKS

In this section we generalize some fundamental results on constructible sheaves in SGA4 from schemes to algebraic stacks.

Let X be an algebraic stack. We let $\text{Et}(X)$ denote the category of étale representable morphisms $E \rightarrow X$. We identify $\text{Et}(X)$ with the category of cartesian lisse-étale sheaves of sets. Under this identification finitely presented étale morphisms correspond to constructible sheaves of sets.

If X is a quasi-compact and quasi-separated algebraic space or Deligne–Mumford stack, then there is an étale presentation by an affine scheme. Using this presentation it is easily seen that every étale sheaf on X is a filtered colimit of constructible sheaves. We will now extend this result to every quasi-compact and quasi-separated algebraic stack.

Recall that if $f: X \rightarrow Y$ is flat of finite presentation with geometrically reduced fibers, then there exists a factorization $X \rightarrow \pi_0(X/Y) \rightarrow Y$ where the first map has connected fibers and the second is representable and étale [Rom11, Thm. 2.5.2]. This construction commutes with arbitrary base change on Y and is functorial in X . The following result is due to J. Wise.

Proposition 5.1 ([Wis16, Thm. 4.5]). *Let $f: X \rightarrow Y$ be flat of finite presentation with geometrically reduced fibers (e.g., f smooth, quasi-compact and quasi-separated). If every étale sheaf on X is a filtered colimit of constructible sheaves (e.g., X is a quasi-compact and quasi-separated algebraic space), then $f^*: \text{Et}(Y) \rightarrow \text{Et}(X)$ admits a left-adjoint $f_!: \text{Et}(X) \rightarrow \text{Et}(Y)$ with the following properties:*

- (1) *If $(E \rightarrow X) \in \text{Et}(X)$, then the unit induces a X -morphism $E \rightarrow f^*f_!E$. This gives a factorization $E \rightarrow f_!E \rightarrow Y$ of the morphism $E \rightarrow X \rightarrow Y$ such that $E \rightarrow f_!E$ has geometrically connected fibers.*
- (2) *$f_!$ preserves constructible sheaves.*
- (3) *$f_!$ commutes with pull-back: $g^*f_! = f_!g'^*$ for any morphism $g: Y' \rightarrow Y$, where $f': X' := X \times_Y Y' \rightarrow Y'$ and $g': X' \rightarrow X$.*

Proof. For constructible sheaves, it is readily seen that $f_!(E \rightarrow X) := (\pi_0(E/Y) \rightarrow Y)$ is a left adjoint of f^* and it commutes with arbitrary base change. It remains to extend the construction to non-constructible étale sheaves $E \rightarrow X$. If $E = \varinjlim E_\lambda$ is a filtered colimit of constructible sheaves, then necessarily $f_!E = \varinjlim f_!E_\lambda$. \square

We may now generalize [SGA4₃, Exp. IX, Cor. 2.7.2, Prop. 2.14] and [SGA4₃, Exp. XII, Prop. 6.5 (i)] to quasi-compact and quasi-separated algebraic stacks.

Proposition 5.2. *Let X be a quasi-compact and quasi-separated algebraic stack. Then every étale sheaf of sets is a filtered colimit of constructible sheaves.*

Proof. The result is known for affine schemes (and quasi-compact and quasi-separated schemes). Pick a smooth presentation $p: U \rightarrow X$ with U affine. Let $F \rightarrow X$ be an étale sheaf. Choose an epimorphism $\coprod_{i \in I} G'_i \rightarrow p^*F$ where the G'_i are constructible. Let $G_i = p_!G'_i$ which is a constructible sheaf. Then $\coprod G_i = p_!(\coprod G'_i) \rightarrow F$ is an epimorphism since $\coprod G'_i \rightarrow p^*p_!(\coprod G'_i) \rightarrow p^*F$ is an epimorphism.

