HILBERT AND CHOW SCHEMES OF POINTS, SYMMETRIC PRODUCTS AND DIVIDED POWERS

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Abstract. Let X be a quasi-projective S-scheme. We explain the relations between the Hilbert scheme of d points on X, the d^{th} symmetric product of X, the scheme of divided powers of X of degree d and the Chow variety of zero-cycles of degree d on X with respect to a given projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$. The last three schemes are shown to be universally homeomorphic with isomorphic residue fields and isomorphic in characteristic zero or outside the degeneracy loci. In arbitrary characteristic, the Chow variety coincides with the reduced scheme of divided powers for a sufficiently ample projective embedding.

Introduction

Let X be a quasi-projective S-scheme. The purpose of this article is to explain the relation between

a) The Hilbert scheme of points $\mathrm{Hilb}^d\big(X/S\big)$ parameterizing zero-dimensional subschemes of X of degree d.
b) The d^{th} symmetric product $\mathrm{Sym}^d(X/S)$.
c) The scheme of divided powers $\Gamma^d(X/S)$ of degree d.
d) The Chow scheme $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ parameterizing zero dimensional cycles of degree d on X with a given projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$.

If X/S is not quasi-projective then none of these objects need exist as schemes but the first three do exist in the category of algebraic spaces separated over S [Ryd07d, Ryd07b, Ryd07c]. The Chow scheme is usually by definition a reduced scheme, but in the case of zero cycles, we will in a natural way give the Chow scheme a possibly non-reduced structure.

There are canonical morphisms

$$\mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^k)\big)$$

where $k \geq 1$ and $X \hookrightarrow \mathbb{P}(\mathcal{E}^k)$ is the Veronese embedding. The last two of these are universal homeomorphisms with trivial residue field extensions and are isomorphisms if S is a $\mathbb{Q}$-scheme. If S is arbitrary and X/S is flat then the second morphism is an isomoprhism. For arbitrary X/S the third morphism is an isomorphism for sufficiently large k. Some aspects of the first

Date: 2006-07-27.
2000 *Mathematics Subject Classification.* 14C05, 14L30.
Key words and phrases. Divided powers, symmetric powers, Chow scheme, Hilbert scheme, Weighted projective spaces.

morphism, known as the "Hilbert-Chow"-morphism or the "Grothendieck-Deligne norm map", are also discussed. Finally, it is shown that all three morphisms are isomorphisms outside the degeneracy locus.

This is a rough draft version.

Contents

1. The algebra of divided powers and symmetric tensors

We begin this section by briefly recalling the definition of polynomial laws in §1.1, the algebra of divided powers $\Gamma_A(M)$ in §1.2 and the multiplicative structure of $\Gamma^d_A(B)$ in §1.4. The only original statement in these sections is Proposition (1.3.2) in which a sufficient and necessary condition for $\Gamma^d_A(M)$ to be generated by $\gamma^d(M)$ is given. This generalizes a result of Ferrand [Fer98, Lemme 2.3.1] where a sufficient condition is given. A condition very similar to the one in Proposition (1.3.2) will be used in Proposition (3.1.7). In §1.5 we recall some explicit degree bound on the generators of $\Gamma^d_A(A[x_1, x_2, \dots, x_r])$.

1.1. **Polynomial laws and symmetric tensors.** We recall the definition of a polynomial law [Rob63, Rob80].

Definition (1.1.1). Let M and N be A-modules. We denote by $\mathcal{F}_M$ the functor

$$\mathcal{F}_M : A\text{–}\mathbf{Alg} \to \mathbf{Sets}, \qquad A' \mapsto M \otimes_A A'$$

A *polynomial law* from M to N is a natural transformation $f : \mathcal{F}_M \to \mathcal{F}_N$. More concretely, a polynomial law is a set of *maps* $f_{A'} : M \otimes_A A' \to N \otimes_A A'$ for every A-algebra A' such that for any homomorphism of A-algebras $g : A' \to A''$ the diagram

$$\begin{array}{ccc} M \otimes_A A' & \xrightarrow{f_{A'}} & N \otimes_A A' \\ {\scriptstyle \mathrm{id}_M \otimes g}\downarrow & \circ & \downarrow{\scriptstyle \mathrm{id}_N \otimes g} \\ M \otimes_A A'' & \xrightarrow{f_{A''}} & N \otimes_A A'' \end{array}$$

commutes. The polynomial law f is *homogeneous of degree* d if for any A-algebra A', the corresponding map $f_{A'} : M \otimes_A A' \to N \otimes_A A'$ is such that $f_{A'}(ax) = a^d f_{A'}(x)$ for any $a \in A'$ and $x \in M \otimes_A A'$. If B and C are A-algebras then a polynomial law from B to C is *multiplicative* if for any A-algebra A', the corresponding map $f_{A'} : B \otimes_A A' \to C \otimes_A A'$ is such that $f_{A'}(xy) = f_{A'}(x) f_{A'}(y)$ for any $x, y \in B \otimes_A A'$.

Notation (1.1.2). Let A be a ring and M and N be A-modules (resp. A-algebras). We let $\mathrm{Pol}^d(M, N)$ (resp. $\mathrm{Pol}^d_{\mathrm{mult}}(M, N)$) denote the polynomial laws (resp. multiplicative polynomial laws) $M \to N$ which are homogeneous of degree d.

Notation (1.1.3). Let A be a ring and M an A-algebra. We denote the d^{th} tensor product of M over A by $\mathrm{T}^d_A(M)$. We have an action of the symmetric group $\mathfrak{S}_d$ on $\mathrm{T}^d_A(M)$ permuting the factors. The invariant ring of this action is the symmetric tensors and is denoted $\mathrm{TS}^d_A(M)$. By $\mathrm{T}_A(M)$ and $\mathrm{TS}_A(M)$ we denote the graded A-modules $\bigoplus_{d \geq 0} \mathrm{T}^d_A(M)$ and $\bigoplus_{d \geq 0} \mathrm{TS}^d_A(M)$ respectively.

(1.1.4) *Shuffle product* — When B is an A-algebra, then $\mathrm{TS}^d_A(B)$ has a natural A-algebra structure induced from the A-algebra structure of $\mathrm{T}^d_A(B)$. The multiplication on $\mathrm{TS}^d_A(B)$ will be written as juxtaposition. For any A-module M, we can equip $\mathrm{T}_A(M)$ and $\mathrm{TS}_A(M)$ with A-algebra structures.

The multiplication on $\mathrm{T}_A(M)$ is the ordinary tensor product and the multiplication on $\mathrm{TS}_A(M)$ is called the *shuffle product* and is denoted by $\times$. If $x \in \mathrm{TS}_A^d(M)$ and $y \in \mathrm{TS}_A^e(M)$ then

$$x \times y = \sum_{\sigma \in \mathfrak{S}_{d,e}} \sigma\,(x \otimes_A y)$$

where $\mathfrak{S}_{d,e}$ is the subset of $\mathfrak{S}_{d+e}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(d)$ and $\sigma(d+1) < \sigma(d+2) < \ldots \sigma(d+e)$.

1.2. **Divided powers.** This section is a quick review of the results needed from [Rob63]. A nice exposition can also be found in [Fer98].

(1.2.1) Let A be a ring and M an A-module. Then there exists a graded A-algebra, the algebra of divided powers, denoted $\Gamma_A(M) = \bigoplus_{d \geq 0} \Gamma_A^d(M)$ equipped with maps $\gamma^d : M \to \Gamma_A^d(M)$ such that, denoting the multiplication with $\times$ as in [Fer98], we have that for every $x, y \in M$, $a \in A$ and $d, e \in \mathbb{N}$

(1.2.1.1) $$\Gamma_A^0(M) = A, \quad \text{and} \quad \gamma^0(x) = 1$$

(1.2.1.2) $$\Gamma_A^1(M) = M, \quad \text{and} \quad \gamma^1(x) = x$$

(1.2.1.3) $$\gamma^d(ax) = a^d \gamma^d(x)$$

(1.2.1.4) $$\gamma^d(x+y) = \textstyle\sum_{d_1+d_2=d} \gamma^{d_1}(x) \times \gamma^{d_2}(y)$$

(1.2.1.5) $$\gamma^d(x) \times \gamma^e(x) = \binom{d+e}{d} \gamma^{d+e}(x)$$

Using (1.2.1.1) and (1.2.1.2) we will identify A with $\Gamma_A^0(M)$ and M with $\Gamma_A^1(M)$. If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a set of elements of M and $\nu \in \mathbb{N}^{(\mathcal{I})}$ then we let

$$\gamma^\nu(x) = \mathop{\times}_{\alpha \in \mathcal{I}} \gamma^{\nu_\alpha}(x_\alpha)$$

which is an element of $\Gamma_A^d(M)$ with $d = |\nu| = \sum_{\alpha \in \mathcal{I}} \nu_\alpha$.

(1.2.2) *Functoriality* — $\Gamma_A(\cdot)$ is a covariant functor from the category of A-modules to the category of graded A-algebras [Rob63, Ch. III §4, p. 251].

(1.2.3) *Base change* — If A' is an A-algebra then there is a natural isomorphism $\Gamma_A(M) \otimes_A A' \to \Gamma_{A'}(M \otimes_A A')$ mapping $\gamma^d(x) \otimes_A 1$ to $\gamma^d(x \otimes_A 1)$ [Rob63, Thm. III.3, p. 262].

(1.2.4) *Universal property* — The map $\mathrm{Hom}_A\big(\Gamma_A^d(M), N\big) \to \mathrm{Pol}^d(M, N)$ given by $f \to f \circ \gamma^d$ is an isomorphism [Rob63, Thm. IV.1, p. 266].

(1.2.5) *Basis* — If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a set of generators of M, then $\big(\gamma^\nu(x)\big)_{\nu \in \mathbb{N}^{(\mathcal{I})}}$ is a set of generators of $\Gamma_A(M)$. If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a basis of M then $\big(\gamma^\nu(x)\big)_{\nu \in \mathbb{N}^{(\mathcal{I})}}$ is a basis of $\Gamma_A(M)$ [Rob63, Thm. IV.2, p. 272].

(1.2.6) *Exactness* — The functor $\Gamma_A(\cdot)$ is a left adjoint [Rob63, Thm. III.1, p. 257] and thus commutes with any (small) direct limit. It is thus *right exact* [GV72, Def. 2.4.1] but note that $\Gamma_A(\cdot)$ is a functor from A–**Mod** to A–**Alg** and that the latter category is not abelian. By [GV72, Rem. 2.4.2] a functor is right exact if and only if it takes the initial object onto the initial object and commutes with finite coproducts and coequalizers. Thus $\Gamma_A(0) = A$ and given an exact diagram of A-modules

$$M' \underset{g}{\overset{f}{\rightrightarrows}} M \xrightarrow{h} M''$$

the diagram

$$\Gamma_A(M') \underset{\Gamma g}{\overset{\Gamma f}{\rightrightarrows}} \Gamma_A(M) \xrightarrow{\Gamma h} \Gamma_A(M'')$$

is exact in the category of A-algebras.

(1.2.7) *Presentation* — Let $M = G/R$ be a presentation of the A-module M. Then $\Gamma_A(M) = \Gamma_A(G)/I$ where I is the ideal of $\Gamma_A(G)$ generated by the images in $\Gamma_A(G)$ of $\gamma^d(x)$ for every $x \in R$ and $d \geq 1$ [Rob63, Prop. IV.8, p. 284]. In fact, denoting the inclusion of R in G by i, we can write M as a coequalizer of A-modules

$$R \underset{0}{\overset{i}{\rightrightarrows}} G \xrightarrow{h} M$$

which by (1.2.6) gives the exact sequence

$$\Gamma_A(R) \underset{\Gamma(0)}{\overset{\Gamma(i)}{\rightrightarrows}} \Gamma_A(G) \xrightarrow{\Gamma(h)} \Gamma_A(M)$$

of A-algebras. Since $\Gamma^0_A(0) = \Gamma^0_A(i) = \mathrm{id}_A$ and $\Gamma^d_A(0) = 0$ for $d > 0$ it follows that $\Gamma_A(M)$ is the quotient of $\Gamma_A(G)$ by the ideal generated by $\Gamma(i)\big(\bigoplus_{d\geq 1}\Gamma^d(R)\big)$.

(1.2.8) Γ *and* TS — The homogeneous polynomial law $M \to \mathrm{TS}^d_A(M)$ of degree d given by $x \mapsto x^{\otimes_A d} = x \otimes_A \cdots \otimes_A x$ corresponds by the universal property (1.2.4) to an A-module homomorphism $\Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$. This extends to an A-algebra homomorphism $\Gamma_A(M) \to \mathrm{TS}_A(M)$, where the multiplication in $\mathrm{TS}_A(M)$ is the shuffle product (1.1.4).

When M is a free A-module the homomorphisms $\Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$ and $\Gamma_A(M) \to \mathrm{TS}_A(M)$ are isomorphisms of A-modules respectively A-algebras [Rob63, Prop. IV.5, p. 272]. The functors Γ^d_A and TS^d_A commute with filtered direct limits [Ryd07c, 1.1.4, 1.2.11]. Since any flat A-module is the filtered direct limit of free A-modules [Laz69, Thm. 1.2], it thus follows that $\Gamma_A(M) \to \mathrm{TS}_A(M)$ is an isomorphism of graded A-algebras for any flat A-module M.

Moreover by [Rob63, Prop. III.3, p. 256], there is a diagram of A-modules

$$\begin{tikzcd} \mathrm{TS}^d_A(M) \arrow[r, hook] & \mathrm{T}^d_A(M) \arrow[d, two heads] \\ \Gamma^d_A(M) \arrow[u] & \mathrm{S}^d_A(M) \arrow[l] \end{tikzcd}$$

such that going around the square is multiplication by $d!$. Thus if $d!$ is invertible then $\Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$ is an isomorphism. In particular, this is the case when A is purely of characteristic zero, i.e. contains the field of rationals.

