THE ÉTALE LOCAL STRUCTURE OF ALGEBRAIC STACKS

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Abstract. We prove that an algebraic stack with affine stabilizers over an arbitrary base is étale-locally a quotient stack around any point with a linearly reductive stabilizer. This generalizes earlier work by the authors of this article (stacks over algebraically closed fields) and by Abramovich, Olsson and Vistoli (stacks with finite inertia). In addition, we prove a number of foundational results, which are new even over a field. These include various coherent completeness and effectivity results for adic sequences of algebraic stacks. Finally, we give several applications of our results and methods, such as structure theorems for linearly reductive group schemes and generalizations to the relative setting of Sumihiro's theorem on torus actions and Luna's étale slice theorem.

Résumé (*Les voisinages étales dans les champs algébriques*)
Soit X un champ algébrique, localement de présentation finie et quasi-séparé sur un espace algébrique quasi-séparé, avec stabilisateurs affines. Nous montrons que tout point de X, avec stabilisateur linéairement réductif, possède un voisinage étale qui est un champ quotient. Ce résultat généralise les travaux précédents des auteurs de cet article (champs sur un corp algébriquement clos) et d'Abramovich, Olsson et Vistoli (champs avec inertie finie). En outre, nous obtenons plusieurs résultats fondamentaux qui sont nouveaux même sur un corps. Ceux-ci comprennent divers résultats de complétion cohérente d'un champ algébrique le long d'un sous-champ fermé, ainsi que d'effectivité de déformations formelles d'un champ algébrique. Enfin, nous fournissons plusieurs applications de ces résultats, notamment des généralisations au cadre relatif du théorème de Sumihiro sur les actions de tores et du théorème du « slice étale » de Luna.

Contents

1. Introduction

This paper offers a broad generalization and extension of our previous work [AHR20], which provided a local structure theorem for algebraic stacks of finite

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type over an algebraically closed field. In addition to establishing a local structure theorem for algebraic stacks defined over an arbitrary base (Theorem 1.1), we prove a number of foundational results that are new even over an algebraically closed field. This includes a general coherent completeness result for algebraic stacks (Theorem 1.6), which becomes particularly powerful when coupled with Tannaka duality (see §1.7). We also prove an effectivity theorem for adic sequences of noetherian algebraic stacks (Theorem 1.10), analogous to Grothendieck's result on algebraization of formal schemes [EGA, III.5.4.5]. While of independent interest, this is one of the key ingredients for the other main theorems in this paper—including the local structure theorem.

We prove several other foundational results and provide numerous applications to equivariant geometry and moduli theory. To highlight a few, we prove that adequate moduli spaces are universal for maps to algebraic spaces (Theorem 3.12); this implies that GIT quotients in positive characteristic are categorical quotients in algebraic spaces, which was formerly unknown—even over an algebraically closed field. We establish that various properties (e.g., the resolution property[1]) and objects (e.g., morphisms, finite étale covers, vector bundles), defined on a closed substack $\mathcal{X}_0$ of $\mathcal{X}$, extend to an étale neighborhood of $\mathcal{X}_0$ if there is an affine good moduli space $\mathcal{X} \to X$ (see Propositions 7.17 to 7.20). These extension results are technical but extremely useful in local-to-global arguments. We also prove a generalization of Sumihiro's theorem on torus actions (Corollary 10.2): an algebraic space X over a base S with an action of a torus $T \to S$ has T-equivariant affine étale neighborhoods.

1.1. **A local structure theorem.** We prove that if $\mathcal{X}$ is an algebraic stack satisfying some mild assumptions, then a point $x \in |\mathcal{X}|$ with linearly reductive stabilizer has an étale neighborhood $([\operatorname{Spec} A/\mathrm{GL}_n], w) \to (\mathcal{X}, x)$ inducing an isomorphism on residual gerbes. This is the conclusion of the following theorem in the special case that $\mathcal{W}_0 = \mathcal{G}_x$.

Theorem 1.1 (Local structure)**.** *Let S be a quasi-separated algebraic space and $\mathcal{X}$ be an algebraic stack, locally of finite presentation, and quasi-separated over S, with affine stabilizers. Let $x \in |\mathcal{X}|$ be a point with residual gerbe $\mathcal{G}_x$ and image $s \in |S|$ such that the residue field extension $\kappa(x)/\kappa(s)$ is finite. Let $h_0 \colon \mathcal{W}_0 \to \mathcal{G}_x$ be a smooth (resp. étale) morphism where $\mathcal{W}_0$ is a gerbe over the spectrum of a field and has linearly reductive stabilizer. Then there exist an algebraic stack $\mathcal{W} = [\operatorname{Spec} A/\mathrm{GL}_n]$ and a point $w \in |\mathcal{W}|$ with an identification $\mathcal{G}_w = \mathcal{W}_0$ together with a cartesian diagram*

$$\begin{array}{ccc} \mathcal{G}_w = \mathcal{W}_0 & \xrightarrow{h_0} & \mathcal{G}_x \\ \downarrow & & \downarrow \\ [\operatorname{Spec} A/\mathrm{GL}_n] = \mathcal{W} & \xrightarrow{h} & \mathcal{X}, \end{array}$$

where $h \colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ is a smooth (resp. étale) pointed morphism and w is closed in its fiber over s. Moreover, if $\mathcal{X}$ has separated (resp. affine) diagonal and h_0 is representable, then h can be arranged to be representable (resp. affine).

The flexibility of allowing $\mathcal{W}_0 \to \mathcal{G}_x$ to be a smooth or étale morphism (rather than an isomorphism) is useful when the stabilizer G_x of x is not linearly reductive, which is a particularly restrictive condition in positive characteristic. Assuming that the residual gerbe $\mathcal{G}_x = BG_x$ is trivial, one can choose any linearly reductive

[1]That is, every quasi-coherent sheaf is a quotient of a direct sum of vector bundles. If the algebraic stack is quasi-compact and quasi-separated with affine stabilizers, then this is equivalent to being expressible as $[U/\mathrm{GL}_n]$, where U is a quasi-affine scheme [Tot04, Gro17].

subgroup $H \subseteq G_x$ such that G_x/H is smooth (resp. étale) and apply the theorem to the smooth (resp. étale) morphism $\mathcal{W}_0 = BH \to BG_x$.

Remark 1.2 (Known results). In the case that $\mathcal{X}$ has finite inertia and h_0 is an isomorphism, this theorem had been established in [AOV11, Thm. 3.2]. When $S = \operatorname{Spec} k$ and k is algebraically closed, this theorem follows from [AHR20, Thm. 1.1], which established the stronger result that $\mathcal{W}$ can be written as $[\operatorname{Spec} A/H]$.

Remark 1.3 (Further generalizations). In Theorem 8.1 and Theorem 8.2, we provide a more refined description of the stack $\mathcal{W}$ as a quotient stack $[\operatorname{Spec} A/G]$ for a specific group scheme G in terms of properties of the gerbe $\mathcal{W}_0$. For instance, if the stabilizer group of $\mathcal{W}_0$ is connected, we can arrange that G is split reductive, and if $\mathcal{W}_0 = BG_0$ is neutral, we can arrange that G is a deformation of G_0 which is linearly reductive under mild characteristic assumptions.

In work with Halpern-Leistner [AHHLR24], this theorem is generalized to allow the case where $\mathcal{W}_0$ is an arbitrary linearly fundamental stack (rather than a gerbe over a field), where $x \in |\mathcal{X}|$ is an arbitrary point (rather than the finiteness of $\kappa(x)/\kappa(s)$), and where $\mathcal{W}_0 \to \mathcal{G}_x$ is syntomic (rather than smooth or étale).

The proof of Theorem 1.1 is given in Section 5 and follows the same general strategy as the proof of [AHR20, Thm. 1.1]:

(1) We begin by constructing smooth infinitesimal deformations $h_n \colon \mathcal{W}_n \to \mathcal{X}_n$ where $\mathcal{X}_n$ is the nth infinitesimal neighborhood of $\mathcal{G}_x$ in $\mathcal{X}$.
(2) We show that the system $\mathcal{W}_n$ effectivizes to a *coherently complete* stack $\widehat{\mathcal{W}}$. This is Theorem 1.10.
(3) Tannaka duality [HR19] (see also §1.7) then gives us a unique formally smooth morphism $\widehat{h} \colon \widehat{\mathcal{W}} \to \mathcal{X}$.
(4) Finally we apply equivariant Artin algebraization [AHR20, App. A] to approximate $\widehat{h}$ with a smooth morphism $h \colon \mathcal{W} \to \mathcal{X}$.

Step (1) follows by standard infinitesimal deformation theory. Step (2) is the main technical result of this paper. Theorem 1.10 is far more general than the related results in [AHR20]—even over an algebraically closed field. Step (3) was handled in [HR19]. Steps (1)–(3) are summarized in Theorem 1.11. The equivariant Artin algebraization results established in [AHR20, App. A] are sufficient for step (4).

Remark 1.4 (Existence theorem). Theorem 1.1 and its refinements are fundamental ingredients in the recent article of the first author with Halpern-Leistner and Heinloth on establishing necessary and sufficient conditions for an algebraic stack to admit a good moduli space [AHLH23].

1.2. **Coherent completeness.** The notion of an algebraic stack that is coherently complete with respect to a closed substack plays an essential role in this paper. It is not only used in Step (2) above but it is featured in many of our other results and techniques. The definition first appeared in [AHR20, Defn. 2.1].

Definition 1.5. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed immersion of noetherian algebraic stacks. We say that the pair $(\mathcal{X}, \mathcal{Z})$ is *coherently complete* (or $\mathcal{X}$ is *coherently complete along* $\mathcal{Z}$) if the natural functor

$$\mathsf{Coh}(\mathcal{X}) \to \varprojlim_n \mathsf{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]}),$$

from the abelian category of coherent sheaves on $\mathcal{X}$ to the category of projective systems of coherent sheaves on the nth nilpotent thickenings $\mathcal{X}_{\mathcal{Z}}^{[n]}$ of $\mathcal{Z} \subseteq \mathcal{X}$, is an equivalence of categories.

A noetherian affine scheme $\operatorname{Spec} A$ is coherently complete along $\operatorname{Spec}(A/I)$ if and only if A is I-adically complete (Example 3.2). The following statement was an essential ingredient in all of the main results of [AHR20]: if A is a noetherian k-algebra, where k is a field, and G is a linearly reductive affine group scheme over k acting on $\operatorname{Spec} A$ such that there is a k-point fixed by G and the ring of invariants A^G is a complete local ring, then the quotient stack $[\operatorname{Spec} A/G]$ is coherently complete along the residual gerbe of its unique closed point [AHR20, Thm. 1.3].

In this article, coherent completeness also features prominently. In fact, we establish the following generalization of [AHR20, Thm. 1.3], where we do not assume a priori that $\mathcal{X}$ has the resolution property, only that the closed substack $\mathcal{Z}$ has it.

Theorem 1.6 (Coherent completeness)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and good moduli space $\pi\colon \mathcal{X} \to X = \operatorname{Spec} A$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a coherent ideal $\mathcal{I}$. Let $I = \Gamma(\mathcal{X}, \mathcal{I})$. If $\mathcal{Z}$ has the resolution property, then $\mathcal{X}$ is coherently complete along $\mathcal{Z}$ if and only if A is I-adically complete. If this is the case, then $\mathcal{X}$ has the resolution property.*

This theorem is proved in Section 4. The difference between the statement above and formal GAGA for good moduli space morphisms is that the statement above asserts that $\mathcal{X}$ is coherently complete along $\mathcal{Z}$ and not merely along $\pi^{-1}(\pi(\mathcal{Z}))$. Indeed, as a consequence of this theorem, we can easily deduce the following version of formal GAGA (Corollary 1.7), which had been established in [GZB15, Thm. 1.1] with the additional hypotheses that $\mathcal{X}$ has the resolution property and I is maximal, and in [AHR20, Cor. 4.14] in the case that $\mathcal{X}$ is defined over a field and I is maximal.

Corollary 1.7 (Formal GAGA)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal. Suppose there exists a good moduli space $\pi\colon \mathcal{X} \to \operatorname{Spec} A$, where A is noetherian and I-adically complete. Suppose that either*

(1) *$I \subseteq A$ is a maximal ideal; or*
(2) *$\mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec}(A/I)$ has the resolution property.*

Then $\mathcal{X}$ has the resolution property and is coherently complete along $\mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec}(A/I)$.

1.3. **Effectivity.** The key method to prove many of the results in this paper is an effectivity result for algebraic stacks. This is similar in spirit to Grothendieck's result on algebraization of formal schemes [EGA, III.5.4.5].

Definition 1.8. An *adic sequence* is a sequence of closed immersions

$$\mathcal{X}_0 \hookrightarrow \mathcal{X}_1 \hookrightarrow \dots$$

of noetherian algebraic stacks such that if $\mathcal{I}_{(j)}$ denotes the coherent sheaf of ideals defining $u_{0j}\colon \mathcal{X}_0 \hookrightarrow \mathcal{X}_j$, then $\mathcal{I}_{(j)}^{i+1}$ defines $u_{ij}\colon \mathcal{X}_i \hookrightarrow \mathcal{X}_j$ for every $i \leq j$.

The sequence of infinitesimal thickenings of a closed substack of a noetherian algebraic stack is adic.

Definition 1.9. Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of algebraic stacks. An algebraic stack $\widehat{\mathcal{X}}$ is a *completion* of $\{\mathcal{X}_n\}$ if

(1) there are compatible closed immersions $\mathcal{X}_n \hookrightarrow \widehat{\mathcal{X}}$ for all n such that $\mathcal{X}_n$ is the nth infinitesimal neighborhood of $\mathcal{X}_0$ in $\widehat{\mathcal{X}}$;
(2) $\widehat{\mathcal{X}}$ is noetherian with affine diagonal; and
(3) $\widehat{\mathcal{X}}$ is coherently complete along $\mathcal{X}_0$.

By Tannaka duality (see §1.7), the completion is unique if it exists. Moreover, Tannaka duality implies that if the completion exists, then it is the colimit of

$\{\mathcal{X}_n\}_{n\geq 0}$ in the category of noetherian stacks with quasi-affine diagonal (and in the category of algebraic stacks with affine stabilizers if $\mathcal{X}_0$ is excellent).

The following result is our main effectivity theorem. An algebraic stack $\mathcal{X}$ is *linearly fundamental* if $\mathcal{X}$ is cohomologically affine and isomorphic to $[U/\mathrm{GL}_{n,\mathbb{Z}}]$ for an affine scheme U. An important example of a linearly fundamental stack is the quotient stack of an affine scheme by a linearly reductive group scheme.

Theorem 1.10 (Effectivity)**.** *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic stacks. If $\mathcal{X}_0$ is linearly fundamental, then the completion $\widehat{\mathcal{X}}$ exists and is linearly fundamental.*

We prove Theorem 1.10 in three stages of increasing generality in Section 4. The case of characteristic zero (Theorem 4.15) is reasonably straightforward whereas the case of positive and mixed characteristic requires a short detour through group schemes (Proposition 4.16). In Section 5, we then use Theorem 1.10 to establish the existence of formally smooth neighborhoods and completions, such as the following results.

Theorem 1.11 (Formal neighborhoods)**.** *Let $\mathcal{X}$ be noetherian algebraic stack with affine stabilizers and $\mathcal{X}_0 \subseteq \mathcal{X}$ be a locally closed substack. Let $h_0\colon \mathcal{W}_0 \to \mathcal{X}_0$ be a syntomic (e.g., smooth) morphism. Assume that $\mathcal{W}_0$ is linearly fundamental and that its good moduli space is quasi-excellent. Then there is a cartesian diagram*

$$\begin{array}{ccc} \mathcal{W}_0 & \dashrightarrow & \widehat{\mathcal{W}} \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{X}_0 & \longrightarrow & \mathcal{X}, \end{array}$$

where $h\colon \widehat{\mathcal{W}} \to \mathcal{X}$ is flat and $\widehat{\mathcal{W}}$ is noetherian, linearly fundamental and coherently complete along $\mathcal{W}_0$.

Applying the theorem above to the case when $\mathcal{W}_0 = \mathcal{X}_0$ is the residual gerbe of a point with linearly reductive stabilizer gives the existence of completions.

Corollary 1.12 (Existence of completions)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine stabilizers. For any point $x \in |\mathcal{X}|$ with linearly reductive stabilizer, the completion of $\mathcal{X}$ at x exists and is linearly fundamental.*

In fact, the last two results are proven more generally for pro-immersions (Theorem 5.1 and Corollary 5.2).

1.4. **Further results and applications.** In the course of establishing the results above, we prove several foundational results of independent interest.

1.4.1. *Local structure of good moduli spaces.* We provide the following refinement of Theorem 1.1: if $\mathcal{X}$ admits a good moduli space X, then étale-locally on X, $\mathcal{X}$ is of the form $[\operatorname{Spec} A/\mathrm{GL}_n]$ (Theorem 6.1). Also see Theorem 8.1. We also prove that a good moduli space $\mathcal{X} \to X$ necessarily has affine diagonal as long as $\mathcal{X}$ has separated diagonal and affine stabilizers (Theorem 6.1); this result is new even over an algebraically closed field.

1.4.2. *Structure of linearly reductive affine group schemes.* We prove that every linearly reductive group scheme $G \to S$ is étale-locally embeddable (Corollary 6.2) and canonically an extension of a finite flat tame group scheme by a smooth linearly reductive group scheme with connected fibers G^0_{sm} (Theorem 9.9). If S is of equal characteristic, then G is canonically an extension of a finite étale tame group scheme by a linearly reductive group scheme G^0 with connected fibers. In equal positive characteristic, G^0 is of multiplicative type and we say that G is nice. We also prove that if (S,s) is henselian and $G_s \to \operatorname{Spec} \kappa(s)$ is linearly reductive, then there exists an embeddable linearly reductive group scheme $G \to S$ extending G_s (Proposition 7.13).

1.4.3. *Representability of local quotient presentations.* We have resolved the issue (see [AHR20, Question 1.10]) of representability of the local quotient presentation in the presence of a separated diagonal (Proposition 5.3(2)).

1.4.4. *Results on adequate moduli spaces.* We prove that adequate moduli spaces are universal for maps to algebraic spaces (Theorem 3.12) and establish Luna's fundamental lemma for adequate moduli spaces (Theorem 3.14). We also prove that an adequate moduli space $\mathcal{X} \to X$, where the closed points of $\mathcal{X}$ have linearly reductive stabilizers, is necessarily a good moduli space (Theorem 4.21 and Corollary 6.12). These foundational results are even new over an algebraically closed field.

1.4.5. *Compact generation and algebraicity.* We prove compact generation of the derived category of an algebraic stack admitting a good moduli space (Proposition 6.15). We also prove algebraicity results for stacks parameterizing coherent sheaves (Theorem 10.14), Quot schemes (Corollary 10.15), and Hom stacks (Theorem 10.17).

1.4.6. *Deforming objects.* We prove that various objects and properties, defined on a closed substack $\mathcal{X}_0$ of $\mathcal{X}$, extend to an étale neighborhood of $\mathcal{X}_0$ if there is an affine good moduli space $\mathcal{X} \to X$ (see Propositions 7.15 to 7.20). These are applications of deformation results for henselian pairs of algebraic stacks (see Propositions 7.8 to 7.10, 7.12 and 7.13). These results are new even over an algebraically closed field.

1.4.7. *Equivariant geometry.* We provide generalizations of Sumihiro's theorem on torus actions (Theorem 10.1 and Corollary 10.2) and establish a relative version of Luna's étale slice theorem (Theorem 10.4).

1.4.8. *Completions and henselizations.* In addition to establishing the existence of completions (Corollary 1.12), we prove the existence of henselizations of algebraic stacks at points with linearly reductive stabilizer (Theorem 10.8). We prove that two algebraic stacks are étale-locally isomorphic near points with linearly reductive stabilizers if and only if they have isomorphic henselizations or completions (Theorem 10.10).

1.5. **Overview.** Sections 2 and 3 consist of the basic setup with some applications to adequate moduli spaces. We provide definitions and properties of reductive group schemes, fundamental stacks and local, henselian and coherently complete pairs.

Sections 4 and 5 contain the proofs of the central theorems of this paper: formal functions for good moduli spaces (Corollary 4.2), coherent completeness for good moduli spaces (Theorem 1.6), effectivity of adic sequences (Theorem 1.10), the existence of formal neighborhoods (Theorem 1.11), the existence of completions (Corollary 1.12), and the local structure theorem (Theorem 1.1).

Section 6 contains our applications to algebraic stacks with good moduli spaces. Section 7 contains technical results on approximation of linearly fundamental stacks, approximation of good moduli spaces, and deformation of objects over henselian pairs. Section 8 uses these results to give refinements of the local structure theorem. Section 9 contains our structure results for linearly reductive group schemes. Section 10 contains our applications to equivariant geometry, henselizations and algebraicity.

1.6. **Conventions on good moduli spaces.** Throughout this paper, we use the concepts of cohomologically affine morphisms and adequately affine morphisms:

Definition 1.13. A quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *cohomologically affine* (resp. *adequately affine*) if

(1) f_* is exact on the category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules (resp. if for every surjection $\mathcal{A} \to \mathcal{B}$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras, then every section s of $f_*(\mathcal{B})$ over a smooth morphism $\operatorname{Spec} A \to \mathcal{Y}$ has a positive power that lifts to a section of $f_*(\mathcal{A})$); and
(2) property (1) holds after arbitrary base change $\mathcal{Y}' \to \mathcal{Y}$.

In the original definitions, [Alp13, Defn. 3.1] and [Alp14, Defn. 4.1.1], condition (2) was not required. If $\mathcal{Y}$ has quasi-affine diagonal (e.g., $\mathcal{Y}$ is a quasi-separated algebraic space), then (2) holds automatically ([Alp13, Prop. 3.10(vii)] and [Alp14, Prop. 4.2.1(6)]).

We also use throughout the concepts of good moduli spaces [Alp13, Defn. 4.1] and adequate moduli spaces [Alp14, Defn. 5.1.1]:

Definition 1.14. A quasi-compact and quasi-separated morphism $\pi\colon \mathcal{X} \to X$ of algebraic stacks, where X is an algebraic space, is a *good moduli space* (resp. an *adequate moduli space*) if π is cohomologically affine (resp. adequately affine) and $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

Definition 1.15. A group algebraic space $G \to S$ is *linearly reductive* (resp. *geometrically reductive*) if

(1) $G \to S$ is flat and of finite presentation;
(2) $G \to S$ is affine; and
(3) $BG \to S$ is a good (resp. adequate) moduli space.

In [Alp13, Defn. 12.1] and [Alp14, Defn. 9.1.1], condition (2) was not required. This is, however, often automatic, see Remark 2.6.

Definition 1.16. An algebraic stack $\mathcal{X}$ is said to have the *resolution property* if every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module of finite type is a quotient of a locally free sheaf of finite rank.

By the main theorems of [Tot04] and [Gro17], a quasi-compact and quasi-separated algebraic stack $\mathcal{X}$ is isomorphic to $[U/\mathrm{GL}_N]$, where U is a quasi-affine scheme and N is a positive integer, if and only if the closed points of $\mathcal{X}$ have affine stabilizers and $\mathcal{X}$ has the resolution property. Note that when this is the case, $\mathcal{X}$ has affine diagonal.

1.7. **Recollection of Tannaka duality.** If $\mathcal{X}$ is a locally noetherian algebraic stack and $\mathcal{Z} \subseteq \mathcal{X}$ is a closed substack, we denote by $\mathcal{X}_{\mathcal{Z}}^{[n]}$ the nth order thickening of $\mathcal{Z}$ in $\mathcal{X}$, that is, if $\mathcal{Z}$ is defined by a sheaf of ideals $\mathcal{I}$, then $\mathcal{X}_{\mathcal{Z}}^{[n]}$ is defined by $\mathcal{I}^{n+1}$. If $i\colon \mathcal{Z} \to \mathcal{X}$ denotes the closed immersion, then we write $i^{[n]}\colon \mathcal{X}_{\mathcal{Z}}^{[n]} \to \mathcal{X}$ for the nth order thickening of i.

We freely use the following form of Tannaka duality, which was established in [HR19]. Let $\mathcal{X}$ be a noetherian algebraic stack and let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack such that $\mathcal{X}$ is coherently complete along $\mathcal{Z}$. Let $\mathcal{Y}$ be a noetherian algebraic stack with affine stabilizers. Suppose that either

(a) $\mathcal{X}$ is locally the spectrum of a G-ring (e.g., quasi-excellent); or
(b) $\mathcal{Y}$ has quasi-affine diagonal.

Then the natural functor

$$\operatorname{Hom}(\mathcal{X}, \mathcal{Y}) \to \varprojlim_n \operatorname{Hom}\big(\mathcal{X}_{\mathcal{Z}}^{[n]}, \mathcal{Y}\big)$$

is an equivalence of categories. This statement follows directly from [HR19, Thms. 1.1 and 8.4]; cf. the proof of [AHR20, Cor. 2.8].

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2. Reductive group schemes and fundamental stacks

In this section, we recall various notions of reductivity for group schemes and introduce certain classes of algebraic stacks that we will refer to as fundamental, linearly fundamental, and nicely fundamental. We also recall various relations between these notions. Besides some approximation results at the end, this section is largely expository.

2.1. **Reductive group schemes.** Recall that an affine, flat, and finitely presented group algebraic space $G \to S$ is linearly (resp. geometrically) reductive if $BG \to S$ is a good (resp. adequate) moduli space (Definition 1.15). We also introduce the following notions.

Definition 2.1. Let G be a group algebraic space which is affine, flat, and of finite presentation over an algebraic space S. We say that $G \to S$ is

(1) *embeddable* if G is a closed subgroup of $\mathrm{GL}(\mathcal{E})$ for a vector bundle $\mathcal{E}$ on S;
(2) *reductive* if $G \to S$ is smooth with reductive and connected geometric fibers [$\mathrm{SGA3}_{\mathrm{III}}$, Exp. XIX, Defn. 2.7]; and
(3) *nice* if there is an open and closed normal subgroup $G^0 \subseteq G$ that is of multiplicative type over S such that the étale group scheme $H = G/G^0$ is finite over S and $|H|$ is invertible on S.

Linearly reductive group schemes are the focus of this paper as their representation theory is semi-simple. Geometrically reductive group schemes appear since in positive characteristic GL_n is not linearly reductive but merely geometrically reductive. They also appear as global deformations of linearly reductive group schemes in mixed characteristic (Remark 2.4). Nice group schemes is a special class of linearly reductive group schemes that deform well also in mixed characteristic.

Remark 2.2 (Relations between the notions). For group schemes, we have the implications:

$$\text{nice} \implies \text{linearly reductive} \implies \text{geometrically reductive} \Longleftarrow \text{reductive}.$$

The first implication follows since a nice group algebraic space G is an extension of the linearly reductive groups G^0 and H, and is thus linearly reductive [Alp13, Prop. 2.17]. The second implication is immediate from the definitions, and is reversible in characteristic 0 [Alp14, Rem. 9.1.3]. The third implication is Seshadri's generalization [Ses77] of Haboush's theorem, and is reversible if $G \to S$ is smooth with geometrically connected fibers [Alp14, Thm. 9.7.5]. If k is a field of characteristic p, then GL_n is reductive over k but not linearly reductive, and a finite non-reduced group scheme (e.g., α_p) is geometrically reductive but not reductive.

Remark 2.3 (Positive characteristic). The notion of niceness is particularly useful in positive characteristic and was introduced in [HR15, Defn. 1.1] for affine group schemes over a field k. If k is a field of characteristic p, an affine group scheme G of finite type over k is nice if and only if the connected component of the identity G^0 is of multiplicative type and p does not divide the number of geometric components of G. In this case, by Nagata's theorem [Nag62] and its generalization to the non-smooth case (cf. [HR15, Thm. 1.2]), G is nice if and only if it is linearly reductive; moreover, this is also true over a base of equal characteristic p (Theorem 9.9).

Remark 2.4 (Mixed characteristic). Consider a scheme S, a point $s \in S$ and a linearly reductive group scheme G_0 over $\kappa(s)$. If G_0 is nice (e.g., if s has positive characteristic), then it deforms to a nice group scheme $G' \to S'$ over an étale neighborhood $S' \to S$ of s (Proposition 4.16).

Deformations of linearly reductive group schemes are more subtle. It is possible for a schemes S to have a closed point $s \in S$ of characteristic 0 which has no open neighborhood of characteristic 0. For instance, let R be the localization $\Sigma^{-1}\mathbb{Z}[x]$ where Σ is the multiplicative submonoid generated by the elements $p+x$ as p ranges over all primes. Then $S = \operatorname{Spec} R$ is a noetherian and excellent integral scheme, and $s = (x) \in S$ is a closed point with residue field $\mathbb{Q}$ which has no characteristic 0 neighborhood. Also see Appendix A.1. In such examples, a linearly reductive group scheme G_0 need not deform to a linearly reductive group scheme $G \to S'$ over an étale neighborhood $S' \to S$ of s. For example, take $G_0 = \mathrm{GL}_{2,\kappa(s)}$. However, G_0 does deform to a geometrically reductive embeddable group scheme over an étale neighborhood of s (Proposition 7.16).

Remark 2.5 (Embeddability and geometric reductivity). Any affine group scheme of finite type over a field is embeddable. It is not known to which extent general affine group schemes are embeddable—even over the dual numbers [Con10]. Thomason proved that certain reductive group schemes are embeddable [Tho87, Cor. 3.2]; in particular, if S is a normal, quasi-projective scheme, then every reductive group scheme $G \to S$ is embeddable. There is an example [$\mathrm{SGA3_{II}}$, Exp. X,§1.6] of a 2-dimensional torus over the nodal cubic curve that is not locally isotrivial and hence not Zariski-locally embeddable. We will eventually show that every linearly reductive group scheme $G \to S$ is embeddable if S is a normal quasi-projective scheme (Corollary 9.10) and always étale-locally embeddable (Corollary 6.2(2)).

If G is a closed subgroup of $\mathrm{GL}(\mathcal{E})$ for a vector bundle $\mathcal{E}$ on an algebraic space S, then a generalization of Matsushima's theorem asserts that $G \to S$ is geometrically reductive if and only if the quotient $\mathrm{GL}(\mathcal{E})/G$ is affine [Alp14, Thm. 9.4.1].

If S is affine and $G \to S$ is embeddable and geometrically reductive, then any quotient stack $\mathcal{X} = [\operatorname{Spec} A/G]$ has the resolution property. Indeed, if G is a closed subgroup of $\mathrm{GL}(\mathcal{E})$ for some vector bundle $\mathcal{E}$ of rank n on S, then the $(\mathrm{GL}(\mathcal{E}), \mathrm{GL}_{n,S})$-bitorsor $\mathrm{Isom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_S^n)$ induces an isomorphism $B_S GL(\mathcal{E}) \cong B_S \mathrm{GL}_n$, and the composition $\mathcal{X} = [\operatorname{Spec} A/G] \to B_S G \to B_S \mathrm{GL}(\mathcal{E}) \cong B_S \mathrm{GL}_n$ is affine, that is $\mathcal{X} \cong [\operatorname{Spec} B/\mathrm{GL}_{n,S}]$. By [Gro17, Thm. 1.1], $\mathcal{X}$ has the resolution property.

Remark 2.6 (Affineness). In contrast to [Alp13], we have only defined linear reductivity for affine group schemes $G \to S$. We will however prove that if $G \to S$ is a separated, flat group scheme of finite presentation with affine fibers such that $BG \to S$ is cohomologically affine, then $G \to S$ is affine (Corollary 6.2(1)).

2.2. **Fundamental stacks.** In [AHR20], we dealt with stacks of the form $[\operatorname{Spec} A/G]$ where G is a linearly reductive group scheme over a field k. In this paper, we are working over an arbitrary base and it will be convenient to introduce the following classes of quotient stacks.

Definition 2.7. Let $\mathcal{X}$ be an algebraic stack. We say that $\mathcal{X}$ is:

(1) *fundamental* if $\mathcal{X} \cong [U/\mathrm{GL}_{n,\mathbb{Z}}]$ for an affine scheme U and some integer n, i.e., $\mathcal{X}$ admits an affine morphism to $B\mathrm{GL}_{n,\mathbb{Z}}$;
(2) *linearly fundamental* if $\mathcal{X}$ is fundamental and cohomologically affine; and
(3) *nicely fundamental* if $\mathcal{X} \cong [U/Q]$ for an affine scheme U and a nice and embeddable group scheme Q, both over some common affine scheme S, i.e., $\mathcal{X}$ admits an affine morphism to $B_S Q$.

Remark 2.8 (Relations between the notions)*.* For algebraic stacks, we have the obvious implications:

$$\text{nicely fundamental} \implies \text{linearly fundamental} \implies \text{fundamental}$$

If $\mathcal{X}$ is fundamental (resp. linearly fundamental), then $\mathcal{X}$ admits an adequate (resp. good) moduli space: $\operatorname{Spec}\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

In characteristic 0, an algebraic stack is linearly fundamental if and only if it is fundamental. We will show that in positive equicharacteristic, a linearly fundamental stack is nicely fundamental étale-locally over its good moduli space (Proposition 6.8).

The additional condition of a fundamental stack to be linearly fundamental is that $\mathcal{X} \cong [\operatorname{Spec} B/\mathrm{GL}_N]$ is cohomologically affine, which means that the adequate moduli space $\mathcal{X} \to \operatorname{Spec} B^{\mathrm{GL}_n}$ is a good moduli space. We will show that this happens precisely when the stabilizer of every closed point is linearly reductive (Corollary 6.11).