The remainder of the proof is standard, cf. [SGA4₃, Exp. IX, Cor. 2.7.2]. For every finite subset $J \subseteq I$, the coproduct $G_J = \coprod_{i \in J} G_i$ is constructible. The fiber product $H_J := G_J \times_F G_J$ is not constructible but at least quasi-separated since it is a subsheaf of the constructible sheaf $G_J \times_X G_J$. Consider the set Λ of pairs (J, H') where $J \subseteq I$ is finite and $H' \subseteq H_J$ is quasi-compact, and hence constructible. For $\lambda = (J, H') \in \Lambda$, let $F_\lambda = \text{coker}(H' \rightrightarrows G_J)$ which is a constructible sheaf. We order Λ by $(J_1, H'_1) \leq (J_2, H'_2)$ if $J_1 \subseteq J_2$ and $g(H'_1) \subseteq H'_2$ where $g: H_{J_1} \rightarrow H_{J_2}$. Then $F = \varinjlim_{\lambda \in \Lambda} F_\lambda$ is a filtered colimit of constructible sheaves. \square

Proposition 5.3. *Let X be a quasi-compact and quasi-separated algebraic stack. Let $F \in \text{Et}(X)$ be a constructible sheaf of sets. Then there exists finite morphisms $p_i: X'_i \rightarrow X$, $i = 1, 2, \dots, n$ and finite sets A_1, A_2, \dots, A_n and a monomorphism $F \hookrightarrow \prod (p_i)_* \underline{A}_{i, X'_i}$.*

Proof. There exists a stratification of X into locally closed constructible substacks Y_i such that $F|_{Y_i}$ is locally constant [Ryd11, Prop. 4.4]. If $u_i: Y_i \rightarrow X$ denotes the corresponding quasi-compact immersion, then $F \rightarrow \prod (u_i)_*(u_i)^*F$ is a monomorphism. After refining the stratification, we can assume that the cardinality of $F|_{Y_i}$ is constant. Let $q_i: Y'_i \rightarrow Y_i$ be a finite étale surjective morphism such that $q_i^*(u_i)^*F$ is a constant sheaf with value A_i .

Let X_i be the closure of Y_i and let $p_i: X'_i \rightarrow X$ be the integral closure of X with respect to $Y'_i \rightarrow Y_i \rightarrow X_i \rightarrow X$. Then p_i is integral and $p_i|_{Y_i} = q_i$. If $v_i: Y'_i \rightarrow X'_i$ denotes the open immersion, then $(v_i)_* \underline{A}_{i, Y'_i} = \underline{A}_{i, X'_i}$ is constant. Thus,

$$F \rightarrow \prod (u_i)_*(u_i)^*F \hookrightarrow \prod (u_i)_*(q_i)_*(q_i)^*(u_i)^*F = \prod (p_i)_* \underline{A}_{i, X'_i}$$

is a monomorphism.

Finally, write $X'_i \rightarrow X_i$ as an inverse limit of finite morphisms [Ryd16]. By an easy limit argument, we can replace p_i by a finite morphism. \square

For an algebraic stack X , we let $\text{OC}(X)$ denote the boolean algebra of closed and open substacks.

Proposition 5.4. *Let $h: Y \rightarrow X$ be a morphism of algebraic stacks. If X is quasi-compact and quasi-separated, then the following conditions are equivalent.*

- (1) *For every sheaf of sets $F \in \text{Et}(X)$, the canonical map*

$$H^0(X, F) \rightarrow H^0(Y, h^*F)$$

is bijective.

- (2) *Condition (1) for constructible sheaves.*

- (3) *For every finite morphism $f: X' \rightarrow X$, the canonical map*

$$\text{OC}(X') \rightarrow \text{OC}(Y \times_X X')$$

is bijective.

Proof. The equivalence between (1) and (2) follows by Proposition 5.2. That (1) implies (3) follows by the following two observations: (a) if A is a two-point set, then $H^0(X, f_* \underline{A}_{X'}) = \text{OC}(X')$, and (b) by finite base change $h^* f_* \underline{A}_{X'} = (f_Y)_* \underline{A}_{Y \times_X X'}$.

To see that (3) implies (1), take a monomorphism $F \hookrightarrow G$ as in Proposition 5.3. Then by (3), $H^0(X, G) \rightarrow H^0(Y, h^* G)$ is bijective. It follows that $H^0(X, F) \rightarrow H^0(Y, h^* F)$ is injective. Finally, take $H = G \amalg_F G$. Then we have a diagram

$$\begin{array}{ccccc} H^0(X, F) & \longrightarrow & H^0(X, G) & \xrightarrow{\cong} & H^0(X, H) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(Y, h^* F) & \longrightarrow & H^0(Y, h^* G) & \xrightarrow{\cong} & H^0(Y, h^* H) \end{array}$$

with exact rows and injective vertical maps and bijective middle map. It follows that the left map is bijective. \square

We recall the following well-known definition.

Definition 5.5 (Henselian pairs). A pair of algebraic stacks (X, X_0) is a *henselian pair* if $i: X_0 \hookrightarrow X$ is a closed immersion and for every finite morphism $X' \rightarrow X$, the natural map

$$\text{OC}(X') \rightarrow \text{OC}(X' \times_X X_0)$$

is bijective.