1.3. **When is $\Gamma^d_A(M)$ generated by $\gamma^d(M)$?** $\Gamma^d_A(M)$ is not always generated by $\gamma^d(M)$ but a result due to Ferrand [Fer98, Lemme 2.3.1], cf. Proposition (1.3.4), shows that there is a finite free base change $A \hookrightarrow A'$ such that $\Gamma^d_{A'}(M \otimes_A A')$ is generated by $\gamma^d(M \otimes_A A')$. We will prove a slightly stronger statement in Proposition (1.3.2).

We let $\bigl(\gamma^d(M)\bigr)$ denote the A-submodule of $\Gamma^d_A(M)$ generated by the subset $\gamma^d(M)$.

Lemma (1.3.1). *Let A be a ring and M an A-module. There is a commutative diagram*

$$\begin{tikzcd}
\bigl(\gamma^d_A(M)\bigr) \otimes_A A' \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circ"] & \Gamma^d_A(M) \otimes_A A' \arrow[d, "\psi"', "\cong"] \\
\bigl(\gamma^d_{A'}(M \otimes_A A')\bigr) \arrow[r, phantom, "\subseteq"] & \Gamma^d_{A'}(M \otimes_A A')
\end{tikzcd}$$

where ψ is the canonical isomorphism of (1.2.3). *If $A \to A'$ is a surjection or a localization then φ is surjective. In particular, if in addition $\bigl(\gamma^d_{A'}(M \otimes_A A')\bigr) = \Gamma^d_{A'}(M \otimes_A A')$ then $\bigl(\gamma^d_A(M)\bigr) \otimes_A A' \to \Gamma^d_A(M) \otimes_A A'$ is surjective.*

Proof. The morphism φ is well-defined as $\psi\bigl(\gamma^d(x) \otimes_A a'\bigr) = a'\gamma^d(x \otimes_A 1)$ if $x \in M$ and $a' \in A'$. If $A' = A/I$ then φ is clearly surjective. If $A' = S^{-1}A$ is a localization then φ is surjective since any element of $M \otimes_A A'$ can be written as $x \otimes_A (1/f)$ and $\varphi\bigl(\gamma^d(x) \otimes_A 1/f^d\bigr) = \gamma^d\bigl(x \otimes_A (1/f)\bigr)$. $\square$

Proposition (1.3.2). *Let M be an A-module. The A-module $\Gamma^d_A(M)$ is generated by the subset $\gamma^d(M)$ if the following condition is satisifed*

(*) *For every $\mathfrak{p} \in \operatorname{Spec}(A)$ the residue field $k(\mathfrak{p})$ has at least d elements or $M_{\mathfrak{p}}$ is generated by one element.*

If M is of finite type, then this condition is also necessary.

Proof. By Lemma (1.3.1) it follows that $\bigl(\gamma^d_A(M)\bigr) = \Gamma^d_A(M)$ if and only if $\bigl(\gamma^d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\bigr) = \Gamma^d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$. We can thus assume that A is a local ring and only need to consider the condition (*) for the maximal ideal $\mathfrak{m}$. If M is generated by one element then it is obvious that $\bigl(\gamma^d_A(M)\bigr) = \Gamma^d_A(M)$.

Further, any element in $\Gamma^d_A(M)$ is the image of an element in $\Gamma^d_A(M')$ for some submodule $M' \subseteq M$ of finite type. It is thus sufficient, but not necessary, that $\Gamma^d_A(M')$ is generated by $\gamma^d(M')$ for every submodule $M' \subseteq M$ of finite type. We can thus assume that M is of finite type. Lemma (1.3.1) applied with $A \twoheadrightarrow A/\mathfrak{m} = k(\mathfrak{m})$ together with Nakayama's lemma then shows that $\bigl(\gamma^d_A(M)\bigr) = \Gamma^d_A(M)$ if and only if $\bigl(\gamma^d_{A/\mathfrak{m}}(M/\mathfrak{m}M)\bigr) = \Gamma^d_{A/\mathfrak{m}}(M/\mathfrak{m}M)$. We can thus assume that $A = k$ is a field.

We will prove by induction on e that $\Gamma^e_k(M)$ is generated by $\gamma^e(M)$ when $0 \leq e \leq d$ if and only if either $\operatorname{rk} M \leq 1$ or $|k| \geq e$. Every element in $\Gamma^e_k(M)$

is a linear combination of elements of the form

$$\gamma^{\nu}(x) = \gamma^{\nu_1}(x_1) \times \gamma^{\nu_2}(x_2) \times \cdots \times \gamma^{\nu_m}(x_m).$$

where $x_i \in M$ and $|\nu| = e$. By induction $\gamma^{\nu_2}(x_2) \times \cdots \times \gamma^{\nu_m}(x_m) \in \big(\gamma^{e-\nu_1}(M)\big)$ and we can thus assume that $m = 2$ and it is enough to show that $\gamma^i(x) \times \gamma^{e-i}(y) \in \big(\gamma^e(M)\big)$ for every $x, y \in M$ and $0 \leq i \leq e$ if and only if either $\operatorname{rk} M \leq 1$ or $|k| \geq e$. If x and y are linearly dependent this is obvious. Thus we need to show that for x and y linearly independent $\gamma^i(x) \times \gamma^{e-i}(y) \in \big(\gamma^e(kx \oplus ky)\big)$ if and only if $|k| \geq e$. A basis for $\Gamma^e_k(kx \oplus ky)$ is given by $z_0, z_1, \dots, z_e$ where $z_i = \gamma^i(x) \times \gamma^{e-i}(y)$, see (1.2.5). For any $a, b \in k$ we let

$$\xi_{a,b} := \gamma^e(ax + by) = \sum_{i=0}^{e} \gamma^i(ax) \times \gamma^{e-i}(by) = \sum_{i=0}^{e} a^i b^{e-i} z_i.$$

Then $\big(\gamma^e_k(kx \oplus ky)\big) = \Gamma^e_k(kx \oplus ky)$ if and only if $\sum_{(a,b)\in k^2} k\xi_{a,b} = \bigoplus_{i=0}^{e} kz_i$. Since $\xi_{\lambda a, \lambda b} = \lambda^e \xi_{a,b}$ this is equivalent to $\sum_{(a:b)\in \mathbb{P}^1_k} k\xi_{a,b} = \bigoplus_{i=0}^{e} kz_i$. It is thus necessary that $\left|\mathbb{P}^1_k\right| = k+1 \geq e+1$. On the other hand if $a_1, a_2, \dots, a_e \in k$ are distinct then $\xi_{a_1,1}, \xi_{a_2,1}, \dots, \xi_{a_e,1}, \xi_{1,0}$ are linearly independent. In fact, this amounts to $(1, a_i, a_i^2, a_i^3, \dots, a_i^e)_{i=1,2,\dots,e}$ and $(0, 0, \dots, 0, 1)$ being linearly independent in k^{e+1}. If they are dependent then there exist a non-zero $(c_0, c_1, \dots, c_{e-1}) \in k^e$ such that $c_0 + c_1 a_i + c_2 a_i^2 + \cdots + c_{e-1} a_i^{e-1} = 0$ for every $1 \leq i \leq e$ but this is impossible since $c_0 + c_1 x + \cdots + c_{e-1} x^{e-1} = 0$ has at most $e - 1$ solutions. □

Lemma (1.3.3). *Let $\Lambda_d = \mathbb{Z}[T]/P_d(T)$ where $P_d(T)$ is the unitary polynomial $\prod_{0\leq i<j\leq d}(T^i - T^j) - 1$. Then every residue field of Λ_d has at least $d+1$ elements. In particular, if A is any algebra, then $A \hookrightarrow A' = A \otimes_{\mathbb{Z}} \Lambda_d$ is a faithfully flat finite extension such that every residue field of A' has at least $d + 1$ elements.*

Proof. The Vandermonde matrix $(T^{ij})_{0\leq i,j\leq d}$ is invertible in $\operatorname{End}_{\Lambda_d}(\Lambda_d^{d+1})$ since it has determinant one. Let k be a field and $\varphi : \Lambda_d \to k$ be any homomorphism. If $t = \varphi(T)$ then $(t^{ij})_{0\leq i,j\leq d}$ is invertible in $\operatorname{End}_k(k^{d+1})$ and it follows that $1, t, t^2, \dots, t^d$ are all distinct and hence that k has at least $d + 1$ elements. □

Proposition (1.3.4). [Fer98, Lemme 2.3.1] *Let Λ_d be as in Lemma* (1.3.3). *If A is a Λ_d-algebra then $\Gamma^d_A(M)$ is generated by $\gamma^d(M)$. In particular, for every A there is a finite faithfully flat extension $A \to A'$, independent of M, such that $\Gamma^d_{A'}(M')$ is generated by $\gamma^d(M')$.*

Proof. Follows immediately from Proposition (1.3.2) and Lemma (1.3.3). □

1.4. **Multiplicative structure.** When B is an A-algebra then the multiplication of B induces a multiplication on $\Gamma^d_A(B)$ which we will denote by juxtaposition [Rob80]. This multiplication is such that $\gamma^d(x)\gamma^d(y) = \gamma^d(xy)$.

(1.4.1) *Universal property* — Let B and C be A-algebras. Then the map $\operatorname{Hom}_{A\text{-}\mathbf{Alg}}\big(\Gamma^d_A(B), C\big) \to \operatorname{Pol}^d_{\text{mult}}(B, C)$ given by $f \to f \circ \gamma^d$ is an isomorphism [Rob80].

(1.4.2) Γ *and* TS — The homogeneous polynomial law $M \to \mathrm{TS}^d_A(M)$ of degree d given by $x \mapsto x^{\otimes_A d} = x \otimes_A \cdots \otimes_A x$ is multiplicative. The homomorphism $\varphi : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ in (1.2.8) is thus an A-algebra homomorphism. It is an isomorphism when B is a flat over A or when A is purely of characteristic zero (1.2.8). Section §3.3. is devoted to a study of φ in the general case.

1.5. **Generators of the ring of divided powers.** In this section we will recall some results of the degree of the generators of $\Gamma^d_A(B)$. For our purposes the results of Fleischmann [Fle98] is sufficient and we will not use the more precise and stronger statements of [Ryd07a] even though some bounds then can be slightly improved.

Definition (1.5.1) (Multidegree)**.** Let $B = A[x_1, x_2, \dots, x_r]$. We define the *multidegree* of a monomial $x^\alpha \in B$ to be α. This makes B into a $\mathbb{N}^r$-graded ring
$$B = \bigoplus_{\alpha \in \mathbb{N}^r} B_\alpha = \bigoplus_{\alpha \in \mathbb{N}^r} A x^\alpha$$
Let $\mathcal{M}$ be the A-module basis of B consisting of the monomials. Recall from paragraph (1.2.5) that a basis of $\Gamma_A(B)$ is given by the elements $\gamma^\nu(x) = \times_\alpha \gamma^{\nu_\alpha}(x^\alpha)$ for $\nu \in \mathbb{N}^{(\mathcal{M})}$. We let $\operatorname{mdeg}\bigl(\gamma^k(x^\alpha)\bigr) = k\alpha$ and $\operatorname{mdeg}(f \times g) = \operatorname{mdeg}(f) + \operatorname{mdeg}(g)$ for $f, g \in \Gamma_A(B)$. Then
$$\operatorname{mdeg}\bigl(\mathop{\times}_\alpha \gamma^{\nu_\alpha}(x^\alpha)\bigr) = \sum_{x^\alpha \in \mathcal{M}} \nu_\alpha \operatorname{mdeg}(x^\alpha) = \sum_{\alpha \in \mathbb{N}^r} \nu_\alpha \alpha.$$
We let $\Gamma^d_A(B)_\alpha$ be the A-module generated by basis elements $\gamma^\nu(x)$ of multidegree α. This makes $\Gamma^d_A(B) = \bigoplus_{\alpha \in \mathbb{N}^r} \Gamma^d_A(B)_\alpha$ into a $\mathbb{N}^r$-graded ring.

Definition (1.5.2) (Degree)**.** Let $B = A[x_1, x_2, \dots, x_r] = \bigoplus_{k \geq 0} B_k$ with the usual grading, i.e. B_k are the homogeneous polynomials of degree k. The graded A-algebra $C = \bigoplus_{k \geq 0} \Gamma^d_A(B_k)$ is a subalgebra of $\Gamma^d_A(B)$. If an element $f \in \Gamma^d_A(B)$ belongs to $C_k = \Gamma^d_A(B_k)$ we say that f is *homogeneous of degree* k. The *degree* of an arbitrary element $f \in \Gamma^d_A(B)$ is the smallest natural number n such that $f \in \Gamma^d_A\left(\bigoplus_{k=0}^n B_k\right)$.

Remark **(1.5.3).** Let $B = A[x_0, x_1, \dots, x_r]$ and let $C = \bigoplus_{k \geq 0} \Gamma^d_A(B_k)$ be the graded subring of $\Gamma^d_A(B)$. The degree in the previous definition is such that there is a relation between the degree of elements in C and the degree of an element in the graded localization $C_{(\gamma^d(s))}$ for $s \in B_1$. To see this, note that
$$C_{\left(\gamma^d(s)\right)} = \Gamma^d_A\left(B_{(s)}\right) = \Gamma^d_A\bigl(A[x_0/s, \dots, x_r/s]\bigr).$$
We let $A[x_0/s, \dots, x_r/s]$ be graded such that x_i/s has degree 1. An element $f \in \Gamma^d_A(A[x_0/s, \dots, x_r/s])$ of degree n can then be written as $g/\gamma^d(s)^n$ where $g \in \Gamma^d_A(B_n)$ is homogeneous of degree n.