Remark 2.9 (Equivalences I)*.* If G is a group scheme which is affine, flat and of finite presentation over an affine scheme S, then:

$$\begin{aligned} BG \text{ fundamental} &\iff G \text{ geometrically reductive and embeddable} \\ BG \text{ linearly fundamental} &\iff G \text{ linearly reductive and embeddable} \\ BG \text{ nicely fundamental} &\iff G \text{ nice and embeddable.} \end{aligned}$$

This follows from Remark 2.5 and the definitions of geometrically reductive and linearly reductive, using that flat closed subgroups of diagonalizable groups are diagonalizable for the nice case. If $\mathcal{X}$ is an algebraic stack, then it also follows from the definitions that:

$$\begin{aligned} \mathcal{X} \text{ fundamental} &\iff \mathcal{X} = [\operatorname{Spec} A/G] \text{ for } G \text{ geom. red. and emb.} \\ \mathcal{X} \text{ lin. fundamental} &\impliedby \mathcal{X} = [\operatorname{Spec} A/G] \text{ for } G \text{ lin. red. and emb.} \\ \mathcal{X} \text{ nicely fundamental} &\iff \mathcal{X} = [\operatorname{Spec} A/G] \text{ for } G \text{ nice and emb.} \end{aligned}$$

The second implication is not an equivalence, see Appendix A.1. We will, however, show that under mild mixed characteristic hypotheses it is an equivalence *étale-locally* over the good moduli space (Corollary 6.9).

Remark 2.10 (Equivalences II)*.* An algebraic stack $\mathcal{X}$ is a *global quotient stack* if $\mathcal{X} \cong [U/\mathrm{GL}_n]$, where U is an algebraic space. Since adequately affine and representable morphisms are necessarily affine ([Alp14, Thm. 4.3.1]), we have the following equivalences for a quasi-compact and quasi-separated algebraic stack $\mathcal{X}$:

$$\begin{aligned} \text{fundamental} &\iff \text{adequately affine and a global quotient} \\ \text{linearly fundamental} &\iff \text{cohomologically affine and a global quotient.} \end{aligned}$$

Proposition 2.11. *Let $\mathcal{G}$ be a gerbe over a field k.*

(1) *If the stabilizer group of the unique point of $\mathcal{G}$ is nice, then $\mathcal{G}$ is nicely fundamental.*
(2) *If the characteristic of k is $p > 0$ and $\mathcal{G}$ has linearly reductive stabilizer group, then $\mathcal{G}$ is nicely fundamental.*

Proof. Claim (2) follows from (1) and Remark 2.3. For claim (1): since $\mathcal{G} \to \operatorname{Spec} k$ is smooth, there is a finite separable extension $k \subseteq k'$ that neutralizes the gerbe. Hence, $\mathcal{G}_{k'} \cong BQ'$, for some nice group scheme Q' over k'. Let Q be the Weil restriction of Q' along $\operatorname{Spec} k' \to \operatorname{Spec} k$; then Q is nice and there is an induced affine morphism $\mathcal{G} \to BQ$. □

2.3. **Approximation of fundamental and nicely fundamental stacks.** Here we establish that standard limit arguments, allowing to reduce arguments to schemes of finite type, admit variants for fundamental and nicely fundamental stacks. These results will be used to reduce from the situation of a complete local ring to an excellent henselian local ring (via Artin approximation), from a henselian local ring to an étale neighborhood, and from (non-)noetherian rings to excellent rings. The linearly fundamental case is more subtle—see Appendix A for some counterexamples—but will eventually be established in equal characteristic and in certain mixed characteristics (Theorem 7.3).

We begin with the following standard limit result, in the style of [EGA, IV.8], for the property of an embeddable group scheme being geometrically reductive or nice.

Lemma 2.12. *Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of quasi-compact and quasi-separated algebraic spaces with affine transition maps and limit S. Let $\alpha \in \Lambda$ and let $G_\alpha \to S_\alpha$ be a flat group algebraic space of finite presentation. For $\lambda \geq \alpha$, let G_λ be the pullback of G_α along $S_\lambda \to S_\alpha$ and let G be the pullback of G_α along $S \to S_\alpha$. If G is geometrically reductive (resp. nice) and embeddable over S, then G_λ is geometrically reductive (resp. nice) and embeddable over S_λ for all $\lambda \gg \alpha$.*

Proof. Let $\mathcal{E}$ be a vector bundle on S and let $G \hookrightarrow \mathrm{GL}(\mathcal{E})$ be a closed embedding. By standard limit methods, there exists a vector bundle $\mathcal{E}_\lambda$ on S_λ and a closed embedding $G_\lambda \hookrightarrow \mathrm{GL}(\mathcal{E}_\lambda)$ for all sufficiently large λ. If G is geometrically reductive, then $\mathrm{GL}(\mathcal{E})/G \to S$ is affine and so is $\mathrm{GL}(\mathcal{E}_\lambda)/G_\lambda \to S_\lambda$ for all sufficiently large λ; hence G_λ is geometrically reductive by Matsushima's theorem (Remark 2.5).

If $G^0 \subseteq G$ is an open and closed normal subgroup as in the definition of a nice group scheme, then by standard limit methods, we can find an open and closed normal subgroup $G^0_\lambda \subseteq G_\lambda$ for all sufficiently large λ satisfying the conditions in the definition of nice group schemes. □

Lemma 2.13. *An algebraic stack $\mathcal{X}$ is nicely fundamental if and only if there exists an affine scheme S of finite presentation over $\operatorname{Spec} \mathbb{Z}$, a nice and embeddable group scheme $Q \to S$ and an affine morphism $\mathcal{X} \to B_S Q$.*

Proof. The condition is sufficient by definition. For necessity, by definition, we have an affine map $\mathcal{X} \to B_S Q$, where Q is a nice and embeddable group scheme over an affine scheme S. We can write $S = \varprojlim_\lambda S_\lambda$ as a limit of affine schemes of finite type over $\mathbb{Z}$, and find a flat group scheme $Q_\alpha \to S_\alpha$ of finite type such that $Q = Q_\alpha \times_{S_\alpha} S$. Let $Q_\lambda = Q_\alpha \times_{S_\alpha} S_\lambda$ so that $\mathcal{X} \to B_S Q \to B_{S_\lambda} Q_\lambda$ is affine for all $\lambda \geq \alpha$. Then Lemma 2.12 implies that Q_λ is nice for $\lambda \gg \alpha$. □

Lemma 2.14. *Let $\mathcal{X}$ be a fundamental (resp. a nicely fundamental) stack. Then there exists an inverse system of fundamental (resp. nicely fundamental) stacks $\mathcal{X}_\lambda$ of finite type over $\operatorname{Spec} \mathbb{Z}$ with affine transition maps such that $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$.*

Proof. If $\mathcal{X}$ is fundamental, then we have an affine morphism $\mathcal{X} \to B\mathrm{GL}_{n,\mathbb{Z}}$. Since every quasi-coherent sheaf on the noetherian stack $B\mathrm{GL}_{n,\mathbb{Z}}$ is a union of its finitely generated subsheaves [LMB, Prop. 15.4], we can write $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$, where the $\mathcal{X}_\lambda \to B\mathrm{GL}_{n,\mathbb{Z}}$ are affine and of finite type. If $\mathcal{X}$ is nicely fundamental, we argue analogously with $B_S Q$ of Lemma 2.13 instead of $B\mathrm{GL}_{n,\mathbb{Z}}$. □

The following proposition shows that the property of being (nicely) fundamental descends under limits and also describes how adequate moduli spaces commute with inverse limits.

Proposition 2.15. *Let $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ be an inverse limit of quasi-compact and quasi-separated algebraic stacks with affine transition maps.*

(1) *If $\mathcal{X}$ is fundamental (resp. nicely fundamental), then so is $\mathcal{X}_\lambda$ for all sufficiently large λ.*
(2) *Let $x \in |\mathcal{X}|$ be a point with image $x_\lambda \in |\mathcal{X}_\lambda|$. If $\mathcal{G}_x$ (resp. $\overline{\{x\}}$) is nicely fundamental, then so is $\mathcal{G}_{x_\lambda}$ (resp. $\overline{\{x_\lambda\}}$) for all sufficiently large λ.*
(3) *If $\mathcal{X} \to X$ and $\mathcal{X}_\lambda \to X_\lambda$ are adequate moduli spaces, then $X = \varprojlim_\lambda X_\lambda$.*

Proof. For the first statement, let $\mathcal{Y} = B\mathrm{GL}_{n,\mathbb{Z}}$ (resp. $\mathcal{Y} = B_S Q$ for Q as in Lemma 2.13). Then there is an affine morphism $\mathcal{X} \to \mathcal{Y}$ and hence an affine morphism $\mathcal{X}_\lambda \to \mathcal{Y}$ for all sufficiently large λ [Ryd15, Prop. B.1, Thm. C]. The second statement follows from the first by noting that $\mathcal{G}_x = \varprojlim_\lambda \mathcal{G}_{x_\lambda}$ and $\overline{\mathcal{G}_x} = \varprojlim_\lambda \overline{\mathcal{G}_{x_\lambda}}$.

The third statement follows directly from the following two facts (a) pushforward of quasi-coherent sheaves along $\pi_\lambda \colon \mathcal{X}_\lambda \to X_\lambda$ preserves filtered colimits and (b) if $\mathcal{A}$ is a quasi-coherent sheaf of algebras, then the adequate moduli space of $\operatorname{Spec}_{\mathcal{X}_\lambda} \mathcal{A}$ is $\operatorname{Spec}_{X_\lambda} (\pi_\lambda)_* \mathcal{A}$. □

Remark 2.16. The analogous statements of Lemma 2.12 (resp. Proposition 2.15(1)) for linearly reductive and embeddable group schemes (resp. linearly fundamental stacks) are false in mixed characteristic. Indeed, $\mathrm{GL}_{2,\mathbb{Q}} = \varprojlim_m \mathrm{GL}_{2,\mathbb{Z}[\frac{1}{m}]}$ and $\mathrm{GL}_{2,\mathbb{Q}}$ is linearly reductive but $\mathrm{GL}_{2,\mathbb{Z}[\frac{1}{m}]}$ is never linearly reductive. Likewise, $B\mathrm{GL}_{2,\mathbb{Q}}$ is linearly fundamental but $B\mathrm{GL}_{2,\mathbb{Z}[\frac{1}{m}]}$ is never linearly fundamental.

It is not true in general that the induced maps $\mathcal{X} \to \mathcal{X}_\lambda \times_{X_\lambda} X$ are isomorphisms for sufficiently large λ. Take, for example, $\mathcal{X}_n = [\operatorname{Spec} A_n / \mathbb{G}_m]$ where $A_n = k[x_1, \dots, x_n]$ with the standard scaling action.

3. Luna's fundamental lemma and universality of adequate moduli spaces

We begin by defining local, henselian, and coherently complete pairs, and stating a general version of Artin approximation (Theorem 3.8). We then prove that a pair of stacks is henselian if and only if their adequate moduli spaces are henselian (Theorem 3.10), which provides a henselian analogue of our main result on coherent completeness (Theorem 1.6). We apply this theorem to give quick proofs of the universality of adequate moduli spaces (Theorem 3.12) and Luna's fundamental lemma (Theorem 3.14).

3.1. Henselian pairs.

Recall that we have defined the notion of coherently complete pairs in Definition 1.5. We now introduce the following weaker notions:

Definition 3.1. Fix a closed immersion of algebraic stacks $\mathcal{Z} \subseteq \mathcal{X}$. The pair $(\mathcal{X}, \mathcal{Z})$ is said to be

(1) *local* if every non-empty closed subset of $|\mathcal{X}|$ intersects $|\mathcal{Z}|$ non-trivially; and

(2) *henselian* if for every finite morphism $\mathcal{X}' \to \mathcal{X}$, the restriction map

$$\operatorname{ClOpen}(\mathcal{X}') \to \operatorname{ClOpen}(\mathcal{Z} \times_{\mathcal{X}} \mathcal{X}'), \tag{3.1}$$

is bijective, where $\operatorname{ClOpen}(\mathcal{X})$ denotes the set of closed and open substacks of $\mathcal{X}$ [EGA, IV.18.5.5].

In addition, we call a pair $(\mathcal{X}, \mathcal{Z})$ *affine* if $\mathcal{X}$ is affine and an affine pair $(\mathcal{X}, \mathcal{Z})$ *(quasi-)excellent* if $\mathcal{X}$ is (quasi-)excellent. The completion of an affine (quasi-)excellent pair is (quasi-)excellent [KS21]. Occasionally, we will also say $\mathcal{X}$ is local, henselian, or coherently complete along $\mathcal{Z}$ if the pair $(\mathcal{X}, \mathcal{Z})$ has the corresponding property.

We list some examples of henselian and coherently complete pairs.

Example 3.2 (Adic rings)**.** Let A be a noetherian ring and let $I \subseteq A$ be an ideal. Then $(\operatorname{Spec} A, \operatorname{Spec} A/I)$ is a coherently complete pair if and only if A is I-adically complete. The sufficiency is trivial. For the necessity, we note that $\varprojlim_n \mathsf{Coh}(A/I^{n+1}) \simeq \mathsf{Coh}(\widehat{A})$, where $\widehat{A}$ denotes the completion of A with respect to the I-adic topology. Hence, the natural functor $\mathsf{Coh}(A) \to \mathsf{Coh}(\widehat{A})$ is an equivalence of abelian tensor categories. It follows from Tannaka duality (see §1.7) that the natural map $A \to \widehat{A}$ is an isomorphism.

Example 3.3 (Proper maps)**.** Let A be a ring and let $I \subseteq A$ be an ideal. Let $f \colon \mathcal{X} \to \operatorname{Spec} A$ be a proper morphism of algebraic stacks. Let $\mathcal{Z} = f^{-1}(\operatorname{Spec} A/I)$.

(1) If A is I-adically complete, then $(\mathcal{X}, \mathcal{Z})$ is coherently complete. This is just the usual Grothendieck existence theorem, see [EGA, III.5.1.4] for the case of schemes and [Ols05, Thm. 1.4] for algebraic stacks.
(2) If A is henselian along I, then $(\mathcal{X}, \mathcal{Z})$ is henselian. This is part of the proper base change theorem in étale cohomology; the case where I is maximal is well-known, see [HR14, Rem. B.6] for further discussion.

Let A be a noetherian ring and let $I \subseteq J \subseteq A$ be ideals. Assume that A is J-adically complete. Recall that A/I is then J-adically complete and A is also I-adically complete. This is analogous to parts (1) and (2), respectively, of the following result. We omit the proof.

Lemma 3.4. *Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed immersion of algebraic stacks. Assume that the pair $(\mathcal{X}, \mathcal{Z})$ is henselian or coherently complete.*

(1) *Let $f \colon \mathcal{X}' \to \mathcal{X}$ be a finite morphism and let $\mathcal{Z}' \subseteq \mathcal{X}'$ be the pullback of $\mathcal{Z}$. Then $(\mathcal{X}', \mathcal{Z}')$ is henselian or coherently complete, respectively.*
(2) *Let $\mathcal{W} \subseteq \mathcal{X}$ be a closed substack. If $|\mathcal{Z}| \subseteq |\mathcal{W}|$, then $(\mathcal{X}, \mathcal{W})$ is henselian or coherently complete, respectively.*

Remark 3.5. For a pair $(\mathcal{X}, \mathcal{Z})$, we have the following sequence of implications:

$$\text{coherently complete} \implies \text{henselian} \implies \text{local.}$$

The second implication is trivial: if $\mathcal{W} \subseteq \mathcal{X}$ is a closed substack, then $\operatorname{ClOpen}(\mathcal{W}) \to \operatorname{ClOpen}(\mathcal{Z} \cap \mathcal{W})$ is bijective. For the first implication, note that we have bijections:

$$\operatorname{ClOpen}(\mathcal{X}) \simeq \varprojlim_n \operatorname{ClOpen}(\mathcal{X}_{\mathcal{Z}}^{[n]}) \simeq \operatorname{ClOpen}(\mathcal{Z})$$

whenever $(\mathcal{X}, \mathcal{Z})$ is coherently complete. The implication now follows from the elementary Lemma 3.4(1).

Remark 3.6. It follows from the main result of [Ryd16] that if $\mathcal{X}$ is quasi-compact and quasi-separated, then $(\mathcal{X}, \mathcal{Z})$ is a henselian pair if and only if (3.1) is bijective for every integral morphism $\mathcal{X}' \to \mathcal{X}$.

Remark 3.7 (Nakayama's lemma for stacks). As seen by descent from a smooth presentation, the following variants of Nakayama's lemma hold for local pairs $(\mathcal{X}, \mathcal{Z})$: (1) if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module of finite type and $\mathcal{F}|_{\mathcal{Z}} = 0$, then $\mathcal{F} = 0$; and (2) if $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules with $\mathcal{G}$ of finite type and $\varphi|_{\mathcal{Z}}$ is surjective, then φ is surjective.

We will frequently use Artin approximation over henselian pairs to pass from completions to henselizations, especially in Section 7. This version of Artin approximation is due to Popescu [Pop86, Thm. 1.3] and follows from his desingularization theorem as we will also explain below. Artin's original approximation theorem [Art69, Cor. 2.2] is valid for henselian pairs (S, s) where S is the spectrum of the henselization of a local ring essentially of finite type over either a field or an excellent Dedekind domain.

Theorem 3.8 (Artin approximation over henselian pairs). *Let A be a G-ring, e.g., quasi-excellent, let $(S, S_0) = (\operatorname{Spec} A, \operatorname{Spec} A/I)$ be an affine henselian pair and let $\widehat{S} = \operatorname{Spec} \widehat{A}$ be its I-adic completion. Let $F\colon (\mathsf{Sch}/S)^{\mathrm{opp}} \to \mathsf{Sets}$ be a limit preserving functor. Given an element $\overline{\xi} \in F(\widehat{S})$ and an integer $n \geq 0$, there exists an element $\xi \in F(S)$ such that ξ and $\overline{\xi}$ have equal images in $F(S_n)$ where $S_n = \operatorname{Spec} A/I^{n+1}$.*

Proof. The completion map $\widehat{S} \to S$ is regular. Hence, by Néron–Popescu desingularization [Pop86, Thm. 1.8], there exists a smooth morphism $S' \to S$ and a section $\xi' \in F(S')$ such that $\xi'|_{\widehat{S}} = \overline{\xi}$. By Elkik [Elk73, Thm., p. 568], there is an element $\xi \in F(S)$ as requested. $\square$

3.2. **Adequate moduli spaces and henselian/complete pairs.** The following proposition gives a generalization of one direction for Theorem 1.6: coherent completeness passes to adequate moduli spaces. The other direction, which we defer until later, is much more involved.

Proposition 3.9. *Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed immersion of noetherian algebraic stacks. Let $\pi\colon \mathcal{X} \to X$ be an adequate moduli space, where X is affine and noetherian. If the pair $(\mathcal{X}, \mathcal{Z})$ is coherently complete, then the pair $(X, \pi(\mathcal{Z}))$ is coherently complete.*

Proof. Let $I \subseteq A$ be the ideal defining $Z = \pi(\mathcal{Z})$, let $A \to \widehat{A}$ be the I-adic completion and let $\widehat{X} = \operatorname{Spec} \widehat{A}$. The composition $\mathcal{X}_{\mathcal{Z}}^{[n]} \to \mathcal{X} \to X$ factors through $X_Z^{[n]}$, hence lifts uniquely to $\widehat{X}$. By Tannaka duality, we obtain a unique lift $\mathcal{X} \to \widehat{X}$. But, by definition of an adequate moduli space, $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = A$, so we obtain a retraction $\widehat{A} \to A$. It follows that A is I-adically complete. $\square$

As we show below, for henselian pairs, the analog of Theorem 1.6 (coherent completeness) and Proposition 3.9 is straightforward.

Theorem 3.10. *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack with adequate moduli space $\pi\colon \mathcal{X} \to X$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack with $Z = \pi(\mathcal{Z})$. The pair $(\mathcal{X}, \mathcal{Z})$ is henselian if and only if the pair (X, Z) is henselian.*

Proof. The induced morphism $\mathcal{Z} \to Z$ factors as the composition of an adequate moduli space $\mathcal{Z} \to \widetilde{Z}$ and an adequate homeomorphism $\widetilde{Z} \to Z$ [Alp14, Lem. 5.2.11]. If $\mathcal{X}' \to \mathcal{X}$ is integral, then $\mathcal{X}'$ admits an adequate moduli space X' and $X' \to X$ is integral. Conversely, if $X' \to X$ is integral, then $\mathcal{X} \times_X X' \to X'$ factors as the composition of an adequate moduli space $\mathcal{X} \times_X X' \to \widetilde{X}'$ and an adequate homeomorphism $\widetilde{X}' \to X'$ [Alp14, Prop. 5.2.9(3)]. It is thus enough to show that

$$\operatorname{ClOpen}(\mathcal{X}) \to \operatorname{ClOpen}(\mathcal{Z})$$

is bijective if and only if

$$\mathrm{ClOpen}(X) \to \mathrm{ClOpen}(Z)$$

is bijective. But $\mathcal{X} \to X$ and $\mathcal{Z} \to Z$ are surjective and closed with connected fibers [Alp14, Thm. 5.3.1]. Thus we have identifications $\mathrm{ClOpen}(\mathcal{X}) = \mathrm{ClOpen}(X)$ and $\mathrm{ClOpen}(\mathcal{Z}) = \mathrm{ClOpen}(Z)$ that are compatible with the restriction maps. The result follows. □

3.3. **Characterization of henselian pairs.** A quasi-compact and quasi-separated pair of schemes (X, X_0) is henselian if and only if for every étale morphism $g\colon X' \to X$, every section of $g_0\colon X' \times_X X_0 \to X_0$ extends to a section of g (for g separated see [EGA, IV.18.5.4] and in general see [SGA4$_3$, Exp. XII,Prop. 6.5]). This is also true for stacks:

Proposition 3.11. *Let $(\mathcal{X}, \mathcal{X}_0)$ be a pair of quasi-compact and quasi-separated algebraic stacks. Then the following are equivalent*

(1) *$(\mathcal{X}, \mathcal{X}_0)$ is henselian.*
(2) *For every* representable *étale morphism $g\colon \mathcal{X}' \to \mathcal{X}$, the induced map*

$$\Gamma(\mathcal{X}'/\mathcal{X}) \to \Gamma(\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_0/\mathcal{X}_0)$$

is bijective.

Proof. This is the equivalence between (1) and (3) of [HR23, Prop. 5.4]. □

We will later prove that (2) holds for non-representable étale morphisms when $\mathcal{X}$ is a stack with a good moduli space and affine diagonal (Proposition 7.9). A henselian pair does not always satisfy (2) for general non-representable morphisms though, see Example 3.16.

3.4. **Application: Universality of adequate moduli spaces.** For noetherian algebraic stacks, good moduli spaces were shown in [Alp13, Thm. 6.6] to be universal for maps to quasi-separated algebraic spaces and adequate moduli spaces were shown in [Alp14, Thm. 7.2.1] to be universal for maps to algebraic spaces which are either locally separated or Zariski-locally have affine diagonal. We now establish this result unconditionally for adequate (and hence good) moduli spaces—see Theorem 7.22 for a generalization to good moduli space morphisms.

Theorem 3.12. *Let $\mathcal{X}$ be an algebraic stack. An adequate moduli space $\pi\colon \mathcal{X} \to X$ is universal for maps to algebraic spaces.*

Proof. We need to show that if Y is an algebraic space, then the natural map

$$\mathrm{Map}(X, Y) \to \mathrm{Map}(\mathcal{X}, Y) \tag{3.2}$$

is bijective. To see the injectivity of (3.2), suppose that $h_1, h_2\colon X \to Y$ are maps such that $h_1 \circ \pi = h_2 \circ \pi$. Let $E \to X$ be the equalizer of h_1 and h_2, that is, the pullback of the diagonal $Y \to Y \times Y$ along $(h_1, h_2)\colon X \to Y \times Y$. The equalizer is a monomorphism and locally of finite type. By assumption $\pi\colon \mathcal{X} \to X$ factors through E, and since $\mathcal{X} \to X$ is universally closed, so is $E \to X$. It follows that $E \to X$ is a closed immersion [Stacks, Tag 04XV]. Since $\mathcal{X} \to X$ is schematically dominant, so is $E \to X$, hence $E = X$.

The surjectivity of (3.2) is an étale-local property on X since the injectivity of (3.2) implies the gluing condition in étale descent. Thus, we may assume that X is affine. In particular, $\mathcal{X}$ is quasi-compact and since any map $\mathcal{X} \to Y$ factors through a quasi-compact open of Y, we may assume that Y is also quasi-compact.

Let $g\colon \mathcal{X} \to Y$ be a morphism and let $p\colon Y' \to Y$ be an étale presentation where Y' is an affine scheme. To show that g factors through $\pi\colon \mathcal{X} \to X$, we claim that after replacing X with an étale cover, the base change $f\colon \mathcal{X}' \to \mathcal{X}$ of p along g admits a section $s\colon \mathcal{X} \to \mathcal{X}'$. If this claim is established, then the map $g\colon \mathcal{X} \to Y$ factors as $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{g'} Y' \xrightarrow{p} Y$. Since X and Y' are affine, the equality $\Gamma(X, \mathcal{O}_X) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ implies that the map $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{g'} Y'$ factors through $\pi\colon \mathcal{X} \to X$.

To show the claim, observe that $f\colon \mathcal{X}' \to \mathcal{X}$ is representable, étale, surjective and induces an isomorphism of stabilizer group schemes at all points. Let $x \in |X|$ be a point, $q \in |\mathcal{X}|$ be the unique closed point over x and $q' \in |\mathcal{X}'|$ any point over q. Note that $\kappa(q)/\kappa(x)$ is a purely inseparable extension. After replacing X with an étale neighborhood of x (with a residue field extension), we may thus assume that $\kappa(q') = \kappa(q)$. Since f induces an isomorphism of stabilizer groups, the induced map $\mathcal{G}_{q'} \to \mathcal{G}_q$ on residual gerbes is an isomorphism. Theorem 3.10 implies that $(\mathcal{X} \times_X \operatorname{Spec} \mathcal{O}^h_{X,x}, \mathcal{G}_q)$ is a henselian pair so Proposition 3.11 gives a section of $\mathcal{X}' \times_X \operatorname{Spec} \mathcal{O}^h_{X,x} \to \mathcal{X} \times_X \operatorname{Spec} \mathcal{O}^h_{X,x}$. Since f is locally of finite presentation, we obtain a section $s\colon \mathcal{X} \to \mathcal{X}'$ of $f\colon \mathcal{X}' \to \mathcal{X}$ after replacing X with an étale neighborhood of x. □

3.5. Application: Luna's fundamental lemma.

Definition 3.13. If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic stacks admitting adequate moduli spaces $\mathcal{X} \to X$ and $\mathcal{Y} \to Y$, we say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is *strongly étale* if the induced morphism $X \to Y$ is étale and $\mathcal{X} \cong X \times_Y \mathcal{Y}$.

The following result generalizes [Alp10, Thm. 6.10] from good moduli spaces to adequate moduli spaces and also removes noetherian and separatedness assumptions.

Theorem 3.14 (Luna's fundamental lemma). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks with adequate moduli spaces $\pi_{\mathcal{X}}\colon \mathcal{X} \to X$ and $\pi_{\mathcal{Y}}\colon \mathcal{Y} \to Y$. Let $x \in |\mathcal{X}|$ be a point, closed in its fiber $\pi_{\mathcal{X}}^{-1}(\pi_{\mathcal{X}}(x))$, such that*

1. *f is étale and representable in a neighborhood of x;*
2. *$y := f(x) \in |\mathcal{Y}|$ is closed in its fiber $\pi_{\mathcal{Y}}^{-1}(\pi_{\mathcal{Y}}(y))$; and*
3. *f induces an isomorphism of stabilizer groups at x.*

Then there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{X}$ of x such that $\pi_{\mathcal{X}}^{-1}(\pi_{\mathcal{X}}(\mathcal{U})) = \mathcal{U}$ and $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{Y}$ is strongly étale. In particular, $X \to Y$ is étale at $\pi_{\mathcal{X}}(x)$.

Remark 3.15. If G a smooth algebraic group over an algebraically closed field k such that G^0 is reductive and $\varphi\colon U \to V$ is a G-equivariant morphism of irreducible normal affine varieties over k, then [BR85, Thm. 4.1] (see also [MFK94, pg. 198]) established the result above for $f\colon [U/G] \to [V/G]$.

Proof of Theorem 3.14. An open subset $\mathcal{U} \subseteq \mathcal{X}$ such that $\pi_{\mathcal{X}}^{-1}(\pi_{\mathcal{X}}(\mathcal{U})) = \mathcal{U}$ is called *saturated* and it has adequate moduli space $\pi_{\mathcal{X}}(\mathcal{U})$. Given any open neighborhood $\mathcal{U} \subseteq \mathcal{X}$ of x, the smaller open neighborhood $|\mathcal{X}| \smallsetminus \pi_{\mathcal{X}}^{-1}(\pi_{\mathcal{X}}(|\mathcal{X}| \smallsetminus \mathcal{U}))$ is saturated.

By (1), we may replace $\mathcal{X}$ with a saturated open neighborhood of x such that f becomes étale and representable. As adequate moduli spaces commute with flat base change, the question is étale-local on Y. We may therefore assume that Y is affine in which case $\mathcal{Y}$ is quasi-compact and quasi-separated.

If Y is strictly henselian with closed point $\pi_{\mathcal{Y}}(y)$, then $(\mathcal{Y}, y)$ is a henselian pair (Theorem 3.10) and $\mathcal{G}_x \to \mathcal{G}_y$ is an isomorphism. Since f is representable, we may apply Proposition 3.11 to construct a section s of f such that $s(y) = x$. For general Y, since f is locally of finite presentation, we obtain a section s of f such that $s(y) = x$ after replacing Y with an étale neighborhood $(Y', y') \to (Y, \pi_{\mathcal{Y}}(y))$. The image of s is an open substack $\mathcal{U} \subseteq \mathcal{X}$ and $f|_{\mathcal{U}}$ is an isomorphism. After replacing $\mathcal{X}$ with a saturated open neighborhood of x contained in $\mathcal{U}$, we can thus assume that f is an open immersion. After repeating the argument we obtain a section s which is open and closed. Then $\mathcal{U} \subseteq \mathcal{X}$ is automatically saturated and we are done. □

The result is not true in general if f is not representable in a neighborhood of x as the following example shows. However, if $\mathcal{Y}$ has separated diagonal, then f is always representable in a neighborhood of x; see Proposition 5.3(2).

Example 3.16. Let $S = \operatorname{Spec} k[\![t]\!]$ where k is an algebraically closed field. Let $G = (\mathbb{Z}/2\mathbb{Z})_S$ and let $G' = G/H$ where $H \subseteq G$ is the open subgroup that is the complement of the non-trivial element over the origin. Let $\mathcal{X} = BG$ and $\mathcal{Y} = BG'$ which both have good moduli space S (adequate if $\operatorname{char} k = 2$) but $\mathcal{Y}$ does not have separated diagonal. The induced morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is étale, but not representable, and induces an isomorphism of the residual gerbes $B\mathbb{Z}/2\mathbb{Z}$ of the unique closed points. But f is not strongly étale and does not admit a section.

4. Coherent completeness and effectivity

In this section we prove Theorem 1.6 (coherent completeness) and Theorem 1.10 (effectivity), which are both essential ingredients in our proof of Theorem 1.1 (local structure).

The proofs of coherent completeness and effectivity are intertwined. First, we prove an important special case of coherent completeness, namely when the resolution property holds for $\mathcal{X}$ (Theorem 4.5). Then we use this special case to prove effectivity in characteristic zero (Theorem 4.15) and in local characteristic p (Corollary 4.18). Then we prove that adequate moduli spaces with linearly reductive stabilizers are good moduli spaces (Theorem 4.21). Then we can prove effectivity in general and finally use it to prove coherent completeness in general.

4.1. **Theorem on formal functions.** The following theorem on formal functions for good moduli spaces is a key component in the proof of the coherent completeness result (Theorem 1.6). This theorem is close in spirit to [EGA, III.4.1.5] and is a generalization of [Alp12, Thm. 1.1]. Surprisingly, we obtain, and will use, a version that also holds for adequate moduli spaces.

Theorem 4.1 (Formal functions, adequate version)**.** *Let $\mathcal{X}$ be an algebraic stack admitting an adequate moduli space $\pi\colon \mathcal{X} \to \operatorname{Spec} A$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a sheaf of ideals $\mathcal{I}$. Let $I = \Gamma(\mathcal{X}, \mathcal{I})$ be the corresponding ideal of $A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If A is noetherian and I-adically complete, and π is of finite type, then for every $\mathcal{F} \in \mathsf{Coh}(\mathcal{X})$ the natural map*

$$\Gamma(\mathcal{X}, \mathcal{F}) \to \varprojlim_n \Gamma(\mathcal{X}, \mathcal{F})/\Gamma(\mathcal{X}, \mathcal{I}^n\mathcal{F}) \tag{4.1}$$

is an isomorphism.

Proof. The A-module $F := \Gamma(\mathcal{X}, \mathcal{F})$ is finitely generated since A is noetherian and π is of finite type [Alp14, Thm. 6.3.3]. Let $I_n = \Gamma(\mathcal{X}, \mathcal{I}^n)$ and $F_n = \Gamma(\mathcal{X}, \mathcal{I}^n\mathcal{F})$. Note that $\mathcal{I}^* := \bigoplus \mathcal{I}^n$ is a finitely generated $\mathcal{O}_{\mathcal{X}}$-algebra and $\mathcal{I}^*\mathcal{F} := \bigoplus \mathcal{I}^n\mathcal{F}$ is a finitely generated $\mathcal{I}^*$-module. If we let $I_* = \bigoplus I_n = \Gamma(\mathcal{X}, \mathcal{I}^*)$, then $\operatorname{Spec}_{\mathcal{X}} \mathcal{I}^* \to \operatorname{Spec} I_*$ is an adequate moduli space [Alp14, Lem. 5.2.11]. Since A is noetherian and $\operatorname{Spec}_{\mathcal{X}} \mathcal{I}^* \to \operatorname{Spec} A$ is of finite type, it follows that I_* is a finitely generated A-algebra and that $F_* := \bigoplus F_n = \Gamma(\mathcal{X}, \mathcal{I}^*\mathcal{F})$ is a finitely generated I_*-module [Alp14, Thm. 6.3.3].