We have the following simple lemma.

Lemma 5.6. *Let (X, X_0) be a henselian pair. Let $X' \rightarrow X$ be an integral morphism. If X is quasi-compact and quasi-separated, then $(X', X' \times_X X_0)$ is a henselian pair.*

Proof. Since $X' \rightarrow X$ is a limit of finite morphisms [Ryd16], the result follows from a simple approximation argument. \square

Remark 5.7 (Proper base change). Let (X, X_0) be a henselian pair, where X is quasi-compact and quasi-separated. If $g: X' \rightarrow X$ is *proper and representable*, then $(X', X' \times_X X_0)$ is a henselian pair (see [HR14b, Cor. B.4] and [Gab94, Cor. 1]). This follows from the existence of the Stein factorization $X' \rightarrow \text{Spec}_{\mathcal{O}_X} g_* \mathcal{O}_{X'} \rightarrow X$ where the first map is proper with geometrically connected fibers and the second map is integral [Stacks, 0A1C]. This is a baby case of the proper base change theorem in étale cohomology.

6. MAYER-VIETORIS SQUARES IN ÉTALE COHOMOLOGY

We will now glue étale morphisms, or equivalently, étale sheaves of sets. It is thus natural to introduce the following squares which are analogous to Mayer-Vietoris D_{qc} -squares.

Definition 6.1. Fix a cartesian square as in (1.1). It is a Mayer-Vietoris Et-square if the following conditions are satisfied:

- (1) the natural transformation $f^* j_* \rightarrow j'_* f'_*$ is an isomorphism for every cartesian sheaf of sets $F \in \text{Et}(U)$; and
- (2) $f^*: \text{Et}_Z(X) \rightarrow \text{Et}_{Z'}(X')$ is an equivalence of categories, where $\text{Et}_Z(X) = \{F \in \text{Et}(X) : j^* F = 0\}$ and similarly for $\text{Et}_{Z'}(X')$.

Note that $\text{Et}_Z(X)$ does not depend on the choice of Z and that $i_*: \text{Et}(Z) \rightarrow \text{Et}_Z(X)$ is an isomorphism.

For Mayer-Vietoris Et-squares, gluing is immediate from recollement.

Theorem 6.2. *Consider a Mayer–Vietoris Et-square. Then the functor*

$$\Phi_{\text{Et}}: \text{Et}(X) \rightarrow \text{Et}(X') \times_{\text{Et}(U')} \text{Et}(U)$$

is an equivalence of categories.

Proof. By recollement [SGA4₁, Exp. IV, Thm. 9.5.4],

$$\text{Et}(X) \cong (\text{Et}(Z), \text{Et}(U), i^*j_*),$$

that is, the category $\text{Et}(X)$ is equivalent to the category of triples $E_Z \in \text{Et}(Z)$, $E_U \in \text{Et}(U)$, $\psi: E_Z \rightarrow i^*j_*E_U$. Similarly,

$$\text{Et}(X') \cong (\text{Et}(Z'), \text{Et}(U'), i'^*j'_*)$$

and

$$\begin{aligned} \text{Et}(X') \times_{\text{Et}(U')} \text{Et}(U) &\cong (\text{Et}(Z'), \text{Et}(U), i'^*j'_*f_U^*) \\ &\cong (\text{Et}(Z), \text{Et}(U), (f_Z)_*i'^*j'_*f_U^*) \end{aligned}$$

where we have used that $(f_Z)_*$ is an equivalence of categories. Since $(f_Z)_*i'^*j'_*f_U^* = (f_Z)_*i'^*f^*j_* = i^*j_*$ the result follows. \square

We will now proceed to show that weak Mayer–Vietoris squares are Mayer–Vietoris Et-squares. We begin with the following result that generalizes [FR70, Cor. 4.4].

Proposition 6.3. *Fix a weak Mayer–Vietoris square as in (1.1). Assume that (X, Z) and (X', Z') are henselian pairs. If X, X', U and U' are all quasi-compact and quasi-separated, then the natural map*

$$f_U^*: \text{OC}(U) \rightarrow \text{OC}(U')$$

is bijective.

Proof. Since X and U are quasi-compact and quasi-separated, we may assume that the complement $i: Z \hookrightarrow X$ is finitely presented [Ryd16, Prop. 8.2]. Thus, we may replace the square with its blow-up so that it becomes a tor-independent Mayer–Vietoris square (Lemma 3.4). Note that (X', Z') and (X, Z) remain henselian pairs (Remark 5.7).