Theorem (1.5.4) ([Ric96, Prop. 2], [Ryd07a, Cor. 6.26])**.** *If $d!$ is invertible in A then $\Gamma^d_A(A[x_1, \dots, x_r])$ is generated by the elementary multisymmetric functions $\gamma^{d_1}(x_1) \times \gamma^{d_2}(x_2) \times \cdots \times \gamma^{d_r}(x_r) \times \gamma^{d - d_1 - d_2 - \cdots - d_r}(1)$, $d_i \in \mathbb{N}$ and $d_1 + d_2 + \cdots + d_r \leq d$.*

Theorem (1.5.5) ([Fle98, Thm. 4.6, 4.7], [Ryd07a, Cor. 6.28])**.** *Let A be an arbitrary ring. Then $\Gamma^d_A(A[x_1,\dots,x_r])$ is generated as an A-algebra by $\gamma^d(x_1),\gamma^d(x_2),\dots,\gamma^d(x_r)$ and the elements $\gamma^k(x^\alpha)\times\gamma^{d-k}(1)$ with $k\alpha\leq(d-1,d-1,\dots,d-1)$. Further, there is no smaller multidegree bound and if $d=p^s$ for some prime p not invertible in A, then $\Gamma^d_A(A[x_1,\dots,x_r])$ is not generated by elements of strictly smaller multidegree.*

Theorems (1.5.4) and (1.5.5) give the following degree bound:

Corollary (1.5.6). *Let A be a ring and $B=k[x_1,x_2,\dots,x_r]$. Then $\Gamma^d_A(B)$ is generated by elements of degree at most $\max\bigl(1,r(d-1)\bigr)$. If $d!$ is invertible in A, then $\Gamma^d_A(B)$ is generated by elements of degree one.*

2. Weighted projective schemes and quotients by finite groups

2.1. **Remarks on projectivity.** We will follow the definitions in EGA. In particular, very ample, ample, quasi-projective and projective will have the meanings of [$\mathrm{EGA}_{\mathrm{II}}$, §4.4, §4.6, §5.3, §5.5]. By definition, a morphism $q : X \to S$ is quasi-projective if it is of finite type and there exists an invertible $\mathcal{O}_X$-sheaf $\mathcal{L}$ ample with respect to q. Note that this does *not* imply that X is a subscheme of $\mathbb{P}_S(\mathcal{E})$ for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$. However, if S is quasi-compact and quasi-separated then there is a quasi-coherent $\mathcal{O}_S$-module of finite type $\mathcal{E}$ and an immersion $X \hookrightarrow \mathbb{P}(\mathcal{E})$ [$\mathrm{EGA}_{\mathrm{II}}$, Prop. 5.3.2]. Similarly, a projective morphism is always quasi-projective and proper but the converse only holds if S is quasi-compact and quasi-separated.

Furthermore, if $q : X \to S$ is a projective morphism and $\mathcal{L}$ a very ample invertible sheaf on X then $\mathcal{L}$ does not necessarily correspond to a closed embedding into a projective space over S. We always have a closed embedding $X \hookrightarrow \mathbb{P}(q_*\mathcal{L})$ as q is proper [$\mathrm{EGA}_{\mathrm{II}}$, Prop. 4.4.4] but $q_*\mathcal{L}$ need not be of finite type. If S is *locally noetherian* however, then $q_*\mathcal{L}$ is of finite type [$\mathrm{EGA}_{\mathrm{III}}$, Thm. 3.2.1]. If S is quasi-compact and quasi-separated then we can find a sub-$\mathcal{O}_S$-module of finite type $\mathcal{E}$ of $q_*\mathcal{L}$ such that we have a closed immersion $i : X \hookrightarrow \mathbb{P}(\mathcal{E})$ and such that $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

We will also need the following stronger notion of projectivity introduced in [AK80, §2]:

Definition (2.1.1). A morphism $X \to S$ is *strongly projective* (resp. *strongly quasi-projective*) if it is of finite type[1] and factors through a closed immersion (resp. an immersion) $X \hookrightarrow \mathbb{P}_S(\mathcal{L})$ where $\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of constant rank.

Remark **(2.1.2).** A strongly (quasi-)projective morphism is (quasi-)projective and the converse holds when S is quasi-compact, quasi-separated and admits an ample sheaf, e.g. S affine [AK80, Ex. 2.2 (i)]. In fact, in this case there is an embedding $X \hookrightarrow \mathbb{P}^n_S$ and thus the notions of projective and strongly projective also agrees with the definition in [Har77].

2.2. **Weighted projective schemes.**

Definition (2.2.1). Let S be a scheme. A *weighted projective scheme* over S is an S-scheme X together with a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ of *finite type*, not necessarily generated by degree one elements, such that $X = \mathrm{Proj}_S(\mathcal{A})$. We let as usual $\mathcal{O}_X(n) = \widetilde{\mathcal{A}(n)}$ for any $n \in \mathbb{Z}$.

If $\mathcal{A}$ is generated by degree one elements then $\mathcal{O}_X(n)$ are invertible for any integer n and very ample if n is positive. Further $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n)$. All these properties may be false if $\mathcal{A}$ is not generated by degree one elements.

It can however be shown, cf. Corollary (2.2.5), that if S is quasi-compact then $q : X \to S$ is projective. To be precise, there is a positive integer n such that $\mathcal{O}_X(n)$ is invertible, the homomorphism $q^*\mathcal{A}_n \to \mathcal{O}_X(n)$

[1] Altman and Kleiman requires $X \to S$ to be of finite presentation, but for the $\Gamma^d(X/S)$-scheme we will not need this.

is surjective and $i_n : X \to \mathbb{P}(\mathcal{A}_n)$ is a closed immersion. In particular, $\mathcal{O}_X(n) = i_n^* \mathcal{O}_{\mathbb{P}(\mathcal{A}_n)}(1)$ is very ample. Another consequence is that if X is a weighted projective scheme over an arbitrary scheme S then $X \to S$ is proper.

We will give a demonstration of the projectivity of $X \to S$ when S is quasi-compact and also show some properties of the sheaves $\mathcal{O}_X(n)$. The results will be somewhat weaker than those in [BR86, §4] but we will also give stronger results in a particular case that will be important in the other sections.

The following lemma is an explicit form of [EGA$_{\text{II}}$, Lemma 2.1.6].

Lemma (2.2.2). *If B is a graded A-algebra generated by elements $f_1, f_2, \dots, f_s \in B$ of degrees $d_1, d_2, \dots, d_s$ and l is the least common multiple of $d_1, d_2, \dots, d_s$ then*

(i) $B_{n+l} = (B_n B_l)$ *for every* $n \geq (s-1)(l-1)$.
(ii) $B_{kn} = (B_n)^k$ *for every* $k \geq 0$ *if* $n = al$ *with* $a \geq s-1$.

Proof. Clearly B_k is generated by $f_1^{a_1} f_2^{a_2} \dots f_s^{a_s}$ such that $\sum_i a_i d_i = k$. Let $g_i = f_i^{l/d_i} \in B_l$. If $k \geq s(l-1)+1$ and $f = f_1^{a_1} f_2^{a_2} \dots f_s^{a_s} \in B_k$ then $g_i | f$ for some i which shows (i). (ii) follows easily from (i). □

Remark **(2.2.3).** In the terminology of [BR86, §4B] (i) of Lemma (2.2.2) says that $D\big((s-1)(l-1)\big)$ holds and (ii) is related to that $D\big((s-1)l\big)$ holds. Hence $F < (s-1)(l-1)$ and $E < s-1$. Using [BR86, Lemma 4B.4] it is easily seen that the bound $F < G$ given in [BR86, Prop 4B.5] is stronger than $F < (s-1)(l-1)$.

Proposition (2.2.4) (cf. [BR86, Cor 4A.5, Thm 4B.7])**.** *Let A be a ring and let B be a graded A-algebra generated by a finite number of elements $f_1, f_2, \dots, f_s$ of degrees $d_1, d_2, \dots, d_s$. Let l be the least common multiple of the d_i:s. Let $S = \operatorname{Spec}(A)$, $X = \operatorname{Proj}(B)$ and $\mathcal{O}_X(n) = \widetilde{B(n)}$. Then*

(i) $X = \bigcup_{f \in B_n} \mathrm{D}_+(f)$ *if* $n = al$ *and* $a \geq 1$.
(ii) $\mathcal{O}_X(n)$ *is invertible if* $n = al$ *and* $a \in \mathbb{Z}$.
(iii) $\mathcal{O}_X(n)$ *is ample and generated by global sections if* $n = al$ *and* $a \geq 1$.
(iv) *The canonical homomorphism* $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \to \mathcal{O}_X(m+n)$ *is an isomorphism if* $m = al$ *and* $a, n \in \mathbb{Z}$.
(v) *If* $n = al$ *with* $a \geq 1$ *then there is a canonical morphism* $i_n : X \to \mathbb{P}(B_n)$. *If* $a \geq \max\{1, s-1\}$ *then* i_n *is a closed immersion and* $\mathcal{O}_X(n) = i_n^* \mathcal{O}_{\mathbb{P}(B_n)}(1)$ *is very ample relative to* S.
(vi) $\mathcal{O}_X(n)$ *is generated by global sections if* $n \geq (s-1)(l-1)$.

Proof. (i) is trivial as $X = \bigcup_{i=1}^s \mathrm{D}_+(f_i) = \bigcup_{i=1}^s \mathrm{D}_+\big(f_i^{al/d_i}\big)$ if $a \geq 1$, cf [EGA$_{\text{II}}$, Cor 2.3.14]. Note that if $f \in B_l$ then

$$B_f = \big(B_{(f)} \oplus B(1)_{(f)} \oplus \dots \oplus B(l-1)_{(f)}\big)[f, f^{-1}]. \tag{2.2.4.1}$$

Thus $\Gamma\big(\mathrm{D}_+(f), \mathcal{O}_X(al)\big) = B(al)_{(f)} = B_{(f)} f^a$ is a free $B_{(f)}$-module of rank one which shows (ii).

(iii) If $a \geq 1$ then $\bigl(\mathrm{D}_+(f)\bigr)_{f\in B_{al}}$ is an affine cover of X. As $\mathcal{O}_X(al)$ is an invertible sheaf it is thus generated by global sections and ample by definition, cf. [EGA$_{\mathrm{II}}$, Def 4.5.3 and Thm 4.5.2 a′)].

(iv) It is enough to show that the homomorphism $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \to \mathcal{O}_X(m+n)$ is an isomorphism locally over $\mathrm{D}_+(f)$ with $f \in B_l$. Locally this homomorphism is $B(al)_{(f)} \otimes_{B_{(f)}} B(n)_{(f)} \to B(al+n)_{(f)}$ which is an isomorphism by equation (2.2.4.1)

(v) If $n = al$ with $a \geq 1$ then by (i) the morphism $i_n : X \to \mathbb{P}(B_n)$ is everywhere defined. If in addition $a \geq s-1$ then $B^{(n)}$ is generated by degree one elements by Lemma (2.2.2, (ii)). Thus we have a closed immersion $X = \mathrm{Proj}(B) \cong \mathrm{Proj}(B^{(n)}) \hookrightarrow \mathbb{P}(B_n)$.

(vi) Assume that $n \geq (s-1)(l-1)$, then $B_{n+kl} = (B_n B_l^k)$ for any positive integer k by Lemma (2.2.2, (i)). If $f \in B_l$ and $b \in B(n)_{(f)}$, then $b = b'/f^k$ for some $b' \in B_{n+kl} = \bigl(B_n B_l^k\bigr)$ and thus $b \in \bigl(B_{(f)} B_n\bigr)$. This shows that $\mathcal{O}_X(n)$ is generated by global sections as $B_n \subseteq \Gamma\bigl(\mathrm{D}_+(f), \mathcal{O}_X(n)\bigr)$. □

Corollary (2.2.5) ([EGA$_{\mathrm{II}}$, Cor 3.1.11])**.** *If S is quasi-compact and $X = \mathrm{Proj}_S(\mathcal{A})$ is a weighted projective scheme then there exists a positive integer n such that $X \to \mathbb{P}(\mathcal{A}_n)$ is everywhere defined and a closed immersion. In particular X is projective and $\mathcal{O}_X(n)$ is very ample relative to S.*

Proof. Let $\{S_i\}$ be a finite affine cover of S and let $A_i = \Gamma(S_i, \mathcal{O}_S)$ and $B_i = \Gamma(S_i, \mathcal{A})$. Then as B_i is a finitely generated graded A_i-algebra, there is by Proposition (2.2.4) a positive integer n_i such that $X \times_S S_i \to \mathbb{P}\bigl((B_i)_{n_i}\bigr)$ is defined and a closed immersion. Choosing n as the least common multiple of the n_i:s we obtain a closed immersion $X \hookrightarrow \mathbb{P}(\mathcal{A}_n)$. □

Remark **(2.2.6).** Note that (2.2.4, (iv), (v), (vi)) implies that the following are equivalent:

(i) $\mathcal{O}_X(n)$ is invertible for all $0 < n < l$.
(ii) $\mathcal{O}_X(n)$ is invertible for all n.
(iii) $\mathcal{O}_X(n)$ is very ample for all sufficiently large n.

As (i) is easily seen to not hold in many examples in particular (iii) is not always true.

The following condition will be important later on as it is satisfied for $\mathrm{Sym}^d(X/S)$ for X/S quasi-projective.

Definition (2.2.7). Let S be a scheme, $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_S$-algebra and $X = \mathrm{Proj}_S(\mathcal{A})$. If there is an affine cover (S_α) of S such that $X \times_S S_\alpha$ is covered by $\bigcup_{f\in\Gamma(S_\alpha,\mathcal{A}_1)} \mathrm{D}_+(f)$, then we say that X/S is *covered in degree one*.