By [EGA, II.2.1.6(v)], there is an integer $N \geq 1$ such that $I_{kN} = (I_N)^k$ for all $k \geq 1$. That is, the topology induced by the non-adic system I_n is equivalent to the I_N-adic topology. Without loss of generality, we can replace $\mathcal{I}$ with $\mathcal{I}^N$ so that $I_* = I^* = \bigoplus_{k \geq 0} I^k$.

Similarly, for sufficiently large n (e.g., larger than all degrees of a set of homogeneous generators of F_*), $F_{n+1} = IF_n$ [AM69, Lem. 10.8]; that is, (F_n) is an I-stable filtration on F. It follows that (F_n) induces the same topology on F as (I^nF) [AM69, Lem. 10.6]. But F is a finite A-module, hence I-adically complete, hence complete with respect to (F_n). □

Corollary 4.2 (Formal functions, good version)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack admitting a good moduli space $\pi\colon \mathcal{X} \to \operatorname{Spec} A$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a sheaf of ideals $\mathcal{I}$. Let $I = \Gamma(\mathcal{X}, \mathcal{I})$ be the corresponding ideal of $A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If A is I-adically complete, then for every $\mathcal{F} \in \mathsf{Coh}(\mathcal{X})$ the natural map*

$$\Gamma(\mathcal{X}, \mathcal{F}) \to \varprojlim_n \Gamma(\mathcal{X}, \mathcal{F}/\mathcal{I}^n\mathcal{F}) \tag{4.2}$$

is an isomorphism.

Proof. By [Alp13, Thm. 4.16(x)], the ring A is noetherian and by [AHR20, Thm. A.1], $\mathcal{X} \to \operatorname{Spec} A$ is of finite type so Theorem 4.1 applies. For good moduli spaces, the natural map $\Gamma(\mathcal{X}, \mathcal{F})/\Gamma(\mathcal{X}, \mathcal{I}^n\mathcal{F}) \to \Gamma(\mathcal{X}, \mathcal{F}/\mathcal{I}^n\mathcal{F})$ is an isomorphism by definition. □

Remark 4.3. The formal functions theorem generalizes the isomorphism of [AHR20, Eqn. (2.1)] from the case of $\mathcal{X} = [\operatorname{Spec} B/G]$ for G linearly reductive and $A = B^G$ complete local, all defined over a field k, to $\mathcal{X} = [\operatorname{Spec} B/\mathrm{GL}_n]$ and $A = B^{\mathrm{GL}_n}$ complete but not necessarily local. This also includes $[\operatorname{Spec} B/G]$ for G geometrically reductive and embeddable since GL_n/G is affine.

Remark 4.4. In the setting of the adequate version, Theorem 4.1, suppose that $\mathcal{X} = [\operatorname{Spec}(B)/G]$ where $G \to S$ is a reductive group scheme. Then van der Kallen has shown that $H^i(\mathcal{X}, -)$ preserves coherence for all i [vdK15, Cor. 1.2]. Using an argument similar to [EGA, III.4.1.5] one can then show that the map (4.2) is an isomorphism, see [AHL23, Cor. 4.9]. We will not use this result.

4.2. **Coherent completeness I: with resolution property.** We can now prove the first version of Theorem 1.6.

Theorem 4.5 (Coherent completeness assuming resolution property)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and good moduli space $\pi\colon \mathcal{X} \to X = \operatorname{Spec} A$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ and let $I = \Gamma(\mathcal{X}, \mathcal{I})$. Assume that $\mathcal{X}$ has the resolution property. If A is I-adically complete, then $\mathcal{X}$ is coherently complete along $\mathcal{Z}$.*

Note that in this theorem, $\mathcal{X}$ is assumed to have the resolution property, whereas in Theorem 1.6 it is only assumed that $\mathcal{Z}$ has the resolution property. The following full faithfulness result does not require any resolution property hypothesis and follows from arguments similar to those of [EGA, III.5.1.3] and [GZB15, Thm. 1.1(i)].

Lemma 4.6. *Let $\mathcal{X}$ be a noetherian algebraic stack that is cohomologically affine. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a sheaf of ideals $\mathcal{I}$. Let $I = \Gamma(\mathcal{X}, \mathcal{I})$ be the corresponding ideal of $A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If A is I-adically complete, then the functor*

$$\mathsf{Coh}(\mathcal{X}) \to \varprojlim_n \mathsf{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]}).$$

is fully faithful.

Proof. Following [Con05, §1], let $\mathcal{O}_{\widehat{\mathcal{X}}}$ denote the sheaf of rings on the lisse-étale site of $\mathcal{X}$ that assigns to each smooth morphism $p\colon \operatorname{Spec} B \to \mathcal{X}$ the ring $\varprojlim_n B/\mathcal{I}^n B$. The sheaf of rings $\mathcal{O}_{\widehat{\mathcal{X}}}$ is coherent and the natural functor

$$\mathsf{Coh}(\widehat{\mathcal{X}}) \to \varprojlim_n \mathsf{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]})$$

is an equivalence of categories [Con05, Thm. 2.3]. Let $c\colon \widehat{\mathcal{X}} \to \mathcal{X}$ denote the induced morphism of ringed topoi and let $\mathcal{F}, \mathcal{G} \in \mathsf{Coh}(\mathcal{X})$; then it remains to prove that the map

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}_{\mathcal{O}_{\widehat{\mathcal{X}}}}(c^*\mathcal{F}, c^*\mathcal{G})$$

is bijective. Now we have the following commutative square, whose vertical arrows are isomorphisms:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{\widehat{\mathcal{X}}}}(c^*\mathcal{F},c^*\mathcal{G}) \\ \downarrow & & \downarrow \\ \Gamma(\mathcal{X},\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F},\mathcal{G})) & \longrightarrow & \Gamma(\widehat{\mathcal{X}},\mathcal{H}om_{\mathcal{O}_{\widehat{\mathcal{X}}}}(c^*\mathcal{F},c^*\mathcal{G})). \end{array}$$

Since c is flat and $\mathcal{F}$ is coherent the natural morphism

$$c^*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F},\mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_{\widehat{\mathcal{X}}}}(c^*\mathcal{F},c^*\mathcal{G})$$

is an isomorphism [GZB15, Lem. 3.2]. Thus, it remains to prove that the map

$$\Gamma(\mathcal{X},\mathcal{Q}) \to \Gamma(\widehat{\mathcal{X}},c^*\mathcal{Q})$$

is an isomorphism whenever $\mathcal{Q} \in \mathsf{Coh}(\mathcal{X})$. But there are natural isomorphisms:

$$\Gamma(\widehat{\mathcal{X}},c^*\mathcal{Q}) \cong \varprojlim_n \Gamma(\widehat{\mathcal{X}},\mathcal{Q}/\mathcal{J}^{n+1}\mathcal{Q}) \cong \varprojlim_n \Gamma(\mathcal{X}^{[n]}_{\mathcal{Z}},\mathcal{Q}/\mathcal{J}^{n+1}\mathcal{Q}) \cong \varprojlim_n \Gamma(\mathcal{X},\mathcal{Q}/\mathcal{J}^{n+1}\mathcal{Q}).$$

The result now follows from Corollary 4.2. □

Proof of Theorem 4.5. By Lemma 4.6 it remains to show that if $\{\mathcal{F}_n\} \in \varprojlim_n \mathsf{Coh}(\mathcal{X}^{[n]}_{\mathcal{Z}})$, then there exists a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ with $(i^{[n]})^*\mathcal{F} \simeq \mathcal{F}_n$ for all n. The following argument is similar to the proof of essential surjectivity of [AHR20, Thm. 1.3]. Since $\mathcal{X}$ has the resolution property, there is a vector bundle $\mathcal{E}$ on $\mathcal{X}$ together with a surjection $\phi_0\colon \mathcal{E} \to \mathcal{F}_0$. We claim that ϕ_0 lifts to a compatible system of morphisms $\phi_n\colon \mathcal{E} \to \mathcal{F}_n$ for every $n > 0$. Indeed, since $\mathcal{E}^\vee \otimes \mathcal{F}_{n+1} \to \mathcal{E}^\vee \otimes \mathcal{F}_n$ is surjective and $\Gamma(\mathcal{X},-)$ is exact, it follows that the natural map $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E},\mathcal{F}_{n+1}) \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E},\mathcal{F}_n)$ is surjective. By Nakayama's lemma (see Remark 3.7), each ϕ_n is surjective.

It follows that we obtain an induced morphism of systems $\{\phi_n\}\colon \{\mathcal{E}_n\} \to \{\mathcal{F}_n\}$, which is surjective. Applying this procedure to the kernel of $\{\phi_n\}$, there is another vector bundle $\mathcal{H}$ and a morphism of systems $\{\psi_n\}\colon \{\mathcal{H}_n\} \to \{\mathcal{E}_n\}$ such that $\operatorname{coker}\{\psi_n\} \cong \{\mathcal{F}_n\}$. By the full faithfulness (Lemma 4.6), the morphism $\{\psi_n\}$ arises from a unique morphism $\psi\colon \mathcal{H} \to \mathcal{E}$. Let $\widetilde{\mathcal{F}} = \operatorname{coker}\psi$; then the universal property of cokernels proves that there is an isomorphism of systems $\{\widetilde{\mathcal{F}}_n\} \cong \{\mathcal{F}_n\}$ and the result follows. □

Theorem 4.5 in particular applies to complete affine quotient stacks.

Example 4.7. Let $S = \operatorname{Spec} B$ where B is a noetherian ring. Let $G \subseteq \mathrm{GL}_{n,S}$ be a linearly reductive closed subgroup scheme acting on a noetherian affine scheme $X = \operatorname{Spec} A$. Then $[\operatorname{Spec} A/G]$ satisfies the resolution property; see Remark 2.5. If $(A^G,\mathfrak{m})$ is an $\mathfrak{m}$-adically complete local ring, then it follows from Theorem 4.5 that $[\operatorname{Spec} A/G]$ is coherently complete along the unique closed point. When S is the spectrum of a field and the unique closed G-orbit is a fixed point, this is [AHR20, Thm. 1.3].

4.3. **Effectivity: general setup.** We will now introduce a general setup for the proof of the effectivity theorem.

Setup 4.8. Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic stacks, i.e., $\mathcal{X}_0 \hookrightarrow \mathcal{X}_1 \hookrightarrow \cdots$ is a sequence of closed immersions such that if $\mathcal{I}_{(j)}$ denotes the coherent sheaf of ideals defining the closed immersion $u_{0j}\colon \mathcal{X}_0 \hookrightarrow \mathcal{X}_j$, then $(\mathcal{I}_{(j)})^{i+1}$ defines $u_{ij}\colon \mathcal{X}_i \hookrightarrow \mathcal{X}_j$ for every $i \leq j$. Let $A_n = \Gamma(\mathcal{X}_n,\mathcal{O}_{\mathcal{X}_n})$, $X_n = \operatorname{Spec} A_n$, $A = \varprojlim_n A_n$, $I_n = \ker(A \to A_{n-1})$, and $X = \operatorname{Spec} A$.

A classical result states that if each $\mathcal{X}_i$ is affine, then $A = \varprojlim_n \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ is a noetherian ring and $\mathcal{X}_i$ is the ith infinitesimal neighborhood of $\mathcal{X}_0$ in $\operatorname{Spec} A$ [EGA, 0_{I}.7.2.8]. The main effectivity theorem (Theorem 1.10) is an analogous result when $\mathcal{X}_0$ is linearly fundamental.

A key observation here is that even if $\mathcal{X}_0$ is linearly fundamental, the sequence of closed immersions of affine schemes:

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots . \tag{4.3}$$

is not adic (this is just as in the proof of Theorem 4.1). The following lemma shows that the sequence (4.3) is equivalent to an adic one, however.

Lemma 4.9. *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence as in Setup 4.8. If $\mathcal{X}_0$ is cohomologically affine, then A is a noetherian I_1-adically complete ring.*

Proof. For $n \geq 0$, let $\mathcal{F}_n = (\mathcal{I}_{(n)})^n$; then $\mathcal{F}_n$ defines the closed immersion $\mathcal{X}_{n-1} \hookrightarrow \mathcal{X}_n$ and may be regarded as a coherent $\mathcal{O}_{\mathcal{X}_0}$-module. Let $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n$ as an $\mathcal{O}_{\mathcal{X}_0}$-module. The truncation $\mathcal{A}_{\leq N} = \bigoplus_{n=0}^{N} \mathcal{F}_n$ coincides with the graded $\mathcal{O}_{\mathcal{X}_0}$-algebra $\operatorname{Gr}_{\mathcal{I}_{(N)}} \mathcal{O}_{\mathcal{X}_N} := \bigoplus_{n=0}^{N} (\mathcal{I}_{(N)})^n/(\mathcal{I}_{(N)})^{n+1}$ which is finitely generated by elements of degree 1 since $\mathcal{X}_N$ is noetherian. The truncation map $\mathcal{A}_{\leq N} \to \mathcal{A}_{\leq M}$ is an algebra homomorphism for every $N \geq M$. We can thus endow $\mathcal{A}$ with the structure of a graded algebra, finitely generated by elements in degree 1.

By cohomological affineness, we have that $I_n/I_{n+1} = \ker(A_n \to A_{n-1}) = \Gamma(\mathcal{X}_0, \mathcal{F}_n)$ and we obtain an isomorphism of graded algebras $\Gamma(\mathcal{X}_0, \mathcal{A}) = \operatorname{Gr}_{I_*} A := \bigoplus I_n/I_{n+1}$. Now by [AHR20, Lem. A.2], the graded A_0-algebra $\operatorname{Gr}_{I_*} A$ is finitely generated (but not necessarily by elements of degree 1). That is, for the filtration $\{I_n\}_{n\geq 0}$ on the ring A, the associated graded ring is a noetherian A_0-algebra. It follows from [God56, Thm. 4] that A is noetherian.

Since A is noetherian and complete with respect to the topology defined by $\{I_n\}_{n\geq 0}$ and $I_1^n \subseteq I_n$ for all n, it is also complete with respect to the I_1-adic topology. Indeed, the ideals I_1^n are finitely generated, hence closed in the (I_n)-topology and we can conclude by [EGA, 0_{I}.7.2.4, $\mathrm{Err}_{\mathrm{III}}$, 3]. Alternatively, we can give a direct argument as follows. If $\widehat{A}$ denotes the I_1-adic completion of A, then there is a natural factorization

$$A \to \widehat{A} = \varprojlim_n A/(I_1)^n \to A = \varprojlim_n A/I_n$$

of the identity. Since $\widehat{A} \to A$ is surjective and $\widehat{A}$ is noetherian and complete with respect to the I_1-adic topology, so is A. □

An easy case of effectivity is when the adic sequence $\{\mathcal{X}_n\}_{n\geq 0}$ embeds into a linearly fundamental stack.

Lemma 4.10. *Let $\mathcal{H}$ be a noetherian linearly fundamental algebraic stack. If $\{\mathcal{X}_n\}_{n\geq 0}$ is an adic sequence of noetherian algebraic stacks with compatible closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{H}$, then the completion of $\{\mathcal{X}_n\}_{n\geq 0}$ exists and is affine over $\mathcal{H}$.*

Proof. Let $\pi\colon \mathcal{H} \to H$ be its good moduli space, $\widehat{H}$ be the completion of H along $\pi(\mathcal{X}_0)$, and $\widehat{\mathcal{H}}$ be the base change $\mathcal{H} \times_H \widehat{H}$. Let $\mathcal{H}_0 = \mathcal{X}_0$ be the induced closed substack of $\widehat{\mathcal{H}}$ and $\mathcal{H}_n$ be its nth infinitesimal neighborhood. We have a compatible sequence of closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{H}_n$. If we let $\mathcal{K}$ be the sheaf of ideals defining $\mathcal{H}_0 \to \widehat{\mathcal{H}}$, then $\mathcal{K}\mathcal{O}_{\mathcal{X}_n} = \mathcal{I}_{(n)}$ and hence $\mathcal{K}^n\mathcal{O}_{\mathcal{X}_n} = \mathcal{I}_{(n)}^n$. This means that the sequence $\{\mathcal{O}_{\mathcal{X}_n}\}_{n\geq 0}$ of coherent $\mathcal{O}_{\mathcal{H}_n}$-modules is adic, i.e., is an element of $\varprojlim_n \mathsf{Coh}(\mathcal{H}_n)$. Since $\widehat{\mathcal{H}} \to \mathcal{H}$ is affine, $\widehat{\mathcal{H}}$ is also linearly fundamental and hence has the resolution property. Thus, Theorem 4.5 implies that $\widehat{\mathcal{H}}$ is coherently complete along $\mathcal{H}_0$. Hence, there exists a closed immersion $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{H}}$ that induces the $\mathcal{X}_n$. □

We now prove the effectivity of an adic sequence $\{\mathcal{X}_n\}$ under the hypothesis that $\mathcal{X}_0$ is cohomologically affine and admits a representable map to a linearly fundamental stack $\mathcal{Y}$ that is smooth over X. We will apply this result in the case that $\mathcal{Y} = B\mathrm{GL}_{n,X}$ or $\mathcal{Y} = BQ$ for a nice and embeddable group scheme $Q \to X$.

Proposition 4.11. *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence as in Setup 4.8. Let $\mathcal{Y}$ be a noetherian linearly fundamental algebraic stack that is smooth over X. If $\mathcal{X}_0$ is cohomologically affine and there is a representable morphism $\mathcal{X}_0 \to \mathcal{Y}$, then the completion of $\{\mathcal{X}_n\}_{n\geq 0}$ exists and is affine over $\mathcal{Y}$.*

Proof. By Lemma 4.10, it suffices to show that there are compatible embeddings $\mathcal{X}_n \hookrightarrow \mathcal{H}$ into an algebraic stack $\mathcal{H}$ affine and of finite type over $\mathcal{Y}$. This is a consequence of the following lemma (which holds more generally if $\mathcal{Y}$ is merely fundamental). □

Lemma 4.12 (Embedding)**.** *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence as in Setup 4.8. Let $\mathcal{Y}$ be a noetherian fundamental algebraic stack that is smooth over X. If $\mathcal{X}_0$ is cohomologically affine and there is a representable morphism $\mathcal{X}_0 \to \mathcal{Y}$ over X, then there exist*

(1) *a smooth affine morphism $\mathcal{H} \to \mathcal{Y}$ of finite type; and*
(2) *compatible closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{H}$ over X.*

Remark 4.13. In the proof of [AHR20, Thm. 1], effectivity was established by embedding the residual gerbe $\mathcal{X}_0$ and its thickenings into the normal space of $\mathcal{X}_0$ in $\mathcal{X}$. The technique here is similar, but we instead first choose a deformation $\mathcal{X}_1 \to \mathcal{Y}$, and then embed $\mathcal{X}_1$ and its thickenings into the affine space $\mathcal{H}$ over $\mathcal{Y}$ defined by a vector bundle resolution of the push-forward of $\mathcal{O}_{\mathcal{X}_1}$.

Proof of Lemma 4.12. By [AHR20, Thm. A.1], $\mathcal{X}_0 \to X_0$ is of finite type. Hence, $\mathcal{X}_0 \to X$ is of finite type and cohomologically affine. But the diagonal of $\mathcal{Y} \to X$ is affine and of finite type, so $\phi_0 \colon \mathcal{X}_0 \to \mathcal{Y}$ is cohomologically affine and of finite type. By assumption, it is representable, so Serre's theorem (e.g., [Alp13, Prop. 3.3]) tells us that $\phi_0 \colon \mathcal{X}_0 \to \mathcal{Y}$ is also affine.

We claim that there is an X-morphism $\phi_1 \colon \mathcal{X}_1 \to \mathcal{Y}$ that lifts ϕ_0. The obstruction to such a lift belongs to the group $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_0}}(\mathsf{L}\phi_0^* L_{\mathcal{Y}/X}, \mathcal{I}_{(1)})$ [Ols06a, Thm. 1.5]. Since $\mathcal{Y} \to X$ is smooth, the cotangent complex $L_{\mathcal{Y}/X}$ is perfect of amplitude $[0,1]$. The assumption that $\mathcal{X}_0$ is cohomologically affine now proves that this obstruction group vanishes. As before, it follows that ϕ_1 is affine and of finite type.

Since $\mathcal{Y}$ is fundamental it has the resolution property, so there exists a vector bundle of finite rank $\mathcal{E}$ on $\mathcal{Y}$ and a surjection of quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-algebras $\mathrm{Sym}_{\mathcal{O}_{\mathcal{Y}}}(\mathcal{E}) \to (\phi_1)_*\mathcal{O}_{\mathcal{X}_1}$. Defining

$$\mathcal{H} := \mathrm{Spec}_{\mathcal{Y}} \mathrm{Sym}_{\mathcal{O}_{\mathcal{Y}}}(\mathcal{E}),$$

there is an induced closed immersion $i_1 \colon \mathcal{X}_1 \hookrightarrow \mathcal{H}$ and $\mathcal{H} \to X$ is smooth. By using the same deformation theory argument as above, we can produce compatible X-morphisms $i_n \colon \mathcal{X}_n \to \mathcal{H}$ lifting i_1. The following Nakayama-like lemma implies that each i_n is a closed immersion. □

The following lemma is a generalization of [AHR20, Prop. A.8(1)] to the case where $|\mathcal{X}_1|$ is not necessarily a single point.

Lemma 4.14. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{I}$ be a nilpotent quasi-coherent sheaf of ideals of $\mathcal{O}_{\mathcal{X}}$. Let $\mathcal{X}_1 \subseteq \mathcal{X}$ be the closed immersion defined by $\mathcal{I}^2$. If the composition $\mathcal{X}_1 \to \mathcal{X} \xrightarrow{f} \mathcal{Y}$ is a closed immersion, then f is a closed immersion.*

Proof. The statement is local on $\mathcal{Y}$ for the smooth topology, so we may assume that $\mathcal{Y} = \operatorname{Spec} A$. Then $\mathcal{X}_1$ is affine, and since $\mathcal{X}$ is an infinitesimal thickening of $\mathcal{X}_1$, it follows that $\mathcal{X}$ is also affine [Ryd15, Cor. 8.2]. Hence, we may assume that $\mathcal{X} = \operatorname{Spec} B$ and $\mathcal{I} = \widetilde{I}$ for some nilpotent ideal I of B. Let $\phi\colon A \to B$ be the induced morphism. The assumptions are that the composition $A \to B \to B/I^2$ is surjective and that $I^{n+1} = 0$ for some $n \geq 0$. Let $K = \ker(A \to B/I)$. Since $KB \to I \to I/I^2$ is surjective and $I^{n+1} = 0$, it follows that $I = KB + I^2 = KB + I^4 = \cdots = KB$. That is, $KB = I$. Further, since $A \to B \to B/KB = B/I$ is surjective and $K^{n+1}B = I^{n+1} = 0$, it follows that $B = \operatorname{im}\phi + KB = \operatorname{im}\phi + K^2B = \cdots = \operatorname{im}\phi$. That is, ϕ is surjective. □

4.4. **Effectivity I: characteristic zero.**

Theorem 4.15 (Effectivity in characteristic zero). *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic $\mathbb{Q}$-stacks. If $\mathcal{X}_0$ is linearly fundamental, then the completion of the sequence exists and is linearly fundamental.*

Proof. Since $\mathcal{X}_0$ is linearly fundamental, it admits an affine morphism to $B\mathrm{GL}_{N,\mathbb{Q}}$ for some $N > 0$. This gives an affine morphism $\mathcal{X}_0 \to \mathcal{Y} := B\mathrm{GL}_{N,X}$. Since $X = \operatorname{Spec}\big(\varprojlim_n \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})\big)$ is a $\mathbb{Q}$-scheme, $\mathcal{Y}$ is linearly fundamental. The conclusion now follows from Proposition 4.11. □

To prove effectivity in positive and mixed characteristic (Theorem 1.10), we will need to make a better choice of group than $\mathrm{GL}_{N,\mathbb{Q}}$. To do this, we will next study the deformations of nice group schemes.

4.5. **Deformation of nice group schemes.** We will now prove that a nice and embeddable group scheme (see Definition 2.1) can be deformed along an affine henselian pair (Definition 3.1). After we have established the general effectivity result, we will prove the corresponding result for linearly reductive group schemes (Proposition 7.13).

Proposition 4.16 (Deformation of nice group schemes). *Let (S, S_0) be an affine henselian pair. If $G_0 \to S_0$ is a nice and embeddable group scheme, then there exists a nice and embeddable group scheme $G \to S$ whose restriction to S_0 is isomorphic to G_0.*

Proof. Let $(S, S_0) = (\operatorname{Spec} A, \operatorname{Spec} A/I)$. By limit methods (Lemma 2.12), we may assume that S is the henselization of an affine scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. Let $S_n = \operatorname{Spec} A/I^{n+1}$. Also, let R be the I-adic completion of A and let $\widehat{S} = \operatorname{Spec} R$.

Let $F\colon (\mathsf{Sch}/S)^{\mathrm{opp}} \to \mathsf{Sets}$ be the functor that assigns to each S-scheme T the set of isomorphism classes of nice and embeddable group schemes over T. By Lemma 2.12, F is limit preserving. Suppose that we have a nice embeddable group scheme $G_{\widehat{S}} \in F(\widehat{S})$ restricting to G_0. By Artin approximation (Theorem 3.8), there exists $G_S \in F(S)$ that restricts to G_0. We can thus replace S by $\widehat{S}$ and assume that A is complete.

Fix a closed immersion of S_0-group schemes $i\colon G_0 \to \mathrm{GL}_{n,S_0}$. By definition, there is an open and closed subgroup $(G_0)^0 \subseteq G_0$ of multiplicative type. By [SGA3$_{\mathrm{II}}$, Exp. XI, Thm. 5.8], there is a lift of i to a closed immersion of group schemes $i_S\colon G_S^0 \to \mathrm{GL}_{n,S}$, where G_S^0 is of multiplicative type. Let $N = \mathrm{Norm}_{\mathrm{GL}_{n,S}}(G_S^0)$ be the normalizer, which is a smooth S-group scheme and closed S-subgroup scheme of $\mathrm{GL}_{n,S}$ [SGA3$_{\mathrm{II}}$, Exp. XI, 5.3 bis].

Since $(G_0)^0$ is a normal S_0-subgroup scheme of G_0, it follows that G_0 is a closed S_0-subgroup scheme of $N \times_S S_0$. In particular, there is an induced closed immersion $q_{S_0}\colon (G_0)/(G_0)^0 \to (N/G_S^0) \times_S S_0$ of group schemes over S_0. Since G_0 is nice, the locally constant group scheme $(G_0)/(G_0)^0$ has order prime to p. Since R is complete, there is a unique locally constant group scheme H over S such that $H \times_S S_0 = (G_0)/(G_0)^0$. Note that H is finite and linearly reductive over S.

Since N/G_S^0 is a smooth and affine group scheme over S, there are compatible closed immersions of S_n-group schemes $q_{S_n} \colon H \times_S S_n \to (N/G_S^0) \times_S S_n$ lifting q_{S_0}, which are unique up to conjugation [SGA3$_{\mathrm{I}}$, Exp. III, Cor. 2.8]. Since H is finite, these morphisms effectivize to a morphism of group schemes $q_S \colon H \to N/G_S^0$. We now define G_S to be the preimage of H under the quotient map $N \to N/G_S^0$. Then G_S is nice and embeddable, and $G_S \times_S S_0 \cong G_0$. □

4.6. **Effectivity II: local case in positive characteristic.** We can now establish the effectivity theorem for nicely fundamental stacks (Definition 2.7).

Theorem 4.17 (Effectivity for nice stacks)**.** *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic stacks. If $\mathcal{X}_0$ is nicely fundamental, then the completion of the sequence exists and is nicely fundamental.*

Proof. Let X_0 be the good moduli space of $\mathcal{X}_0$. Since $\mathcal{X}_0$ is nicely fundamental, it admits an affine morphism to $B_{X_0}Q_0$, for some nice and embeddable group scheme $Q_0 \to X_0$. By Lemma 4.9, the affine scheme $X = \operatorname{Spec}\big(\varprojlim_n \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})\big)$ is noetherian and complete along X_0. It follows from Proposition 4.16 that there is a nice and embeddable group scheme $Q \to X$ lifting $Q_0 \to X_0$. Let $\mathcal{Y} = B_X Q$; then $\mathcal{Y}$ is linearly fundamental and smooth over X. The result now follows from Proposition 4.11. □

The following corollary will shortly be subsumed by Theorem 1.10, but is sufficient for many applications, e.g., it is sufficient for Theorem 1.1.

Corollary 4.18 (Effectivity for local stacks)**.** *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic stacks. Assume that $\mathcal{X}_0$ is a gerbe over a field k. If $\mathcal{X}_0$ is linearly fundamental (i.e., has linearly reductive stabilizer), then the completion of the sequence exists and is linearly fundamental.*

Proof. If $\mathcal{X}$ is a $\mathbb{Q}$-stack, then we are already done by Theorem 4.15. If not, then k has characteristic $p > 0$ and $\mathcal{X}_0$ is nicely fundamental by Proposition 2.11(2). Theorem 4.17 completes the proof. □

4.7. **Adequate moduli spaces with linearly reductive stabilizers are good.** We prove that adequate moduli spaces of stacks with linearly reductive stabilizers at closed points are good (Theorem 4.21) by using the adequate version of the formal function theorem (Theorem 4.1) and the effectivity theorem in the form of Corollary 4.18.

Lemma 4.19. *Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack defined by the sheaf of ideals $\mathcal{I}$. Assume that $\mathcal{X}$ has an adequate moduli space $\pi \colon \mathcal{X} \to \operatorname{Spec} A$ of finite type, where A is noetherian and I-adically complete along $I = \Gamma(\mathcal{X}, \mathcal{I})$. Let $B_n = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1})$ for $n \geq 0$ and $B = \varprojlim_n B_n$. If $\mathcal{Z}$ is cohomologically affine with affine diagonal, then the induced homomorphism $A \to B$ is finite.*

Proof. Since $\mathcal{Z} = \mathcal{X}^{[0]}$ is cohomologically affine with affine diagonal, so are its infinitesimal neighborhoods $\mathcal{X}^{[n]}$. The surjection $\mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1} \to \mathcal{O}_{\mathcal{X}}/\mathcal{I}^n$ therefore induces a surjection $B_n \to B_{n-1}$ of rings with kernel $\Gamma(\mathcal{X}, \mathcal{I}^n/\mathcal{I}^{n+1})$ for all n. Thus, if we let $J_{n+1} = \ker(B \to B_n)$; then $J_n/J_{n+1} = \Gamma(\mathcal{X}, \mathcal{I}^n/\mathcal{I}^{n+1})$ and B is complete with respect to the topology given by the filtration (J_n).

Let $I_n = \Gamma(\mathcal{X}, \mathcal{I}^n)$; then Theorem 4.1 implies that A is also complete with respect to the filtration given by (I_n), that is, $A = \varprojlim_n A/I_n$. Taking global sections of the short exact sequence $0 \to \mathcal{I}^{n+1} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1} \to 0$ induces an injection

$A/I_{n+1} \to B_n$, which is an adequate ring homomorphism, i.e., every element of B_n has a positive power contained in the image. Passing to inverse limits, we see that the homomorphism $A \to B$ is an injective continuous map between complete topological rings.

Taking global sections of the exact sequence $0 \to \mathcal{J}^{n+1} \to \mathcal{J}^n \to \mathcal{J}^n/\mathcal{J}^{n+1} \to 0$ induces an injective map $I_n/I_{n+1} \to J_n/J_{n+1}$. Taking direct sums gives a surjection of algebras $\bigoplus \mathcal{J}^n \to \operatorname{Gr}_{\mathcal{J}}(\mathcal{O}_{\mathcal{X}})$ with kernel $\bigoplus \mathcal{J}^{n+1}$. Since π is an adequate moduli space, taking global sections provides an injective adequate map $\operatorname{Gr}_{I_*} A = \bigoplus I_n/I_{n+1} \to \operatorname{Gr}_{J_*} B = \bigoplus J_n/J_{n+1}$.

Also, $\operatorname{Gr}_{\mathcal{J}}(\mathcal{O}_{\mathcal{X}})$ is a finitely generated algebra and since $\operatorname{Spec}\big(\operatorname{Gr}_{J_*} B\big)$ is the adequate moduli space of $\operatorname{Spec}_{\mathcal{X}}\big(\operatorname{Gr}_{\mathcal{J}}(\mathcal{O}_{\mathcal{X}})\big)$, it follows that $\operatorname{Gr}_{J_*} B$ is a finitely generated A-algebra [Alp14, Thm. 6.3.3]. Thus $\operatorname{Gr}_{I_*} A \to \operatorname{Gr}_{J_*} B$ is an injective adequate map of finite type, hence finite. It follows that $A \to B$ is finite [God56, Lem. on p. 6]. □

Remark 4.20. It is, a priori, not clear that $A \to B$ is adequate. Consider the following example: $A = \mathbb{F}_2[\![x]\!]$, $B = A[y]/(y^2 - x^2 y - x)$. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ is a ramified, generically étale, finite flat cover of degree 2, so not adequate. But the induced map on graded rings $\mathbb{F}_2[x] \to \mathbb{F}_2[x,y]/(y^2 - x)$ is adequate. Nevertheless, it follows from Theorem 4.21, proven below, that $A = B$ in Lemma 4.19. If the formal functions theorem (Corollary 4.2) holds for stacks with adequate moduli spaces, then $A = B$ without assuming that $\mathcal{Z}$ is cohomologically affine.

Theorem 4.21. *Let S be a noetherian algebraic space. Let $\mathcal{X}$ be an algebraic stack of finite type over S with an adequate moduli space $\pi\colon \mathcal{X} \to X$. Assume that π has affine diagonal. Then π is a good moduli space if and only if every closed point of $\mathcal{X}$ has linearly reductive stabilizer.*

Remark 4.22. See Corollaries 6.11 and 6.12 for non-noetherian versions.

Proof. By [Alp14, Thm. 6.3.3], X is of finite type over S. We can thus replace S with X. If π is a good moduli space, then every closed point has linearly reductive stabilizer [Alp13, Prop. 12.14]. For the converse, we need to prove that π_* is exact. This can be verified after replacing S with the completion at every closed point (since adequate moduli spaces commute with flat base change). We may thus assume that $X = S$ is a complete local scheme. Since π is an adequate moduli space, $\mathcal{X}$ has a unique closed point and we let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be the corresponding closed immersion.