By Corollary 4.11, we may replace X and X' by \overline{X} and \overline{X}' and assume that X and X' are integrally closed with respect to U and U' respectively. Since the open and closed subsets of an algebraic stack W are in bijection with idempotents of $\Gamma(W, \mathcal{O}_W)$, it follows that $\text{OC}(X) \rightarrow \text{OC}(U)$ and $\text{OC}(X') \rightarrow \text{OC}(U')$ are bijections. The corollary thus follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} \text{OC}(Z) & \xleftarrow{\cong} & \text{OC}(X) & \xrightarrow{\cong} & \text{OC}(U) \\ & & \downarrow & & \downarrow \\ \text{OC}(Z) & \xleftarrow{\cong} & \text{OC}(X') & \xrightarrow{\cong} & \text{OC}(U'). \end{array} \quad \square$$

We can now prove Gabber’s rigidity theorem. For affine henselian pairs, this is proven in [ILO14, Exp. 20, Thm. 2.1.1]. See Remark 6.7 for some history of this result.

Theorem 6.4 (Rigidity theorem). *Fix a weak Mayer–Vietoris square as in (1.1). Assume that (X, Z) and (X', Z') are henselian pairs. If X, X', U and U' are all quasi-compact and quasi-separated, then the natural map:*

$$H^0(U, F) \rightarrow H^0(U', F)$$

is a bijection for all sheaves of sets $F \in \text{Et}(U)$.

Proof. It is enough to prove that $\mathrm{OC}(V) \rightarrow \mathrm{OC}(U' \times_U V)$ is bijective for every finite morphism $V \rightarrow U$ (Proposition 5.4). By Zariski's main theorem [Ryd16, Thm. 8.1], we can extend the finite morphism $V \rightarrow U$ to a finite morphism $\bar{V} \rightarrow X$. Since weak Mayer–Vietoris squares are stable under arbitrary base change (Lemma 3.1(1)), it is enough to prove that $\mathrm{OC}(U) \rightarrow \mathrm{OC}(U')$ is bijective, which is Proposition 6.3. \square

Corollary 6.5. *Fix a weak Mayer–Vietoris square as in (1.1). If j is quasi-compact, then it is a Mayer–Vietoris \mathbf{Et} -square.*

Proof. We need to verify that the natural morphism $f^*j_* \rightarrow j'_*f_U^*$ is an isomorphism. This equality certainly holds over U' since the counits of the adjunctions (j^*, j_*) and (j'^*, j'_*) are isomorphisms and hence $j'^*j'_*f_U^* = f_U^*$ and $j'^*f^*j_* = f_U^*j^*j_* = f_U^*$. It is thus enough to verify the equality over points of Z . We can first assume that X is affine by working smooth-locally on X and then replace X with the henselization at a point $z \in Z$. Then X' is Deligne–Mumford in a neighborhood of Z and we can thus replace X' with the henselization at $z \in Z$; in particular, X and X' are quasi-compact and quasi-separated. Then the equality $f^*j_* = j'_*f_U^*$ becomes $H^0(U, F) = H^0(U', f_U^*F)$, which follows by the rigidity theorem. \square

We can now prove Theorem C.

Proof of Theorem C. Combine Corollary 6.5 with Theorem 6.2. \square

Corollary 6.6. *Fix a weak Mayer–Vietoris square as in (1.1). If j is quasi-compact, then $X' \amalg U \rightarrow X$ is universally submersive and $|X| = |X'| \amalg_{|U'|} |U|$ is a pushout of topological spaces.*

Proof. Since weak Mayer–Vietoris squares are preserved under arbitrary base change it is enough to prove the latter statement. Set-theoretically, $|X| = |X'| \amalg_{|U'|} |U|$ holds since $f_Z: Z' \rightarrow Z$ is an isomorphism. It is thus enough to prove that $|X|$ has the correct topology. Now a morphism of stacks is an open immersion if and only if it is an étale monomorphism. That an étale morphism is a monomorphism can be checked pointwise; thus, we have a bijection

$$\Phi_{\mathrm{Op}}: \mathrm{Op}(X) \rightarrow \mathrm{Op}(X') \times_{\mathrm{Op}(U')} \mathrm{Op}(U).$$

It follows that a subset $W \subseteq |X|$ is open if and only if $j^{-1}(W)$ and $f^{-1}(W)$ are open. \square