Proposition (2.2.8). *Let A be a ring and let B be a graded A-algebra generated by elements of degree $\leq d$. Let $S = \mathrm{Spec}(A)$, $X = \mathrm{Proj}(B)$ and $\mathcal{O}_X(n) = \widetilde{B(n)}$. If X/S is covered in degree one then*

(i) *$X = \bigcup_{f\in B_n} \mathrm{D}_+(f)$ if $n \geq 1$.*
(ii) *$\mathcal{O}_X(n)$ is invertible for $n \in \mathbb{Z}$ and ample and generated by global sections if $n \geq 1$.*
(iii) *$\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n)$ for every $m, n \in \mathbb{Z}$.*

(iv) *The canonical morphism* $i_n : X \to \mathbb{P}(B_n)$ *is defined for every* $n \geq 1$. *If* $n \geq d$ *then* i_n *is a closed immersion and* $\mathcal{O}_X(n) = i_n^* \mathcal{O}_{\mathbb{P}(B_n)}(1)$ *is very ample relative* S.

Proof. (i) is equivalent to X/S being covered in degree one. Using the cover $X = \bigcup_{f \in B_1} \mathrm{D}_+(f)$ instead of the cover $X = \bigcup_{f \in B_l} \mathrm{D}_+(f)$ we may then prove (ii) and (iii) exactly as (ii), (iii) and (iv) in Proposition (2.2.4).

(iv) Let $n \geq d$ and let B' be the sub-A-algebra of B generated by B_n. It is enough to show that the inclusion $B' \hookrightarrow B$ induces an isomorphism $\mathrm{Proj}(B) \cong \mathrm{Proj}(B')$. We will show this using the cover $X = \bigcup_{f \in B_1} \mathrm{D}_+(f^n)$. Let $f \in B_1$ and $g \in B_{(f^n)}$ such that $g = b/f^{nk}$ for some $b \in B_{nk}$. To show that $g \in B'_{(f^n)}$ we can assume that $b = b_1 b_2 \dots b_s$ is a product of elements of degree $d_i \leq d$, as every element of B_{nk} are sums of such. Then $g = \left(\prod_{i=1}^{s} b_i f^{n-d_i}\right) / f^{ns}$ which is an element of $B'_{(f^n)}$. □

Corollary (2.2.9). *Let* S *be any scheme and* $\mathcal{A}$ *a graded quasi-coherent* $\mathcal{O}_S$*-algebra such that* $\mathcal{A}$ *is generated by elements of degree at most* d. *Let* $X = \mathrm{Proj}(\mathcal{A})$, $\mathcal{O}_X(n) = \widetilde{\mathcal{A}(n)}$ *and assume that* X/S *is covered in degree one. Then*

(i) $\mathcal{O}_X(n) = \mathcal{O}_X(1)^{\otimes n}$ *and is invertible for every* $n \in \mathbb{Z}$.
(ii) *If* $n \geq 1$ *then* $\mathcal{O}_X(n)$ *is ample and* $q^*\mathcal{A}_n \to \mathcal{O}_X(n)$ *is surjective.*
(iii) *For every* $n \geq 1$ *the canonical morphism* $i_n : X \to \mathbb{P}(\mathcal{A}_n)$ *is everywhere defined. If* $n \geq d$ *it is a closed immersion.*

In particular, if $X = \mathrm{Proj}_S(\mathcal{A})$ *also is a weighted projective scheme, i.e. if* $\mathcal{A}$ *is of finite type, then* X *is projective.*

Example (2.2.10) (Standard weighted projective spaces)**.** Let $A = k$ be an algebraically closed field of characteristic zero and $B = k[x_0, x_1, \dots, x_r]$. Let $d_0, d_1, \dots, d_r$ be positive integers and consider the action of $G = \mu_{d_0} \times \cdots \times \mu_{d_r} \cong \mathbb{Z}/d_0\mathbb{Z} \times \cdots \times \mathbb{Z}/d_r\mathbb{Z}$ given by $(n_0, n_1, \dots, n_r) \cdot x_i = \xi_{d_i}^{n_i} x_i$ where ξ_{d_i} is a d_i^{th} primitive root of unity. Then $B^G = k[x_0^{d_0}, x_1^{d_1}, \dots, x_r^{d_r}]$ and $\mathrm{Proj}\left(B^G\right)$ is a *weighted projective space* of type $(d_0, d_1, \dots, d_r)$.

It can be seen, cf. Proposition (2.3.4), that $\mathrm{Proj}\left(B^G\right)$ is the geometric quotient of $\mathrm{Proj}(B) = \mathbb{P}^r$ by G. More generally, if S is a noetherian scheme and X/S projective with an action of a finite group G linear with respect to a very ample sheaf $\mathcal{O}_X(1)$, then a geometric quotient X/G exists and can be given a structure as a weighted projective scheme.

The weighted projective space $\mathrm{Proj}\left(B^G\right)$ is often denoted $\mathbb{P}(d_0, d_1, \dots, d_r)$. It can also be constructed as the quotient of $\mathbb{A}^{r+1} - 0$ by $\mathbb{G}_m$ where $\mathbb{G}_m$ acts on $\mathbb{A}^{r+1}$ by $\lambda \cdot x_i = \lambda^{d_i} x_i$. The closed points of $\mathbb{P}(d_0, d_1, \dots, d_r)$ are thus $\{x = (x_0 : x_1 : \cdots : x_r)\} = k^{r+1}/\sim$ where $x \sim y$ if there is a $\lambda \in k^*$ such that $\lambda^{d_i} x_i = y_i$ for every i.

2.3. **Quotients of projective schemes by finite groups.** Let X be an S-scheme and G a discrete group acting on X/S, i.e. there is a group homomorphism $G \to \mathrm{Aut}_S(X)$. In the category of ringed spaces we can construct a quotient $Y = (X/G)_{\mathrm{rs}}$ as following. Let Y as a topological space be X/G with the quotient topology, and quotient map $q : X \to Y$. Further let the sheaf of sections $\mathcal{O}_Y$ be the subsheaf $(q_*\mathcal{O}_X)^G \hookrightarrow q_*\mathcal{O}_X$ of

G-invariant sections. Note that G acts on $q_*\mathcal{O}_X$ since for any open subset $U \subseteq Y$ the inverse image $q^{-1}(U)$ is G-stable and hence has an induced action of G. Thus we obtain a ringed S-space $(Y, \mathcal{O}_Y)$ together with a morphism of ringed S-spaces $q : X \to Y$. The ringed space $(Y, \mathcal{O}_Y)$ is not always a scheme, in fact not always even a locally ringed space. But when it exists as a scheme it is called the *geometrical quotient* and is also the *categorial quotient* in the category of schemes over S. For general existence results we refer to [Ryd07b]. The existence of a geometric quotient of an affine schemes by a finite group is not difficult to show:

Proposition (2.3.1) ([SGA_1, Exp. V, Prop. 1.1, Cor. 1.5])**.** *Let S be a scheme, $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras and $X = \mathrm{Spec}_S(\mathcal{A})$. An action of G on X/S induces an action of G on $\mathcal{A}$. If G is a finite group then $Y = \mathrm{Spec}_S\left(\mathcal{A}^G\right)$ is the geometric quotient of X by G. If S is* locally noetherian *and $X \to S$ is of finite type, then $Y \to S$ is of finite type.*

From this local result it is not difficult to show the following result:

Theorem (2.3.2) ([SGA_1, Exp. V, Prop. 1.8])**.** *Let $f : X \to S$ be a morphism of arbitrary schemes and G a finite discrete group acting on X by S-morphisms. Assume that every G-orbit of X is contained in an affine open subset. Then the geometrical quotient $q : X \to Y = X/G$ exists.*

It can also be shown, from general existence results, that if X/S is separated then this is also a necessary condition [Ryd07b, Rmk. 4.9].

Remark **(2.3.3).** If $X \to S$ is quasi-projective, then every G-orbit is contained in an affine open set. In fact, we can assume that $S = \mathrm{Spec}(A)$ is affine and thus that we have an embedding $X \hookrightarrow \mathbb{P}^n_S$. For any orbit Gx we can then choose a section $f \in \mathcal{O}_{\mathbb{P}^n}(m)$ for some sufficiently large m such that $\mathrm{V}(f)$ does not intersect Gx. The affine subset $\mathrm{D}(f)$ then contains the orbit Gx. More generally [$\mathrm{EGA}_{\mathrm{II}}$, Cor. 4.5.4] shows that every finite set, in particulary every G-orbit, is contained in an affine open set if X/S is such that there is an ample invertible sheaf on X relative S.

In Corollary (2.3.6) we will show that if S is *noetherian* and $X \to S$ is (quasi-)projective, then so is $X/G \to S$. In fact if X is projective we will give a weighted projective structure on X/G.

Proposition (2.3.4). *Let S be a scheme and let $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ be a graded quasi-coherent $\mathcal{O}_S$-algebra, generated by degree one elements. Let G be a finite group acting on $\mathcal{A}$ by graded $\mathcal{O}_S$-algebra automorphisms. Then G acts on $X = \mathrm{Proj}_S(\mathcal{A})$ linearly with respect to $\mathcal{O}_X(1)$. As X admits a very ample invertible sheaf relative to S, a geometric quotient $Y = X/G$ exists, cf. Remark* (2.3.3)*. There is an isomorphism $Y \cong \mathrm{Proj}_S\left(\mathcal{A}^G\right)$ and under this isomorphism, the quotient map $q : X \to Y$ is induced by $\mathcal{A}^G \hookrightarrow \mathcal{A}$.*

Proof. Everything is local over S so we can assume that $S = \mathrm{Spec}(A)$, $\mathcal{A} = \widetilde{B}$ and $X = \mathrm{Proj}(B)$. We can cover X by G-stable affine subsets of the form $\mathrm{D}_+(f)$ with $f \in B^G$ homogeneous. In fact, if Z is a G-orbit of X then the demonstration of [$\mathrm{EGA}_{\mathrm{II}}$, Cor. 4.5.4] shows that there is a homogeneous $f' \in B$ such that $Z \subseteq \mathrm{D}_+(f')$. If we let $f = \prod_{\sigma \in G} \sigma(f')$, then $Z \subseteq \mathrm{D}_+(f)$

and $f \in B^G$ is homogeneous. Over such an open set we have that

$$X|_{\mathrm{D}_+(f)}/G = \operatorname{Spec}\bigl(\bigl(B_{(f)}\bigr)^G\bigr) = \operatorname{Spec}\bigl((B^G)_{(f)}\bigr) = \operatorname{Proj}\left(B^G\right)|_{\mathrm{D}_+(f)}.$$

It is thus clear that $Y = \operatorname{Proj}\left(B^G\right)$. □

Remark **(2.3.5).** Note that $\mathcal{A}^G$ is not always generated by $\mathcal{A}_1^G$ even though $\mathcal{A}$ is generated by $\mathcal{A}_1$. Also, if $S = \operatorname{Spec}(A)$ is affine and $\mathcal{A} = \widetilde{B}$, we may not be able to cover $X = \operatorname{Proj}(B)$ with G-stable affine subsets of the form $\mathrm{D}_+(f)$ with $f \in B_1^G$. This is demonstrated by example (2.2.10) if we choose $d_i > 1$ for some i.

Corollary (2.3.6) ([Knu71, Ch. IV, Prop 1.5]). *Let S be* noetherian*, $X \to S$ be projective (resp. quasi-projective) and G a finite group acting on X by S-morphisms. Then the geometrical quotient X/G is projective (resp. quasi-projective).*

Proof. Let $X \hookrightarrow (X/S)^m = X \times_S X \times_S \cdots \times_S X$ be the closed immersion given by $x \to (\sigma_1 x, \sigma_2 x, \ldots, \sigma_m x)$ where $G = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$. As $X \to S$ is quasi-projective and S is noetherian, there is an immersion $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ for some quasi-coherent $\mathcal{O}_S$-module of finite type $\mathcal{E}$, see [EGA$_{\mathrm{II}}$, Prop. 5.3.2]. This immersion together with the immersion $X \hookrightarrow (X/S)^m$ given above, gives a G-equivariant immersion $X \hookrightarrow \bigl(\mathbb{P}_S(\mathcal{E})/S\bigr)^n$ if we let G permute the factors of $\bigl(\mathbb{P}_S(\mathcal{E})/S\bigr)^n$. Following this immersion by the Segre embedding we get a G-equivariant immersion $f : X \hookrightarrow \mathbb{P}_S(\mathcal{E}^{\otimes m})$ where G acts linearly on $\mathbb{P}_S(\mathcal{E}^{\otimes m})$, i.e. by automorphisms of $\mathcal{E}^{\otimes m}$.

Let $Y = \overline{f(X)}$ be the schematic image of f. As Y is clearly G-stable we have an action of G on Y and a geometric quotient $q : Y \to Y/G$. Then, as $X \hookrightarrow Y$ is an open immersion and q is open, we have that $X/G = (Y/G)|_{q(Y)}$. Thus it is enough to show that Y/G is projective. Let $\mathcal{A} = \mathrm{S}(\mathcal{E}^{\otimes m})/I$ such that $Y = \operatorname{Proj}(\mathcal{A})$. Then there is an action of G on $\mathcal{A}_1$ which induces the action Y. By Proposition (2.3.4) we have that $Y/G = \operatorname{Proj}\left(\mathcal{A}^G\right)$. The scheme Y/G is a weighted projective scheme as $\mathcal{A}$ is an $\mathcal{O}_S$-algebra of finite type by Proposition (2.3.1). It then follows by Corollary (2.2.5) that Y/G is projective. □

2.4. **Finite quotients, base change and closed subschemes.** A geometric quotient is always *uniform*, i.e., it commutes with flat base change [GIT, Rmk. (7), p. 9]. It is also a *universal topological quotient*, i.e., the fibers corresponds to the orbits and the quotient has the quotient topology and this holds after any base change. However, in positive characteristic a geometric quotient is not necessarily a *universal* geometric quotient, i.e., it need not commute with arbitrary base change. This is shown by the following example:

Example (2.4.1). Let $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$, $S' = \operatorname{Spec}(A/I)$ with $A = k[\epsilon]/\epsilon^2$ where k is a field of characteristic $p > 0$, $B = k[\epsilon, x]/(\epsilon^2, \epsilon x)$ and $I = (\epsilon)$. We have an action of $G = \mathbb{Z}/p = \langle \tau \rangle$ on B given by $\tau(x) = x + \epsilon$ and $\tau(\epsilon) = \epsilon$. Then $\tau(x^n) = x^n$ for all $n \geq 2$ and thus $B^G = k[\epsilon, x^2, x^3]/(\epsilon^2, \epsilon x^2, \epsilon x^3)$. Further, we have that $(B \otimes_A A')^G = k[x]$ and $B^G \otimes_A A' = k[x^2, x^3]$.