By Corollary 4.18, the adic sequence $\mathcal{X}_{\mathcal{Z}}^{[0]} \hookrightarrow \mathcal{X}_{\mathcal{Z}}^{[1]} \hookrightarrow \dots$ has completion $\widehat{\mathcal{X}}$ that has a good moduli space X'. By Tannaka duality (see §1.7), there is a natural map $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ and it suffices to prove it is an isomorphism. Now f induces a map $g\colon X' \to X$ of adequate moduli spaces. In the notation of Lemma 4.19, $X' = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, and we conclude that $X' \to X$ is finite. In particular, $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ is also of finite type since the good moduli map $\widehat{\mathcal{X}} \to X'$ is of finite type [AHR20, Thm. A.1]. The morphism $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ is formally étale, hence étale, and also affine [AHR20, Prop. 3.2], hence representable. Moreover, $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ induces an isomorphism of stabilizer groups at the unique closed points so we may apply Luna's fundamental lemma (Theorem 3.14) to conclude that $X' \times_X \mathcal{X} = \widehat{\mathcal{X}}$ and thus $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ is finite. But f is an isomorphism over the unique closed point of $\mathcal{X}$, hence f is a closed immersion. But f is also étale, hence a closed and open immersion, hence an isomorphism. □

Corollary 4.23. *Let S be a noetherian algebraic space and let $G \to S$ be an affine flat group scheme of finite presentation. Then $G \to S$ is linearly reductive if and only if $G \to S$ is geometrically reductive and every closed fiber is linearly reductive.* □

Proof. Apply Theorem 4.21 to $BG \to S$, which is an adequate (resp. good) moduli space if and only if $G \to S$ is geometrically (resp. linearly) reductive. □

The corollary also holds in the non-noetherian case by Corollary 6.12.

4.8. **Effectivity III: the general case.** We now finally come to the proof of the general effectivity result for adic systems of algebraic stacks. Recall that this says that the completion $\widehat{\mathcal{X}}$ of an adic sequence $\{\mathcal{X}_n\}_{n\geq 0}$ of noetherian algebraic stacks, such that $\mathcal{X}_0$ is linearly fundamental, exists and is linearly fundamental.

Proof of Theorem 1.10. Let X be as in Setup 4.8. Since $\mathcal{X}_0$ is (linearly) fundamental, it admits an affine morphism to $\mathcal{Y} = B\mathrm{GL}_{N,X}$. By Lemma 4.12, there is an affine morphism $\mathcal{H} \to \mathcal{Y}$ of finite type and compatible closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{H}$. Let $H = \operatorname{Spec}\Gamma(\mathcal{H}, \mathcal{O}_{\mathcal{H}})$ be the adequate moduli space of $\mathcal{H}$. Since the composition $\mathcal{H} \to \mathcal{Y} \to X$ is of finite type and X is noetherian (Lemma 4.9), $H \to X$ is of finite type [Alp14, Thm. 6.3.3] and so $\mathcal{H} \to H$ is of finite type and H is noetherian.

Since $\mathcal{X}_n \to X_n$ is a good moduli space, there are uniquely induced morphisms $X_n \to H$. Passing to limits, we produce a unique morphism $X \to H$, which is a closed immersion as the composition $X \to H \to X$ is the identity. Take $\mathcal{H}'$ to be the base change of $\mathcal{H} \to H$ along $X \to H$. Let $H' = \operatorname{Spec}\Gamma(\mathcal{H}', \mathcal{O}_{\mathcal{H}'})$; then the induced morphism $H' \to X$ is a finite type adequate universal homeomorphism, hence finite. Since $\mathcal{H}' \to X$ is universally closed, the closed points of $\mathcal{H}'$ are identified with the closed points of $\mathcal{X}_0$ and thus have linearly reductive stabilizer. By Theorem 4.21, $\mathcal{H}'$ is cohomologically affine. We may now apply Lemma 4.10 to the induced closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{H}'$ to conclude that the completion of $\{\mathcal{X}_n\}_{n\geq 0}$ exists. □

Remark 4.24 (Quasi-excellence). In Theorem 1.10, if $\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ is quasi-excellent, then $\widehat{\mathcal{X}}$ is locally quasi-excellent. Indeed, using the notation of Setup 4.8, we know that $A = \Gamma(\widehat{\mathcal{X}}, \mathcal{O}_{\widehat{\mathcal{X}}})$ is an I_1-adically complete noetherian ring (Lemma 4.9). Since $\mathcal{X}_0$ is cohomologically affine, $A/I_1 = \Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$, which is quasi-excellent by assumption. Hence, A is quasi-excellent by the Gabber–Kurano–Shimomoto theorem [KS21, Main Thm. 1]. But $\widehat{\mathcal{X}} \to \operatorname{Spec} A$ is of finite type, so $\widehat{\mathcal{X}}$ is locally quasi-excellent.

4.9. **Coherent completeness II: general case.** We can now finish the general coherent completeness theorem. Recall that this says that a noetherian algebraic stack $\mathcal{X}$ with affine good moduli space X is coherently complete along $\mathcal{Z}$ if and only if X is coherently complete along the image of $\mathcal{Z}$. This is under the assumption that $\mathcal{Z}$ has the resolution property and it also follows that $\mathcal{X}$ has the resolution property.

Proof of Theorem 1.6. The necessity of the condition follows from Proposition 3.9. For the sufficiency: the completion $\widehat{\mathcal{X}}$ of $\{\mathcal{X}_{\mathcal{Z}}^{[n]}\}$ exists and is linearly fundamental (Theorem 1.10). By formal functions (Corollary 4.2),

$$A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \varprojlim_n \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}_{\mathcal{Z}}^{[n]}}) = \varprojlim_n \Gamma(\widehat{\mathcal{X}}, \mathcal{O}_{\mathcal{X}_{\mathcal{Z}}^{[n]}}) \simeq \Gamma(\widehat{\mathcal{X}}, \mathcal{O}_{\widehat{\mathcal{X}}}).$$

Hence, the good moduli space of $\widehat{\mathcal{X}}$ is X. By Tannaka duality, there is an induced morphism $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ and it is affine [AHR20, Prop. 3.2]. The composition $\widehat{\mathcal{X}} \to \mathcal{X} \to X$ is a good moduli space and hence of finite type [AHR20, Thm. A.1]. It follows that f is of finite type. Since f is formally étale, it is thus étale. Luna's fundamental lemma (Theorem 3.14) now implies that $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ is an isomorphism. In particular, $\mathcal{X}$ is linearly fundamental, i.e., has the resolution property. □

We are now in position to prove Formal GAGA (Corollary 1.7).

Proof of Corollary 1.7. The first case follows from the second since if $I \subseteq A$ is a maximal ideal, $\mathcal{X}\times_{\operatorname{Spec} A}\operatorname{Spec}(A/I)$ necessarily has the resolution property [AHR20, Cor. 4.14]. The corollary then follows from applying Theorem 1.6 with $\mathcal{Z} = \mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec}(A/I)$. □

5. The local structure of algebraic stacks

In this section, we first prove Theorem 1.11, which establishes the existence of formally syntomic neighborhoods of locally closed substacks. We then use this theorem to prove Corollary 1.12 establishing the existence of completions at points with linearly reductive stabilizers. Finally, we prove the local structure of algebraic stacks (Theorem 1.1) in a slightly more general form, see Theorem 5.6.

The results of this section establish the local structure of algebraic stacks near not necessarily closed points or immersions. It turns out to be convenient to work in the more general setting of *pro-unramified* morphisms. Recall that if $\mathcal{X}$ is a noetherian algebraic stack, then a morphism $\mathcal{V} \to \mathcal{X}$ is *pro-unramified* (resp. a *pro-immersion*) if it can be written as a composition $\mathcal{V} \hookrightarrow \mathcal{V}' \to \mathcal{X}$, where $\mathcal{V} \hookrightarrow \mathcal{V}'$ is a flat quasi-compact monomorphism and $\mathcal{V}' \hookrightarrow \mathcal{X}$ is unramified and of finite type (resp. a closed immersion). Clearly, pro-immersions are pro-unramified. Note that residual gerbes on quasi-separated algebraic stacks are pro-immersions [Ryd11b, Thm. B.2]. Moreover, every monomorphism of finite type is pro-unramified.

5.1. Existence of formally syntomic neighborhoods.

Recall that a morphism is *syntomic* if it is flat and locally of finite presentation, with fibers that are local complete intersections (e.g., smooth). In particular, if a morphism is syntomic and representable, then its cotangent complex is perfect of tor-amplitude $[-1, 0]$ [Stacks, Tag 0FK3], which is the only property of syntomic morphisms that we will use. As promised, we now establish the following generalization of Theorem 1.11.

Theorem 5.1 (Formal neighborhoods)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack. Let $\mathcal{X}_0 \to \mathcal{X}$ be pro-unramified. Let $h_0\colon \mathcal{W}_0 \to \mathcal{X}_0$ be a syntomic (e.g., smooth) morphism. Assume that $\mathcal{W}_0$ is linearly fundamental. If either*

(1) *$\mathcal{X}$ has quasi-affine diagonal; or*
(2) *$\mathcal{X}$ has affine stabilizers and $\Gamma(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0})$ is quasi-excellent;*

then there is a flat morphism $h\colon \widehat{\mathcal{W}} \to \mathcal{X}$, where $\widehat{\mathcal{W}}$ is noetherian, linearly fundamental, $h|_{\mathcal{X}_0} \simeq h_0$, and $\widehat{\mathcal{W}}$ is coherently complete along $\mathcal{W}_0 = h^{-1}(\mathcal{X}_0)$. Moreover if h_0 is smooth (resp. étale), then h is unique up to non-unique 1-isomorphism (resp. unique up to unique 2-isomorphism).

Proof. We first reduce to the situation where $\mathcal{X}_0 \to \mathcal{X}$ is a pro-immersion. Since $\mathcal{X}_0 \to \mathcal{X}$ is pro-unramified, it factors as $\mathcal{X}_0 \xrightarrow{j} \mathcal{V}_0 \xrightarrow{u} \mathcal{X}$, where j is a flat quasi-compact monomorphism and u is unramified and of finite type. By [Ryd11a, Thm. 1.2], there is a further factorization $\mathcal{V}_0 \xrightarrow{i} \mathcal{X}' \xrightarrow{p} \mathcal{X}$ of u, where i is a closed immersion and p is étale, representable and finitely presented. In particular, $\mathcal{X}_0 \xrightarrow{j} \mathcal{V}_0 \xrightarrow{i} \mathcal{X}'$ is a pro-immersion. Since p has quasi-affine diagonal, $\mathcal{X}'$ inherits the conditions (1) or (2) from $\mathcal{X}$; hence, we may replace $\mathcal{X}$ by $\mathcal{X}'$ and assume that $\mathcal{X}_0 \to \mathcal{X}$ is a pro-immersion.

We thus have a factorization $\mathcal{X}_0 \xrightarrow{j} \mathcal{V}_0 \xrightarrow{i} \mathcal{X}$ where j is a flat quasi-compact monomorphism and i is a closed immersion. Note that j is schematic [Stacks, Tag 0B8A] and even quasi-affine [Ray68, Prop. 1.5] and that $\mathcal{X}_0$ is noetherian [Ray68, Prop. 1.2]. In particular, $\mathcal{W}_0$ is also noetherian.

Let $g_0 = j \circ h_0 \colon \mathcal{W}_0 \to \mathcal{V}_0 = \mathcal{X}^{[0]}_{\mathcal{V}_0}$ which is flat. We claim that it suffices to prove, using induction on $n \geq 1$, that there are compatible cartesian diagrams:

$$\begin{array}{ccc} \mathcal{W}_{n-1} & \longrightarrow & \mathcal{W}_n \\ {\scriptstyle g_{n-1}}\downarrow & & \downarrow{\scriptstyle g_n} \\ \mathcal{X}^{[n-1]}_{\mathcal{V}_0} & \longrightarrow & \mathcal{X}^{[n]}_{\mathcal{V}_0}, \end{array}$$

where each g_n is flat and the $\mathcal{W}_n$ are noetherian. Indeed, the flatness of the g_n implies that the resulting system $\{\mathcal{W}_n\}_{n\geq 0}$ is adic. By Theorem 1.10, the completion $\widehat{\mathcal{W}}$ of the sequence $\{\mathcal{W}_n\}_{n\geq 0}$ exists and is noetherian and linearly fundamental. If $\mathcal{X}$ has quasi-affine diagonal, then the morphisms $\mathcal{W}_n \to \mathcal{X}$ induce a unique morphism $\widehat{\mathcal{W}}$ by Tannaka duality (case (b) of §1.7). If $\mathcal{X}$ only has affine stabilizers, however, then Tannaka duality (case (a) of §1.7) has the additional hypothesis that $\widehat{\mathcal{W}}$ is locally the spectrum of a G-ring, e.g., locally quasi-excellent, which follows from the quasi-excellency of $\Gamma(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0})$ (Remark 4.24). The flatness of $\widehat{\mathcal{W}} \to \mathcal{X}$ is just the local criterion for flatness [EGA, 0_{III}.10.2.1].

We now get back to solving the lifting problem. If g_0 is not representable, choose an affine morphism $\mathcal{W}_0 \to B\mathrm{GL}_N$ for some N. Since $B\mathrm{GL}_N$ has smooth diagonal, the induced *representable* morphism $\mathcal{W}_0 \to \mathcal{X}_0 \times B\mathrm{GL}_N$ is syntomic. Hence, we may replace $\mathcal{X}$ with $\mathcal{X} \times B\mathrm{GL}_N$ and assume that g_0 is representable. By [Ols06a, Thm. 1.4], the obstruction to lifting g_{n-1} to g_n belongs to the group $\mathrm{Ext}^2_{\mathcal{O}_{\mathcal{W}_0}}(L_{\mathcal{W}_0/\mathcal{V}_0}, g_0^*(\mathcal{I}^n/\mathcal{I}^{n+1}))$, where $\mathcal{I}$ is the coherent ideal sheaf defining the closed immersion $i \colon \mathcal{V}_0 \hookrightarrow \mathcal{X}$.

Now since $\mathcal{X}_0 \to \mathcal{V}_0$ is a flat monomorphism, $L_{\mathcal{X}_0/\mathcal{V}_0} \simeq 0$ [LMB, Prop. 17.8]. Hence, $L_{\mathcal{W}_0/\mathcal{V}_0} \simeq L_{\mathcal{W}_0/\mathcal{X}_0}$. But $\mathcal{W}_0 \to \mathcal{X}_0$ is syntomic, so $L_{\mathcal{W}_0/\mathcal{X}_0}$ is perfect of amplitude $[-1, 0]$ and $\mathcal{W}_0$ is cohomologically affine. Thus, the Ext-group vanishes, and we have the required lift. That $\mathcal{W}_n$ is noetherian is clear: it is a thickening of a noetherian stack by a coherent sheaf of ideals.

For the uniqueness statement: Let $h \colon \widehat{\mathcal{W}} \to \mathcal{X}$ and $h' \colon \widehat{\mathcal{W}'} \to \mathcal{X}$ be two different morphisms as in the theorem. Let $g_n = j_n \circ h_n \colon \mathcal{W}_n \to \mathcal{X}^{[n]}_{\mathcal{V}_0}$ and $g'_n = j_n \circ h'_n \colon \mathcal{W}'_n \to \mathcal{X}^{[n]}_{\mathcal{V}_0}$ be the induced nth infinitesimal neighborhoods. By Tannaka duality, it is enough to show that an isomorphism $f_{n-1} \colon \mathcal{W}_{n-1} \to \mathcal{W}'_{n-1}$ lifts (resp. lifts up to a unique 2-isomorphism) to an isomorphism $f_n \colon \mathcal{W}_n \to \mathcal{W}'_n$. The obstruction to a lift lies in $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{W}_0}}(f_0^* L_{\mathcal{W}'_0/\mathcal{V}_0}, g_0^*(\mathcal{I}^n/\mathcal{I}^{n+1}))$, which vanishes if $h_0 = h'_0$ is smooth. The obstruction to the existence of a 2-isomorphism between two lifts lies in $\mathrm{Ext}^0_{\mathcal{O}_{\mathcal{W}_0}}(f_0^* L_{\mathcal{W}'_0/\mathcal{V}_0}, g_0^*(\mathcal{I}^n/\mathcal{I}^{n+1}))$ and the 2-automorphisms of a lift lies in $\mathrm{Ext}^{-1}_{\mathcal{O}_{\mathcal{W}_0}}(f_0^* L_{\mathcal{W}'_0/\mathcal{V}_0}, g_0^*(\mathcal{I}^n/\mathcal{I}^{n+1}))$. All three groups vanish if h_0 is étale. □

5.2. **Existence of completions.** If $\mathcal{X}_0 \to \mathcal{X}$ is a morphism of algebraic stacks, we say that a morphism of pairs $(\mathcal{W}, \mathcal{W}_0) \to (\mathcal{X}, \mathcal{X}_0)$, that is, compatible maps $\mathcal{W} \to \mathcal{X}$ and $\mathcal{W}_0 \to \mathcal{X}_0$, is the *completion of* $\mathcal{X}$ *along* $\mathcal{X}_0$ if $(\mathcal{W}, \mathcal{W}_0)$ is a coherently complete pair (Definition 1.5) and $(\mathcal{W}, \mathcal{W}_0) \to (\mathcal{X}, \mathcal{X}_0)$ is final among morphisms from coherently complete pairs. That is, if $(\mathcal{Z}, \mathcal{Z}_0) \to (\mathcal{X}, \mathcal{X}_0)$ is any other morphism of pairs from a coherently complete pair, there exists a morphism $(\mathcal{Z}, \mathcal{Z}_0) \to (\mathcal{W}, \mathcal{W}_0)$ over $\mathcal{X}$ unique up to unique 2-isomorphism. In particular, the pair $(\mathcal{W}, \mathcal{W}_0)$ is unique up to unique 2-isomorphism.

We prove the following generalization of Corollary 1.12.

Corollary 5.2 (Existence of completions)**.** *Let* $\mathcal{X}$ *be a noetherian algebraic stack. Let* $\mathcal{X}_0 \to \mathcal{X}$ *be a pro-immersion such that* $\mathcal{X}_0$ *is linearly fundamental, e.g., the residual gerbe at a point with linearly reductive stabilizer. If either*

(1) *$\mathcal{X}$ has quasi-affine diagonal; or*
(2) *$\mathcal{X}$ has affine stabilizers and $\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ is quasi-excellent;*

then the completion of $\mathcal{X}$ along $\mathcal{X}_0$ exists and is linearly fundamental.

Proof. Applying Theorem 5.1 to the pro-immersion $\mathcal{X}_0 \to \mathcal{X}$ with $\mathcal{W}_0 = \mathcal{X}_0$, we obtain a flat morphism $h\colon \widehat{\mathcal{X}} \to \mathcal{X}$ where $\widehat{\mathcal{X}}$ is a linearly fundamental stack, coherently complete along $h^{-1}(\mathcal{X}_0) \cong \mathcal{X}_0$. Let $(\mathcal{Z}, \mathcal{Z}_0)$ be any other coherently complete stack with a morphism $\varphi\colon \mathcal{Z} \to \mathcal{X}$ such that $\varphi|_{\mathcal{Z}_0}$ factors through $\mathcal{X}_0$. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ be the sheaf of ideals defining the closure of $\mathcal{X}_0$. Then $\mathcal{X}_n = V(\mathcal{I}^{n+1}\mathcal{O}_{\widehat{\mathcal{X}}})$ and $\mathcal{Z}_n \subseteq V(\mathcal{I}^{n+1}\mathcal{O}_{\mathcal{Z}})$. Since $\mathcal{X}_n \to V(\mathcal{I}^{n+1})$ is a flat monomorphism, it follows that $\mathcal{Z}_n \to \mathcal{X}$ factors uniquely through $\mathcal{X}_n$. By coherent completeness of $\mathcal{Z}$ and Tannaka duality (using that $\widehat{\mathcal{X}}$ has affine diagonal), there is a unique morphism $\mathcal{Z} \to \widehat{\mathcal{X}}$. ☐

Let $\mathcal{X}$ be a noetherian algebraic stack and let x be a point of $\mathcal{X}$ with linearly reductive stabilizer. Applying Corollary 5.2 to the pro-immersion corresponding to the residual gerbe $\mathcal{G}_x \to \mathcal{X}$ of x, we obtain a flat morphism $\widehat{\mathcal{X}}_x \to \mathcal{X}$, which we refer to as the *completion* at the point x. Note that when $\mathcal{G}_x = V(\mathcal{I})$ is a closed point, then $\widehat{\mathcal{X}}_x = \varinjlim_n V(\mathcal{I}^{n+1})$ in the category of noetherian algebraic stacks with affine stabilizers.

5.3. **Representability properties of presentations.** We will now give a representability criteria for morphisms from fundamental stacks to algebraic stacks, generalizing [AHR20, Prop. 3.2 and Prop. 3.4] and partially answering [AHR20, Question 1.10]. This will then be used in the local structure theorem.

Proposition 5.3. *Let $f\colon \mathcal{W} \to \mathcal{X}$ be a morphism of algebraic stacks such that $\mathcal{W}$ is adequately affine with affine diagonal (e.g., fundamental). Suppose $\mathcal{W}_0 \subseteq \mathcal{W}$ is a closed substack such that $f|_{\mathcal{W}_0}$ is representable.*

(1) *If $\mathcal{X}$ has affine diagonal, then there exists an adequately affine open neighborhood $\mathcal{U} \subseteq \mathcal{W}$ of $\mathcal{W}_0$ such that $f|_{\mathcal{U}}$ is affine.*
(2) *If $\mathcal{X}$ has separated diagonal and $\mathcal{W}$ is fundamental, then there exists an adequately affine open neighborhood $\mathcal{U} \subseteq \mathcal{W}$ of $\mathcal{W}_0$ such that $f|_{\mathcal{U}}$ is representable.*

To prove Proposition 5.3, we require the following Lemma.

Lemma 5.4. *Let $\mathcal{W}$ be a fundamental stack and let $G \hookrightarrow I_{\mathcal{W}}$ be a closed subgroup. If $G \to \mathcal{W}$ is quasi-finite, then $G \to \mathcal{W}$ is finite.*

Proof. Note that $I_{\mathcal{W}} \to \mathcal{W}$ is affine so $G \to \mathcal{W}$ is also affine. If $h \in |G|$ is a point, then the order of h is finite. It is thus enough to prove the following: if $h \in |I_{\mathcal{W}}|$ is a point of finite order such that $\mathcal{Z} := \overline{\{h\}} \to \mathcal{W}$ is quasi-finite, then $\mathcal{Z} \to \mathcal{W}$ is finite. Using approximation of fundamental stacks (Lemma 2.14) we reduce this question to the case where $\mathcal{W}$ is of finite presentation over $\operatorname{Spec} \mathbb{Z}$.

By [Alp14, Lem. 8.3.1], it is enough to prove that $\mathcal{Z} \to \mathcal{W}$ takes closed points to closed points and that the morphism on their adequate moduli spaces $Z \to W$ is universally closed. This can be checked using DVRs as follows: for every DVR R with fraction field K, every morphism $f\colon \operatorname{Spec} R \to W$ and every lift $h\colon \operatorname{Spec} K \to \mathcal{Z}$, there exists a lift $\widetilde{h}\colon \operatorname{Spec} R \to \mathcal{Z}$ such that the closed point $0 \in \operatorname{Spec} R$ maps to a point in $\mathcal{W}$ that is closed in the fiber over $f(0)$.

Since $\mathcal{W} \to W$ is universally closed, we can start with a lift $\xi\colon \operatorname{Spec} R \to \mathcal{W}$, such that $\xi(0)$ is closed in the fiber over $f(0)$. We can then identify h with an automorphism $h \in \operatorname{Aut}_{\mathcal{W}}(\xi)(K)$ of finite order. Applying [AHLH23, Prop. 5.11 and Lem. 5.10] gives us an extension of DVRs $R \hookrightarrow R'$ and a new lift $\xi'\colon \operatorname{Spec} R' \to \mathcal{W}$ such that $\xi'(0) = \xi(0)$ together with an automorphism $\widetilde{h} \in \operatorname{Aut}_{\mathcal{W}}(\xi')(R')$. Since $\mathcal{Z}$

is closed in $I_{\mathcal{W}}$, this is a morphism $\widetilde{h}\colon \operatorname{Spec} R \to \mathcal{Z}$ as requested. Note that while the paper [AHLH23] cites this paper on several occasions, the proofs of [AHLH23, Prop. 5.11 and Lem. 5.10 (when $\mathcal{X}$ is fundamental)] do not rely on it. □

Remark 5.5. If $h \in |I_{\mathcal{W}}|$ is any element of finite order, then every element of $\mathcal{Z} = \overline{\{h\}} \subseteq I_{\mathcal{W}}$ is of finite order but $\mathcal{Z}$ is not always quasi-finite. For example, consider the action of $\mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{A}^2$ as in [AHLH23, Ex. 3.56]. Then the generic point has stabilizer group $\mathbb{Z}/2\mathbb{Z}$. If we let $h = \tau$ be the non-trivial element of the generic point, then $\overline{\{h\}}$ has fiber $\{(x/y, \tau)\}$ outside $xy = 0$, is empty along $xy = 0$ outside $x = y = 0$ and is $\mathbb{G}_m \times \{\tau\}$ over $x = y = 0$.

Note that subgroups of inertia stacks are automatically normal and the corresponding result for non-normal quasi-finite subgroup schemes of geometrically reductive group schemes is false. Indeed, take $H \subseteq \mathrm{GL}_{2,k[T]}$ to be the closed subgroup with only non-trivial element $\begin{bmatrix} 0 & T^{-1} \\ T & 0 \end{bmatrix}$, which has order 2. Then H is quasi-finite over $\operatorname{Spec} k[T]$, not finite, but is also not normal in $\mathrm{GL}_{2,k[T]}$.

Proof of Proposition 5.3. Since $f|_{\mathcal{W}_0}$ is representable, we can after replacing $\mathcal{W}$ with an open, adequately affine, neighborhood of $\mathcal{W}_0$, assume that f has quasi-finite diagonal (or in fact, even unramified diagonal). For (1) we argue exactly as in [AHR20, Prop. 3.2] but replace [Alp13, Prop. 3.3] with [Alp14, Cor. 4.3.2].

For (2), we note that the subgroup $G := I_{\mathcal{W}/\mathcal{X}} \hookrightarrow I_{\mathcal{W}}$ is closed because $\mathcal{X}$ has separated diagonal and is quasi-finite over $\mathcal{W}$ because f has quasi-finite diagonal. We conclude by Lemma 5.4 and Nakayama's lemma. □

5.4. **The local structure theorem.** We will now prove the following local structure theorem. Note that Theorem 1.1 is the case where $\mathcal{W}_0$ is a gerbe over a field.

Theorem 5.6 (Local structure)**.** *Suppose that S is a quasi-separated algebraic space; $\mathcal{X}$ is an algebraic stack, locally of finite presentation and quasi-separated over S, with affine stabilizers; $x \in |\mathcal{X}|$ is a point with residual gerbe $\mathcal{G}_x$ and image $s \in |S|$ such that the residue field extension $\kappa(x)/\kappa(s)$ is finite; and $h_0\colon \mathcal{W}_0 \to \mathcal{G}_x$ is a smooth (resp. étale) morphism, where $\mathcal{W}_0$ is linearly fundamental and $\Gamma(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0})$ is a field. Then there exists a cartesian diagram of algebraic stacks*

$$\begin{array}{ccc} & \mathcal{W}_0 & \xrightarrow{h_0} & \mathcal{G}_x \\ & \big\downarrow & & \big\downarrow \\ [\operatorname{Spec} A/\mathrm{GL}_n] = & \mathcal{W} & \xrightarrow{h} & \mathcal{X} \end{array}$$

where $h\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ is a smooth (resp. étale) pointed morphism and w is closed in its fiber over s. Moreover, if $\mathcal{X}$ has separated (resp. affine) diagonal and h_0 is representable, then h can be arranged to be representable (resp. affine).

Remark 5.7. In Theorems 1.1 and 5.6, the condition that $\kappa(x)/\kappa(s)$ is finite is equivalent to the condition that the morphism $\mathcal{G}_x \to \mathcal{X}_s$ is of finite type. In particular, it holds if x is closed in its fiber $\mathcal{X}_s = \mathcal{X} \times_S \operatorname{Spec} \kappa(s)$.

To prove Theorem 5.6, we will need the following version of equivariant Artin algebraization (cf. [AHR20, Thm. A.18]).

Theorem 5.8 (Equivariant Artin algebraization)**.** *Let S be an excellent scheme. Let $\mathcal{X}$ be an algebraic stack, locally of finite presentation over S. Let $\mathcal{Z}$ be a noetherian fundamental stack with adequate moduli space map $\pi\colon \mathcal{Z} \to Z$ of finite type (automatic if $\mathcal{Z}$ is linearly fundamental). Let $z \in |\mathcal{Z}|$ be a closed point such that $\mathcal{G}_z \to S$ is of finite type. Let $\eta\colon \mathcal{Z} \to \mathcal{X}$ be a morphism over S that is formally versal at z. Then there exist*

(1) *an algebraic stack* $\mathcal{W}$ *which is fundamental and of finite type over* S*;*
(2) *a closed point* $w \in |\mathcal{W}|$*;*
(3) *a morphism* $\xi \colon \mathcal{W} \to \mathcal{X}$ *over* S*;*
(4) *isomorphisms* $\varphi^{[n]} \colon \mathcal{W}^{[n]} \to \mathcal{Z}^{[n]}$ *over* $\mathcal{X}$ *for every* n*; and*
(5) *if* $\operatorname{Stab}(z)$ *is linearly reductive, an isomorphism* $\widehat{\varphi} \colon \widehat{\mathcal{W}} \to \widehat{\mathcal{Z}}$ *over* $\mathcal{X}$*, where* $\widehat{\mathcal{W}}$ *and* $\widehat{\mathcal{Z}}$ *denote the completions of* $\mathcal{W}$ *at* w *and* $\mathcal{Z}$ *at* z *which exist by Corollary 1.12.*

In particular, ξ *is formally versal at* w*.*

Proof. Since $\mathcal{Z}$ is a fundamental stack, by definition there exists an affine morphism $\mathcal{Z} \to B\mathrm{GL}_m$ for some $m > 0$. We now apply [AHR20, Thm. A.18] with $T = Z$ and $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_2 = B\mathrm{GL}_m$, which gives (1)–(4). Claim (5) is an immediate consequence of (4). □

Proof of Theorem 5.6. **Step 1: Reduction to S an excellent scheme.** It is enough to find a solution $(\mathcal{W}, w) \to (\mathcal{X}, x)$ after replacing S with an étale neighborhood of s so we can assume that S is affine. We can also replace $\mathcal{X}$ with a quasi-compact neighborhood of x and assume that $\mathcal{X}$ is of finite presentation.

Write S as a limit of affine schemes S_λ of finite type over $\operatorname{Spec} \mathbb{Z}$. For sufficiently large λ, we can find $\mathcal{X}_\lambda \to S_\lambda$ of finite presentation such that $\mathcal{X} = \mathcal{X}_\lambda \times_{S_\lambda} S$. Let $w_0 \in |\mathcal{W}_0|$ be the unique closed point and let $x_\lambda \in |\mathcal{X}_\lambda|$ be the image of x. Since $\mathcal{G}_x$ is the limit of the $\mathcal{G}_{x_\lambda}$, we can, for sufficiently large λ, also find a smooth (or étale if h_0 is étale) morphism $h_{0,\lambda} \colon (\mathcal{W}_{0,\lambda}, w_{0,\lambda}) \to (\mathcal{G}_{x_\lambda}, x_\lambda)$ with pull-back h_0. For sufficiently large λ:

(1) $\mathcal{X}_\lambda$ has affine stabilizers [HR15, Thm. 2.8];
(2) if $\mathcal{X}$ has separated (resp. affine) diagonal, then so has $\mathcal{X}_\lambda$;
(3) $\operatorname{Stab}(x_\lambda) = \operatorname{Stab}(x)$ (because $\operatorname{Stab}(x_\mu) \to \operatorname{Stab}(x_\lambda)$ is a closed immersion for every $\mu > \lambda$); and
(4) $\mathcal{W}_{0,\lambda}$ is fundamental (Proposition 2.15).

That $\mathcal{G}_x \to \mathcal{G}_{x_\lambda}$ is stabilizer-preserving implies that $\mathcal{G}_x = \mathcal{G}_{x_\lambda} \times_{\operatorname{Spec} \kappa(x_\lambda)} \operatorname{Spec} \kappa(x)$ and, in particular, $\mathcal{W}_0 = \mathcal{W}_{0,\lambda} \times_{\operatorname{Spec} \kappa(x_\lambda)} \operatorname{Spec} \kappa(x)$. It follows, by flat descent, that $\mathcal{W}_{0,\lambda}$ is cohomologically affine and that $\Gamma(\mathcal{W}_{0,\lambda}, \mathcal{O}_{\mathcal{W}_{0,\lambda}})$ is the spectrum of a field. We can thus replace S, $\mathcal{X}$, $\mathcal{W}_0$ with S_λ, $\mathcal{X}_\lambda$, $\mathcal{W}_{0,\lambda}$ and assume that S is an excellent scheme. By standard limit arguments, it is also enough to find a solution after replacing S with $\operatorname{Spec} \mathcal{O}_{S,s}$. We can thus assume that s is closed.