Remark 6.7. The rigidity theorem holds more generally for cohomology as well. Fix a weak Mayer–Vietoris square as in (1.1) and assume that (X, Z) and (X', Z') are affine henselian pairs. If $n = 0$ (resp. $n \leq 1$, resp. n an integer), then

$$H^n(U, F) \rightarrow H^n(U', F)$$

is a bijection for all sheaves of sets $F \in \mathbf{Et}(U)$ (resp. sheaves of ind-finite groups, resp. sheaves of torsion abelian groups). When X is noetherian, this is Gabber–Fujiiwara's rigidity theorem [Fuj95, Cor. 6.6.4]. For $n = 0, 1$, this was extended to non-noetherian schemes by Gabber [Gab05, Thm. 7.1], cf. [ILO14, Exp. 20, Thm. 2.1.1]. For $n \geq 2$, the non-noetherian case is sketched by Gabber in [ILO14, Exp. 20, §4.4, CTC]. Note that the general case reduces to the case where X' is the completion of X in Z . Indeed, such a completion is a weak Mayer–Vietoris square and the completions of X in Z and X' in Z are equal by definition.

7. GLUING OF ALGEBRAIC SPACES ALONG MAYER–VIETORIS SQUARES

In this section, we prove the main theorems of the article. We begin with a slight strengthening of Theorem E.

Proposition 7.1. *Fix an algebraic stack S and a tor-independent Mayer–Vietoris square as in (1.1) over S with j quasi-compact. Let $W \rightarrow S$ be an algebraic stack. Then*

$$\Phi_{\mathrm{Hom}_S(-, W)} : \mathrm{Hom}_S(X, W) \rightarrow \mathrm{Hom}_S(X', W) \times_{\mathrm{Hom}_S(U', W)} \mathrm{Hom}_S(U, W)$$

is fully faithful. If either

- (1) $W \rightarrow S$ is Deligne–Mumford; or
- (2) $\Delta_{W/S}$ is quasi-finite and $\Delta_{\Delta_{W/S}}$ is a quasi-compact immersion (e.g., $\Delta_{W/S}$ separated);

then $\Phi_{\mathrm{Hom}_S(-, W)}$ is an equivalence of groupoids. In particular, the square is co-cartesian in the category of Deligne–Mumford stacks.

Proof. The question is fppf-local on X , so we may assume that X is affine. We may also replace S and W with X and $W \times_S X \rightarrow X$ and assume that $X = S$. Further, we may replace X' with a quasi-compact open neighborhood of Z . Then, we may also assume that W is quasi-compact.

If $W \rightarrow X$ is arbitrary (resp. representable, resp. a monomorphism), then $\Delta_{W/X}$ is representable (resp. a monomorphism, resp. an isomorphism). Fully faithfulness of $\Phi_{\mathrm{Hom}_X(-, W)}$ follows if $\Phi_{\mathrm{Hom}_X(-, W \times_W \times_X W)}$ is an equivalence for every morphism $X \rightarrow W \times_X W$. By induction on the diagonal, we may thus assume that $\Phi_{\mathrm{Hom}_X(-, W)}$ is fully faithful.

If W is Deligne–Mumford, then there exists an étale presentation $W' \rightarrow W$. If W is as in the second case, then by [Ryd11, Thm. 7.2] there exists an étale representable morphism $W' \rightarrow W$ and a finite faithfully flat morphism $V \rightarrow W'$ such that V is affine.

Given maps $U \rightarrow W$ and $X' \rightarrow W$ that agree on U' , we obtain, by pulling back $W' \rightarrow W$, an element of $\mathrm{Et}(X') \times_{\mathrm{Et}(U')} \mathrm{Et}(U)$, hence a unique element of $(E \rightarrow X) \in \mathrm{Et}(X)$ by Corollary 6.5 and Theorem 6.2. Pulling-back the square along $E \rightarrow X$, we may replace X by E and assume that $W = W'$.

In the second case, we pull-back $V \rightarrow Z' = Z$ to finite flat morphisms over X' , U' and U . These glue to a unique finite faithfully flat morphism $F \rightarrow X$ (Corollary 4.9 and Theorem 4.4(3b)). We may thus replace X with F and assume that $V = W = W'$.