Recall that a morphism of schemes is a *universal homeomorphism* if the underlying morphism of topological spaces is a homeomorphism after any base change.

Proposition (2.4.2) ([EGA$_{\mathrm{IV}}$, Cor. 18.12.11])**.** *Let $f : X \to Y$ be a morphism of schemes. Then f is a universal homeomorphism if and only if f is integral, universally injective and surjective.*

Proposition (2.4.3). *Let X/S be a scheme with an action of a finite group G such that every G-orbit of X is contained in an affine open subset. Let $S' \to S$ be any morphism and let $X' = X \times_S S'$. Then geometric quotients $q : X \to X/G$ and $r : X' \to X'/G$ exists. Let $(X/G)' = (X/G) \times_S S'$. As r is a categorical quotient we have a canonical morphism $X'/G \to (X/G)'$. This morphism is a universal homeomorphism.*

Proof. The geometric quotients q and r exists by Theorem (2.3.2). As q and r are universal topological quotients it follows that $X'/G \to (X/G)'$ is universally bijective. As $X' \to X'/G$ is surjective and $X' \to (X/G)'$ is universally open it follows that $X'/G \to (X/G)'$ is universally open and hence a universal homeomorphism. $\square$

If G acts on X and $U \subseteq X$ is a G-stable open subscheme, then U/G is an open subscheme of X/G. In fact U/G is the image of U by the open morphism $q : X \to X/G$. If $Z \hookrightarrow X$ is a closed G-stable subscheme, then Z/G is not always the image of Z by q. In fact Z/G need not even be a subscheme of X/G. We have the following result:

Proposition (2.4.4). *Let G be a finite group, X/S a scheme with an action of G such that the geometric quotient $q : X \to X/G$ exists. Let $Z \hookrightarrow X$ be a closed G-stable subscheme. Then the geometric quotient $r : Z \to Z/G$ exist. Let $q(Z)$ be the scheme-theoretic image of the morphism $Z \hookrightarrow X \to X/G$. As r is a categorical quotient, the morphism $Z \to q(Z) \hookrightarrow X/G$ factors canonically as $Z \to Z/G \to q(Z) \hookrightarrow X/G$. The morphism $Z/G \to q(Z)$ is a* schematically dominant universal homeomorphism*.*

Proof. As Z/G and $q(Z)$ both are universal topological quotients of Z, the canonical morphism $Z/G \to q(Z)$ is universally bijective. Since $Z \to q(Z)$ is universally open and $Z \to Z/G$ is surjective we have that $Z/G \to q(Z)$ is universally open and thus a universal homeomorphism. Further as $Z \to q(Z)$ is schematically dominant the morphism $Z/G \to q(Z)$ is also schematically dominant. $\square$

Corollary (2.4.5). *Let G and X/S be as in Proposition* (2.4.4)*. There is a canonical universal homeomorphism $(X_{\mathrm{red}})/G \to (X/G)_{\mathrm{red}}$.*

We can say even more about the exact structure of $Z/G \to q(Z)$. For ease of presentation we state the result in the affine case.

Proposition (2.4.6). *Let A be a ring with an action by a finite group G and let $I \subset A$ be a G-stable ideal. Let $X = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(A/I)$. Then $Z/G = \operatorname{Spec}\big((A/I)^G\big)$ and $q(Z) = \operatorname{Spec}\left(A^G/I^G\right)$. We have an injection $A^G/I^G \hookrightarrow (A/I)^G$. If $f \in (A/I)^G$ then there is an $n \mid \operatorname{card}(G)$ such that $f^n \in A^G/I^G$. To be more precise we have that*

(i) *If A is a $\mathbb{Z}_{(p)}$-algebra with p a prime, e.g. a local ring with residue field k or a k-algebra with* $\operatorname{char} k = p$*, then n can be chosen as a power of p.*
(ii) *If A is purely of characteristic zero, i.e. a $\mathbb{Q}$-algebra, then $A^G/I^G \hookrightarrow (A/I)^G$ is an isomorphism.*

Proof. Let $f \in A$ such that its image $\overline{f} \in A/I$ is G-invariant. To show that $\overline{f}^n \in A^G/I^G$ for some positive integer n it is enough to show that $\overline{f}^n \in A^G/I^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec}(\mathbb{Z})$. As $\mathbb{Z} \to \mathbb{Z}_{\mathfrak{p}}$ is flat, we have that

$$\begin{aligned} A^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (A \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G \\ I^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (I \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G \\ A^G/I^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (A \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G/(I \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G \\ (A/I)^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (A/I \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G. \end{aligned}$$

Thus we can assume that A is a $\mathbb{Z}_{\mathfrak{p}}$-algebra.

Let q be the characteristic exponent of $\mathbb{Z}_{\mathfrak{p}}/\mathfrak{p}\mathbb{Z}_{\mathfrak{p}}$, i.e. $q = p$ if $\mathfrak{p} = (p)$, $p > 0$ and $q = 1$ if $\mathfrak{p} = (0)$. Choose positive integers k and m such that $\operatorname{card}(G) = q^k m$ and $q \nmid m$ if $q \neq 1$. Then choose a Sylow subgroup H of G of order q^k, or $H = (e)$ if $q = 1$, and let $\sigma_1 H, \sigma_2 H, \dots, \sigma_m H$ be its cosets. Then

$$g = \frac{1}{m} \sum_{i=1}^{m} \prod_{\sigma \in \sigma_i H} \sigma(f)$$

is G-invariant and its image $\overline{g} \in A^G/I^G$ maps to $\overline{f}^{q^k} \in (A/I)^G$. □

Proposition (2.4.6) can also be extended to the case where G is any reductive group [GIT, Lemma A.1.2].

Remark **(2.4.7).** Let X/S be a scheme with an action of a finite group G with geometric quotient $q : X \to X/G$. Then

(i) If S is a $\mathbb{Q}$-scheme and $S' \to S$ is any morphism then $(X \times_S S')/G = X/G \times_S S'$.
(ii) If S is arbitrary and $U \subseteq X$ is an open immersion then $U/G = q(U)$.
(iii) If S is a $\mathbb{Q}$-scheme and $Z \hookrightarrow X$ is a closed immersion then $Z/G = q(Z)$.
(iv) If S is a $\mathbb{Q}$-scheme then $(X/G)_{\mathrm{red}} = X_{\mathrm{red}}/G$.

(ii) follows from the uniformity of geometric quotients, (i) and (iv) follows from the universality of geometric quotients in characteristic zero and (iii) follows from Proposition (2.4.6).

Statement (iii) can also be proven as follows. We can assume that $X = \operatorname{Spec}(A)$ is affine. Then the homomorphism $A^G \hookrightarrow A$ has an A^G-module retraction, the Reynolds-operator R, given by $R(a) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(a)$. This implies that $A^G \hookrightarrow A$ is *universally injective*, i.e. injective after tensoring with any A-module M. In particular $A^G \hookrightarrow A$ is *cyclically pure*, i.e., $I^G A = I$, where $I^G = I \cap A^G$, for any ideal $I \subseteq A$. If we let $S = \operatorname{Spec}\left(A^G\right)$ and $S' = \operatorname{Spec}\left(A^G/I^G\right)$ then $Z = X \times_S S' = \operatorname{Spec}(A/I)$ and (iii) follows from (i).

3. The main section

3.1. The symmetric product.

Definition (3.1.1). Let X be a scheme over S and d a positive integer. We let the symmetric group on d letters $\mathfrak{S}_d$ act by permutations on $(X/S)^d = X \times_S X \times_S \cdots \times_S X$. When a geometric quotient of $(X/S)^d$ by $\mathfrak{S}_d$ exists, we let $\mathrm{Sym}^d(X/S) := (X/S)^d/\mathfrak{S}_d$. The scheme $\mathrm{Sym}^d(X/S)$ is called the d^{th} *symmetric product* of X over S and is also denoted $\mathrm{Symm}^d(X/S)$, $(X/S)^{(d)}$ or $X^{(d)}$ by some authors.

Definition (3.1.2). Let X/S be a scheme. We say that X/S is AF if the following condition is satisfied:

(AF) Every finite set of points $x_1, x_2, \dots, x_n$ over the same point $s \in S$ is contained in an affine open subset of X.

Remark **(3.1.3).** If X has an ample sheaf relative to S, then X/S is AF, cf. [EGA_{II}, Cor. 4.5.4]. It is also clear from [EGA_{II}, Cor. 4.5.4] that if X/S is AF then so is $X \times_S S'/S'$ for any base change $S' \to S$. It can further be seen that if X/S is AF then X/S is separated.

Remark **(3.1.4).** Let X/S be an AF-scheme and d be a natural integer. By Theorem (2.3.2) a geometric quotient $\mathrm{Sym}^d(X/S)$ exists. Let (S_α) be an affine cover of S and let $(U_{\alpha\beta})$ be an affine cover of $X \times_S S_\alpha$ such that any set of d points of X lying over the same point $s \in S_\alpha$ is included in some $U_{\alpha\beta}$ then $(U_{\alpha\beta}/S_\alpha)^d$ is an affine cover of $(X/S)^d$. Thus $\coprod_{\alpha,\beta} \mathrm{Sym}^d(U_{\alpha\beta}/S_\alpha) \to \mathrm{Sym}^d(X/S)$ is an open covering by affines.

In the remainder of this section we will study the symmetric product when $S = \mathrm{Spec}(A)$ is an affine scheme and X/S is projective. We will use the following notation:

Notation (3.1.5). Let A be a ring and $B = \bigoplus_{k \geq 0} B_k$ a graded A-algebra finitely generated by elements in degree one. Let $S = \mathrm{Spec}(A)$ and $X = \mathrm{Proj}(B)$ with very ample sheaf $\mathcal{O}_X(1) = \widetilde{B(1)}$ and canonical morphism $q : X \to S$.

Further we let $C = \bigoplus_{k \geq 0} \mathrm{T}^d_A(B_k) \subset \mathrm{T}^d_A(B)$. Then $(X/S)^d = \mathrm{Proj}(C)$ and $\mathrm{Proj}(C) \hookrightarrow \mathbb{P}(C_1) = \mathbb{P}\big(\mathrm{T}^d_A(B_1)\big)$ is the Segre embedding of $(X/S)^d$ corresponding to the embedding $X = \mathrm{Proj}(B) \hookrightarrow \mathbb{P}(B_1)$. The permutation of the factors induces an action of the symmetric group $\mathfrak{S}_d$ on C and we let $D = C^{\mathfrak{S}_d} = \bigoplus_{k \geq 0} \mathrm{TS}^d_A(B_k)$ be the graded invariant ring. By Proposition (2.3.4) we have that $\mathrm{Sym}^d(X/S) := \mathrm{Proj}(C)/\mathfrak{S}_d = \mathrm{Proj}(D)$.

Lemma (3.1.6). *Let $x_1, x_2, \dots, x_d \in X$ be points such that $q(x_1) = q(x_2) = \cdots = q(x_d) = s$. Then there exists a positive integer n and an element $f \in B_n \subseteq \Gamma\big(X, \mathcal{O}_X(n)\big)$ such that $x_1, x_2, \dots, x_d \in X_f = \mathrm{D}_+(f)$. If the residue field $k(s)$ has at least d elements then it is possible to take $n = 1$.*

Proof. The existence of f for some n follows from [EGA_{II}, Cor. 4.5.4]. We will prove the lemma when $k(s)$ has at least d elements. As we can lift any element $\overline{f} \in B_n \otimes_A k(s)$ to an element $f \in B_n$ after multiplying with an invertible element of $k(s)$, we can assume that $A = k(s)$. Replacing B with the symmetric product $\mathrm{S}(B_1) = k[t_0, t_1, \dots, t_r]$ we can further assume that B is a polynomial ring and $X = \mathbb{P}^r_{k(s)}$.