Step 2: An effective formally smooth solution. Since $\mathcal{W}_0$ is linearly fundamental and $\mathcal{X}$ has affine stabilizers, we can find a formal neighborhood of $\mathcal{W}_0 \to \mathcal{X}_0 := \mathcal{G}_x \hookrightarrow \mathcal{X}$, that is, deform the smooth morphism $\mathcal{W}_0 \to \mathcal{X}_0$ to a flat morphism $\widehat{\mathcal{W}} \to \mathcal{X}$ where $\widehat{\mathcal{W}}$ is a linearly fundamental stack which is coherently complete along $\mathcal{W}_0$ (Theorem 1.11). Since $\mathcal{W}_n \to \mathcal{X}_n$ is smooth, $\widehat{\mathcal{W}} \to \mathcal{X}$ is formally smooth at $\mathcal{W}_0$ [AHR20, Prop. A.14]. Since the good moduli space of $\mathcal{W}_0$ is a field, hence equals the good moduli space of the residual gerbe $\mathcal{G}_{w_0}$, it follows that $\mathcal{W}_0$ and $\widehat{\mathcal{W}}$ are coherently complete along w_0 (Theorem 1.6).

Step 3: Algebraization. We now apply equivariant Artin algebraization (Theorem 5.8), with $\mathcal{Z} = \widehat{\mathcal{W}}$, to obtain a fundamental stack $\mathcal{W}$, a closed point $w \in |\mathcal{W}|$, a morphism $h \colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ smooth at w, and an isomorphism $\widehat{\mathcal{W}}_w \cong \widehat{\mathcal{W}}$ over

$\mathcal{X}$. Let $\widetilde{\mathcal{W}}_0 = h^{-1}(\overline{\mathcal{X}}_0)$. Then we have the cartesian diagrams

$$\begin{array}{ccccc} \mathcal{W}_0 & \xrightarrow{c_0} & \widetilde{\mathcal{W}}_0 & \longrightarrow & \overline{\mathcal{X}}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathcal{W}} & \xrightarrow{c} & \mathcal{W} & \xrightarrow{h} & \mathcal{X} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{W}_0 & \xrightarrow{c_0} & \widetilde{\mathcal{W}}_0 \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ \operatorname{Spec} \widehat{\mathcal{O}}_{\widetilde{W}_0, \pi_0(w)} & \longrightarrow & \widetilde{W}_0 \end{array}$$

where c and c_0 are completions at w and π_0 is an adequate moduli spaces. But $\mathcal{W}_0$ has good moduli space $\operatorname{Spec} k$ so $\widetilde{W}_0$ is the disjoint union of $\{\pi_0(w)\} = \operatorname{Spec} k$ and its complement Q. If $\pi\colon \mathcal{W} \to W$ is the adequate moduli space, then $\widetilde{W}_0 \to W$ is closed and injective so after replacing W with an open neighborhood V of $\pi(w)$ and replacing $\mathcal{W}$, $\widetilde{W}_0$, $\widetilde{W}_0$ with the inverse images of V, we can assume that $\widetilde{\mathcal{W}}_0 = \mathcal{W}_0$.

Note that if $U \subseteq \mathcal{W}$ is an open neighborhood of w, we can shrink to the smaller open neighborhood $\pi^{-1}(V)$, where V is an open affine neighborhood of $\pi(w)$ contained in $W \smallsetminus \pi(\mathcal{W} \smallsetminus U)$; then $\pi^{-1}(V) \to V$ remains adequately affine.

If $h_0\colon \mathcal{W}_0 \to \mathcal{X}_0$ is étale, then h is étale at w. After shrinking $\mathcal{W}$ as above, we can assume that h is smooth (resp. étale). If $\mathcal{X}$ has separated (resp. affine) diagonal, then we can shrink $\mathcal{W}$ as above so that h becomes representable (resp. affine), see Proposition 5.3. □

6. Applications to stacks with good moduli spaces

In this section, we prove that if $\pi\colon \mathcal{X} \to X$ is a good moduli space, with affine stabilizers and separated diagonal, then $\mathcal{X}$ has the resolution property étale-locally on X (Theorem 6.1). This generalizes [AHR20, Thm. 4.12] to the relative case. As a consequence, linearly reductive groups are Nisnevich-locally embeddable. We also give a version for adequate moduli spaces (Theorem 6.5). It follows that the derived category of a stack with a good moduli space is compactly generated (Proposition 6.15).

6.1. **Good moduli spaces and linearly reductive groups.**

Theorem 6.1. *Let $\mathcal{X}$ be an algebraic stack with good moduli space $\pi\colon \mathcal{X} \to X$. Assume that $\mathcal{X}$ has affine stabilizers, separated diagonal and is of finite presentation over a quasi-compact and quasi-separated algebraic space S.*

1. *There is a Nisnevich covering $X' \to X$ such that the pull-back $\mathcal{X}' = \mathcal{X} \times_X X'$ is linearly fundamental.*
2. *$\pi\colon \mathcal{X} \to X$ has affine diagonal.*
3. *$\mathcal{X} \to X$ and $X \to S$ are of finite presentation.*
4. *$\pi_* \mathcal{F}$ is finitely presented if $\mathcal{F}$ is a finitely presented $\mathcal{O}_{\mathcal{X}}$-module.*

Moreover, if every closed point $x \in |X|$ either has $\operatorname{char} \kappa(x) > 0$ or has an open neighborhood of characteristic zero, then we can arrange that $\mathcal{X}' \cong [\operatorname{Spec} A/G]$ where $G \to X'$ is linearly reductive and embeddable.

We will prove Theorem 6.1 at the end of §6.3 after establishing the adequate version and some auxiliary results on étale neighborhoods of points with nice stabilizers.

If there are closed points of characteristic zero without characteristic zero neighborhoods, then it is sometimes impossible to find a linearly reductive G; see Appendix A. See Theorem 8.1 for some variants in mixed characteristic, however.

Applying Theorem 6.1 to the classifying stack BG of a linearly reductive group scheme we obtain:

Corollary 6.2. *Let S be a quasi-separated algebraic space.*

(1) *If $G \to S$ is a separated group algebraic space, flat and of finite presentation, with affine fibers such that $BG \to S$ is a good moduli space, then $G \to S$ is affine, that is, G is linearly reductive.*
(2) *If $G \to S$ is linearly reductive, then there exists a Nisnevich covering $S' \to S$ such that $G' = G \times_S S'$ is embeddable.* □

Remark 6.3. A consequence of Corollary 6.2 is that in the definition of a tame group scheme given in [Hoy17, Defn. 2.26], if we assume that $G \to B$ is separated with affine fibers, then the condition on having the G-resolution property Nisnevich-locally is automatic.

Over a field, normal subgroups, quotients and extensions of reductive groups are reductive. The analogous statement holds for linearly reductive groups over a base.

Corollary 6.4. *Let S be an algebraic space. Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of flat group algebraic spaces of finite presentation over S. Then the following are equivalent:*

(1) *$G \to S$ is linearly reductive and $G' \to G$ is a closed immersion.*
(2) *$G' \to S$ and $G'' \to S$ are linearly reductive.*

Proof. (1) $\implies$ (2): Since $G' \to G$ is closed and $G \to S$ is affine, the quotient $G'' \to S$ is separated with affine fibers. Since $BG \to S$ is cohomologically affine, so is $BG'' \to S$ [Alp14, Prop. 12.17(i)]. By Corollary 6.2(1), $G'' \to S$ is linearly reductive and in particular affine. Since G'' is affine, the G''-torsor $BG' \to BG$ is affine so $BG' \to S$ is also cohomologically affine, hence linearly reductive.

(2) $\implies$ (1): Since $G' \to S$ is affine, so is the G'-torsor $G \to G''$. Since $G'' \to S$ is affine, so is $G \to S$. The result then follows by [Alp14, Prop. 12.17(ii)]. □

6.2. **Adequate moduli spaces and geometrically reductive groups.**

Theorem 6.5. *Let $\mathcal{X}$ be an algebraic stack with adequate moduli space $\pi \colon \mathcal{X} \to X$ and let $x \in X$ be a point. Assume that*

(1) *$\mathcal{X}$ has affine stabilizers and separated diagonal;*
(2) *$\mathcal{X}$ is of finite presentation over a quasi-separated algebraic space; and*
(3) *the unique closed point y in $\pi^{-1}(x)$ has linearly reductive stabilizer.*

Then there exists an étale neighborhood $(X', x') \to (X, x)$ with $\kappa(x') = \kappa(x)$ such that the pull-back $\mathcal{X}'$ of $\mathcal{X}$ is fundamental. That is, there is a cartesian diagram

$$\begin{array}{ccc} [\operatorname{Spec} A/\mathrm{GL}_n] = \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle \pi'} & \square & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} B = X' & \longrightarrow & X. \end{array}$$

where π' is an adequate moduli space (i.e., $B = A^{\mathrm{GL}_n}$). In particular, π has affine diagonal in an open neighborhood of x.

Proof. Applying Theorem 1.1 with $h_0 \colon \mathcal{W}_0 \to \mathcal{G}_y$ an isomorphism yields an étale representable morphism $f \colon ([\operatorname{Spec} A/\mathrm{GL}_n], w) \to (\mathcal{X}, y)$ inducing an isomorphism $\mathcal{G}_w \to \mathcal{G}_y$. The result follows from Luna's fundamental lemma (Theorem 3.14). □

Corollary 6.6. *Let S be a quasi-separated algebraic space. Let $G \to S$ be a flat and separated group algebraic space of finite presentation with affine fibers such that $BG \to S$ is adequately affine (e.g., $G \to S$ is geometrically reductive). If $s \in S$ is a point such that G_s is linearly reductive, then there exists an étale neighborhood $(S', s') \to (S, s)$, with trivial residue field extension, such that $G' = G \times_S S'$ is embeddable.*

Proof. From Theorem 6.5 we obtain an étale neighborhood $S' \to S$ such that BG' is fundamental. Then G' is affine and embeddable (Remark 2.9). □

Remark 6.7. If $G \to S$ is a reductive group scheme (i.e., geometrically reductive, smooth, and with connected fibers) then $G \to S$ is étale-locally split reductive. A split reductive group is a pull-back from $\operatorname{Spec} \mathbb{Z}$ [SGA3_{II}, Exp. XXV, Thm. 1.1, Cor. 1.2], hence embeddable.

6.3. **Nice neighborhoods.** Parts (1) and (2) of Theorem 6.1 follow directly from Theorem 6.5. To deduce the rest of the theorem, we need to study the structure around points of positive characteristic. The following proposition shows that fundamental stacks (resp. geometrically reductive and embeddable group schemes) are nicely fundamental (resp. nice) in an étale neighborhood of a nice point.

Proposition 6.8 (Niceness is étale-local)**.** *Let $\mathcal{X}$ be a fundamental algebraic stack with adequate moduli space $\mathcal{X} \to X$. Let $x \in |X|$ be a point and let $y \in |\mathcal{X}|$ be the unique closed point in the fiber of x. If the stabilizer of y is nice, then there exists an étale neighborhood $(X', x') \to (X, x)$, with $\kappa(x') = \kappa(x)$, such that $\mathcal{X} \times_X X'$ is nicely fundamental.*

Proof. Since nicely fundamental stacks can be approximated (Proposition 2.15(1)), we may assume that X is henselian with closed point x. Then y is the unique closed point of $|\mathcal{X}|$. By Proposition 2.11(1), the residual gerbe $\mathcal{G}_y = \overline{\{y\}}$ is nicely fundamental.

Now Lemma 2.14 says that we can write $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$, where the $\mathcal{X}_\lambda$ are fundamental and of finite type over $\operatorname{Spec} \mathbb{Z}$ with adequate moduli space X_λ of finite type over $\operatorname{Spec} \mathbb{Z}$. Let $x_\lambda \in |X_\lambda|$ be the image of x and let $y_\lambda \in |\mathcal{X}_\lambda|$ be the unique closed point above x_λ. Then y_λ is contained in the closure of the image of y. Thus, for all sufficiently large λ, the point y_λ has nice stabilizer (Proposition 2.15(2)).

Let X^h_λ denote the henselization of X_λ at x_λ and $\mathcal{X}^h_\lambda = \mathcal{X}_\lambda \times_{X_\lambda} X^h_\lambda$. Then the canonical map $X \to X_\lambda$ factors uniquely through X^h_λ and the induced map $\mathcal{X} \to \mathcal{X}^h_\lambda$ is affine. It is thus enough to prove that $\mathcal{X}^h_\lambda$ is nicely fundamental. By Theorem 4.21, the adequate moduli space $\mathcal{X}^h_\lambda \to X^h_\lambda$ is good, that is, $\mathcal{X}^h_\lambda$ is linearly fundamental.

We can thus assume that X is excellent and henselian and that $\mathcal{X}$ is linearly fundamental. Let $\mathcal{X}_n$ be the nth infinitesimal neighborhood of x. Let $Q_0 \to \operatorname{Spec} \kappa(x)$ be a nice group scheme such that there exists an affine morphism $f_0 \colon \mathcal{X}_0 \to B_{\kappa(x)} Q_0$. By the existence of deformations of nice group schemes (Proposition 4.16), there exists a nice and embeddable group scheme $Q \to X$. Let $\mathcal{I} \subseteq \mathcal{X}$ denote the sheaf of ideals defining $\mathcal{X}_0$. By [Ols06a, Thm. 1.5], the obstruction to lifting a morphism $\mathcal{X}_{n-1} \to B_X Q$ to $\mathcal{X}_n \to B_X Q$ is an element of $\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{X}_0}}(Lf_0^* L_{B_X Q/X}, \mathcal{I}^n/\mathcal{I}^{n+1})$. The obstruction vanishes because the cotangent complex $L_{B_X Q/X}$ is perfect of amplitude $[0,1]$, since $B_X Q \to X$ is smooth, and $\mathcal{X}_0$ is cohomologically affine.

Let $\widehat{X} = \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{X}} = \mathcal{X} \times_X \widehat{X}$. Since $\widehat{\mathcal{X}}$ is linearly fundamental, it is coherently complete along $\mathcal{X}_0$ (Theorem 4.5). By Tannaka duality (see §1.7), we may thus extend $\mathcal{X}_0 \to B_{X_0} Q_0$ to a morphism $\widehat{\mathcal{X}} \to B_X Q$. Applying Artin approximation (Theorem 3.8) to the functor $\operatorname{Hom}_X(\mathcal{X} \times_X -, B_X Q) \colon (\mathsf{Sch}/X)^{\text{opp}} \to \mathsf{Sets}$ yields a morphism $\mathcal{X} \to B_X Q$, which is affine by Proposition 5.3(1). □

Note that if $\mathcal{X}$ is linearly fundamental and $\operatorname{char} \kappa(x) > 0$, then y has nice stabilizer. We thus have the following corollaries:

Corollary 6.9. *Let $\mathcal{X}$ be a linearly fundamental algebraic stack with good moduli space $\mathcal{X} \to X$ and let $x \in |X|$ be a point. If either* $\operatorname{char} \kappa(x) > 0$ *or x has an open neighborhood of characteristic zero, then there exists an étale neighborhood*

$(X', x') \to (X, x)$, *with* $\kappa(x') = \kappa(x)$, *such that* $\mathcal{X} \times_X X' = [\operatorname{Spec} A/G]$ *where* $G \to X'$ *is a linearly reductive embeddable group scheme.* □

Corollary 6.10. *Let* (S, s) *be a Henselian local scheme such that* $\operatorname{char} \kappa(s) > 0$.

(1) *If* $\mathcal{X}$ *is a linearly fundamental algebraic stack with good moduli space* $\mathcal{X} \to S$, *then* $\mathcal{X}$ *is nicely fundamental.*
(2) *If* $G \to S$ *is a linearly reductive and embeddable group scheme, then* $G \to S$ *is nice.* □

We also obtain the following non-noetherian variants of Theorem 4.21 at the expense of assuming that either $\mathcal{X}$ has the resolution property or is of finite presentation over some base.

Corollary 6.11. *Let* $\mathcal{X}$ *be a fundamental algebraic stack. Then the following are equivalent.*

(1) $\mathcal{X}$ *is linearly fundamental.*
(2) *Every closed point of* $\mathcal{X}$ *has linearly reductive stabilizer.*
(3) *Every closed point of* $\mathcal{X}$ *with positive characteristic has nice stabilizer.*

Proof. The only non-trivial implication is (3) $\implies$ (1). Let $\pi \colon \mathcal{X} \to X$ be the adequate moduli space. It is enough to prove that π is a good moduli space after base change to the henselization at a closed point. We may thus assume that X is the spectrum of a henselian local ring. If X is a $\mathbb{Q}$-scheme, then the notions of adequate and good coincide. If not, then the closed point of X has positive characteristic, hence the unique closed point of $\mathcal{X}$ has nice stabilizer. We conclude that $\mathcal{X}$ is nicely fundamental by Proposition 6.8. □

Corollary 6.12. *Let* $\mathcal{X}$ *be an algebraic stack of finite presentation over a quasi-compact and quasi-separated algebraic space* S. *Suppose that there exists an adequate moduli space* $\pi \colon \mathcal{X} \to X$. *Then* π *is a good moduli space with affine diagonal if and only if*

(1) $\mathcal{X}$ *has separated diagonal and affine stabilizers; and*
(2) *every closed point of* $\mathcal{X}$ *has linearly reductive stabilizer.*

Proof. The conditions are clearly necessary. If they are satisfied, then Theorem 6.5 implies that π has affine diagonal. To verify that π is a good moduli space, we may replace X with the henselization at a closed point. Then $\mathcal{X}$ is fundamental by Theorem 6.5 and the result follows from Corollary 6.11. □

We will now finish the proof of Theorem 6.1.

Corollary 6.13. *Let* $\mathcal{X}$ *be a fundamental stack with adequate moduli space* $\pi \colon \mathcal{X} \to X$. *Let* $g \colon X' \to X$ *be a morphism of algebraic spaces such that* $\mathcal{X}' := \mathcal{X} \times_X X'$ *has a good moduli space. Then* $\pi' \colon \mathcal{X}' \to X'$ *is its good moduli space and the natural transformation* $g^*\pi_* \to \pi'_* g'^*$ *is an isomorphism on all quasi-coherent* $\mathcal{O}_{\mathcal{X}}$*-modules.*

Proof. Both claims can be checked on stalks so we may assume that $X' = \operatorname{Spec} A'$ and $X = \operatorname{Spec} A$ are spectra of local rings and that the closed point $x' \in X'$ maps to the closed point $x \in X$. Since $\mathcal{X}'$ has a good moduli space, it follows that the unique closed point of $\mathcal{X}$ has linearly reductive stabilizer. Hence $\mathcal{X}$ is linearly fundamental (Corollary 6.11) and the result follows from [Alp13, Prop. 4.7]. □

Corollary 6.14. *Let* $\mathcal{X}$ *be a linearly fundamental stack of finite presentation over a quasi-separated algebraic space* S *with good moduli space* $\pi \colon \mathcal{X} \to X$. *Then* X *is of finite presentation over* S *and* π_* *takes finitely presented* $\mathcal{O}_{\mathcal{X}}$*-modules to finitely presented* $\mathcal{O}_X$*-modules.*

Proof. We may assume that S is quasi-compact and can thus write S as an inverse limit of algebraic spaces S_λ of finite presentation over $\operatorname{Spec}\mathbb{Z}$ with affine transition maps [Ryd15, Thm. D]. For sufficiently large λ, we can find $\mathcal{X}_\lambda \to S_\lambda$ of finite presentation that pulls back to $\mathcal{X} \to S$. After increasing λ, we can assume that $\mathcal{X}_\lambda$ is fundamental by Proposition 2.15(1) and that a given $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ of finite presentation is the pull-back of a coherent $\mathcal{O}_{\mathcal{X}_\lambda}$-module $\mathcal{F}_\lambda$. Then $\mathcal{X}_\lambda$ has an adequate moduli space X_λ of finite presentation over S_λ and the push-forward of $\mathcal{F}_\lambda$ is a coherent $\mathcal{O}_{X_\lambda}$-module [Alp14, Thm. 6.3.3]. The result now follows from Corollary 6.13. In particular, $X = X_\lambda \times_{S_\lambda} S$ is the good moduli space of $\mathcal{X}$. □

Proof of Theorem 6.1. By Theorem 6.5, we obtain a Nisnevich covering $X' \to X$ such that $\pi' \colon \mathcal{X}' = \mathcal{X} \times_X X'$ is fundamental. Since $\pi \colon \mathcal{X} \to X$ is a good moduli space, it follows that π' is also a good moduli space and so linearly fundamental. This proves (1) and (2). By Corollary 6.14, we see that (3) and (4) hold for π' and, by étale descent, also for π. The final claim follows from Corollary 6.9. □

6.4. **Compact generation of derived categories.** Here we prove a variant of [AHR20, Thm. 5.1] in the mixed characteristic situation.

Proposition 6.15. *Let $\mathcal{X}$ be a quasi-compact algebraic stack with good moduli space $\pi \colon \mathcal{X} \to X$. If $\mathcal{X}$ has affine stabilizers, separated diagonal and is of finite presentation over a quasi-separated algebraic space S, then $\mathcal{X}$ has the Thomason condition; that is,*

1. *$\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ is compactly generated by a countable set of perfect complexes; and*
2. *for every quasi-compact open immersion $\mathcal{U} \subseteq \mathcal{X}$, there exists a compact and perfect complex $P \in \mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ with support precisely $\mathcal{X} \setminus \mathcal{U}$.*

Proof. By [HR17, Thm. C] and [HR17, Prop. 8.4], it suffices to construct an étale, separated and representable covering $p \colon \mathcal{W} \to \mathcal{X}$ such that $\mathcal{W} = [\operatorname{Spec} C/\mathrm{GL}_n]$ (note that $\mathcal{W}$ is automatically concentrated because $\mathcal{X}$ and p are). This follows from Theorem 6.1. □

7. Approximation and deformation of linearly fundamental stacks

In this section we use the local structure of good moduli stacks (Theorem 6.1) and nice points (Proposition 6.8) to extend the approximation results for fundamental and nicely fundamental stacks in Section 2.3 to linearly fundamental stacks (Theorem 7.3) and good moduli spaces (Corollary 7.5). We will then deform objects over henselian pairs (Definition 3.1), using the approximation results to reduce from the henselian case to the excellent henselian case. These approximation and deformation results will be used prominently in §8–10, and in this section we give a first consequence by generalizing the universal property of good moduli spaces (Theorem 3.12) to good moduli space morphisms (Theorem 7.22).

To this end, we introduce the following mild mixed characteristic assumptions on an algebraic stack $\mathcal{W}$:

(FC) There is only a finite number of different characteristics in $\mathcal{W}$.
(PC) Every closed point of $\mathcal{W}$ has positive characteristic.
(N) Every closed point of $\mathcal{W}$ has nice stabilizer.

Remark 7.1. Note that if $\eta \rightsquigarrow s$ is a specialization in $\mathcal{W}$, then the characteristic of η is 0 or agrees with that of s. In particular, if $(\mathcal{W}, \mathcal{W}_0)$ is a local pair (Definition 3.1), so that every closed point of $\mathcal{W}$ belongs to $\mathcal{W}_0$, it follows that $\mathcal{W}$ satisfies (FC), (PC) or (N), respectively, if and only if $\mathcal{W}_0$ does so.

7.1. **Approximation.** Let $\mathcal{X}$ be a fundamental stack with adequate moduli space $\pi\colon \mathcal{X} \to X$. Let $X_{\mathrm{nice}} \subseteq |X|$ be the locus of points $x \in |X|$ such that the unique closed point in the fiber $\pi^{-1}(x)$ has nice stabilizer. If $x \in X_{\mathrm{nice}}$, then there exists an étale neighborhood $X' \to X$ of x such that $\mathcal{X} \times_X X'$ is nicely fundamental (Proposition 6.8). It follows that X_{nice} is open and that $\mathcal{X} \times_X X_{\mathrm{nice}} \to X_{\mathrm{nice}}$ is a good moduli space.

Lemma 7.2. *Let $\mathcal{X}$ be a fundamental stack with adequate moduli space X. Let $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ be an inverse limit of fundamental stacks with affine transition maps. Let X_λ denote the adequate moduli space of $\mathcal{X}_\lambda$. Then*

(1) *$X \times_{X_\lambda} (X_\lambda)_{\mathrm{nice}} \subseteq X_{\mathrm{nice}}$ for every λ; and*
(2) *if $V \subseteq X_{\mathrm{nice}}$ is a quasi-compact open subset, then $V \subseteq X \times_{X_\lambda} (X_\lambda)_{\mathrm{nice}}$ for every sufficiently large λ.*

Proof. Note that the subtlety is that while the map $\mathcal{X} \to \mathcal{X}_\lambda \times_{X_\lambda} X$ is always affine, it is not necessarily an isomorphism.

For $x \in (X_\lambda)_{\mathrm{nice}}$, let $U_\lambda \to X_\lambda$ be an étale neighborhood of x such that $\mathcal{X}_\lambda \times_{X_\lambda} U_\lambda$ is nicely fundamental (Proposition 6.8). Then $\mathcal{X} \times_{X_\lambda} U_\lambda$ is also nicely fundamental as it is affine over the former. Thus $X \times_{X_\lambda} (X_\lambda)_{\mathrm{nice}} \subseteq X_{\mathrm{nice}}$. This proves (1).

For (2), let $U \to V$ be an étale surjective morphism such that $\mathcal{X} \times_X U$ is nicely fundamental (Proposition 6.8). Since $X = \varprojlim_\lambda X_\lambda$ (Proposition 2.15(3)) and $U \to X$ is affine, we can for all sufficiently large λ find $U_\lambda \to X_\lambda$ affine étale such that $U = U_\lambda \times_{X_\lambda} X$. Since $\mathcal{X} \times_X U = \varprojlim_\lambda \mathcal{X}_\lambda \times_{X_\lambda} U_\lambda$ is nicely fundamental, so is $\mathcal{X}_\lambda \times_{X_\lambda} U_\lambda$ for all sufficiently large λ (Proposition 2.15(1)). It follows that $(X_\lambda)_{\mathrm{nice}}$ contains the image of U_λ so $V \subseteq X \times_{X_\lambda} (X_\lambda)_{\mathrm{nice}}$. ☐

The main theorem of this section is the following variant of Proposition 2.15(1) for linearly fundamental stacks.

Theorem 7.3 (Approximation of linearly fundamental)**.** *Let $\mathcal{Y}$ be a quasi-compact and quasi-separated algebraic stack. Let $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ where $\mathcal{X}_\lambda$ is an inverse system of quasi-compact and quasi-separated algebraic stacks over $\mathcal{Y}$ with affine transition maps. Assume that* (1) *$\mathcal{Y}$ is* (FC)*, or* (2) *$\mathcal{X}$ is* (PC)*, or* (3) *$\mathcal{X}$ is* (N)*. Then, if $\mathcal{X}$ is linearly fundamental, so is $\mathcal{X}_\lambda$ for all sufficiently large λ.*

Proof. By Proposition 2.15(1) we can assume that the $\mathcal{X}_\lambda$ are fundamental. Since $\mathcal{X}$ is linearly fundamental, (PC) $\implies$ (N). If $\mathcal{X}$ satisfies (N), then $X_{\mathrm{nice}} = X$ and it follows from Lemma 7.2 that $(X_\lambda)_{\mathrm{nice}} = X_\lambda$ for all sufficiently large λ; hence that $\mathcal{X}_\lambda$ is linearly fundamental. Thus, it remains to prove the theorem when $\mathcal{Y}$ satisfies (FC). In this case, $\mathcal{Y}_{\mathbb{Q}} := \mathcal{Y} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q}$ is open in $\mathcal{Y}$. Similarly for the other stacks. In particular, if X denotes the good moduli space of $\mathcal{X}$, then X is the union of the two open subschemes X_{nice} and $X_{\mathbb{Q}}$. In addition, since $X \smallsetminus X_{\mathbb{Q}}$ is closed, hence quasi-compact, we may find a quasi-compact open subset $V \subseteq X_{\mathrm{nice}}$ such that $X = V \cup X_{\mathbb{Q}}$. For sufficiently large λ, we have that $V \subseteq (X_\lambda)_{\mathrm{nice}} \times_{X_\lambda} X$ (Lemma 7.2(2)) and thus, after possibly increasing λ, that $X_\lambda = (X_\lambda)_{\mathrm{nice}} \cup (X_\lambda)_{\mathbb{Q}}$. It follows that $\mathcal{X}_\lambda$ is linearly fundamental. ☐

Corollary 7.4. *Let $\mathcal{X}$ be a linearly fundamental stack. Assume that $\mathcal{X}$ satisfies* (FC)*,* (PC) *or* (N)*. Then we can write $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ as an inverse limit of linearly fundamental stacks, with affine transition maps, such that each $\mathcal{X}_\lambda$ is essentially of finite type over* $\operatorname{Spec}\mathbb{Z}$*.*

Proof. If $\mathcal{X}$ satisfies (FC), let S be the semi-localization of $\operatorname{Spec}\mathbb{Z}$ in all characteristics that appear in $\mathcal{X}$. Then there is a canonical map $\mathcal{X} \to S$. If $\mathcal{X}$ satisfies (PC) or (N), let $S = \operatorname{Spec}\mathbb{Z}$. Since $\mathcal{X}$ is fundamental, we can write $\mathcal{X}$ as an inverse limit of algebraic stacks $\mathcal{X}_\lambda$ that are fundamental and of finite presentation over S. The result then follows from Theorem 7.3. ☐

Corollary 7.4 is not true unconditionally, even if we merely assume that the $\mathcal{X}_\lambda$ are noetherian, see Appendix A.

Corollary 7.5 (Approximation of good moduli spaces)**.** *Let $X = \varprojlim_\lambda X_\lambda$ be an inverse system of quasi-compact algebraic spaces with affine transition maps. Let α be an index, let $f_\alpha\colon \mathcal{X}_\alpha \to X_\alpha$ be a morphism of finite presentation and let $f_\lambda\colon \mathcal{X}_\lambda \to X_\lambda$, for $\lambda \geq \alpha$, and $f\colon \mathcal{X} \to X$ denote the base changes of f_α. Assume that X_α satisfies (FC) or $\mathcal{X}$ satisfies (PC) or (N). Then if $\mathcal{X} \to X$ is a good moduli space with affine diagonal, so is $\mathcal{X}_\lambda \to X_\lambda$ for all sufficiently large λ.*

Proof. Theorem 6.1 gives an étale and surjective morphism $X' \to X$ such that $\mathcal{X}' = \mathcal{X} \times_X X'$ is linearly fundamental. For sufficiently large λ, we can find an étale surjective morphism $X'_\lambda \to X_\lambda$ that pulls back to $X' \to X$. For sufficiently large λ, we have that $\mathcal{X}'_\lambda := \mathcal{X}_\lambda \times_{X_\lambda} X'_\lambda$ is linearly fundamental by Theorem 7.3. Its good moduli space $\overline{X}'_\lambda$ is of finite presentation over X'_λ (Corollary 6.14). It follows that $\overline{X}'_\lambda \to X'_\lambda$ is an isomorphism for all sufficiently large λ. By descent, it follows that $\mathcal{X}_\lambda \to X_\lambda$ is a good moduli space for all sufficiently large λ. ☐

7.2. **Deformation: setup.** For most of the remainder of this section, we will be in the following situation.

Setup 7.6. Let $\mathcal{X}$ be a quasi-compact algebraic stack with affine diagonal and affine good moduli space X. Let $\mathcal{X}_0 \hookrightarrow \mathcal{X}$ be a closed substack with good moduli space X_0. Assume that (X, X_0) is an affine henselian pair and one of the following conditions holds:

(a) $\mathcal{X}_0$ has the resolution property, $\mathcal{X}$ is noetherian and (X, X_0) is complete;
(b) $\mathcal{X}_0$ has the resolution property, $\mathcal{X}$ is noetherian and (X, X_0) is quasi-excellent;
(c) $\mathcal{X}$ has the resolution property and $\mathcal{X}_0$ satisfies (FC), (PC), or (N); or
(c′) $\mathcal{X}_0$ has the resolution property, $\mathcal{X} \to X$ is of finite presentation, and $\mathcal{X}_0$ satisfies (FC), (PC), or (N).

Remark 7.7. Note that (FC) and (PC) for $\mathcal{X}_0$ are equivalent to the corresponding properties for X_0. Since the pair (X, X_0) is henselian and so local, it follows that these are equivalent to the corresponding properties for X and $\mathcal{X}$ (Remark 7.1).

7.3. **Deformation of the resolution property.** The first result of this section is the following remarkable proposition. It is a simple consequence of some results proved several sections ago.

Proposition 7.8 (Deformation of the resolution property)**.** *Let $(\mathcal{X}, \mathcal{X}_0)$ be as in Setup 7.6 (a), (b) or (c′). Then $\mathcal{X}$ has the resolution property; in particular, $\mathcal{X}$ is linearly fundamental and (c′) implies (c).*

Proof. Case (a) is part of the coherent completeness result (Theorem 1.6). For (b), let $\widehat{X}$ denote the completion of X along X_0 and $\widehat{\mathcal{X}} = \mathcal{X} \times_X \widehat{X}$. By the complete case, $\widehat{\mathcal{X}}$ has the resolution property. Equivalently, there is a quasi-affine morphism $\widehat{\mathcal{X}} \to B\mathrm{GL}_n$ for some n. The functor parametrizing quasi-affine morphisms to $B\mathrm{GL}_n$ is locally of finite presentation [Ryd15, Thm. C] so by Artin approximation (Theorem 3.8), there exists a quasi-affine morphism $\mathcal{X} \to B\mathrm{GL}_n$. In particular, $\mathcal{X}$ has the resolution property.

For (c′), let $S = \operatorname{Spec} \mathbb{Z}$ be the semi-localization of $\operatorname{Spec} \mathbb{Z}$ in all characteristics that appear in X. Since (X, X_0) is an affine henselian pair over S, we may write it as an inverse limit of affine excellent henselian pairs $(X_\lambda, X_{\lambda,0})$ over S. For a sufficiently large λ, we have a morphism of finite presentation $\mathcal{X}_\lambda \to X_\lambda$ such that the pull-back along $X \to X_\lambda$ is $\mathcal{X} \to X$. After increasing λ we can assume that $\mathcal{X}_\lambda \to X_\lambda$ is a good moduli space (Corollary 7.5) and that $\mathcal{X}_\lambda \times_{X_\lambda} X_{\lambda,0}$ has the resolution property (Proposition 2.15(1)). Since X_λ is excellent, $\mathcal{X}_\lambda$ has the resolution property by case (b) and we conclude that $\mathcal{X}$ has the resolution property since $\mathcal{X} \to \mathcal{X}_\lambda$ is affine. ☐

7.4. **Deformation of sections.** If $f\colon \mathcal{X}' \to \mathcal{X}$ is a morphism of algebraic stacks, we will denote the groupoid of sections $s\colon \mathcal{X} \to \mathcal{X}'$ of f as $\Gamma(\mathcal{X}'/\mathcal{X})$ [EGA, I.2.5.5] (the notation $\mathrm{Sec}(\mathcal{X}'/\mathcal{X})$ is also used [Ols06b]).