Now W is affine. Let $A_W = \Gamma(W, \mathcal{O}_W)$ for any algebraic stack W . The map $\Phi_{\mathrm{Hom}(-, W)}$ then becomes

$$\begin{aligned} \mathrm{Hom}(A_W, A_X) &\rightarrow \mathrm{Hom}(A_W, A_{X'}) \times_{\mathrm{Hom}(A_W, A_{U'})} \mathrm{Hom}(A_W, A_U) \\ &= \mathrm{Hom}(A_W, A_{X'} \times_{A_{U'}} A_U). \end{aligned}$$

This is an isomorphism since $A_X \rightarrow A_{X'} \times_{A_{U'}} A_U$ is an isomorphism by Theorem 4.4 applied to the structure sheaf \mathcal{O}_X . \square

We can now prove Theorem E.

Proof of Theorem E. This is the last statement of Proposition 7.1. \square

We can now also generalize Corollary 4.6 from quasi-coherent sheaves to algebraic spaces.

Corollary 7.2. *Fix a tor-independent Mayer–Vietoris square as in (1.1) with j quasi-compact. Let $Y \rightarrow X$ and $Z \rightarrow X$ be algebraic spaces. If $Y \rightarrow X$ is f -flat, then*

$$\mathrm{Hom}_X(Y, Z) \rightarrow \mathrm{Hom}_X(Y \times_X X', Z) \times_{\mathrm{Hom}_X(Y \times_X U', Z)} \mathrm{Hom}_X(Y \times_X U, Z)$$

is bijective.

Proof. Since $Y \rightarrow X$ is f -flat, the pull-back of the square along $Y \rightarrow X$ is a tor-independent Mayer–Vietoris square (Lemma 3.1(2)). The result thus follows from Proposition 7.1. \square

We now have proved Theorem D in its entirety.

Proof of Theorem D. Claim (1) is Theorem 4.4 and claim (2) is Corollary 7.2. \square

Remark 7.3. The map in Corollary 7.2 need not be injective if the square is a weak Mayer–Vietoris square. Indeed, Example 3.9 is an example of a weak Mayer–Vietoris square such that $\Gamma(X) \rightarrow \Gamma(X') \times_{\Gamma(U')} \Gamma(U)$ is not injective. If $f, g \in \Gamma(X)$ are two element that have equal images, then the corresponding maps $f, g: X \rightarrow \mathbb{A}^1$ become equal after restricting to X' and U .

We can also now prove Theorem B.

Proof of Theorem B. That Φ_{QCoh} is an equivalence is Corollary 4.7. That Φ_{Aff} and Φ_{Qaff} are equivalences is Corollary 4.10. That Φ_{AlgSp} is fully faithful is a special case of Corollary 7.2. That $\Phi_{\mathrm{Hom}(-, W)}$ is fully faithful for every algebraic stack W is Proposition 7.1. That $\Phi_{\mathrm{Hom}(-, W)}$ is an equivalence when W has quasi-affine diagonal follows from Corollary 4.10 and an identical argument to [MB96, Cor. 6.5.1(a)].

It remains to prove (4): $\Phi_{\mathrm{AlgSp}_{\mathrm{fpp}}}$ is an equivalence when X is locally excellent. For quasi-separated algebraic spaces the essential surjectivity of $\Phi_{\mathrm{AlgSp}_{\mathrm{fpp}, \mathrm{qs}}}$ follows as in [MB96, Thm. 5.2 (ii), Cor. 5.6 (iii), Thm. 5.7] but since we are working in a slightly more general setting let us write out the details. For brevity, we let $\Phi = \Phi_{\mathrm{AlgSp}_{\mathrm{fpp}}}$.

Since algebraic spaces satisfy descent for the fppf topology, we may use Proposition 3.7 and the étale gluing result [Ryd11, Thm. A], and so assume that X is affine and X' is quasi-affine.

If P is a property of morphisms of algebraic spaces, then we say that a morphism of triples is P if the three components are P . Since $X' \amalg U \rightarrow X$ is faithfully flat and quasi-compact, a morphism $f: W \rightarrow Z$ in $\mathrm{AlgSp}_{\mathrm{fpp}}(X)$ is quasi-compact (resp. quasi-separated, resp. étale, resp. open, resp. a monomorphism) if and only if $\Phi(f)$ has the same property [EGA, IV.2.7.1, IV.17.7.3 (ii)].

We now prove essential surjectivity of Φ . Thus, consider a triple $W' \rightarrow X'$, $W_U \rightarrow U$, $W_{U'} \rightarrow U'$ on the right-hand side.