An element of $B_1 = \Gamma\big(X, \mathcal{O}_X(1)\big)$ is then a linear form $f = a_0t_0 + a_1t_1 + \dots + a_rt_r$ with $a_i \in k(s)$ and can be thought of as a $k(s)$-rational point of $\mathbb{P}^r_{k(s)}$. The linear forms zero in one of the x_i:s is a proper closed linear subset of all linear forms. Thus if $k(s)$ is infinite then there is a $k(s)$-rational point corresponding to a linear form non-zero in every x_i. If $k = k(s)$ is finite, then at most $(|k|^r - 1)/|k^*|$ linear forms are zero at a certain x_i and equality holds when x_i is k-rational. Thus at most

$$\begin{aligned} d(|k|^r - 1)/(|k| - 1) &\leq (|k|^{r+1} - |k|)/(|k| - 1) \\ &= (|k|^{r+1} - 1)/(|k| - 1) - 1 \end{aligned}$$

linear forms contain at least one of the $x_1, x_2, \dots, x_d$ and hence there is at least one linear form which does not vanish on any of the points. □

Proposition (3.1.7). *The product $X^d = X \times_S X \times_S \dots \times_S X$ is covered by $\mathfrak{S}_d$-stable affine open subsets of the form $X_f \times_S X_f \times_S \dots \times_S X_f$ where $f \in B_n \subseteq \Gamma\big(X, \mathcal{O}_X(n)\big)$ for some n. If every residue field of S has at least d elements then the open subsets with $f \in B_1 \subseteq \Gamma\big(X, \mathcal{O}_X(1)\big)$ cover X^d.*

Proof. Follows immediately from Lemma (3.1.6). □

Corollary (3.1.8). *The symmetric product $\mathrm{Sym}^d(X/S)$ is covered by open affine subsets $\mathrm{Sym}^d\big(X_f/S\big)$ with $f \in B_n \subseteq \Gamma\big(X, \mathcal{O}_X(n)\big)$ for some n. If every residue field of S has at least d elements then those affine subsets with $n = 1$ cover $\mathrm{Sym}^d(X/S)$.*

Corollary (3.1.9). *The symmetric product $Y = \mathrm{Sym}^d(X/S) = \mathrm{Proj}(D)$ is covered by Y_g where $g \in D_1 \subseteq \Gamma\big(Y, \mathcal{O}_Y(1)\big)$, i.e. $Y = \mathrm{Proj}(D)$ is covered in degree one.*

Proof. Let $A \hookrightarrow A'$ be a finite flat extension such that every residue field of A' has at least d elements, e.g. the extension $A' = A \otimes_{\mathbb{Z}} \Lambda_d$ suffices by Lemma (1.3.3). Let $B' = B \otimes_A A'$ and $C' = C \otimes_A A'$ and let $D' = D \otimes_A A' = \bigoplus_{n\geq 0} \mathrm{TS}^d_A(B_n) \otimes_A A'$. Then $D' = \bigoplus_{n\geq 0} \mathrm{TS}^d_{A'}(B'_n)$ as $A \hookrightarrow A'$ is flat. Note that if $f' \in B'_n$ then $g' = f' \otimes f' \otimes \dots \otimes f' \in D'_n$ and $\mathrm{Sym}^d(X'_{f'}/S) = \mathrm{D}_+(g')$ as open subsets of $\mathrm{Sym}^d(X'/S')$. Thus Corollary (3.1.8) shows that $\sqrt{D'_1 D'_+} = D'_+$. As $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is surjective it follows that $\sqrt{D_1 D_+} = D_+$. □

We now use the degree bound on the generators of $\mathrm{TS}^d_A(A[x_0, x_1, \dots, x_r])$ obtained in Corollary (1.5.6) to get something very close to a degree bound on the generators of $D = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_k)$ when $B = A[x_0, x_1, \dots, x_r]$ is the polynomial ring.

Proposition (3.1.10). *Let N be a positive integer and $D_{\leq N}$ be the subring of $D = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_k)$ generated by elements of degree at most N. Then the inclusion $D_{\leq N} \hookrightarrow D$ induces a morphism $\psi_N : \mathrm{Proj}(D) \to \mathrm{Proj}(D_{\leq N})$. Further we have that:*

(i) *If* $B = A[x_0, x_1, \dots, x_r]$ *is a polynomial ring and* $N \geq r(d-1)$ *then* ψ_N *is an isomorphism.*
(ii) *If* A *is purely of characteristic zero, i.e. a* $\mathbb{Q}$*-algebra, then* ψ_N *is an isomorphism for any* N.

Proof. By Corollary (3.1.9) the morphism ψ_N is everywhere defined for $N \geq 1$. Let $A \hookrightarrow A'$ be a finite flat extension such that every residue field of A' has at least d elements, e.g. $A' = A \otimes_{\mathbb{Z}} \Lambda_d$ as in Lemma (1.3.3). If we let $C' = C \otimes_A A'$ then $D' = D \otimes_A A' = C'^{\mathfrak{S}_d}$ and $D'_{\leq N} = D_{\leq N} \otimes_A A'$ as $A \hookrightarrow A'$ is flat. If $\psi'_N : \operatorname{Proj}(D') \to \operatorname{Proj}(D'_{\leq N})$ is an isomorphism then so is ψ_N as $A \hookrightarrow A'$ is faithfully flat. Replacing A with A', it is thus enough to prove the corollary when every residue field of S has at least d elements. Hence we can assume that we have a cover of $\operatorname{Proj}(D)$ by $\mathrm{D}_+\left(f^{\otimes d}\right)$ with $f \in B_1$ by Corollary (3.1.8).

We have that $D_{(f^{\otimes d})} = \mathrm{TS}^d_A(B_{(f)})$ and this latter ring is generated by elements of degree $\leq \max\{r(d-1), 1\}$ for arbitrary A and by elements of degree one when A is purely of characteristic zero by Corollary (1.5.6). As noted in Remark (1.5.3) this implies that $D_{(f^{\otimes d})} = D_{\leq N(f^{\otimes d})}$ which shows (i) and (ii). □

Corollary (3.1.11). *Let* N *be a positive integer and* D_N *be the subring of* $D^{(N)} = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_{Nk})$ *generated by* $\mathrm{TS}^d_A(B_N)$. *Then the inclusion* $D_N \hookrightarrow D^{(N)}$ *induces a morphism* $\psi_N : \operatorname{Proj}(D) \to \operatorname{Proj}(D_N)$. *Further we have that:*

(i) *If* $B = A[x_0, x_1, \dots, x_r]$ *is a polynomial ring and* $N \geq r(d-1)$ *then* ψ_N *is an isomorphism.*
(ii) *If* A *is purely of characteristic zero, i.e. a* $\mathbb{Q}$*-algebra, then* ψ_N *is an isomorphism for any* N.

Proof. Let $D_{\leq N}$ be the subring of $D = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_k)$ generated by elements of degree at most N. As $\operatorname{Proj}(D)$ is covered in degree one by Corollary (3.1.9) then so is $\operatorname{Proj}(D_{\leq N})$. By Proposition (2.2.8, (iv)) it then follows that $D_N \hookrightarrow D^{(N)}_{\leq N}$ induces an isomorphism $\operatorname{Proj}(D^{(N)}_{\leq N}) \to \operatorname{Proj}(D_N)$. The corollary then follows from Proposition (3.1.10) which shows that $D_{\leq N} \hookrightarrow D$ induces a morphism $\operatorname{Proj}(D) \to \operatorname{Proj}(D_{\leq N})$ with properties as in (i) and (ii). □

Corollary (3.1.12). *Let* S *be any scheme and* $\mathcal{E}$ *a quasi-coherent* $\mathcal{O}_S$*-sheaf of finite type. Then for any* $N \geq 1$, *there is a canonical morphism* $\operatorname{Sym}^d(\mathbb{P}(\mathcal{E})/S) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{E})\big)$. *If* $\mathcal{L}$ *is a locally free* $\mathcal{O}_S$*-sheaf of constant rank* $r+1$ *then the canonical morphism* $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S) \hookrightarrow \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{L})\big)$ *is a closed immersion for* $N \geq r(d-1)$. *In particular, it follows that* $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S) \to S$ *is strongly projective.*

Proof. The existence of the morphism follows by Corollary (3.1.11). Part (i) of the same corollary shows that $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S) \hookrightarrow \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{L})\big)$ is a closed immersion when $N \geq r(d-1)$. As $\mathrm{S}^N\mathcal{L}$ is locally free of constant rank it follows by paragraph (1.2.5) that $\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{L})$ is locally free of constant rank which shows that $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S)$ is strongly projective. □

3.2. **The scheme of divided powers.** Let S be any scheme and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. As the construction of $\Gamma^d_A(B)$ commutes with localization with respect to multiplicatively closed subsets of A we may define a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\Gamma^d_{\mathcal{O}_S}(\mathcal{A})$. We let $\Gamma^d(\mathrm{Spec}(\mathcal{A})/S) = \mathrm{Spec}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{A})\big)$. The scheme $\Gamma^d(X/S)$ is thus defined for any scheme X affine over S. Similary we obtain for any homomorphism of quasi-coherent $\mathcal{O}_S$-algebras $\mathcal{A} \to \mathcal{B}$ a morphism of schemes $\Gamma^d(\mathrm{Spec}(\mathcal{B})/S) \to \Gamma^d(\mathrm{Spec}(\mathcal{A})/S)$. This defines a covariant functor $X \mapsto \Gamma^d(X/S)$ from affine schemes over S to affine schemes over S.

It is more difficult to define $\Gamma^d(X/S)$ for any X-scheme S since $\Gamma^d_A(B)$ does not commute with localization with respect to B. In fact, it is not even a B-algebra. In [Ryd07c] a certain functor $\underline{\Gamma}^d_{X/S}$ is defined which is represented by $\Gamma^d(X/S)$ when X/S is affine. When X/S is quasi-projective, or more generally AF, cf. Definition (3.1.2), then $\underline{\Gamma}^d_{X/S}$ is represented by a scheme. More generally it is shown that this functor is representable by a separated algebraic space for any separated algebraic space X/S. We will briefly state some facts about the representing scheme $\Gamma^d(X/S)$ used in the other sections.

Theorem (3.2.1) ([Ryd07c, ?]). *For any algebraic scheme X separated above S, there is an algebraic space $\Gamma^d(X/S)$ over S with the following properties*

(i) *For any morphism $S' \to S$, there is a canonical isomorphism $\Gamma^d(X/S)\times_S S' \cong \Gamma^d(X \times_S S'/S')$.*
(ii) *If X/S is an* AF*-scheme, then $\Gamma^d(X/S)$ is an* AF*-scheme.*
(iii) *If $\mathcal{A}$ is a quasi-coherent sheaf on S such that $X = \mathrm{Spec}_S(\mathcal{A})$ is affine S, then $\Gamma^d(X/S) = \mathrm{Spec}_S\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{A})\big)$ is affine over S.*
(iv) *If $X = \coprod_{i=1}^n X_i$ then $\Gamma^d(X/S)$ is the disjoint union*

$$\coprod_{\substack{d_1,d_2,\dots,d_n\geq 0\\ d_1+d_2+\dots+d_n=d}} \Gamma^{d_1}(X_i/S)\times_S \Gamma^{d_2}(X_2/S)\times_S \dots \times_S \Gamma^{d_n}(X_n/S).$$

(v) *If $X \to S$ is of finite type (resp. of finite presentation, resp. locally of finite type, resp. locally of finite presentation, resp. quasi-compact, resp. finite, resp. integral, resp. flat) then so is $\Gamma^d(X/S) \to S$.*

Proposition (3.2.2) ([Ryd07c, ?]). *Let $f : X \to Y$ be any morphism of algebraic schemes separated over S. There is then a natural morphism, push-forward of cycles, $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ which for affine schemes is given by the covariance of the functor $\Gamma^d_A(\cdot)$. If $f : X \to Y$ is an immersion (resp. a closed immersion, resp. an open immersion) then $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ is an immersion (resp. closed immersion, resp. open immersion).*

Corollary (3.2.3). *If $X \to S$ is strongly projective (resp. strongly quasi-projective) then $\Gamma^d(X/S) \to S$ is strongly projective (resp. strongly quasi-projective). If $X \to S$ is projective (resp. quasi-projective) and S is quasi-compact and quasi-separated then $\Gamma^d(X/S) \to S$ is projective (resp. quasi-projective).*

Proof. In the strongly projective (resp. strongly quasi-projective) case we immediately reduce to the case where $X = \mathbb{P}_S(\mathcal{L})$ for some locally free $\mathcal{O}_S$-module $\mathcal{L}$ of finite rank $r+1$, using Proposition (3.2.2), and the result follows from Corollary (3.1.12).

If S is quasi-compact and quasi-separated and $X \to S$ is projective (resp. quasi-projective) then there is a closed immersion (resp. immersion) $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type. Let $S = \bigcup_i S_i$ be a finite open cover by affines. There are then closed immersions $\mathbb{P}(\mathcal{E}|_{S_i}) \hookrightarrow \mathbb{P}^{r_i}_{S_i}$ for some positive integers r_i. It follows from Proposition (3.2.2) and Corollary (3.1.11) that we have closed immersions

$$\Gamma^d(X|_{S_i}/S_i) \hookrightarrow \Gamma^d(\mathbb{P}(\mathcal{E}|_{S_i})/S_i) \hookrightarrow \Gamma^d(\mathbb{P}^{r_i}_{S_i}/S_i) \hookrightarrow \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_{S_i}}(\mathrm{S}^N \mathcal{O}^{r_i+1}_{S_i})\big)$$

for $N \geq r_i(d-1)$. Taking a sufficiently large N, we then obtain a closed immersion $\Gamma^d(X/S) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N \mathcal{E})\big)$. □

Proposition (3.2.4) ([Ryd07c, ?])**.** *Let X/S be an* AF*-scheme and d be a natural integer. Let (S_α) be an affine cover of S and let $(U_{\alpha\beta})$ be an affine cover of $X \times_S S_\alpha$ such that any d points of $X \times_S S_\alpha$ lying over the same point $s \in S_\alpha$ is included in some $U_{\alpha\beta}$. Then the morphism $\coprod_{\alpha,\beta} \Gamma^d(U_{\alpha\beta}/S_\alpha) \to \Gamma^d(X/S)$, given by push-forward, is an open covering by affines.*

Definition (3.2.5). Let d, e be positive integers. The composition of the open and closed immersion $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \hookrightarrow \Gamma^{d+e}(X \coprod X/S)$ given by Proposition (3.2.1) (iv) and the push-forward $\Gamma^{d+e}(X \coprod X/S) \to \Gamma^{d+e}(X/S)$ along the canonical morphism $X \coprod X \to X$ is called "addition of cycles".

Proposition (3.2.6) ([Ryd07c, ?])**.** *Let X/S be an* AF*-scheme and let $(X/S)^d = X \times_S X \times_S \cdots \times_S X$. There is an integral surjective morphism $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$, given by addition of cycles, invariant under the permutation of the factors. This gives a factorization $(X/S)^d \to \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$. If $U \hookrightarrow X$ is an open immersion then there is a cartesian diagram*

$$\begin{tikzcd}
(U/S)^d \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (X/S)^d \arrow[d] \\
\mathrm{Sym}^d(U/S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sym}^d(X/S) \arrow[d] \\
\Gamma^d(U/S) \arrow[r] & \Gamma^d(X/S)
\end{tikzcd}$$

In particular, if $U_{\alpha\beta}$ is a open covering of X as in Proposition (3.2.4) *then*

$$\begin{tikzcd} \coprod_{\alpha,\beta} \mathrm{Sym}^d(U_{\alpha\beta}/S_\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sym}^d(X/S) \arrow[d] \\ \coprod_{\alpha,\beta} \Gamma^d(U_{\alpha\beta}/S_\alpha) \arrow[r] & \Gamma^d(X/S) \end{tikzcd}$$

is cartesian and the horizontal maps are open coverings.