Proposition 7.9 (Deformation of sections)**.** *Let $(\mathcal{X}, \mathcal{X}_0)$ be as in Setup 7.6. Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a quasi-separated and smooth (resp. smooth gerbe, resp. étale) morphism with affine stabilizers. In case Setup 7.6 (a), also assume that f has quasi-affine diagonal. Then $\Gamma(\mathcal{X}'/\mathcal{X}) \to \Gamma(\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_0/\mathcal{X}_0)$ is essentially surjective (resp. essentially surjective and full, resp. an equivalence of groupoids).*

Proof. Any section s_0 of $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_0 \to \mathcal{X}_0$ has quasi-compact image. In particular, we may immediately reduce to the situation where f is finitely presented.

We first handle case (a): By Theorem 1.6, $\mathcal{X}$ is coherently complete along $\mathcal{X}_0$. Let $\mathcal{I}$ be the ideal sheaf defining $\mathcal{X}_0 \subseteq \mathcal{X}$ and let $\mathcal{X}_n := \mathcal{X}^{[n]}_{\mathcal{X}_0}$ be its nilpotent thickenings. Set $\mathcal{X}'_n = \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_n$. Let $s_0\colon \mathcal{X}_0 \to \mathcal{X}'_0$ be a section of $\mathcal{X}'_0 \to \mathcal{X}_0$. Given a section s_{n-1} of $\mathcal{X}'_{n-1} \to \mathcal{X}_{n-1}$, lifting s_0, the obstruction to deforming s_{n-1} to a section s_n of $\mathcal{X}'_n \to \mathcal{X}_n$ is an element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_0}}(Ls_0^* L_{\mathcal{X}'/\mathcal{X}}, \mathcal{I}^n/\mathcal{I}^{n+1})$ by [Ols06a, Thm. 1.5].[2] Since $\mathcal{X}' \to \mathcal{X}$ is smooth (resp. a smooth gerbe, resp. étale), the cotangent complex $L_{\mathcal{X}'/\mathcal{X}}$ is perfect of amplitude $[0, 1]$ (resp. perfect of amplitude 1, resp. zero). Further $\mathcal{X}_0$ is cohomologically affine, so there exists a lift (resp. a unique lift up to non-unique 2-isomorphism, resp. a unique lift up to unique 2-isomorphism). By Tannaka duality (see §1.7), these sections lift to a unique section $s\colon \mathcal{X} \to \mathcal{X}'$. Here we need that f has quasi-affine diagonal unless X is quasi-excellent.

We now handle case (b). Let $\widehat{X}$ be the completion of X along X_0 and set $\widehat{\mathcal{X}} = \mathcal{X} \times_X \widehat{X}$ and $\widehat{\mathcal{X}}' = \mathcal{X}' \times_X \widehat{X}$. Case (a) yields a section $\widehat{s}\colon \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}'$ extending s_0. The functor assigning an X-scheme T to the set of sections $\Gamma(\mathcal{X}' \times_X T/\mathcal{X} \times_X T)$ is limit preserving, and we may apply Artin approximation (Theorem 3.8) to obtain a section of $s\colon \mathcal{X}' \to \mathcal{X}$ restricting to s_0.

Finally, we handle case (c). By Corollary 7.4, we may write the linearly fundamental stack $\mathcal{X}$ as an inverse limit of linearly fundamental excellent stacks $\mathcal{X}_\lambda$, with affine transition maps. Since $\mathcal{X}_0$ is the intersection of finitely presented closed substacks [Ryd16], we can also write the pair $(\mathcal{X}, \mathcal{X}_0)$ as an inverse limit of pairs $(\mathcal{X}_\lambda, \mathcal{X}_{\lambda,0})$. Since the good moduli space (X, X_0) is a henselian pair, the induced map $(X, X_0) \to (X_\lambda, X_{\lambda,0})$ on good moduli spaces factors through the henselization $(X^h_\lambda, X_{\lambda,0})$. After replacing $\mathcal{X}_\lambda$ with $\mathcal{X}_\lambda \times_{X_\lambda} X^h_\lambda$, we may thus assume that $(X_\lambda, X_{\lambda,0})$ is henselian for every λ.

Since $\mathcal{X}' \to \mathcal{X}$ is smooth (resp. a smooth gerbe, resp. étale) and finitely presented, after possibly increasing λ it descends to $\mathcal{X}'_\lambda \to \mathcal{X}_\lambda$ and retains its properties of being smooth (resp. a smooth gerbe, resp. étale) [Ryd15, Prop. B.3]. If $s_0\colon \mathcal{X}_0 \to \mathcal{X}'_0$ is a section of $\mathcal{X}'_0 \to \mathcal{X}_0$, then for sufficiently large λ, we have a section $s_{\lambda,0}\colon \mathcal{X}_{\lambda,0} \to \mathcal{X}'_{\lambda,0}$. From case (b), we obtain a section $s_\lambda\colon \mathcal{X}_\lambda \to \mathcal{X}'_\lambda$, hence a section $s\colon \mathcal{X} \to \mathcal{X}'$ as requested.

[2]Note that [Ols06a, Thm. 1.5] only treats the case of embedded deformations over a base *scheme*. In the case of a relatively flat target morphism, however, this can be generalized to a base algebraic stack by deforming the graph and employing [Ols06a, Thm. 1.1], together with the tor-independent base change properties properties of the cotangent complex. In the situation at hand we may also simply apply [Ols06a, Thm. 1.1] to $s_n\colon \mathcal{X}_n \to \mathcal{X}'$ and $\mathcal{X}_n \hookrightarrow \mathcal{X}$.

The full and full faithfulness statements in (b) and (c) can be deduced using similar methods: the quasi-excellent case can be reduced to the complete case using Artin approximation, and the non-excellent case can be reduced to the excellent case using limits. □

7.5. **Deformation of morphisms.** A simple application of Proposition 7.9 yields a deformation result of morphisms.

Proposition 7.10. *Let $(\mathcal{X}, \mathcal{X}_0)$ be as in Setup 7.6. Let $\mathcal{Y} \to X$ be a quasi-separated and smooth (resp. smooth gerbe, resp. étale) morphism with affine stabilizers. In case Setup 7.6 (a), also assume that $\mathcal{Y} \to X$ has quasi-affine diagonal. Then any morphism $\mathcal{X}_0 \to \mathcal{Y}$ can be extended (resp. extended uniquely up to non-unique 2-isomorphism, resp. extended uniquely up to unique 2-isomorphism) to a morphism $\mathcal{X} \to \mathcal{Y}$. In particular,*

1. *the natural functor $\mathsf{F\acute{E}T}(\mathcal{X}) \to \mathsf{F\acute{E}T}(\mathcal{X}_0)$ between the categories of finite étale covers is an equivalence;*
2. *the natural functor $\mathsf{VB}(\mathcal{X}) \to \mathsf{VB}(\mathcal{X}_0)$ between the categories of vector bundles is essentially surjective and full;*
3. *if $G \to X$ is an affine flat group scheme of finite presentation and $\mathcal{X}_0 = [\operatorname{Spec} A/G]$, then there is a G-equivariant closed immersion $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} B$ over X that induces $\mathcal{X}_0 \hookrightarrow \mathcal{X} = [\operatorname{Spec} B/G]$; and*
4. *if $\mathcal{X}_0$ is nicely fundamental, then so is $\mathcal{X}$.*

Proof. For the main statement, apply Proposition 7.9 with $\mathcal{X}' = \mathcal{X} \times_X \mathcal{Y}$. For (1), apply the result to $\mathcal{Y} = \coprod_n BS_{n,X}$ noting that BS_n classifies finite étale covers of degree n. Similarly, for (2), apply the result to $\mathcal{Y} = \coprod_n B\mathrm{GL}_{n,X}$. For (3), apply the result to $\mathcal{Y} = BG$ together with Proposition 5.3(1) to ensure that the induced morphism $\mathcal{X} \to BG$ is affine. For (4), note that, by definition, $\mathcal{X}_0 = [\operatorname{Spec} A/G_0]$ where $G_0 \to S_0$ is nice and embeddable. We next deform G_0 to a nice and embeddable group scheme $G \to S$ (Proposition 4.16) and then apply (3). □

7.6. **Deformation of linearly fundamental stacks.** If (S, S_0) is an affine complete noetherian pair and $\mathcal{X}_0$ is a linearly fundamental stack with a syntomic morphism $\mathcal{X}_0 \to S_0$ that is a good moduli space, Theorem 1.11 constructs a noetherian and linearly fundamental stack $\mathcal{X}$ that is flat over S, such that $\mathcal{X}_0 = \mathcal{X} \times_S S_0$ and $\mathcal{X}$ is coherently complete along $\mathcal{X}_0$. The following lemma shows that $\mathcal{X} \to S$ is also a good moduli space. We also consider non-noetherian generalizations.

Lemma 7.11. *Let $\mathcal{X}$ be a quasi-compact algebraic stack with affine diagonal and affine good moduli space X. Let $\pi \colon \mathcal{X} \to S$ be a flat morphism. Let $S_0 \hookrightarrow S$ be a closed immersion. Let $\mathcal{X}_0 = \mathcal{X} \times_S S_0$ and assume $\pi_0 \colon \mathcal{X}_0 \to S_0$ is a good moduli space and $(\mathcal{X}, \mathcal{X}_0)$ is a local pair. In addition, assume that (S, S_0) is an affine local pair and*

(a) *$\mathcal{X}$ is noetherian and (S, S_0) is complete;*
(b) *$\mathcal{X}$ is noetherian and π is of finite type; or*
(c) *π is of finite presentation and $\mathcal{X}_0$ satisfies (FC), (PC), or (N).*

Then π is a good moduli space morphism of finite presentation. Moreover,

1. *if π_0 is syntomic (resp. smooth, resp. étale), then so is π; and*
2. *if π_0 is an fppf gerbe (resp. a smooth gerbe, resp. an étale gerbe), then so is π.*

Proof. We first show that π is a good moduli space morphism of finite presentation.

Let $S = \operatorname{Spec} A$, $S_0 = \operatorname{Spec}(A/I)$ and $X = \operatorname{Spec} B$. Since $(\mathcal{X}, \mathcal{X}_0)$ is a local pair, it follows that $(\operatorname{Spec} B, \operatorname{Spec} B/IB)$ is a local pair. In particular, IB is contained in the Jacobson radical of B. Note that if $\mathcal{X}$ is noetherian, then $\mathcal{X} \to \operatorname{Spec} B$ is of finite type [AHR20, Thm. A.1]. Moreover, in the commuting diagram:

$$\begin{array}{ccccc} \mathcal{X}_0 & \longrightarrow & \operatorname{Spec}(B/IB) & \longrightarrow & S_0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathcal{X} & \longrightarrow & X & \longrightarrow & S, \end{array}$$

the outer rectangle is cartesian, as is the right square, so it follows that the left square is cartesian. Since the formation of good moduli spaces is compatible with arbitrary base change, it follows that the morphism $A/I \to B/IB$ is an isomorphism.

Case (a): let $A_n = A/I^{n+1}$ and $\mathcal{X}_n = V(I^{n+1}\mathcal{O}_{\mathcal{X}})$. Since $\mathcal{X}_n$ is noetherian and $\pi_n \colon \mathcal{X}_n \to S_n := \operatorname{Spec} A/I^{n+1}$ is flat, it follows that $B_n = \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}) = B/I^{n+1}B$ is a noetherian and flat $A_n = A/I^{n+1}$-algebra [Alp13, Thm. 4.16(ix)]. But $B_n/IB_n = A/I$ so $A_n \to B_n$ is surjective and hence an isomorphism. Let $\widehat{B}$ be the IB-adic completion of B; then the composition $A \to B \to \widehat{B}$ is an isomorphism and $B \to \widehat{B}$ is faithfully flat because IB is contained in the Jacobson radical of B. It follows immediately that $A \to B$ is an isomorphism.

Case (b): now the image of π contains S_0 and by flatness is stable under generizations; it follows immediately that π is faithfully flat. Since $\mathcal{X}$ is noetherian, it follows that S is noetherian. We may now base change everything along the faithfully flat morphism $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} A$, where $\widehat{A}$ is the I-adic completion of A. By faithfully flat descent of good moduli spaces, we are now reduced to Case (a).

Case (c): the good moduli space $X \to S$ is also of finite presentation (Theorem 6.1). The result then follows from (b) using an approximation argument similar to that employed in the proof of Proposition 7.8.

Now claim (1) follows because the conditions are open and all closed points of $\mathcal{X}$ lie in $\mathcal{X}_0$. For claim (2), since $\mathcal{X} \to S$ and $\mathcal{X} \times_S \mathcal{X} \to S$ are flat and $\mathcal{X}_0$ contains all closed points, the fiberwise criterion of flatness shows that $\Delta_{\mathcal{X}/S}$ is flat if and only if $\Delta_{\mathcal{X}_0/S_0}$ is flat. It then follows that $\Delta_{\mathcal{X}/S}$ is smooth (resp. étale) if $\Delta_{\mathcal{X}_0/S_0}$ is so. ☐

Combining Theorem 1.11/Theorem 5.1 and Lemma 7.11 with Artin approximation yields the following result.

Proposition 7.12 (Deformation of linearly fundamental stacks)**.** *Let $\pi_0 \colon \mathcal{X}_0 \to S_0$ be a good moduli space, where $\mathcal{X}_0$ is linearly fundamental. Let (S, S_0) be an affine henselian pair and assume one of the following conditions:*

(a) *(S, S_0) is a noetherian complete pair;*
(b) *S is quasi-excellent; or*
(c) *$\mathcal{X}_0$ satisfies (FC), (PC), or (N).*

If π_0 is syntomic, then there exists a syntomic morphism $\pi \colon \mathcal{X} \to S$ that is a good moduli space such that:

(1) *$\mathcal{X} \times_S S_0 \cong \mathcal{X}_0$;*
(2) *$\mathcal{X}$ is linearly fundamental;*
(3) *$\mathcal{X}$ is coherently complete along $\mathcal{X}_0$ if (S, S_0) is a noetherian complete pair;*
(4) *π is smooth (resp. étale) if π_0 is smooth (resp. étale); and*
(5) *π is an fppf (resp. smooth, resp. étale) gerbe if π_0 is such a gerbe.*

Moreover, if π_0 is smooth (resp. a smooth gerbe, resp. étale), then π is unique up to non-unique isomorphism (resp. non-unique 2-isomorphism, resp. unique 2-isomorphism).

Proof. In case (a): the existence of a flat morphism $\mathcal{X} \to S$ satisfying (1)–(3) is immediate from Theorem 5.1(1) applied to $\mathcal{X}_0 \to S_0 \to S$. Lemma 7.11(a) implies that $\mathcal{X} \to S$ is syntomic, a good moduli space, and satisfies (4)–(5). If $\mathcal{X}' \to S$ is another lift, the uniqueness statements follow by applying Proposition 7.10 with $\mathcal{Y} = \mathcal{X}'$.

In case (b): consider the functor assigning an S-scheme T to the set of isomorphism classes of fundamental stacks $\mathcal{Y}$ over T such that $\pi\colon \mathcal{Y} \to T$ is syntomic. This functor is limit preserving by Proposition 2.15(1), so we may use the construction in the complete case and Artin approximation (Theorem 3.8) to obtain a fundamental stack $\mathcal{X}$ over S such that $\mathcal{X} \times_S S_0 = \mathcal{X}_0$ and $\mathcal{X} \to S$ is syntomic. An application of Lemma 7.11(b) completes the argument again.

Case (c) follows from case (b) by approximation (similar to that used in the proof of Proposition 7.8). □

7.7. **Deformation of linearly reductive groups.** As a direct consequence of Proposition 7.12, we can prove the following result, cf. Proposition 4.16.

Proposition 7.13 (Deformation of linearly reductive group schemes)**.** *Let (S, S_0) be an affine henselian pair and $G_0 \to S_0$ a linearly reductive and embeddable group scheme. Assume one of the following conditions:*

(a) *(S, S_0) is a noetherian complete pair;*
(b) *S is quasi-excellent; or*
(c) *G_0 has nice fibers at closed points or S_0 satisfies (PC) or (FC).*

Then there exists a linearly reductive and embeddable group scheme $G \to S$ such that $G_0 = G \times_S S_0$. If, in addition, $G_0 \to S_0$ is smooth (resp. étale), then $G \to S$ is smooth (resp. étale) and unique up to non-unique (resp. unique) isomorphism.

Proof. Applying Proposition 7.12 to $BG_0 \to S_0$ yields a linearly fundamental fppf gerbe $\mathcal{X} \to S$ such that $BG_0 = \mathcal{X} \times_S S_0$. By Proposition 7.9, we may extend the canonical section $S_0 \to BG_0$ to a section $S \to \mathcal{X}$ with the stated uniqueness property. We conclude that $\mathcal{X}$ is isomorphic to BG for an fppf affine group scheme $G \to S$ extending G_0. Since BG is linearly fundamental, $G \to S$ is linearly reductive and embeddable (see Remark 2.9). □

Remark 7.14. When $G_0 \to S_0$ is a split reductive group scheme, then the existence of $G \to S$ follows from the classification of reductive groups: $G_0 \to S_0$ is the pullback of a split reductive group over $\operatorname{Spec} \mathbb{Z}$ [SGA3_{II}, Exp. XXV, Thm. 1.1, Cor. 1.2]. Our methods require linear reductivity but also work for non-connected, non-split and non-smooth group schemes.

7.8. **Extension over étale neighborhoods.** In this subsection, we consider the problem of extending objects over étale neighborhoods. Recall that if $\pi\colon \mathcal{X} \to X$ is an adequate moduli space, then a morphism $\mathcal{X}' \to \mathcal{X}$ is *strongly étale* if $\mathcal{X}' = \mathcal{X} \times_X X'$ for some étale morphism $X' \to X$ (Definition 3.13).

Proposition 7.15 (Extension of gerbes)**.** *Let (S, S_0) be an affine pair. Let $\pi_0\colon \mathcal{X}_0 \to S_0$ be an fppf gerbe (resp. smooth gerbe, resp. étale gerbe). Suppose that $\mathcal{X}_0$ is linearly fundamental and satisfies (PC), (N) or (FC). Then, there exists an étale neighborhood $S' \to S$ of S_0 and a fundamental fppf gerbe (resp. smooth gerbe, resp. étale gerbe) $\pi\colon \mathcal{X}' \to S'$ extending π_0.*

Proof. The henselization S^h of (S, S_0) is the limit of the affine étale neighborhoods $S' \to S$ of S_0 so the result follows from Proposition 7.12 and Proposition 2.15(1). □

Proposition 7.16 (Extension of groups). *Let (S, S_0) be an affine pair. Let $G_0 \to S_0$ be a linearly reductive and embeddable group scheme. Suppose that G_0 has nice fibers or that S_0 satisfies (PC) or (FC). Then, there exists an étale neighborhood $S' \to S$ of S_0 and a geometrically reductive embeddable group $G' \to S'$ extending G_0.*

Proof. Argue as before, using Proposition 7.13 and Lemma 2.12. □

Proposition 7.17 (Extension of the resolution property). *Let X be an affine scheme and let $\mathcal{X} \to X$ be an adequate moduli space of finite presentation and affine diagonal. Let $\mathcal{X}_0$ be a closed substack which is linearly fundamental. Suppose that X is quasi-excellent or that $\mathcal{X}_0$ satisfies (PC), (N) or (FC). Then there exists a strongly étale neighborhood $\mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}_0$ such that $\mathcal{X}'$ is fundamental.*

Proof. Let X^h be the henselization of X along $X_0 = \pi_0(\mathcal{X}_0)$. Since $X_0 \hookrightarrow X^h$ contains all closed points, it follows that $\mathcal{X} \times_X X^h$ is linearly fundamental (Corollary 6.12 and Proposition 7.8). Since X^h is the limit of all affine étale neighborhoods of X_0 the result follows from Proposition 2.15(1). □

Proposition 7.18 (Extension of sections and morphisms). *Let $(\mathcal{X}, \mathcal{X}_0)$ be a fundamental pair over an algebraic stack S. Suppose that $\mathcal{X}_0$ is linearly fundamental and satisfies (PC), (N) or (FC). Given one of the following:*

1. *a section $s_0 \colon \mathcal{X}_0 \to \mathcal{Y}_0$ of a smooth morphism $\mathcal{Y} \to \mathcal{X}$ that is quasi-separated with affine stabilizers;*
2. *an S-morphism $f_0 \colon \mathcal{X}_0 \to \mathcal{Y}$ where $\mathcal{Y} \to S$ is a smooth morphism that is quasi-separated with affine stabilizers;*
3. *an affine S-morphism $f_0 \colon \mathcal{X}_0 \to \mathcal{Y}$ where $\mathcal{Y} \to S$ is a smooth morphism with affine diagonal;*
4. *a vector bundle $\mathcal{E}_0$ on $\mathcal{X}_0$; or*
5. *a finite étale morphism $\mathcal{W}_0 \to \mathcal{X}_0$.*

Then there exists a strongly étale neighborhood $\mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}_0$ such that the object over $\mathcal{X}_0$ (s_0, f_0, $\mathcal{E}_0$ or $\mathcal{W}_0$) extends to a corresponding object over $\mathcal{X}'$.

Proof. Let X be the adequate moduli space of $\mathcal{X}$ and $X_0 \subseteq X$ the image of $\mathcal{X}_0$. Then (X, X_0) is an affine pair and its henselization X^h is the limit of étale neighborhoods $X' \to X$ of X_0. Since $X_0 \hookrightarrow X^h$ contains all closed points, it follows that $\mathcal{X}^h := \mathcal{X} \times_X X^h$ is linearly fundamental by Corollary 6.11. The result follows from Propositions 7.9 and 7.10, Proposition 5.3(1) and standard limit methods. □

Proposition 7.19 (Extension of nicely fundamental). *Let $(\mathcal{X}, \mathcal{X}_0)$ be a fundamental pair. If $\mathcal{X}_0$ is nicely fundamental, then there exists a strongly étale neighborhood $\mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}_0$ such that $\mathcal{X}'$ is nicely fundamental.*

Proof. As in the previous proof, it follows that $\mathcal{X}^h$ is linearly fundamental, hence nicely fundamental by Proposition 7.10(4). By Proposition 2.15(1), there exists an étale neighborhood $X' \to X$ of X_0 such that $\mathcal{X}' := \mathcal{X} \times_X X'$ is nicely fundamental. □

Proposition 7.20 (Extension of linearly fundamental). *Let $(\mathcal{X}, \mathcal{X}_0)$ be a fundamental pair. Suppose that $\mathcal{X}_0$ satisfies (PC), or (N), or that $\mathcal{X}$ satisfies (FC) in an open neighborhood of $\mathcal{X}_0$. If $\mathcal{X}_0$ is linearly fundamental, then there exists a saturated open neighborhood $\mathcal{X}' \subseteq \mathcal{X}$ of $\mathcal{X}_0$ such that $\mathcal{X}'$ is linearly fundamental.*

Proof. Let X be the adequate moduli space of $\mathcal{X}$ and X_0 the image of $\mathcal{X}_0$. The Zariskification X^Z of X is the limit of all affine open neighborhoods $X' \to X$ of X_0. Since $X_0 \hookrightarrow X^Z$ contains all closed points, the stack $\mathcal{X}^Z := \mathcal{X} \times_X X^Z$ is linearly fundamental (Corollary 6.11). By Theorem 7.3, there exists an open neighborhood $X' \to X$ of X_0 such that $\mathcal{X}' := \mathcal{X} \times_X X'$ is linearly fundamental. □

Remark 7.21. Note that when S_0 is a single point, then (FC) always holds for S_0 and for objects over S_0. In the results of this subsection, the substacks $S_0 \subseteq S$ and $\mathcal{X}_0 \subseteq \mathcal{X}$ are by definition closed substacks. The results readily generalize to the following situation: $S_0 = \{s\}$ is any point and $\mathcal{X}_0 = \mathcal{G}_x$ is the residual gerbe of a point x closed in its fiber over the adequate moduli space.

7.9. **Universal property of good moduli space morphisms.**

Theorem 7.22 (Universal property). *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a good moduli space morphism of finite presentation between algebraic stacks. Let $\mathcal{Z}$ be an algebraic stack with quasi-separated diagonal and let $f\colon \mathcal{X} \to \mathcal{Z}$ be a morphism. Then f factors through π if and only if the induced map on inertia, $I_\pi \to f^*I_{\mathcal{Z}}$, factors through the identity section. Moreover, the factorization of f through π is unique up to unique 2-isomorphism and if $\mathcal{Y}$ is quasi-compact and quasi-separated, then the condition is equivalent to:*

$$\ker\big(\mathrm{Aut}_{\mathcal{X}}(x) \to \mathrm{Aut}_{\mathcal{Y}}(\pi(x))\big) \;\subseteq\; \ker\big(\mathrm{Aut}_{\mathcal{X}}(x) \to \mathrm{Aut}_{\mathcal{Z}}(f(x))\big)$$

for every closed point $x \in |\mathcal{X}|$.

We begin with the uniqueness, which also holds for adequate moduli space morphisms.

Lemma 7.23. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be an adequate moduli space morphism between algebraic stacks. Then π is a categorical epimorphism, that is, if $f, g\colon \mathcal{Y} \to \mathcal{Z}$ are two morphisms then every 2-isomorphism $f \circ \pi \simeq g \circ \pi$ descends to a unique 2-isomorphism $f \simeq g$.*

Proof. Two morphisms $f, g\colon \mathcal{Y} \to \mathcal{Z}$ gives rise to a morphism $(f, g)\colon \mathcal{Y} \to \mathcal{Z} \times \mathcal{Z}$. Let $I := \mathrm{Isom}(f, g) = \mathcal{Z} \times_{\Delta, \mathcal{Z}\times\mathcal{Z}, (f,g)} \mathcal{Y}$. Then $I \to \mathcal{Y}$ is representable and its sections correspond to 2-isomorphisms between f and g. Similarly, a 2-isomorphism between $f \circ \pi$ and $g \circ \pi$ corresponds to a $\mathcal{Y}$-morphism $\mathcal{X} \to I$. That every $\mathcal{Y}$-morphism $\mathcal{X} \to I$ descends to a unique $\mathcal{Y}$-morphism $\mathcal{Y} \to I$ can be checked smooth-locally on $\mathcal{Y}$ and thus follows directly from Theorem 3.12. □

Proof of Theorem 7.22. The uniqueness is Lemma 7.23. For the existence, we may work smooth-locally on $\mathcal{Y}$ and assume that $X = \mathcal{Y}$ is an affine scheme. Let $p\colon U \to \mathcal{Z}$ be a smooth presentation where U is an algebraic space. This gives a smooth, representable and quasi-separated morphism $q\colon U \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{X}$. Let $x \in |\mathcal{X}|$ be a point, closed in its fiber over X. The assumption on inertia shows that $\mathcal{G}_x \to \mathcal{Z}$ factors through the structure morphism $\mathcal{G}_x \to \operatorname{Spec} \kappa(x)$. It follows that $q|_{\mathcal{G}_x}$ has a section after passing to a separable field extension of $\kappa(x)$ and that can be accomplished étale-locally on X. By Propositions 7.17 and 7.18(1) we obtain a section of q after replacing X with an étale neighborhood of $\pi(x)$. We can thus factor $f\colon \mathcal{X} \to \mathcal{Z}$ through p and hence also through π by Theorem 3.12. □

Theorem 7.22 generalizes [ATW20, Thm. 2.3.6] from tame stacks to good moduli space morphisms. The analogous result for adequate moduli space morphisms does not hold. In fact, the result is false even if $\mathcal{X}$ is a wild Deligne–Mumford stack and $\mathcal{X} \to \mathcal{Y}$ is its coarse moduli space [ATW20, A.2.3].

8. Refinements of local structure

In Theorem 1.1, we have seen that for an algebraic stack $\mathcal{X}$ satisfying mild hypotheses and a point $x \in |\mathcal{X}|$ with linearly reductive stabilizer with image $s \in |S|$ such that $\kappa(x)/\kappa(s)$ is finite, there exists an étale morphism

$$(\mathcal{W}, w) \to (\mathcal{X}, x),$$

where $\mathcal{W} = [\operatorname{Spec} A/\mathrm{GL}_n]$ is a fundamental stack, $w \in |\mathcal{W}|$ is closed in its fiber $\mathcal{W}_s$, and the induced map $\mathcal{G}_w \to \mathcal{G}_x$ on residual gerbes is an isomorphism.

We now prove two theorems providing étale neighborhoods $\mathcal{W}' \to \mathcal{W}$ of a point w of a fundamental stack $\mathcal{W}$ such that $\mathcal{W}' = [\operatorname{Spec} A'/G]$ and the group scheme G has a specific form. When applied to the output of Theorem 1.1, these theorems yield refinements of the local structure theorem. In Theorem 8.1, $\mathcal{W}' \to \mathcal{W}$ is even finite étale and $G \to \operatorname{Spec} \mathbb{Z}$ is split reductive such that the stabilizer of the action of G at a point $u \in \operatorname{Spec} A'$ over w is the connected component G_w^0. On the other hand, in Theorem 8.2, if the residual gerbe $\mathcal{G}_w = BG_0$ is neutral, then $G \to S'$ is a geometrically reductive group scheme defined over an étale neighborhood $S' \to S$ and is a deformation of G_0. Moreover, under mild characteristic hypothesis, $G \to S'$ is linearly reductive. When the gerbe $\mathcal{G}_w$ is not neutral, then $\mathcal{W}'$ can be arranged to be affine over a fundamental gerbe $\mathcal{H}' \to S'$ which is a deformation of $\mathcal{G}_w$.

8.1. Split local structure of fundamental stacks.

Theorem 8.1 (Split local structure)**.** *Let S be a quasi-separated algebraic space. Let $\mathcal{W}$ be a fundamental stack of finite presentation over S. Let $w \in |\mathcal{W}|$ be a point with linearly reductive stabilizer and image $s \in |S|$ such that w is closed in its fiber $\mathcal{W}_s$. Then there exists a finite étale morphism $f \colon \mathcal{W}' \to \mathcal{W}$ such that:*

1. *$\mathcal{W}' = [U/G]$ where U is affine and $G \to \operatorname{Spec} \mathbb{Z}$ is a geometrically reductive embeddable group scheme;*
2. *there is a point $u \in |U|$ above w fixed by G and f identifies G_u with the connected component of $\operatorname{Aut}_{\mathcal{W}}(w)$;*
3. *if $\operatorname{char}(\kappa(w)) = 0$, then $G \to \operatorname{Spec} \mathbb{Z}$ is split reductive; and*
4. *if $\operatorname{char}(\kappa(w)) = p$, then $G \to \operatorname{Spec} \mathbb{Z}$ is diagonalizable.*

In other words, there is a commutative diagram of adequate moduli spaces

$$\begin{array}{ccc} [U/G] = \mathcal{W}' & \xrightarrow{f} & \mathcal{W} = [\operatorname{Spec} A/\mathrm{GL}_n] \\ \downarrow & & \downarrow \pi \\ U/\!\!/G = W' & \longrightarrow & W = \operatorname{Spec} A/\!\!/\mathrm{GL}_n \end{array}$$

where f is finite étale. Note that $W' \to W$ is finite but not necessarily étale, and that the diagram is not necessarily cartesian. If $\operatorname{char}(\kappa(w)) = 0$, then $G \to \operatorname{Spec} \mathbb{Z}$ is smooth with geometrically connected fibers. If $\operatorname{char}(\kappa(w)) = p$, then $G \to \operatorname{Spec} \mathbb{Z}$ need neither be smooth nor have connected fibers, e.g., $G = \boldsymbol{\mu}_{p,\mathbb{Z}}$.

Proof of Theorem 8.1. Using standard limit methods, we may replace the adequate moduli space W of $\mathcal{W}$ with its henselization at $\pi(w)$. In this case, $w \in |\mathcal{W}|$ is the unique closed point and $\mathcal{W}$ is *linearly* fundamental by Corollary 6.11.

The structure morphism of the residual gerbe $\mathcal{G}_w \to \operatorname{Spec} \kappa(w)$ is smooth. Thus, there exists a separable field extension $k/\kappa(w)$ that neutralizes the gerbe. Let G_w be the stabilizer group of a lift $\operatorname{Spec} k \to \mathcal{G}_w$. Let $(G_w)^0$ be its connected component. If $\operatorname{char} k > 0$, then $(G_w)^0$ is of multiplicative type and if $\operatorname{char} k = 0$, then $(G_w)^0$ is smooth, connected and reductive. In either case, we may thus find a further separable field extension k'/k such that $(G_w)^0$ becomes diagonalizable or a split reductive group. That is, $(G_w^0)_{k'} = G_{k'}$ where G is a group scheme over $\mathbb{Z}$ which is either diagonalizable or split reductive. We now have a group G as in (3) or (4).

We now apply Proposition 7.10(1) to the finite étale morphism $\mathcal{W}'_0 := B(G_w^0)_{k'} \to \mathcal{G}_w$ and obtain a finite étale morphism $\mathcal{W}' \to \mathcal{W}$. Then $(\mathcal{W}', \mathcal{W}'_0)$ is a local linearly fundamental pair by construction and its good moduli space W' is finite over W by Theorem 6.1(4), hence henselian. By Proposition 7.10 and Proposition 5.3(1), we may extend the affine morphism $\mathcal{W}'_0 \to BG$ to an affine morphism $\mathcal{W}' \to BG$.