We will begin by showing that it is enough to prove essential surjectivity of Φ for the subcategories of *quasi-compact* algebraic spaces (cf. [MB96, Thm. 5.7]). Write W' and W_U as filtered unions of quasi-compact open subspaces W'_λ and $W_{U, \mu}$ respectively. Since $j': U' \rightarrow X'$ is quasi-compact, for every λ , the open subspace $W'_\lambda \cap W_{U'}$ is quasi-compact. Hence, for sufficiently large $\mu = \mu(\lambda)$, the inverse image $W_{U', \mu} := f_U^{-1}(W_{U, \mu})$ contains $W'_\lambda \cap W_{U'}$. We may thus form the triple $(W'_\lambda \cup W_{U', \mu}, W_{U, \mu}, W_{U', \mu})$ of quasi-compact algebraic spaces. By assumption, this triple is in the essential image of Φ and descends to an algebraic space $W_{\lambda, \mu}$. We then let $W = \bigcup_{\lambda, \mu} W_{\lambda, \mu}$ where the union runs over all λ and $\mu \geq \mu(\lambda)$.

We next assume that these morphisms are also quasi-compact and quasi-separated. In this case, we claim that we are free to replace X with any flat covering $(X_i \rightarrow X)$ such that every $X_i \rightarrow X$ is a filtered limit of flat and finitely presented morphisms $X_{i, \lambda} \rightarrow X$. Indeed, assume that the result holds for the X_i , that is,

there exists an algebraic space $W_i \rightarrow X_i$ of finite presentation such that $\Phi(W_i) \cong (W', W_{U'}, W_U) \times_X X_i$. Then, by standard limit arguments, there is for every i and every sufficiently large $\lambda = \lambda(i)$ an algebraic space $W_{i,\lambda} \rightarrow X_{i,\lambda}$ of finite presentation such that $\Phi_{X_{i,\lambda}}(W_{i,\lambda}) \cong (W', W_{U'}, W_U) \times_X X_{i,\lambda}$. Since Φ is fully faithful over $X_{i,\lambda} \times_X X_{i,\lambda}$ and $X_{i,\lambda} \times_X X_{i,\lambda} \times_X X_{i,\lambda}$ there is a canonical gluing datum for $W_{i,\lambda} \rightarrow X_{i,\lambda}$ along $X_{i,\lambda} \rightarrow X$ which is flat and of finite presentation. So by fppf descent, $W_{i,\lambda} \rightarrow X_{i,\lambda}$ descends to an algebraic space over the open image of $X_{i,\lambda} \rightarrow X$. Since we can find a finite number of such $X_{i,\lambda}$ that cover X , the claim follows.

Since X is excellent the completion map $\widehat{X}_x \rightarrow X$ is a regular morphism. Hence, by Popescu's theorem [Pop85], it is a limit of smooth morphisms. Since $(\widehat{X}_x \rightarrow X)_{x \in X}$ is a flat cover, we may replace X with \widehat{X}_x for some x and assume that X is the spectrum of a complete local ring. The completion of X' at z equals the completion of X at z ; hence, $X' \rightarrow X$ has a section $s: X \rightarrow X'$. By Lemma 3.5, this gives rise to a new tor-independent Mayer-Vietoris square. Corollary 7.2 for this square implies that

$$\begin{aligned} & \mathrm{Hom}(X', W') \times_{\mathrm{Hom}(U', W_{U'})} \mathrm{Hom}(U, W_U) \\ &= (\mathrm{Hom}(X, s^*W') \times_{\mathrm{Hom}(U, W_U)} \mathrm{Hom}(U', W_{U'})) \times_{\mathrm{Hom}(U', W_{U'})} \mathrm{Hom}(U, W_U) \\ &= \mathrm{Hom}(X, s^*W'), \end{aligned}$$

which implies that $\Phi(s^*W') \cong (W', W_{U'}, W_U)$. Thus, Φ is essentially surjective for finitely presented algebraic spaces. In fact, by the initial reduction to the quasi-compact case, we have proved that Φ is essentially surjective for triples of quasi-separated algebraic spaces.

Let us finally prove that Φ is also essentially surjective for algebraic spaces that are not quasi-separated. It is enough to prove that it is essentially surjective for quasi-compact algebraic spaces. By the previous argument, it is enough to prove that if $\overline{X} = \varprojlim_{\lambda} X_{\lambda}$ is a limit of affine schemes, and $(\overline{W}', \overline{W}_{U'}, \overline{W}_U) := (W', W_{U'}, W_U) \times_X \overline{X}$ is in the essential image of Φ , then so is $(W', W_{U'}, W_U) \times_X X_{\lambda}$ for sufficiently large λ .