That Ψ_X is integral can also be seen directly from the following result:

Proposition (3.2.7). *Let A be a ring and B an A-algebra. The natural morphism $\Gamma^d_A(B) \to \mathrm{T}^d_A(B)$ is integral.*

Proof. Let $b \in B$ be any element and let $b_i = 1\otimes_A\cdots\otimes_A 1\otimes_A b\otimes_A 1\otimes_A\cdots\otimes_A 1$. Then b_i satisfies the following equation: $x^d - \sigma_1(b)x^{d-1} + \cdots + (-1)^d\sigma_d(b) = 0$ where $\sigma_k(b)$ is the k^{th} elementary symmetric function in the b_i:s. As $\sigma_k(b)$ is the image of $\gamma^k(b) \times \gamma^{d-k}(1)$ by the homomorphism $\Gamma^d_A(B) \to \mathrm{T}^d_A(B)$, the proposition follows. $\square$

3.3. **The Sym-Gamma morphism.** In this section we deduce some properties of the canonical morphism $\Psi_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ defined in Proposition (3.2.6).

Proposition (3.3.1). [Ryd07c, ?] *Let X/S be an* AF*-scheme. The canonical morphism $\Psi_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a universal homeomorphism with trivial residue field extensions. If S is purely of characteristic zero or X/S is flat, then Ψ_X is an isomorphism.*

From Proposition (3.3.1) we obtain the following results which only concerns $\mathrm{Sym}^d(X/S)$ but relies on the existence of the well-behaved functor Γ^d and the morphism $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$.

Corollary (3.3.2). *Let $S \to S'$ be a morphism of schemes and X/S an* AF*-scheme. The induced morphism $\mathrm{Sym}^d(X'/S') \to \mathrm{Sym}^d(X/S) \times_S S'$ is a universal homeomorphism with trivial residue field extensions. If S' is of characteristic zero then this morphism is an isomorphism. If X'/S' is flat then the morphism is a nil-immersion.*

Proof. Follows from Proposition (3.3.1) and the commutative diagram

$$\begin{tikzcd} \mathrm{Sym}^d(X'/S') \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Sym}^d(X/S) \times_S S' \arrow[d] \\ \Gamma^d(X'/S') \arrow[r, "\cong"] & \Gamma^d(X/S) \times_S S'. \end{tikzcd}$$

$\square$

Corollary (3.3.3). *Let X/S be an* AF*-scheme and $Z \hookrightarrow X$ a closed subscheme. Let $q : (X/S)^d \to \mathrm{Sym}^d(X/S)$ be the quotient morphism. The induced morphism $\mathrm{Sym}^d(Z/S) \to q\big((Z/S)^d\big)$ is subintegral. If S is of characteristic zero then this morphism is an isomorphism. If Z/S is flat then the morphism is a closed immersion.*

Proof. Follows from Proposition (3.3.1) and the commutative diagram

$$\begin{tikzcd} \mathrm{Sym}^d(Z/S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Sym}^d(X/S) \arrow[d] \\ \Gamma^d(Z/S) \arrow[r, hook] & \Gamma^d(X/S). \end{tikzcd}$$

□

In order to invesigate the homomorphism $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ more closely we introduce the following setup:

Notation (3.3.4). Let A be any ring and B an A-algebra. Choose a flat A-algebra C such that $B = C/I$ for some ideal I. Let $\varphi_i : C \hookrightarrow \mathrm{T}^d_A(C)$ be the homomorphism onto the i^{th} factor and let $J \subseteq \mathrm{T}^d_A(C)$ be the ideal generated by $\big(\varphi_i(I)\big)_{i=1,2,\dots,d}$ such that $\mathrm{T}^d_A(B) = \mathrm{T}^d_A(C/I) = \mathrm{T}^d_A(C)/J$. Let $G = \mathfrak{S}_d$ act on $\mathrm{T}^d_A(B)$ by permutations. Then $\mathrm{TS}^d_A(B) = (\mathrm{T}^d_A(C)/J)^G$. As we saw in §2.4 there is a canonical injective homomorphism $\mathrm{T}^d_A(C)^G/J^G \to (\mathrm{T}^d_A(C)/J)^G$. At the end of this section we will give some examples that show that this need not be an isomorphism.

Proposition (3.3.5). *Let A, B, C, I and J be as above. Let K be the kernel of the surjective homomorphism $\Gamma^d_A(C) \to \Gamma^d_A(C/I) = \Gamma^d_A(B)$. Then K is in the kernel of the canonical homomorphism $\Gamma^d_A(C) \cong \mathrm{TS}^d_A(C) \twoheadrightarrow \mathrm{TS}^d_A(B)/J^G$. Thus the canonical homomorphism $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ factors as*

$$\Gamma^d_A(B) = \Gamma^d_A(C)/K \twoheadrightarrow \mathrm{TS}^d_A(C)/J^G \hookrightarrow \mathrm{TS}^d_A(C/I) = \mathrm{TS}^d_A(B).$$

Furthermore, the kernel of $\psi : \Gamma^d_A(C)/K \twoheadrightarrow \mathrm{TS}^d_A(C)/J^G$ consists of nilpotent elements with orders dividing $|G|$. More precisely:

(i) *If A is a $\mathbb{Z}_{(p)}$-algebra with $p > 0$, e.g. a local ring with residue field k or a k-algebra with $\operatorname{char} k = p$, then every element in the kernel of ψ has an order equal to a power of p.*

(ii) *If A is purely of characteristic zero, i.e. a $\mathbb{Q}$-algebra, then ψ is an isomorphism.*

Proof. From (1.2.7) it follows that $K \subseteq \Gamma^d_A(B)$ is the ideal generated by elements of the form $\gamma^s(x) \times y$ where $x \in I$, $1 \leq s \leq d$ and $y \in \Gamma^{d-s}_A(B)$. Clearly $K \subseteq J^G$ and thus ψ is well-defined and surjective.

The question about nilpotency is local over A so we can assume that A is local with residue field k of characteristic exponent p.

Let c be any element of J^G. Write c as a sum $\sum_i^m a_i f_i$ where $f_i = f_{i1} \otimes_A \dots \otimes_A f_{id}$ such that for some j, depending on i, we have that $f_{ij} \in I$. Let $\hat{f}_i = (f_{i1}, f_{i2}, \dots, f_{id}) \in B^d$ and let $\hat{c}$ be the formal sum $\sum_i^m a_i \hat{f}_i \in A^{(B^d)}$. The action of $G = \mathfrak{S}_d$ on B^d induces an action on $A^{(B^d)}$.

Let $\mathrm{card}(G) = p^n r$ with r relatively prime to p and choose a subgroup H of order p^n. Let $D = \{1, 2, \dots, m\}$ and $\mathcal{I} = D^H = D^{p^n}$. Then

$$\prod_{h \in H} h(\hat{c}) = \prod_{h \in H} a_i h(\hat{f}_i) = \sum_{\alpha \in \mathcal{I}} \prod_{h \in H} a_{\alpha_h} h\left(\hat{f}_{\alpha_h}\right).$$

We let $a_\alpha = \prod_{h\in H} a_{\alpha_h}$ and let H act on $\mathcal{I}$ by $(h'\alpha)_h = \alpha_{h'h}$. Then $a_\alpha = a_{h'\alpha}$ and

$$\prod_{h\in H} a_{(h'\alpha)_h} h\left(\hat{f}_{(h'\alpha)_h}\right) = a_\alpha h'^{-1} \prod_{h\in H} h\left(\hat{f}_{\alpha_h}\right).$$

Thus, if we sum over all the elements of an orbit in $\mathcal{I}$ we obtain an H-invariant element:

$$\sum_{\beta\in H\alpha}\prod_{h\in H} a_{\beta_h} h\left(\hat{f}_{\beta_h}\right) = a_\alpha \sum_{\beta=h'\alpha\in H\alpha} h'^{-1}\prod_{h\in H} h\left(\hat{f}_{\alpha_h}\right).$$

Choosing representatives $g_1H,\dots,g_rH$ of the left cosets of H and summing over the cosets we obtain a G-invariant element

$$\frac{1}{r}\sum_{i=1}^{r} g_i\Big(\sum_{\beta\in H\alpha}\prod_{h\in H} a_{\beta_h} h\left(\hat{f}_{\beta_h}\right)\Big) \in A^{(B^d)}$$

independent on the choice of representatives g_i. This gives an element $t_\alpha \in K \subset \Gamma^d_A(B)$ which image by the canonical isomorphism $\Gamma^d_A(B) \cong \mathrm{TS}^d_A(B)$ is

$$\frac{1}{r}\sum_{i=1}^{r} g_i\Big(\sum_{\beta\in H\alpha}\prod_{h\in H} a_{\beta_h} h\left(f_{\beta_h}\right)\Big) \in \mathrm{TS}^d_A(B).$$

Finally, summing over all the orbits $H\alpha$ of $\mathcal{I}$ gives an element $t \in K \subset \Gamma^d_A(B)$ which image in $\mathrm{TS}^d_A(B)$ is

$$\frac{1}{r}\sum_{i=1}^{r} g_i\left(\prod_{h\in H} h(c)\right) = c^{p^n}.$$

□

Theorem (3.3.6). *Let A be any ring (resp. a $\mathbb{Z}_{(p)}$-algebra), B an A-algebra and d a positive integer. Let $f : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ be the canonical homomorphism. Then:*

- (i) *The associated morphism on the spectra ${}^a f$ is a universal homeomorphism.*
- (ii) *The kernel of f is a nilideal and any element in the kernel has an order dividing $d!$ (resp. any element has an order a power of p dividing $d!$).*
- (iii) *If $x \in \mathrm{TS}^d_A(B)$ then x^n is in the image of f for some n dividing $d!$ (resp. n a power of p dividing $d!$).*

Thus f is an isomorphism if $d!$ is invertible in A, e.g. A is purely of characteristic zero.

Proof. Let $B = C/I$ where C is a flat A-algebra and let $J \in \mathrm{T}^d(C)$ such that $\mathrm{T}^d(B) = \mathrm{T}^d(C)/J$ exactly as in the setup (3.3.4). By Proposition (3.3.5) we have a factorization

$$\Gamma^d_A(B) \twoheadrightarrow \mathrm{TS}^d_A(C)/J^G \hookrightarrow \mathrm{TS}^d_A(B)$$

where the first homomorphism is surjective and the second is injective. By Proposition (3.3.5) the kernel of the first homomorphism is as in (ii) of the theorem. By Proposition (2.4.6) the second homomorphism is as in (iii) of the theorem and a universal homeomorphism on the spectra. On the associated spectra we thus obtain a factorization of the morphism

$\operatorname{Spec}\bigl(\mathrm{TS}^d_A(B)\bigr) \to \operatorname{Spec}\bigl(\Gamma^d_A(B)\bigr)$ into a universal homeomorphism followed by a nilimmersion. This shows that the composition $f : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ has the properties stated in the theorem.

If $d!$ is invertible in A, then $f \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is an isomorphism for every p as it is the trivial map between two zero rings for any p dividing $d!$. This shows the last part of the theorem. □

Remark **(3.3.7).** From Theorem (3.3.6) we obtain another, more complicated, proof of the fact that $\operatorname{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a universal homeomorphism with trivial residue field extensions which is independent of Proposition (3.3.1).

Examples (3.3.8). The following examples are due to C. Lundkvist [Lun07]:

(i) An A-algebra B such that $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ is not injective
(ii) An A-algebra B such that $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ is not surjective
(iii) A surjection $B \to C$ of A-algebras such that $\mathrm{TS}^d_A(B) \to \mathrm{TS}^d_A(C)$ is not surjective
(iv) An A-algebra B such that $\Gamma^d_A(B)_{\mathrm{red}} \hookrightarrow \mathrm{TS}^d_A(B)_{\mathrm{red}}$ is not an isomorphism
(v) An A-algebra B and a base change $A \to A'$ such that the canonical homomorphism $\mathrm{TS}^d_A(B) \otimes_A A' \to \mathrm{TS}^d_{A'}(B')$ is not injective.
(vi) An A-algebra B and a base change $A \to A'$ such that the canonical homomorphism $\mathrm{TS}^d_A(B) \otimes_A A' \to \mathrm{TS}^d_{A'}(B')$ is not surjective.

Remark **(3.3.9).** The *seminormalization* of a scheme X is a universal homemomorphism with trivial residue fields $X^{\mathrm{sn}} \to X$ such that any universal homeomorphism with trivial residue field $X' \to X$ factors uniquely through $X^{\mathrm{sn}} \to X$ [Swa80]. If $X^{\mathrm{sn}} = X$ then we say that X is seminormal. If $X \to Y$ is a morphism and X is seminormal then $X \to Y$ factors canonically through $Y^{\mathrm{sn}} \to Y$.

Using Proposition (3.3.1) it is not difficult to show that $\operatorname{Sym}^d(X/S)^{\mathrm{sn}} = \operatorname{Sym}^d(X^{\mathrm{sn}}/S)^{\mathrm{sn}}$. Corollaries (3.3.2) and (3.3.3) then show that in the fibered category of seminormal schemes $\mathbf{Sch}^{\mathrm{sn}}$, taking symmetric products commutes with arbitrary base change and closed subschemes. This is a special property for Sym^d which does not hold for arbitrary quotients.

3.4. **The Chow scheme.**

Let k be a field and let E be a vector space over k with basis $x_0, x_1, \dots, x_n$. Let $E^\vee$ be the dual vector space with dual basis $y_0, y_1, \dots, y_n$. Let $X = \mathbb{P}(E) = \mathbb{P}^n_k$. If k'/k is a field extension then a point $x : \operatorname{Spec}(k') \to X$ is given by coordinates $(x_0 : x_1 : \dots : x_n)$ in k'. To x we associate the *Chow form* $F_x(y_0, y_1, \dots, y_n) = \sum_{i=0}^n x_i y_i \in k'[y_0, y_1, \dots, y_n]$ which is defined up to a constant.