That is, $\mathcal{W}' = [U/G]$ for some affine scheme U and the unique point $u \in |U|$ above $\mathcal{W}'_0$ is fixed by G. □

8.2. Refinements on the local structure theorem.

Theorem 8.2 (Local structure refinement)**.** *Let S be a quasi-separated algebraic space. Let $\mathcal{W}$ be a fundamental stack of finite presentation over S. Let $w \in |\mathcal{W}|$ be a point with linearly reductive stabilizer and image $s \in |S|$ such that w is closed in its fiber $\mathcal{W}_s$. Then there exist a commutative diagram of algebraic stacks*

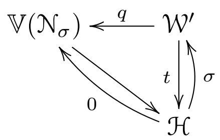
$$\begin{array}{ccc} [\operatorname{Spec} A'/G] = \mathcal{W}' & \xrightarrow{\quad h \quad} & \mathcal{W} = [\operatorname{Spec} A/\mathrm{GL}_n] \\ \downarrow{\scriptstyle t} & & \downarrow \\ \mathcal{H} \xrightarrow{r} BG \longrightarrow & S' & \xrightarrow{g} S \end{array}$$

and a point $w' \in |\mathcal{W}'|$ over $w \in |\mathcal{W}|$ with image $s' \in |S'|$ where

1. *$h\colon (\mathcal{W}', w') \to (\mathcal{W}, w)$ is a strongly étale (see Definition 3.13) neighborhood of w such that $\mathcal{G}_{w'} \to \mathcal{G}_w$ is an isomorphism;*
2. *$g\colon (S', s') \to (S, s)$ is a smooth (étale if $\kappa(w)/\kappa(s)$ is separable) morphism such that there is a $\kappa(s)$-isomorphism $\kappa(w) \cong \kappa(s')$;*
3. *$G \to S'$ is a geometrically reductive embeddable group scheme;*
4. *$\mathcal{H} \to S'$ is a gerbe such that $\mathcal{H}$ is fundamental and $\mathcal{H}_{s'} \cong \mathcal{G}_w$; and*
5. *$t\colon \mathcal{W}' \to \mathcal{H}$ and $r\colon \mathcal{H} \to BG$ are affine morphisms, so $\mathcal{W}' = [\operatorname{Spec} A'/G]$.*

Moreover, we can arrange so that:

6. *if $\mathcal{G}_w$ is neutral, then $\mathcal{H} \to BG$ is an isomorphism;*
7. *if s has an open neighborhood of characteristic zero, then $\mathcal{H}$ is linearly fundamental and G is linearly reductive;*
8. *if w has nice stabilizer (e.g., $\operatorname{char} \kappa(s) > 0$), then $\mathcal{H}$ is nicely fundamental and G is nice; and*
9. *if $\mathcal{W} \to S$ is smooth at w and $\kappa(w)/\kappa(s)$ is separable, then there exists a commutative diagram*



where q is strongly étale and σ is a section of t such that $\sigma(s') = w'$.

Proof. We can replace (S, s) with an étale neighborhood and assume that S is an affine scheme. For (2), we note that we have a tower of finite extensions $\kappa(s) \subseteq \kappa(w)^s \subseteq \kappa(w)$, where the first extension is separable and the latter is purely inseparable. By [EGA, IV.18.1.1], there is an étale neighborhood $(S', s') \to (S, s)$ such that $\kappa(s') \cong \kappa(w)^s$ over $\kappa(s)$. Hence, we are reduced to the situation where $\kappa(s) \subseteq \kappa(w)$ is purely inseparable; then $\kappa(w) = \kappa(s)[a_1^{p^{-r_1}}, \dots, a_n^{p^{-r_n}}]$, where p is the characteristic of $\kappa(s)$ and the $a_i \in \kappa(s)$. There is a point s' of $S' = \mathbb{A}^n_S$ such that $\kappa(s') = \kappa(w)$ and $(S', s') \to (S, s)$ is a smooth morphism as desired. In the sequel, we will repeatedly replace (S', s') by further étale neighborhoods but without residue field extensions.

For (4), we can by Proposition 7.15 (and Remark 7.21) extend the residual gerbe $\mathcal{G}_w$ over $\kappa(w) = \kappa(s')$ to a fundamental gerbe $\mathcal{H} \to S'$ after replacing S' by an étale neighborhood of s'. If s' has an open neighborhood of characteristic zero, then $\mathcal{H}$ is linearly fundamental after restricting to this neighborhood. If w has nice stabilizer, then $\mathcal{H}$ is nicely fundamental after replacing S' with an étale neighborhood by Proposition 7.19.

If $\mathcal{G}_w$ is neutral, that is, has a section σ_0, then after replacing S' with an étale neighborhood, we obtain a section σ of $\mathcal{H} \to S'$ (Proposition 7.18(1) with $\mathcal{X} = S'$) and then $\mathcal{H} = BG$ where $G = \operatorname{Aut}(\sigma)$. This gives (6).

If $\mathcal{G}_w$ is not neutral, there exists, after replacing S' with an étale neighborhood of s', a finite étale surjective morphism $S'' \to S'$ such that $\mathcal{H} \times_{S'} S'' \to S''$ has a section σ'. The group scheme $H' = \operatorname{Aut}(\sigma') \to S''$ is geometrically reductive and embeddable. We let G be the Weil restriction of H' along $S'' \to S'$. It comes equipped with a morphism $\mathcal{H} \to BG$ which is representable, hence affine by [Alp14, Cor. 4.3.2]. It can be seen that $G \to S'$ is geometrically reductive and embeddable and also linearly reductive (resp. nice) if $\mathcal{H}$ is linearly fundamental (resp. nicely fundamental). This establishes (3), (7) and (8).

Since $\mathcal{H} \to S' \to S$ is smooth, we may apply Proposition 7.18 to obtain a strongly étale neighborhood $h\colon (\mathcal{W}', w') \to (\mathcal{W}, w)$ and an affine morphism $t\colon \mathcal{W}' \to \mathcal{H}$. This establishes (1) and (5).

Finally, for (9), if $\mathcal{W} \to S$ is smooth at w and $\kappa(w)/\kappa(s)$ is separable, then $S' \to S$ is étale, so $\mathcal{W}' \to S'$ is also smooth at w'. The result now follows from Proposition 8.3 below. □

Proposition 8.3 (Smooth refinement)**.** *Let S be a quasi-separated algebraic space. Let $\mathcal{H} \to S$ be an fppf gerbe such that $\mathcal{H}$ is fundamental and let $t\colon \mathcal{W} \to \mathcal{H}$ be an affine morphism of finite presentation. Let $w \in |\mathcal{W}|$ be a point with linearly reductive stabilizer and image $s \in |S|$ such that w is closed in its fiber $\mathcal{W}_s$. Suppose that the induced map $\mathcal{G}_w \to \mathcal{H}_s$ is an isomorphism. If $\mathcal{W} \to S$ is smooth at w, then after replacing S with an étale neighborhood of s (without residue field extension), there exist*

(1) *a section $\sigma\colon \mathcal{H} \to \mathcal{W}$ of t such that $\sigma(s) = w$; and*
(2) *a morphism $q\colon \mathcal{W} \to \mathbb{V}(\mathcal{N}_\sigma)$, where $\mathcal{N}_\sigma = t_*(\mathcal{I}/\mathcal{I}^2)$ and $\mathcal{I}$ is the sheaf of ideals in $\mathcal{W}$ defining σ, which is strongly étale in a saturated open neighborhood of σ and such that $q \circ \sigma$ is the zero-section.*

Proof. Since $\mathcal{H} \to S$ is a gerbe, $\mathcal{W} \to S$ is smooth at w and $\mathcal{G}_w \cong \mathcal{H}_s$, it follows that $t\colon \mathcal{W} \to \mathcal{H}$ is smooth at w. The existence of the section σ thus follows from Proposition 7.18(1). Note that since t is affine and smooth, the section σ is a regular closed immersion. An easy approximation argument allows us to replace S by the henselization at s. Then $\mathcal{H}$ is linearly fundamental (Corollary 6.11). Let $\mathcal{I} \subseteq \mathcal{O}_\mathcal{W}$ be the ideal sheaf defining σ. Since $\mathcal{N}_\sigma = t_*(\mathcal{I}/\mathcal{I}^2)$ is locally free and $\mathcal{H}$ is cohomologically affine, the surjection $t_*\mathcal{I} \to \mathcal{N}_\sigma$ of $\mathcal{O}_\mathcal{H}$-modules admits a section. The composition $\mathcal{N}_\sigma \to t_*\mathcal{I} \to t_*\mathcal{O}_\mathcal{W}$ gives a morphism $q\colon \mathcal{W} \to \mathbb{V}(\mathcal{N}_\sigma)$. By definition, q maps σ to the zero-section and induces an isomorphism of normal spaces along σ, hence is étale along σ, hence is strongly étale in a neighborhood by Luna's fundamental lemma (Theorem 3.14). □

9. The structure of linearly reductive groups

Recall from Definition 2.1 that a linearly reductive (resp. geometrically reductive) group scheme $G \to S$ is flat, affine and of finite presentation such that $BG \to S$ is a good moduli space (resp. an adequate moduli space). In this section we will show that a group algebraic space is linearly reductive if and only if it is flat, separated, of finite presentation, has linearly reductive fibers, and has a finite component group (Theorem 9.9).

9.1. **Extension of closed subgroups.**

Lemma 9.1 (Anantharaman)**.** *Let S be the spectrum of a DVR. If $G \to S$ is a separated group algebraic space of finite type, then G is a scheme. If in addition $G \to S$ has affine fibers or is flat with affine generic fiber G_η, then G is affine.*

Proof. The first statement is [Ana73, Thm. 4.B]. For the second statement, it is enough to show that the flat group scheme $\overline{G} = \overline{G_\eta}$ is affine. This is [Ana73, Prop. 2.3.1]. □

Proposition 9.2. *Let $H \to S$ be a geometrically reductive group scheme that is embeddable fppf-locally on S, e.g., H linearly reductive (Corollary 6.2).*

(1) *If $N \subseteq H$ is a closed normal subgroup such that $N \to S$ is quasi-finite, then $N \to S$ is finite.*

Let $G \to S$ be a separated group algebraic space of finite presentation and let $u\colon H \to G$ be a homomorphism.

(2) *If u is a monomorphism, then u is a closed immersion.*
(3) *If $u_s\colon H_s \to G_s$ is a monomorphism for a point $s \in S$, then $u_U\colon H_U \to G_U$ is a closed immersion for some open neighborhood U of s.*

Proof. The questions are local on S so we can assume that H is embeddable. For (1) we note that a normal closed subgroup $N \subseteq H$ gives rise to a closed subgroup $[N/H]$ of the inertia stack $[H/H] = I_{BH}$ (where H acts on itself via conjugation). The result thus follows from Lemma 5.4.

For (2), it is enough to prove that u is proper. After noetherian approximation, we can assume that S is noetherian. By the valuative criterion for properness, we can further assume that S is the spectrum of a DVR. We can also replace G with the closure of $u(H_\eta)$. Then G is an affine group scheme (Lemma 9.1) so $G/H \to BH \to S$ is adequately affine. As $G/H \to S$ is also representable, it follows from [Alp14, Thm. 4.3.1] that it is also affine. It follows that u is a closed immersion.

For (3), we apply (1) to $\ker(u)$ which is quasi-finite, hence finite, in an open neighborhood of s. By Nakayama's lemma u is thus a monomorphism in an open neighborhood and we conclude by (2). □

Remark 9.3. If $G \to S$ is flat, then (2) says that any representable morphism $BH \to BG$ is separated. When H is of multiplicative type then Proposition 9.2 is [SGA3$_{\mathrm{II}}$, Exp. IX, Thm. 6.4 and Exp. VIII, Rmq. 7.13b]. When H is reductive (i.e., smooth with connected reductive fibers) it is [SGA3$_{\mathrm{II}}$, Exp. XVI, Prop. 6.1 and Cor. 1.5a].

Proposition 9.4. *Let (S, s) be a henselian local ring, let $G \to S$ be a flat group algebraic space of finite presentation with affine fibers and let $u_s\colon H_s \hookrightarrow G_s$ be a closed subgroup. If H_s is linearly reductive and G_s/H_s is smooth, then there exists a linearly reductive and embeddable group scheme $H \to S$ and a homomorphism $u\colon H \to G$ extending u_s.*

(1) *If G_s/H_s is étale (i.e., if u_s is open and closed), then u is étale and the pair (H, u) is unique: if H' is a quasi-separated group algebraic space and $u'\colon H' \to G$ is an étale homomorphism extending u_s, then there exists a unique homomorphism $H \to H'$ over G.*
(2) *If $G \to S$ is separated, then u is a closed immersion.*

Proof. Note that since (S, s) is local, condition (FC) is satisfied. By Proposition 7.12, the gerbe BH_s extends to a unique linearly fundamental gerbe $\mathcal{H} \to S$. Since $BG \to S$ is smooth, we can extend the morphism $\varphi_s\colon BH_s \to BG_s$ to a morphism $\varphi\colon \mathcal{H} \to BG$ (Proposition 7.10). The morphism φ is flat and the special fiber φ_s is smooth since G_s/H_s is smooth. Thus φ is smooth. Similarly, if G_s/H_s is étale, then φ is étale.

Let $p\colon S \to BG$ and $f_s\colon \operatorname{Spec}\kappa(s) \to BH_s \cong \mathcal{H}_s$ denote the tautological sections. Since φ is smooth and $\varphi_s \circ f_s = p_s$, we obtain a section $f\colon S \to \mathcal{H}$ such that $f|_s = f_s$ and $\varphi \circ f = p$ by Proposition 7.9 (applied to $\mathcal{X} = S$ and $\mathcal{X}' = \mathcal{H} \times_{BG} S$). We let $H = \operatorname{Aut}(f)$ and let $u\colon H \to G = \operatorname{Aut}(\varphi \circ f)$ be the induced morphism, extending u_s.

For (1), we have already seen that if G_s/H_s is étale, then φ is étale so that u is étale. Let $u'\colon H' \to G$ be another étale homomorphism extending u_s. We have a unique map $\psi\colon BH \to BH'$ over BG extending the isomorphism $BH_s \to BH'_s$ (apply Proposition 7.9 to $\mathcal{X} = BH$ and $\mathcal{X}' = BH' \times_{BG} BH$). The tautological section of BH is then mapped by ψ to the tautological section of BH' (apply Proposition 7.9 to $\mathcal{X} = S$ and $\mathcal{X}' = BH' \times_{BG} S$). That is, ψ is induced by a unique homomorphism $H \to H'$ over G.

For (2), if G is separated, then u is a closed immersion by Proposition 9.2. □

Remark 9.5. If G_s/H_s is not smooth, then the tautological section of BH_s still extends to a section of $\mathcal{H} \to S$ so $\mathcal{H} = BH$ where H is an extension of H_s. But φ merely induces a homomorphism $H \to \widetilde{G}$ where $\widetilde{G}$ is a twisted form of G.

9.2. **The smooth identity component of linearly reductive groups.** Recall that if $G \to S$ is a smooth group scheme, then there is an open characteristic subgroup $G^0 \subseteq G$ such that $G^0 \to S$ is smooth with connected fibers [SGA3_{II}, Exp. 6B, Thm. 3.10]. This is also true when $G \to S$ is a smooth group algebraic space, cf [LMB, 6.8]. The subgroup G^0 is not always closed, not even when G is affine [Ray70, VII, §3]. For a (not necessarily smooth) group scheme of finite type over a field, the identity component G^0 exists and is open and closed.

When (S,s) is henselian and $(G_s)^0$ is linearly reductive but not necessarily smooth, then Proposition 9.4 gives the existence of a unique linearly reductive group scheme G^0_{loc} together with an étale homomorphism $u\colon G^0_{\text{loc}} \to G$ extending $u_s\colon (G_s)^0 \hookrightarrow G_s$. There are at least three subtleties:

(a) u need not be injective, even if G is smooth and of characteristic zero.
(b) G^0_{loc} need not have connected fibers.
(c) Even if u is injective, G^0_{loc} need not be a normal subgroup.

As we will see, the first problem only happens when G is not separated and the latter two only in mixed characteristic when G is not smooth.

Example 9.6. We give two examples in mixed characteristic and one in equal characteristic:

(1) Let $G = \boldsymbol{\mu}_{p,\mathbb{Z}_p} \to \operatorname{Spec}\mathbb{Z}_p$ which is a finite linearly reductive group scheme. Then $G^0_{\text{loc}} = G$ but the generic geometric fiber is not connected, illustrating (b). If we let G' be the gluing of G and a finite group over $\mathbb{Q}_p$ containing $\boldsymbol{\mu}_p$ as a non-normal subgroup, then $G'^0_{\text{loc}} = G^0_{\text{loc}} \subseteq G'$ is not normal, illustrating (c).
(2) Let G be as in the previous example and consider the étale group scheme $H \to \operatorname{Spec}\mathbb{Z}_p$ given as extension by zero from $\boldsymbol{\mu}_{p,\mathbb{Q}_p} \to \operatorname{Spec}\mathbb{Q}_p$. Then we have a bijective monomorphism $H \to G$ which is not an immersion and $G' = G/H$ is a quasi-finite group algebraic space with connected fibers which is not locally separated. Note that $(G')^0_{\text{loc}} = G = \boldsymbol{\mu}_{p,\mathbb{Z}_p}$ and the étale morphism $(G')^0_{\text{loc}} \to G'$ is not injective, illustrating (a).
(3) Let $G = \mathbb{G}_m \times S \to S = \operatorname{Spec} k[\![t]\!]$ and let $H \to S$ be $\boldsymbol{\mu}_{r,k(\!(t)\!)}$ extended by zero for some invertible $r > 1$. Let $G' = G/H$. Then G' is a smooth locally separated algebraic space, $G'^0_{\text{loc}} = G$ and $G'^0_{\text{loc}} \to G'$ is not injective, illustrating (a).

From now on, we only consider separated group schemes. Then $G^0_{\mathrm{loc}} \to G$ is an open and closed subgroup by Proposition 9.4 so (a) does not occur. The subgroup G^0_{loc} exists over the henselization but not globally in mixed characteristic due to problem (b). We remedy this by considering a slightly smaller subgroup G^0_{sm} which is closed but not open.

Lemma 9.7 (Identity component: nice case). *Let S be an algebraic space and let $G \to S$ be a flat and separated group algebraic space of finite presentation with affine fibers.*

(1) *The locus of $s \in S$ such that $(G_s)^0$ is nice is open in S.*

Now assume that $(G_s)^0$ is nice for all $s \in S$.

(2) *There exist a unique characteristic closed subgroup $G^0_{\mathrm{sm}} \hookrightarrow G$ smooth over S that restricts to $(G_s)^0_{\mathrm{red}}$ on fibers.*
(3) *$G^0_{\mathrm{sm}} \to S$ is a torus, $G/G^0_{\mathrm{sm}} \to S$ is quasi-finite and separated, and $G \to S$ is quasi-affine.*

Now assume in addition that S has equal characteristic.

(4) *There exist a unique characteristic open and closed subgroup $G^0 \hookrightarrow G$ that restricts to $(G_s)^0$ on fibers.*
(5) *$G^0 \to S$ is of multiplicative type with connected fibers and $G/G^0 \to S$ is étale and separated.*

Proof. The questions are étale-local on S. For (1), if $(G_s)^0$ is nice, i.e., of multiplicative type, then over the henselization at s we can find an open and closed subgroup $G^0_{\mathrm{loc}} \subseteq G$ such that G^0_{loc} is of multiplicative type (Proposition 9.4). After replacing S with an étale neighborhood of s, we can thus find an open and closed subgroup $H \subseteq G$ where H is of multiplicative type. It follows that $(G_s)^0$ is of multiplicative type for all s in S.

For an H as above, we have a characteristic closed subgroup $H_{\mathrm{sm}} \hookrightarrow H$ such that H_{sm} is a torus and H/H_{sm} is finite. Indeed, the Cartier dual of H is an étale sheaf of abelian groups and its torsion is a characteristic subgroup. It follows that G/H_{sm} is quasi-finite and separated and that G is quasi-affine.

It remains to prove that H_{sm} is characteristic and independent on the choice of H so that it glues to a characteristic subgroup G^0_{sm}. This can be checked after base change to henselian local schemes. If (S, s) is henselian, then $G^0_{\mathrm{loc}} \subseteq H$ and since these are group schemes of multiplicative type of the same dimension, it follows that $(G^0_{\mathrm{loc}})_{\mathrm{sm}} = H_{\mathrm{sm}}$. This shows that H_{sm} is independent on the choice of H. Since any automorphism of G leaves G^0_{loc} fixed, any automorphism leaves $(G^0_{\mathrm{loc}})_{\mathrm{sm}}$ fixed as well. This shows that H_{sm} is characteristic.

If S has equal characteristic, then H is an open and closed subgroup with connected fibers, hence clearly unique. □

Lemma 9.8 (Identity component: smooth case). *Let S be an algebraic space and let $G \to S$ be a flat and separated group algebraic space of finite presentation with affine fibers. Suppose that $G \to S$ is smooth and that $(G_s)^0$ is linearly reductive for all s.*

(1) *The open subgroup $G^0 \subseteq G$ is also closed and linearly reductive (and in particular affine).*
(2) *$G/G^0 \to S$ is étale and separated and $G \to S$ is quasi-affine.*

Proof. This follows immediately from Proposition 9.4 since in the henselian case G^0_{loc} is the smallest open subscheme containing $(G_s)^0$, hence equal to G^0. □

Theorem 9.9 (Identity component)**.** *Let S be an algebraic space and let $G \to S$ be a flat and separated group algebraic space of finite presentation with affine fibers. Suppose that $(G_s)^0$ is linearly reductive for every $s \in S$.*

(1) *There exist a unique linearly reductive and characteristic closed subgroup $G^0_{\mathrm{sm}} \hookrightarrow G$ smooth over S that restricts to $(G_s)^0_{\mathrm{red}}$ on fibers, and $G/G^0_{\mathrm{sm}} \to S$ is quasi-finite and separated.*
(2) *If S is of equal characteristic, then there exists a unique linearly reductive characteristic open and closed subgroup $G^0 \hookrightarrow G$ that restricts to $(G_s)^0$ on fibers, and $G/G^0 \to S$ is étale and separated.*
(3) *$G \to S$ is quasi-affine.*

The following are equivalent:

(4) *$G \to S$ is linearly reductive (in particular affine).*
(5) *$G/G^0_{\mathrm{sm}} \to S$ is finite and tame.*
(6) *(if S of equal characteristic) $G/G^0 \to S$ is finite and tame.*

In particular, if $G \to S$ is linearly reductive and S is of equal characteristic $p > 0$, then $G \to S$ is nice.

Proof. Let $S_1 \subseteq S$ be the open locus where $(G_s)^0$ is nice and let $S_2 \subseteq S$ be the open locus where G_s is smooth. Then $S = S_1 \cup S_2$. Over S_1, we define G^0_{sm} as in Lemma 9.7. Over S_2, we define $G^0_{\mathrm{sm}} = G^0$ as in Lemma 9.8. In equal characteristic, we define G^0 as in Lemmas 9.7 and 9.8. The first three statements follow.

Since $G^0_{\mathrm{sm}} \to S$ is linearly reductive, it follows that $BG \to S$ is cohomologically affine if and only if $B(G/G^0_{\mathrm{sm}}) \to S$ is cohomologically affine [Alp13, Prop. 12.17]. If $B(G/G^0_{\mathrm{sm}}) \to S$ is cohomologically affine, then G/G^0_{sm} is finite [Alp14, Thm. 8.3.2]. Conversely, if G/G^0_{sm} is finite and tame then $BG \to S$ is cohomologically affine and $G \to S$ is affine. □

Corollary 9.10. *If S is a normal noetherian scheme with the resolution property (e.g., S is regular and separated, or S is quasi-projective) and $G \to S$ is linearly reductive, then G is embeddable.*

Proof. The stack BG^0_{sm} has the resolution property [Tho87, Cor. 3.2]. Since $BG^0_{\mathrm{sm}} \to BG$ is finite and faithfully flat, it follows that BG has the resolution property [Gro17], hence that G is embeddable. □

Corollary 9.11. *Let S be an algebraic space and let $G \to S$ be a flat and separated group algebraic space of finite presentation with affine, connected and linearly reductive fibers. Then $G \to S$ is affine and linearly reductive.*

Proof. By Theorem 9.9(1), there is a closed (characteristic) smooth linearly reductive group scheme G^0_{sm} and $G/G^0_{\mathrm{sm}} \to S$ is a quasi-finite flat separated morphism with connected fibers since $G \to S$ has connected fibers. Such a morphism is finite, as for example can be seen by passing to henselizations. Moreover, $G/G^0_{\mathrm{sm}} \to S$ is tame since this can be checked on fibers and G_s is linearly reductive. Thus $G \to S$ is linearly reductive by the equivalence of (4) and (5) in Theorem 9.9. □

Remark 9.12. Let $G \to S$ be as in Theorem 9.9. When G/G^0_{sm} is merely finite, then $G \to S$ is geometrically reductive. This happens precisely when $G \to S$ is *pure* in the sense of Raynaud–Gruson [RG71, Défn. 3.3.3]. In particular, $G \to S$ is geometrically reductive if and only if $\pi\colon G \to S$ is affine and $\pi_*\mathcal{O}_G$ is a locally projective $\mathcal{O}_S$-module [RG71, Thm. 3.3.5].

10. Further applications

In this section we give generalizations of Sumihiro's theorem and Luna's étale slice theorem in equivariant geometry. These are obtained by applying the local structure theorem to $\mathcal{X} = [X/G]$. We also show that the henselization $\mathcal{X}_x^h$ exists if $\mathcal{X}$ has affine stabilizers and x is a closed point with linearly reductive stabilizer. Finally, we deduce that several stacks, including $\underline{\mathsf{Coh}}_X(\mathcal{X})$, $\underline{\mathrm{Hilb}}_{\mathcal{X}/X}$ and $\underline{\mathrm{Hom}}_X(\mathcal{X}, \mathcal{Y})$, are algebraic if $\mathcal{X} \to X$ is a good moduli space.

10.1. **Generalization of Sumihiro's theorem on torus actions.** Sumihiro's theorem on torus actions in the relative case is the following. Let S be a noetherian scheme and $X \to S$ a morphism of schemes satisfying Sumihiro's condition (N), that is, $X \to S$ is flat and of finite type, X_s is geometrically normal for all generic points $s \in S$ and X_s is geometrically integral for all codimension 1 points $s \in S$ (which by a result of Raynaud implies that X is normal); see [Sum75, Defn. 3.4 and Rem. 3.5]. If S is normal and $T \to S$ is a smooth and Zariski-locally diagonalizable group scheme acting on X over S, then there exists a T-equivariant affine open neighborhood of any point of X [Sum75, Cor. 3.11]. We provide the following generalization of this result which simultaneously generalizes [AHR20, Thm. 4.4] to the relative case.

Theorem 10.1. *Let S be a quasi-separated algebraic space. Let G be an affine and flat group scheme over S of finite presentation. Let X be a quasi-separated algebraic space locally of finite presentation over S with an action of G. Let $x \in X$ be a point with image $s \in S$ such that $\kappa(x)/\kappa(s)$ is finite. Assume that x has linearly reductive stabilizer. Then there exists a G-equivariant étale neighborhood $(\operatorname{Spec} A, w) \to (X, x)$ that induces an isomorphism of residue fields and stabilizer groups at w.*

Proof. By applying Theorem 1.1 to $\mathcal{X} = [X/G]$ with $\mathcal{W}_0 = \mathcal{G}_x$ (the residual gerbe of x), we obtain an étale morphism $h\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ with $\mathcal{W}$ fundamental and $h|_{\mathcal{G}_x}$ an isomorphism. By applying Proposition 5.3(1) to the composition $\mathcal{W} \to \mathcal{X} \to BG$, we may shrink $\mathcal{W}$ around w so that $\mathcal{W} \to BG$ is affine. It follows that $W := \mathcal{W} \times_{\mathcal{X}} X$ is affine and $W \to X$ is G-equivariant. If we let $w' \in W$ be the unique preimage of x, then $(W, w') \to (X, x)$ is the desired étale neighborhood. □

Corollary 10.2. *Let S be a quasi-separated algebraic space, $T \to S$ be a group scheme of multiplicative type over S (e.g., a torus), and X be a quasi-separated algebraic space locally of finite presentation over S with an action of T. Let $x \in X$ be a point with image $s \in S$ such that $\kappa(x)/\kappa(s)$ is finite. Then there exists a T-equivariant étale morphism $(\operatorname{Spec} A, w) \to (X, x)$ that induces an isomorphism of residue fields and stabilizer groups at w.*

Proof. This follows immediately from Theorem 10.1 as any subgroup of a fiber of $T \to S$ is linearly reductive. □

Remark 10.3. In [Bri15], Brion establishes several powerful structure results for actions of connected algebraic groups on varieties. In particular, [Bri15, Thm. 4.8] recovers the result above when S is the spectrum of a field, T is a torus and X is quasi-projective without the final conclusion regarding residue fields and stabilizer groups.

10.2. **Relative version of Luna's étale slice theorem.** We provide the following generalization of Luna's étale slice theorem [Lun73] (see also [AHR20, Thm. 4.5]) to the relative case.

Theorem 10.4. *Let S be a quasi-separated algebraic space. Let $G \to S$ be a smooth, affine group scheme. Let X be a quasi-separated algebraic space locally of finite presentation over S with an action of G. Let $x \in X$ be a point with image $s \in S$ such that $\kappa(x)/\kappa(s)$ is a finite separable extension. Assume that x has linearly reductive stabilizer G_x. Then there exist*

(1) *an étale morphism* $(S',s') \to (S,s)$ *and a* $\kappa(s)$*-isomorphism* $\kappa(s') \cong \kappa(x)$*;*
(2) *a geometrically reductive (linearly reductive if* $\operatorname{char}\kappa(s) > 0$ *or* s *has an open neighborhood of characteristic zero) closed subgroup* $H \subseteq G' := G \times_S S'$ *over* S' *such that* $H_{s'} \cong G_x$*; and*
(3) *an unramified* H*-equivariant* S'*-morphism* $(W,w) \to (X',x')$ *of finite presentation with* W *affine and* $\kappa(w) \cong \kappa(x')$ *such that* $W \times^H G' \to X'$ *is étale. Here* $x' \in X' := X \times_S S'$ *is the unique* $\kappa(x)$*-point over* $x \in X$ *and* $s' \in S'$*.*

Moreover, it can be arranged that

(4) *if* $X \to S$ *is smooth at* x*, then* $W \to S'$ *is smooth and there exists an* H*-equivariant section* $\sigma\colon S' \to W$ *such that* $\sigma(s') = w$*, and there exists a strongly étale* H*-equivariant morphism* $W \to \mathbb{V}(\mathcal{N}_\sigma)$*;*
(5) *if* X *admits an adequate GIT quotient by* G *(e.g.,* X *is affine over* S *and* G *is geometrically reductive over* S*), and* Gx *is closed in* X_s*, then* $W \times^H G' \to X'$ *is strongly étale; and*
(6) *if, étale-locally on* S*, either*
   (a) BG *has the resolution property and* $X \to S$ *is affine;*
   (b) BG *has the resolution property,* $G \to S$ *has connected fibers,* S *is a normal noetherian scheme,* X *is a scheme, and* $X \to S$ *is flat of finite type with geometrically normal fibers; or*
   (c) *there exists a* G*-equivariant locally closed immersion* $X \hookrightarrow \mathbb{P}(V)$ *where* V *is a locally free* $\mathcal{O}_S$*-module of finite rank with a* G*-action;*

   then $W \to X'$ *is a locally closed immersion.*

In the statement above, $W \times^H G'$ denotes the quotient $(W \times G')/H$ which inherits a natural action of G', and $\mathcal{N}_\sigma$ is the conormal bundle $\mathcal{I}/\mathcal{I}^2$ (where $\mathcal{I}$ is the sheaf of ideals in W defining σ) which inherits an action of H. If $H \to S$ is a flat and affine group scheme of finite presentation over an algebraic space S, and X and Y are algebraic spaces over S with an action of H which admit adequate GIT quotients (i.e. $[X/H]$ and $[Y/H]$ admit adequate moduli spaces), then an H-equivariant morphism $f\colon X \to Y$ is called *strongly étale* if $[X/H] \to [Y/H]$ is.

The section $\sigma\colon S' \to W$ of (4) induces an H-equivariant section $\widetilde{\sigma}\colon S' \to X'$. This factors as $S' \to G'/H \to W \times^H G' \to X'$. Since the last map is étale, we have that $L_{(G'/H)/X'} = \mathcal{N}_\sigma[1]$. The map $G'/H \to X'$ is unramified and its image is the orbit of $\widetilde{\sigma}$. We can thus think of $\mathcal{N}_\sigma$ as the *conormal bundle for the orbit of* $\widetilde{\sigma}$. We also have an exact sequence:

$$0 \to \mathcal{N}_\sigma \to \mathcal{N}_{\widetilde{\sigma}} \to \mathcal{N}_e \to 0$$

where $e\colon S' \to G'/H$ is the unit section.

Remark 10.5. A considerably weaker variant of this theorem had been established in [Alp10, Thm. 2], which assumed the existence of a section $\sigma\colon S \to X$ such that $X \to S$ is smooth along σ, the stabilizer group scheme G_σ of σ is smooth, and the induced map $G/G_\sigma \to X$ is a closed immersion.

Proof of Theorem 10.4. We start by picking an étale morphism $(S',s') \to (S,s)$ realizing (1) with S' affine. After replacing S' with an étale neighborhood, Proposition 9.4 yields a geometrically reductive closed subgroup scheme $H \subseteq G'$ such that $H_{s'} \cong G_x$. This can be made linearly reductive if $\operatorname{char}\kappa(s) > 0$ or s has an open neighborhood of characteristic zero (Proposition 7.20). This settles (2).

We apply the main theorem (Theorem 1.1) to $([X'/G'],x')$ and $h_0\colon \mathcal{W}_0 = BG_x \cong \mathcal{G}_{x'}$ where x' also denotes the image of x' in $[X'/G']$. This gives us a fundamental stack $\mathcal{W}$ and an étale morphism $h\colon (\mathcal{W},w) \to ([X'/G'],x')$ such that $\mathcal{G}_w = BG_x$.