Thus, let $\overline{W} \rightarrow \overline{X}$ be an algebraic space such that $\Phi(\overline{W}) = (\overline{W}', \overline{W}_{U'}, \overline{W}_U)$ and pick an affine presentation $\overline{V} \rightarrow \overline{W}$. Note that $\overline{V} \rightarrow \overline{W} \rightarrow \overline{X}$ is finitely presented. This induces morphisms of triples

$$\Phi(\overline{V}) = (\overline{V}', \overline{V}_{U'}, \overline{V}_U) \rightarrow (\overline{W}', \overline{W}_{U'}, \overline{W}_U) \rightarrow (\overline{X}', \overline{U}', \overline{U})$$

where the first map is surjective and étale and the composition is of finite presentation. For sufficiently large λ , we may thus descend this to a morphism of triples

$$(V'_{\lambda}, V_{U',\lambda}, V_{U,\lambda}) \rightarrow (W'_{\lambda}, W_{U',\lambda}, W_{U,\lambda}) \rightarrow (X'_{\lambda}, U'_{\lambda}, U_{\lambda})$$

over X_{λ} where the first map is étale and the composition is of finite presentation. Thus, there exists an algebraic space $V_{\lambda} \rightarrow X_{\lambda}$, unique up to unique isomorphism, such that $\Phi(V_{\lambda}) \cong (V'_{\lambda}, V_{U',\lambda}, V_{U,\lambda})$.

Let $R'_{\lambda} = V'_{\lambda} \times_{W'_{\lambda}} V'_{\lambda}$ and similarly over U'_{λ} and U_{λ} . Then the triple $(R'_{\lambda}, R_{U',\lambda}, R_{U,\lambda})$ is locally of finite presentation and quasi-separated over $(X'_{\lambda}, U'_{\lambda}, U_{\lambda})$ and hence isomorphic to $\Phi(R_{\lambda})$ for an essentially unique $R_{\lambda} \rightarrow X_{\lambda}$. By fully faithfulness, we obtain an étale equivalence relation $R_{\lambda} \rightrightarrows V_{\lambda}$ and we let W_{λ} be its quotient algebraic space. By fully faithfulness, $\Phi(W_{\lambda})$ is isomorphic to $(W'_{\lambda}, W_{U',\lambda}, W_{U,\lambda})$ and the theorem follows. \square

Finally, we prove Theorem A.

Proof of Theorem A. We must show that for every algebraic stack W , the functor

$$\Phi_{\mathrm{Hom}(-, W)}: \mathrm{Hom}(X, W) \rightarrow \mathrm{Hom}(X', W) \times_{\mathrm{Hom}(U', W)} \mathrm{Hom}(U, W)$$

is an equivalence of groupoids. Using Proposition 3.7 and the étale gluing result [Ryd11, Thm. A], we may assume that X is affine and X' is quasi-affine. In particular, we are free to assume that W is quasi-compact. We have already seen that $\Phi_{\mathrm{Hom}(-, W)}$ is fully faithful in Proposition 7.1. To see that it is essentially surjective, pick a smooth presentation $W_0 \rightarrow W$ where W_0 is an affine scheme. Pulling back, we obtain a triple in $(X'_0, U'_0, U_0) \in \mathrm{AlgSp}_{\mathrm{lfp}}(X') \times_{\mathrm{AlgSp}_{\mathrm{lfp}}(U')} \mathrm{AlgSp}_{\mathrm{lfp}}(U)$; hence, a representable morphism $X_0 \rightarrow X$ by Theorem B. Since $X' \amalg U' \rightarrow X$ is faithfully flat and quasi-compact, it follows that $X_0 \rightarrow X$ is smooth. Also,

$$\begin{array}{ccc} U'_0 & \longrightarrow & X'_0 \\ \downarrow & \square & \downarrow \\ U_0 & \longrightarrow & X_0, \end{array}$$

is a flat Mayer–Vietoris square. Since W_0 is affine, we obtain a unique morphism $X_0 \rightarrow W_0$ compatible with $X'_0 \rightarrow W_0$ and $U_0 \rightarrow W_0$ (Proposition 7.1). By the full faithfulness of $\Phi_{\mathrm{Hom}(-, Z)}$, the induced morphisms $X_0 \times_X X_0 \xrightarrow{\cong} X_0 \rightarrow W_0 \rightarrow W$ coincide, so there is a unique morphism $X \rightarrow W$ and the result follows. \square

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