Since $G \to S$ is smooth, so is $G'/H \to S'$ and $[X'/H] \to [X'/G']$. The point $x' \in X'$ gives a canonical lift of $\mathcal{G}_w = BG_x \to [X'/G']$ to $\mathcal{G}_w = BG_x \to [X'/H]$. After replacing S' with an étale neighborhood, we can thus lift h to a map $q\colon (\mathcal{W}, w) \to ([X'/H], x')$ (Proposition 7.18). This map is unramified since h is étale and $[X'/H] \to [X'/G']$ is representable. After replacing $\mathcal{W}$ with an open neighborhood, we can also assume that $\mathcal{W} \to [X'/H] \to BH$ is affine by Proposition 5.3(1). Thus $\mathcal{W} = [W/H]$ where W is affine and q corresponds to an H-equivariant unramified map $W \to X'$. Note that since $w \in |\mathcal{W}|$ has stabilizer $H_{s'}$, there is a unique point $w \in |W|$ above $w \in |\mathcal{W}|$. This establishes (1)–(3).

If $X \to S$ is smooth at x, then $\mathcal{W} \to S'$ is smooth at w and (4) follows from Proposition 8.3 applied to $\mathcal{W} \to BH \to S'$. Note that unless H is smooth it is a priori not clear that $W \to S'$ is smooth. But the section $\sigma\colon S' \to W$ is a regular closed immersion since it is a pull-back of the regular closed immersion $BH \hookrightarrow \mathcal{W}$ given by Proposition 8.3. It follows that W is smooth in a neighborhood of σ.

If $[X/G]$ has an adequate moduli space, then $\mathcal{W} \to [X/G]$ becomes strongly étale after replacing $\mathcal{W}$ with a saturated open neighborhood by Luna's fundamental lemma (Theorem 3.14). This establishes (5).

Finally, for (6) we will construct a new W, not relying on the main theorem via (3). By a limit argument we may assume that (S, s) is henselian local. In particular, H is linearly reductive. If (6b) holds, then there exists a G-quasi-projective G-invariant open neighborhood $U \subseteq X$ of x [Sum75, Thms. 3.9 and 2.5]. Thus, cases (6a) and (6b) both reduce to case (6c) after resolving a coherent sheaf by a vector bundle.

As H is linearly reductive, there exists an H-semi-invariant function $f \in V = \Gamma(\mathbb{P}(V), \mathcal{O}(1))$ not vanishing at x. Then $\mathbb{P}(V)_f$ is an H-invariant affine open neighborhood. Applying Proposition 8.3 to $[\mathbb{P}(V)_f/H] \to BH \to S$ gives an affine open H-invariant neighborhood $U \subseteq \mathbb{P}(V)_f$, a section $\widetilde{\sigma}\colon BH \to [U/H]$ and a strongly étale morphism $U \to \mathbb{V}(\mathcal{N}_{\widetilde{\sigma}})$. We now consider the composition $\sigma\colon BH \to [U/H] \to [\mathbb{P}(V)/G]$ which is unramified since σ_s is a closed immersion and S is local. This gives the exact sequence

$$0 \to \mathcal{N}_\sigma \to \mathcal{N}_{\widetilde{\sigma}} \to \Omega_{BH/BG} \to 0.$$

Since H is linearly reductive, this sequence splits. After choosing a splitting, we obtain an H-equivariant closed subscheme $\mathbb{V}(\mathcal{N}_\sigma) \hookrightarrow \mathbb{V}(\mathcal{N}_{\widetilde{\sigma}})$ and by pull back, an H-equivariant closed subscheme $W \hookrightarrow U$. By construction $[W/H] \to [U/H] \to [\mathbb{P}(V)/G]$ is étale at x. Finally, we replace W with an affine open H-saturated neighborhood of x in the quasi-affine scheme $W \cap X$. □

10.3. **Existence of henselizations.** Let $\mathcal{X}$ be an algebraic stack with affine stabilizers and let $x \in |\mathcal{X}|$ be a point with linearly reductive stabilizer. We have already seen that the completion $\widehat{\mathcal{X}}_x$ exists if $\mathcal{X}$ is noetherian (Corollary 5.2). In this section we will prove that there also is a henselization $\mathcal{X}_x^h$ when $\mathcal{X}$ is of finite presentation over an algebraic space S and $\kappa(x)/\kappa(s)$ is finite.

We say that an algebraic stack $\mathcal{G}$ is a *one-point gerbe* if $\mathcal{G}$ is noetherian and an fppf-gerbe over the spectrum of a field k, or, equivalently, if $\mathcal{G}$ is reduced, noetherian and $|\mathcal{G}|$ is a one-point space. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is called *pro-étale* if $\mathcal{X}$ is the inverse limit of a system of quasi-separated étale morphisms $\mathcal{X}_\lambda \to \mathcal{Y}$ such that $\mathcal{X}_\mu \to \mathcal{X}_\lambda$ is affine for all sufficiently large λ and all $\mu \geq \lambda$.

Let $\mathcal{X}$ be an algebraic stack and let $x \in |\mathcal{X}|$ be a point. Consider the inclusion $i\colon \mathcal{G}_x \hookrightarrow \mathcal{X}$ of the residual gerbe of x. Let $\nu\colon \mathcal{G} \to \mathcal{G}_x$ be a pro-étale morphism of one-point gerbes. The *henselization of* $\mathcal{X}$ *at* ν is by definition an initial object in the 2-category of 2-commutative diagrams

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{X}' \\ & \overset{\nu}{\searrow} & \downarrow{\scriptstyle f} \\ & & \mathcal{X} \end{array} \tag{10.1}$$

where f is pro-étale (but not necessarily representable even if ν is representable). If $\nu\colon \mathcal{G}_x \to \mathcal{G}_x$ is the identity, we say that $\mathcal{X}^h_x := \mathcal{X}^h_\nu$ is the *henselization at x*.

Proposition 10.6 (Henselizations for stacks with good moduli spaces)**.** *Let $\mathcal{X}$ be an algebraic stack with affine diagonal and good moduli space $\pi\colon \mathcal{X} \to X$ of finite presentation (e.g., $\mathcal{X}$ noetherian). If $x \in |\mathcal{X}|$ is a point such that $x \in |\mathcal{X}_{\pi(x)}|$ is closed, then the henselization $\mathcal{X}^h_x$ of $\mathcal{X}$ at x exists. Moreover*

(1) $\mathcal{X}^h_x = \mathcal{X} \times_X \operatorname{Spec} \mathcal{O}^h_{X,\pi(x)}$*;*
(2) $\mathcal{X}^h_x$ *is linearly fundamental; and*
(3) $(\mathcal{X}^h_x, \mathcal{G}_x)$ *is a henselian pair.*

Proof. Let $\mathcal{X}^h_x := \mathcal{X} \times_X \operatorname{Spec} \mathcal{O}^h_{X,\pi(x)}$ which has good moduli space $\operatorname{Spec} \mathcal{O}^h_{X,\pi(x)}$. The pair $(\mathcal{X}^h_x, \mathcal{G}_x)$ is henselian (Theorem 3.10) and linearly fundamental (Theorem 6.1). It thus satisfies the hypotheses of Setup 7.6(c). To see that it is the henselization, we note that Proposition 7.9 trivially extends to pro-étale morphisms $\mathcal{X}' \to \mathcal{X}$ and implies that a section $\mathcal{G}_x \to \mathcal{X}' \times_{\mathcal{X}} \mathcal{G}_x$ extends to a unique $\mathcal{X}$-morphism $\mathcal{X}^h_x \to \mathcal{X}'$. □

Remark 10.7. Recall that if $\mathcal{X}$ has merely separated diagonal, then it has affine diagonal (Theorem 6.1). If $\mathcal{X}$ does not have separated diagonal, it is still true that $(\mathcal{X} \times_X \operatorname{Spec} \mathcal{O}^h_{X,\pi(x)}, \mathcal{G}_x)$ is a henselian pair but it need not be the henselization. In Example 3.16 the pair $(\mathcal{Y}, B\mathbb{Z}/2\mathbb{Z})$ is henselian with non-separated diagonal and the henselization map $\mathcal{X} \to \mathcal{Y}$ is non-representable.

Theorem 10.8 (Existence of henselizations)**.** *Let S be a quasi-separated algebraic space. Let $\mathcal{X}$ be an algebraic stack, locally of finite presentation and quasi-separated over S, with affine stabilizers. Let $x \in |\mathcal{X}|$ be a point such that the residue field extension $\kappa(x)/\kappa(s)$ is finite and let $\nu\colon \mathcal{G} \to \mathcal{G}_x$ be a pro-étale morphism such that $\mathcal{G}$ is a one-point gerbe with linearly reductive stabilizer. Then the henselization $\mathcal{X}^h_\nu$ of $\mathcal{X}$ at ν exists. Moreover, $\mathcal{X}^h_\nu$ is a linearly fundamental algebraic stack and $(\mathcal{X}^h_\nu, \mathcal{G})$ is a henselian pair.*

Remark 10.9. If $x \in |\mathcal{X}|$ has linearly reductive stabilizer, the theorem above shows that the henselization $\mathcal{X}^h_x$ of $\mathcal{X}$ at x exists and moreover that $\mathcal{X}^h_x$ is linearly fundamental and $(\mathcal{X}^h_x, \mathcal{G}_x)$ is a henselian pair.

Proof of Theorem 10.8. By definition, we can factor ν as $\mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_x$ where $\nu_1 : \mathcal{G} \to \mathcal{G}_1$ is pro-étale and affine and $\mathcal{G}_1 \to \mathcal{G}_x$ is étale. We can also arrange so that $\mathcal{G}_1$ is a one-point gerbe and $\mathcal{G} \to \mathcal{G}_1$ is stabilizer-preserving. Then $\mathcal{G} = \mathcal{G}_1 \times_{k_1} \operatorname{Spec} k$ where k/k_1 is a separable algebraic field extension and $\mathcal{G}_1$ has linearly reductive stabilizer.

By Theorem 1.1 we can find a fundamental stack $\mathcal{W}$, a closed point $w \in |\mathcal{W}|$ and an étale morphism $(\mathcal{W}, w) \to (\mathcal{X}, x)$ such that $\mathcal{G}_w = \mathcal{G}_1$. Then $\mathcal{W}^h_w = \mathcal{W} \times_W \operatorname{Spec} \mathcal{O}^h_{W,\pi(w)}$, where $\pi\colon \mathcal{W} \to W$ is the adequate moduli space. Indeed, $\mathcal{W} \times_W \operatorname{Spec} \mathcal{O}^h_{W,\pi(w)}$ is linearly fundamental (Corollary 6.11) so Proposition 10.6 applies. Finally, we obtain $\mathcal{X}^h_\nu = \mathcal{W}^h_{\nu_1}$ by base changing $\mathcal{W}^h_w \to W^h_{\pi(w)}$ along a pro-étale morphism $W' \to W^h_{\pi(w)}$ extending k/k_1. □

10.4. **Étale-local equivalences.**

Theorem 10.10. *Let S be a quasi-separated algebraic space. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks, locally of finite presentation and quasi-separated over S, with affine stabilizers. Suppose $x \in |\mathcal{X}|$ and $y \in |\mathcal{Y}|$ are points with linearly reductive stabilizers above a point $s \in |S|$ such that $\kappa(x)/\kappa(s)$ and $\kappa(y)/\kappa(s)$ are finite. Then the following are equivalent:*

(1) *There exists an isomorphism $\mathcal{X}^h_x \to \mathcal{Y}^h_y$ of henselizations.*
(2) *There exists a diagram of étale pointed morphisms*

$$(\mathcal{X}, x) \xleftarrow{\ f\ } ([\operatorname{Spec} A/\mathrm{GL}_n], w) \xrightarrow{\ g\ } (\mathcal{Y}, y)$$

such that both f and g induce isomorphisms of residual gerbes at w.

If S is quasi-excellent, then the conditions above are also equivalent to:

(1′) *There exists an isomorphism $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{Y}}_y$ of completions.*

Proof. The implications (2) $\implies$ (1) and (2) $\implies$ (1′) are clear. For the converses, we may assume that S, $\mathcal{X}$ and $\mathcal{Y}$ are quasi-compact. For (1) $\implies$ (2), since $\mathcal{Y} \to S$ is locally of finite presentation, we obtain a factorization $\mathcal{X}^h_x \to \mathcal{W} \to \mathcal{X}$, where the second map is étale, such that $\mathcal{X}^h_x \cong \mathcal{Y}^h_y \to \mathcal{Y}$ factors via $\mathcal{W}$. The induced map $\mathcal{W} \to \mathcal{Y}$ is flat and unramified at the image w of x, hence étale at w. After passing to a further étale neighborhood, $\mathcal{W}$ is fundamental and (2) follows. For (1′) $\implies$ (2), the argument of [AHR20, Thm. 4.19] is valid if one applies Theorem 1.1 instead of [AHR20, Thm. 1.1]. □

10.5. **Algebraicity results.** Here we generalize the algebraicity results of [AHR20, §5.3] to the setting of mixed characteristic. We will do this using the formulation of Artin's criterion in [Hal17, Thm. A]. This requires us to prove that certain deformation and obstruction functors are coherent, in the sense of [Aus66] (cf. [Hal14, Har98, Hal17]), a definition that we briefly recall. Let X be an affine scheme; then an additive functor $F\colon \mathsf{QCoh}(X) \to \mathsf{Ab}$ is *coherent* if there exists a morphism of quasi-coherent $\mathcal{O}_X$-modules $\phi\colon M \to N$ such that

$$F(-) \simeq \operatorname{coker}(\operatorname{Hom}_{\mathcal{O}_X}(N, -) \to \operatorname{Hom}_{\mathcal{O}_X}(M, -)).$$

The following result generalizes [AHR20, Prop 5.14] to the setting of mixed characteristic.

Proposition 10.11. *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and affine good moduli space X. If $\mathcal{F} \in \mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ and $\mathcal{G} \in \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$, then the functor*

$$\operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_\mathcal{X}} \mathsf{L}\pi^*(-))\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(X)$$

is coherent.

Proof. The proof is identical to [AHR20, Prop. 5.14]: by Proposition 6.15, $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ is compactly generated. Also, the restriction of $\mathsf{R}(f_{\mathrm{qc}})_*\colon \mathsf{D}_{\mathrm{qc}}(\mathcal{X}) \to \mathsf{D}_{\mathrm{qc}}(X)$ to $\mathsf{D}^+_{\mathsf{Coh}}(\mathcal{X})$ factors through $\mathsf{D}^+_{\mathsf{Coh}}(X)$ [Alp13, Thm. 4.16(x)]. By [HR17, Cor. 4.19], the result follows. □

For this subsection, we will now assume that we are in the following situation:

Setup 10.12. Fix an algebraic space X and an algebraic stack $\mathcal{X}$ with affine diagonal over X, such that $\mathcal{X} \to X$ is a good moduli space of finite presentation. Assume that X is quasi-excellent or $\mathcal{X}$ satisfies one of the conditions (FC), (PC) or (N). Note that if $\mathcal{X}$ is noetherian, then $\mathcal{X} \to X$ is automatically of finite type [AHR20, Thm. A.1].

The following corollary is a mixed characteristic variant of [AHR20, Cor. 5.15].

Corollary 10.13 (Hom scheme)**.** *Let $\mathcal{X} \to X$ be a good moduli space as in Setup 10.12. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_{\mathcal{X}}$-module. If $\mathcal{G}$ is flat over X, then the X-presheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}/X}(\mathcal{F}, \mathcal{G})$, whose objects over $T \xrightarrow{\tau} X$ are homomorphisms $\tau_{\mathcal{X}}^*\mathcal{F} \to \tau_{\mathcal{X}}^*\mathcal{G}$ of $\mathcal{O}_{\mathcal{X}\times_X T}$-modules (where $\tau_{\mathcal{X}} \colon \mathcal{X} \times_X T \to \mathcal{X}$ is the projection), is representable by an affine X-scheme.*

Proof. The question is étale-local on X, so we may assume that X is an affine scheme. Since $\mathcal{X} \to X$ is of finite presentation, we can write $\mathcal{F}$ as a filtered colimit of quasi-coherent modules $\mathcal{F}_\lambda$ of finite presentation. Then $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}/X}(\mathcal{F}, \mathcal{G}) = \varprojlim_\lambda \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}/X}(\mathcal{F}_\lambda, \mathcal{G})$ so we may assume that $\mathcal{F}$ is of finite presentation. After a standard limit argument, using Corollary 7.5, we can assume that X is noetherian. The result now follows directly from the coherence (Proposition 10.11) and left-exactness of the functor $\mathrm{Hom}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}}} \pi^*(-)) \colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(X)$, cf. [AHR20, Cor. 5.15] or [Hal14, Thm. D]. □

Theorem 10.14 (Stacks of coherent sheaves)**.** *Let $\mathcal{X} \to X$ be a good moduli space as in Setup 10.12. The X-stack $\underline{\mathsf{Coh}}_{\mathcal{X}/X}$, whose objects over $T \to X$ are finitely presented quasi-coherent sheaves on $\mathcal{X} \times_X T$ flat over T, is an algebraic stack, locally of finite presentation over X, with affine diagonal over X.*

Proof. After approximating to the quasi-excellent situation using Corollary 7.5, the proof is identical to [AHR20, Thm. 5.7], which is a small modification of [Hal17, Thm. 8.1]: the formal GAGA statement of Corollary 1.7 implies that formally versal deformations are effective and Proposition 10.11 implies that the automorphism, deformation and obstruction functors are coherent. Therefore, Artin's criterion (as formulated in [Hal17, Thm. A]) is satisfied and the result follows. Corollary 10.13 implies that the diagonal is affine. □

Just as in [AHR20], the following corollaries follow immediately from Theorem 10.14 appealing to the observation that Corollary 10.13 implies that $\underline{\mathsf{Quot}}_{\mathcal{X}/X}(\mathcal{F}) \to \underline{\mathsf{Coh}}_{\mathcal{X}/X}$ is quasi-affine.

Corollary 10.15 (Quot schemes)**.** *Let $\mathcal{X} \to X$ be a good moduli space as in Setup 10.12. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module, then the X-sheaf $\underline{\mathsf{Quot}}_{\mathcal{X}/X}(\mathcal{F})$, whose objects over $T \xrightarrow{\tau} X$ are quotients $\tau_{\mathcal{X}}^*\mathcal{F} \to \mathcal{G}$ (where $\tau_X \colon \mathcal{X} \times_X T \to \mathcal{X}$ is the projection) such that $\mathcal{G}$ is a finitely presented $\mathcal{O}_{\mathcal{X}\times_X T}$-module that is flat over T, is a separated algebraic space over X. If $\mathcal{F}$ is finitely presented, then $\underline{\mathsf{Quot}}_{\mathcal{X}/X}(\mathcal{F})$ is locally of finite presentation over X.* □

Corollary 10.16 (Hilbert schemes)**.** *Let $\mathcal{X} \to X$ be a good moduli space as in Setup 10.12. The X-sheaf $\underline{\mathrm{Hilb}}_{\mathcal{X}/X}$, whose objects over $T \to X$ are closed substacks $\mathcal{Z} \subseteq \mathcal{X} \times_X T$ such that $\mathcal{Z}$ is flat and of finite presentation over T, is a separated algebraic space locally of finite presentation over X.* □

We now establish algebraicity of Hom stacks. Related results were established in [HLP23] under other hypotheses.

Theorem 10.17 (Hom stacks)**.** *Let $\mathcal{X} \to X$ be a good moduli space as in Setup 10.12. Let $\mathcal{Y}$ be an algebraic stack, quasi-separated and locally of finite presentation over X with affine stabilizers. If $\mathcal{X} \to X$ is flat, then the X-stack $\underline{\mathrm{Hom}}_X(\mathcal{X}, \mathcal{Y})$, whose objects are pairs consisting of a morphism $T \to X$ of algebraic spaces and a morphism $\mathcal{X} \times_X T \to \mathcal{Y}$ of algebraic stacks over X, is an algebraic stack, locally of finite presentation over X with quasi-separated diagonal. If $\mathcal{Y} \to X$ has affine (resp. quasi-affine, resp. separated) diagonal, then the same is true for $\underline{\mathrm{Hom}}_X(\mathcal{X}, \mathcal{Y}) \to X$.*

Proof. As before we may first assume that X is affine and then reduce to the situation where $\mathcal{Y}$ is quasi-compact. After approximating to the quasi-excellent case using Corollary 7.5, the proof becomes identical to the proof of [AHR20, Thm. 5.10], which is a variant of [HR19, Thm. 1.2]. □

Remark 10.18 (G-equivariant Hom stacks). Let $G \to S$ be a group scheme acting on algebraic spaces or stacks X and Y over S. Then G-equivariant morphisms $X \to Y$ are equivalent to morphisms of stacks $[X/G] \to [Y/G]$ over BG. It follows that the G-equivariant Hom-stack $\underline{\mathrm{Hom}}_S^G(X,Y)$ fits into a cartesian square

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_S^G(X,Y) & \longrightarrow & \underline{\mathrm{Hom}}_S([X/G],[Y/G]) \\ \downarrow & \square & \downarrow \\ S & \longrightarrow & \underline{\mathrm{Hom}}_S([X/G],BG). \end{array}$$

Thus, if G is linearly reductive and $X \to X/\!\!/G \cong S$ is a flat good quotient, then we obtain algebraicity results for $\underline{\mathrm{Hom}}_S^G(X,Y)$, cf. [AHR20, Cor. 5.11].

Appendix A. Counterexamples in mixed characteristic

We first recall the following conditions on an algebraic stack $\mathcal{W}$ introduced in Section 7.

(FC) There is only a finite number of different characteristics in $\mathcal{W}$.
(PC) Every closed point of $\mathcal{W}$ has positive characteristic.
(N) Every closed point of $\mathcal{W}$ has nice stabilizer.

We also introduce the following condition which is implied by (FC) or (PC).

($\mathbb{Q}_{\mathrm{open}}$) Every closed point of $\mathcal{W}$ that is of characteristic zero has a neighborhood of characteristic zero.

In this appendix we will give examples of schemes and linearly fundamental stacks in mixed characteristic with various bad behavior.

(1) A noetherian linearly fundamental stack $\mathcal{X}$ with good moduli space $\mathcal{X} \to X$ such that X does not satisfy condition ($\mathbb{Q}_{\mathrm{open}}$) and we cannot write $\mathcal{X} = [\mathrm{Spec}(B)/G]$ with G linearly reductive étale-locally on X or étale-locally on $\mathcal{X}$ (Appendix A.1). In particular, condition ($\mathbb{Q}_{\mathrm{open}}$) is necessary in Theorem 6.1 and the similar condition is necessary in Theorem 8.2(7).
(2) A non-noetherian linearly fundamental stack $\mathcal{X}$ that cannot be written as an inverse limit of noetherian linearly fundamental stacks (Appendices A.2 and A.3).
(3) A noetherian scheme satisfying ($\mathbb{Q}_{\mathrm{open}}$) but neither (FC) nor (PC) (Appendix A.4).

Such counterexamples must have infinitely many different characteristics and closed points of characteristic zero.

Throughout this appendix, we work over the base scheme $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{2}]$. Let SL_2 act on $\mathfrak{sl}_2$ by conjugation. Then $\mathcal{Y} = [\mathfrak{sl}_2/\mathrm{SL}_2]$ is a fundamental stack with adequate moduli space $\mathcal{Y} \to Y := \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 = \mathrm{Spec}\,\mathbb{Z}[\frac{1}{2}, t]$ given by the determinant. Indeed, this follows from Zariski's main theorem and the following description of the orbits over algebraically closed fields. For $t \neq 0$, there is a unique orbit with Jordan normal form

$$\begin{bmatrix} \sqrt{-t} & 0 \\ 0 & -\sqrt{-t} \end{bmatrix}$$

and stabilizer $\mathbb{G}_m$. For $t = 0$, there are two orbits, one closed and one open, with Jordan normal forms

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and stabilizers SL_2 and $\boldsymbol{\mu}_2 \times \mathbb{G}_a$ respectively. The nice locus is $Y_{\mathrm{nice}} = \{t \neq 0\}$. The linearly reductive locus is $\{t \neq 0\} \cup \mathbb{A}^1_{\mathbb{Q}}$.

A.1. **A noetherian example.** Let $A = \mathbb{Z}[\frac{1}{2}, t, \frac{1}{t+p} \ : \ p \in P] \subseteq \mathbb{Q}[t]$, where p ranges over the set P of all odd primes.

- A is a noetherian integral domain: the localization of $\mathbb{Z}[\frac{1}{2}, t]$ in the multiplicative submonoid generated by $(t + p)$ for all $p \in P$.
- $A/(t) = \mathbb{Q}$.

We let $X = \operatorname{Spec} A$, let $X \to Y$ be the natural map (a flat monomorphism) and let $\mathcal{X} = \mathcal{Y} \times_Y X$. Then $\mathcal{X}$ is linearly fundamental with good moduli space X.

The nice locus of X is $\{t \neq 0\}$ and the complement consists of a single closed point x of characteristic zero. Any neighborhood of this point contains points of positive characteristic. It is thus impossible to write $\mathcal{X} = [\operatorname{Spec} B/G]$, with a linearly reductive group G, after restricting to any étale neighborhood of $x \in X$, or more generally, after restricting to any étale neighborhood in $\mathcal{X}$ of the unique closed point above x.

A.2. **A non-noetherian example.** Let $A = \mathbb{Z}[\frac{1}{2}, t, \frac{t-1}{p} \ : \ p \in P] \subseteq \mathbb{Q}[t]$, where p ranges over the set P of all odd primes. Note that

- A is a non-noetherian integral domain,
- $A = \mathbb{Z}[\frac{1}{2}, t, (x_p)_{p \in P}]/(p x_p - t + 1)_{p \in P}$,
- $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[x_p]$ is regular, and thus noetherian, for every $p \in P$,
- $A \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[t]$,
- $A/(t) = \mathbb{Q}$,
- $A/(t-1) = \mathbb{Z}[\frac{1}{2}, (x_p)_{p \in P}]/(p x_p)_{p \in P}$ has infinitely many irreducible components: the spectrum is the union of $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ and $\mathbb{A}^1_{\mathbb{F}_p}$ for every $p \in P$, and
- $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ admits a section: $t = 1$, $x_p = 0$ for all $p \in P$.

We let $X = \operatorname{Spec} A$, let $X \to Y$ be the natural map and let $\mathcal{X} = \mathcal{Y} \times_Y X$. Then $\mathcal{X}$ is linearly fundamental with good moduli space X. Note that $\mathcal{X} \to X$ is of finite presentation as it is a pull-back of $\mathcal{Y} \to Y$.

Proposition A.1. *There does not exist any noetherian linearly fundamental stack $\mathcal{X}_\alpha$ with an affine morphism $\mathcal{X} \to \mathcal{X}_\alpha$.*

Proof. Suppose that such an $\mathcal{X}_\alpha$ exists. Then we may write $\mathcal{X} = \varprojlim \mathcal{X}_\lambda$ where the $\mathcal{X}_\lambda$ are affine and of finite presentation over $\mathcal{X}_\alpha$. Let $\mathcal{X}_\lambda \to X_\lambda$ denote the good moduli space which is of finite type [AHR20, Thm. A.1]. Thus, $\mathcal{X} \to \mathcal{X}_\lambda \times_{X_\lambda} X$ is affine and of finite presentation. For all sufficiently large λ we can thus find an affine finitely presented morphism $\mathcal{X}'_\lambda \to \mathcal{X}_\lambda$ such that $\mathcal{X} \to \mathcal{X}'_\lambda \times_{X_\lambda} X$ is an isomorphism. Since also $\mathcal{X} \to \mathcal{Y} \times_Y X$ is an isomorphism and $Y \to \operatorname{Spec} \mathbb{Z}$ is of finite presentation, it follows that there is an isomorphism $\mathcal{X}'_\lambda \to \mathcal{Y} \times_Y X_\lambda$ for all sufficiently large λ.

To prove the proposition, it is thus enough to show that there does not exist a factorization $X \to X_\lambda \to Y$ with X_λ noetherian and affine such that $\mathcal{Y} \times_Y X_\lambda$ is linearly fundamental. This follows from the following lemma. ☐

Lemma A.2. *Let Z be an integral affine scheme together with a morphism $f \colon Z \to Y = \operatorname{Spec} \mathbb{Z}[\frac{1}{2}, t]$ such that*

(1) $f_{\mathbb{Q}} \colon Z_{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}[t]$ *is an isomorphism;*

(2) $f^{-1}(0)$ *is of pure characteristic zero; and*
(3) $f^{-1}(1)$ *admits a section* s.

Then Z *is not noetherian.*

Proof. For $a \in \mathbb{Z}$ and $p \in P$, let a_p (resp. $a_{\mathbb{Q}}$) denote the point in Y corresponding to the prime ideal $(p, t-a)$ (resp. $(t-a)$). Similarly, let η_p (resp. η) denote the points corresponding to the prime ideals (p) (resp. 0). Let $W = \operatorname{Spec} \mathbb{Z}[\frac{1}{2}] \hookrightarrow Z$ be the image of the section s and let $1_p \in Z$ also denote the unique point of characteristic p on W.

Suppose that the local rings of Z are noetherian. We will prove that $f^{-1}(1)$ then has infinitely many irreducible components. Since $f^{-1}(1)$ is the union of the closed subschemes W and $W_p := f^{-1}(1_p)$, $p \in P$, it is enough to prove that W_p has (at least) dimension 1 for every p.

Note that $\mathcal{O}_{Z,1_p}$ is (at least) 2-dimensional since there is a chain $1_p \leq 1_{\mathbb{Q}} \leq \eta$ of length 2 (here we use (1) and (3)). By Krull's Hauptidealsatz, $\mathcal{O}_{Z,1_p}/(p)$ has (at least) dimension 1 (here we use that the local ring is noetherian). The complement of $\operatorname{Spec} \mathcal{O}_{W_p,1_p} \hookrightarrow \operatorname{Spec} \mathcal{O}_{Z,1_p}/(p)$ maps to η_p. It is thus enough to prove that $f^{-1}(\eta_p) = \emptyset$.

Consider the local ring $\mathcal{O}_{Y,0_p}$. This is a regular local ring of dimension 2. Since $f_{\mathbb{Q}}$ is an isomorphism, $Z \times_Y \operatorname{Spec} \mathcal{O}_{Y,\eta_p} \to \operatorname{Spec} \mathcal{O}_{Y,\eta_p}$ is a birational affine morphism to the spectrum of a DVR. Thus, either $f^{-1}(\eta_p) = \emptyset$ or $Z \times_Y \operatorname{Spec} \mathcal{O}_{Y,\eta_p} \to \operatorname{Spec} \mathcal{O}_{Y,\eta_p}$ is an isomorphism. In the latter case, $f^{-1}(\operatorname{Spec} \mathcal{O}_{Y,0_p}) = f^{-1}(\operatorname{Spec} \mathcal{O}_{Y,0_p} \smallsetminus 0_p) \cong \operatorname{Spec} \mathcal{O}_{Y,0_p} \smallsetminus 0_p$ (here we use (1) and (2)) which contradicts that f is affine. □

A.3. **A variant of the non-noetherian example.** Let $A = \mathbb{Z}[\frac{1}{2}, t, \frac{t-1}{p^a} : p \in P, a \geq 1] \subseteq \mathbb{Q}[t]$, where p ranges over the set P of all odd primes. Note that

- A is a non-noetherian integral domain,
- $A = \mathbb{Z}[\frac{1}{2}, t, (x_{p,a})_{p \in P, a \geq 1}]/(px_{p,1} - t + 1, px_{p,a+1} - x_{p,a})_{p \in P, a \geq 1}$,
- $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[(x_{p,a})_{a \geq 1}]/(px_{p,a+1} - x_{p,a})_{a \geq 1}$ is two-dimensional and integral but not noetherian, for every $p \in P$,
- $A \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[t]$,
- $A/(t) = \mathbb{Q}$,
- $A/(t-1) = \mathbb{Z}[\frac{1}{2}, (x_{p,a})_{p \in P, a \geq 1}]/(px_{p,1}, px_{p,a+1} - x_{p,a})_{p \in P, a \geq 1}$ is non-reduced with one irreducible component: the nil-radical is $(x_{p,a})_{p \in P, a \geq 1}$.
- $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ admits a section: $t = 1$, $x_{p,a} = 0$ for all $p \in P$, $a \geq 1$.

As in the previous subsection, the fiber product $\operatorname{Spec} A \times_Y \mathcal{Y}$ (where $\mathcal{Y} = [\mathfrak{sl}_2/\mathrm{SL}_2]$ and $Y = \mathfrak{sl}_2 /\!\!/ \mathrm{SL}_2 = \operatorname{Spec} \mathbb{Z}[\frac{1}{2}, t]$) provides an example of a linearly fundamental stack that cannot be written as an inverse limit of noetherian linearly fundamental stacks.

A.4. **Condition ($\mathbb{Q}_{\mathrm{open}}$).** We provide examples illustrating that condition ($\mathbb{Q}_{\mathrm{open}}$), introduced in the beginning of the appendix, is slightly weaker than conditions (FC) and (PC) even in the noetherian case. A non-connected example is given by $S = \operatorname{Spec}(\mathbb{Z} \times \mathbb{Q})$ which has infinitely many different characteristics and a closed point of characteristic zero. A connected counterexample is given by the push-out $S = \operatorname{Spec}\big(\mathbb{Z} \times_{\mathbb{F}_p} (\mathbb{Z}_{(p)}[x])\big)$ for any choice of prime number p. Note that the irreducible component $\operatorname{Spec}(\mathbb{Z}_{(p)}[x])$ has closed points of characteristic zero, e.g., the prime ideal $(px - 1)$. The push-out is noetherian by Eakin–Nagata's theorem.

For an irreducible noetherian scheme, condition ($\mathbb{Q}_{\mathrm{open}}$) implies (FC) or (PC). That is, an irreducible noetherian scheme with a dense open of equal characteristic zero, has only a finite number of characteristics. This follows from Krull's Hauptidealsatz. We also note that for a scheme of finite type over $\operatorname{Spec} \mathbb{Z}$, there are no closed points of characteristic zero so ($\mathbb{Q}_{\mathrm{open}}$) and (PC) hold trivially.

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