A zero cycle on $X = \mathbb{P}^n_k$ is a formal sum of closed points. To any zero-dimensional subscheme $Z \hookrightarrow X$ we associate the zero cycle $[Z]$ defined as the sum of its points with multiplicities. If $\mathcal{Z} = \sum_j a_j[z_j]$ is a zero cycle on X and k'/k a field extension then we let $\mathcal{Z}_{k'} = \mathcal{Z} \times_k k' = \sum_j a_j[z_j \times_k k']$. It is clear that if $Z \hookrightarrow X$ is a zero-dimensional subscheme then $[Z] \times_k k' = [Z \times_k k']$.

We say that a cycle is effective if its coefficients are positive. The degree of a cycle $\mathcal{Z} = \sum_j a_j[z_j]$ is defined as $\deg(\mathcal{Z}) = \sum_j a_j \deg\bigl(k(z_j)/k\bigr)$. It is clear that $\deg(\mathcal{Z}_{k'}) = \deg(\mathcal{Z})$ for any field extension k'/k.

Let $\mathcal{Z}$ be an effective zero cycle on X and choose a field extension k'/k such that $\mathcal{Z}_{k'} = \sum_j a_j[z'_j]$ is a sum of k'-points, i.e. $k(z'_j) = k'$. We then define its Chow form as $F_{\mathcal{Z}} = \prod_j F_{z'_j}^{a_j}$. It is easily seen that

(i) $F_{\mathcal{Z}}$ does not depend on the choice of field extension k'/k.
(ii) $F_{\mathcal{Z}}$ has coefficients in k.
(iii) The degree of $F_{\mathcal{Z}}$ coincides with the degree of $\mathcal{Z}$.
(iv) $\mathcal{Z}$ is determined by $F_{\mathcal{Z}}$.

Further, if k is perfect there is a correspondence between zero cycles of degree d on X and Chow forms of degree d, i.e. homogeneous polynomials, $F \in k[y_0, y_1, \dots, y_n]$ which splits into d linear forms after a field extension. The Chow forms of degree d with coefficients in k is a subset of the linear forms on $\mathbb{P}\bigl(\mathrm{S}^d(E^\vee)\bigr)$ and thus a subset of the k-points of $\mathbb{P}\bigl(\mathrm{S}^d(E^\vee)^\vee\bigr) = \mathbb{P}\bigl(\mathrm{TS}^d(E)\bigr)$.

(3.4.1) *The Chow variety* — Classically it is shown that for $r \geq 0$ and $d \geq 1$ there is a closed subset of $\mathbb{P}\bigl(\mathrm{T}^{r+1}(\mathrm{TS}^d(E))\bigr)$ parameterizing r-cycles of degree d on $\mathbb{P}(E)$. The Chow variety $\mathrm{Chow}_{r,d}\bigl(\mathbb{P}(E)\bigr)$ is then taken as the *reduced* scheme corresponding to this subset. More generally, if S is any scheme and $\mathcal{E}$ is a locally free sheaf then there is a closed subset of $\mathbb{P}_S\bigl(\mathrm{T}^{r+1}(\mathrm{TS}^d(\mathcal{E}))\bigr)$ parameterizing r-cycles of degree d on $\mathbb{P}_S(\mathcal{E})$. In the case of zero cycles, however, we can find a canonical closed *subscheme* of $\mathbb{P}\bigl(\mathrm{TS}^d(E)\bigr)$ which parameterizes zero cycles of degree d as follows:

(3.4.2) *The Chow scheme for $\mathbb{P}(E)/k$* — Let k'/k be a field extension such that k' is algebraically closed. As $(\mathbb{P}(E)/k)^d \to \mathrm{Sym}^d(\mathbb{P}(E)/k)$ is integral, it is easily seen that a k'-point of $\mathrm{Sym}^d(\mathbb{P}(E)/k)$ corresponds to an unordered tuple $(x_1, x_2, \dots, x_d)$ of k'-points of $\mathbb{P}(E)$. Assigning such a tuple the Chow form of the cycle $[x_1] + [x_2] + \dots + [x_d]$ gives a map $\mathrm{Hom}\bigl(k', \mathrm{Sym}^d(\mathbb{P}(E)/k)\bigr) \to \mathrm{Hom}\bigl(k', \mathbb{P}(\mathrm{TS}^d(E))\bigr)$. It is easily seen to be compatible with the homomorphism of algebras

$$\bigoplus_{k\geq 0} \mathrm{S}^k\bigl(\mathrm{TS}^d(E)\bigr) \to \bigoplus_{k\geq 0} \mathrm{TS}^d\bigl(\mathrm{S}^k(E)\bigr)$$

and thus extends to a morphism of schemes

$$\mathrm{Sym}^d\bigl(\mathbb{P}(E)/k\bigr) \to \mathbb{P}\bigl(\mathrm{TS}^d(E)\bigr).$$

It is further clear that the image of this morphism consists of the Chow forms of degree d and that $\mathrm{Sym}^d\bigl(\mathbb{P}(E)/k\bigr) \to \mathbb{P}\bigl(\mathrm{TS}^d(E)\bigr)$ is universally injective and hence a universal homeomorphism onto its image as $\mathrm{Sym}^d\bigl(\mathbb{P}(E)/k\bigr)$ is projective. We let $\mathrm{Chow}_{0,d}\bigl(\mathbb{P}(E)\bigr)$ be the scheme-theoretical image of this morphism.

More generally, we define $\mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{E})/S\bigr)$ for any locally free sheaf $\mathcal{E}$ on S as follows:

Definition-Proposition (3.4.3). *Let S be any scheme and $\mathcal{E}$ a locally free $\mathcal{O}_S$-sheaf of finite type. Then the homomorphism $\bigoplus_{k\geq 0} \mathrm{S}^k\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{E}) \to \bigoplus_{k\geq 0} \mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^k\mathcal{E})$ induces a morphism*

$$\varphi_{\mathcal{E}} : \mathrm{Sym}^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{E})\big)$$

which is a universal homeomorphism onto its image. We let $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ *be its scheme-theoretic image.*

Proof. The question is local so we can assume that $S = \mathrm{Spec}(A)$ and $\mathcal{E} = \widetilde{M}$ where M is a free A-module of finite rank. Corollary (3.1.11), with $N = 1$ and $B = \bigoplus_{k\geq 0} \mathrm{S}^k M$, shows that $\bigoplus_{k\geq 0} \mathrm{S}^k\mathrm{TS}^d_A(M) \to \bigoplus_{k\geq 0} \mathrm{TS}^d_A(\mathrm{S}^k M)$ induces a well-defined morphism $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{E})\big)$.

To show that $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{E})\big)$ is a universal homeomorphism onto its image it is enough to show that it is universally injective as $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{E})/S\big) \to S$ is universally closed. As $\mathcal{E}$ is flat over S the symmetric product commutes with base change and it is enough to show that $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{E})\big)$ is injective when S is a field. This was discussed above. □

If $X \hookrightarrow \mathbb{P}(\mathcal{E})$ is a closed immersion (resp. an immersion) then the subset of $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ parameterizing cycles with support in X is closed (resp. locally closed). In fact, it is the image of $\mathrm{Sym}^d(X/S) \to \mathrm{Sym}^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$. Note that this morphism factors through $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ as $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{E})/S\big) = \Gamma^d\big(\mathbb{P}(\mathcal{E})/S\big)$ and that the morphism $\Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ has the same image by Proposition (3.3.1). As Γ^d is more well-behaved, e.g. commutes with base change $S' \to S$, the following definition makes sense:

Definition (3.4.4). Let S be any scheme and $\mathcal{E}$ a locally free sheaf on S. If $X \hookrightarrow \mathbb{P}(\mathcal{E})$ is a closed immersion we let $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ be the scheme-theoretic image of $\Gamma^d(X/S) \hookrightarrow \Gamma^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$. If $X \hookrightarrow \mathbb{P}(\mathcal{E})$ is an immersion we let $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ be the open subscheme of $\mathrm{Chow}_{0,d}\Big(\overline{X} \hookrightarrow \mathbb{P}(\mathcal{E})\Big)$ corresponding to cycles with support in X.

Remark **(3.4.5).** Classically $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is defined as the reduced subscheme of $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big) \hookrightarrow \mathbb{P}\big(\mathrm{TS}^d(\mathcal{E})\big)$ parameterizing zero cycles of degree d with support in X. It is clear that this is the reduction of the scheme $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ as defined in Definition (3.4.4).

Remark **(3.4.6).** If $\mathcal{E}$ is a locally free sheaf on S of finite type then by definition $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ is $\mathrm{Proj}(\mathcal{A})$ where $\mathcal{A}$ is the image of

$$\bigoplus_{k\geq 0} \mathrm{S}^k\big(\mathrm{TS}^d(\mathcal{E})\big) \to \bigoplus_{k\geq 0} \mathrm{TS}^d\big(\mathrm{S}^k(\mathcal{E})\big)$$

i.e. $\mathcal{A}$ is the subalgebra of $\bigoplus_{k\geq 0} \mathrm{TS}^d\big(\mathrm{S}^k(\mathcal{E})\big)$ generated by degree one elements. If $X \hookrightarrow \mathbb{P}(\mathcal{E})$ is a closed immersion then $X = \mathrm{Proj}(\mathcal{A})$ where $\mathcal{A}$ is a quotient of $\mathrm{S}(\mathcal{E})$. The Chow scheme $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is then $\mathrm{Proj}(\mathcal{B})$

where $\mathcal{B}$ is the subalgebra of $\bigoplus_{k\geq 0}\Gamma^d_{\mathcal{O}_S}(\mathcal{A}_k)$ generated by degree one elements.

Proposition (3.4.7). *Let S be any scheme and let $\mathcal{B}$ be a graded quasi-coherent $\mathcal{O}_S$-algebra of finite type generated in degree one. Then $\Gamma^d\big(\mathrm{Proj}(\mathcal{B})/S\big) = \mathrm{Proj}(\mathcal{D})$ where $\mathcal{D}=\bigoplus_{k\geq 0}\Gamma^d_A(\mathcal{B}_k)$. Let N be a positive integer and let $\mathcal{D}_N$ be the subring of $\mathcal{D}^{(N)}=\bigoplus_{k\geq 0}\Gamma^d_A(\mathcal{B}_{Nk})$ generated by $\Gamma^d_A(\mathcal{B}_N)$. The inclusion $\mathcal{D}_N\hookrightarrow\mathcal{D}^{(N)}$ induces a morphism $\psi_N : \Gamma^d\big(\mathrm{Proj}(\mathcal{B})/S\big)\cong\mathrm{Proj}(\mathcal{D}^{(N)})\to\mathrm{Proj}(\mathcal{D}_N)$. Furthermore ψ_N is a universal homeomorphism and*

(i) *If $\mathcal{B}$ is locally generated by at most $r+1$ elements and $N\geq r(d-1)$ then ψ_N is an isomorphism.*
(ii) *If S is purely of characteristic zero, i.e. a $\mathbb{Q}$-scheme, then ψ_N is an isomorphism for every N.*

Proof. The statements are local on S so we may assume that $S=\mathrm{Spec}(A)$ is affine and $\mathcal{B}=\widetilde{B}$ where B is a graded A-algebra finitely generated in degree one. Choose a surjection $B'=A[x_0,x_1,\dots,x_r]\twoheadrightarrow B$. Let $D=\bigoplus_{k\geq 0}\Gamma^d_A(B_k)$, $D'=\bigoplus_{k\geq 0}\Gamma^d_A(B'_k)$ and let D_N and D'_N be the subrings of $D^{(N)}$ and $D'^{(N)}$ generated by degree one elements. Then we have a commutative diagram

(3.4.7.1)
$$\begin{tikzcd} D'_N \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\circ"] & D_N \arrow[d, hook] \\ D'^{(N)} \arrow[r, two heads] & D^{(N)}. \end{tikzcd}$$

By Corollary (3.1.11) the inclusion $D'_N\hookrightarrow D'^{(N)}$ induces a morphism $\psi'_N : \mathrm{Proj}(D'^{(N)})\to\mathrm{Proj}(D'_N)$ with the properties (i) and (ii) and by Definition-Proposition (3.4.3) it is a universal homeomorphism. From the commutative diagram (3.4.7.1) it follows that the inclusion $D_N\hookrightarrow D^{(N)}$ induces a morphism $\psi_N : \mathrm{Proj}(D^{(N)})\to\mathrm{Proj}(D_N)$ with the same properties. □

Corollary (3.4.8). *Let S be any scheme and let $\mathcal{B}$ be a graded quasi-coherent $\mathcal{O}_S$-algebra of finite type generated in degree one. Then there is a canonical morphism $\varphi_{\mathcal{B}} : \Gamma^d\big(\mathrm{Proj}(\mathcal{B})/S\big)\to\mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{B}_1)\big)$ which is a universal homeomorphism onto its image. This morphism commutes with base change $S'\to S$ and surjections $\mathcal{B}\twoheadrightarrow\mathcal{B}'$.*

Remark (3.4.6) and Corollary (3.4.8) shows that we may extend the definition of $\mathrm{Chow}_{0,d}\big(X\hookrightarrow\mathbb{P}_S(\mathcal{E})\big)$ to include the case where $\mathcal{E}$ need not be locally free:

Definition (3.4.9). Let X/S be quasi-projective morphism of schemes and let $X\hookrightarrow\mathbb{P}_S(\mathcal{E})$ be an immersion for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type. Let $\overline{X}$ be the scheme-theoretic image of X in $\mathbb{P}_S(\mathcal{E})$ which can be written as $\overline{X}=\mathrm{Proj}(\mathcal{B})$ where $\mathcal{B}$ is a quotient of $\mathrm{S}(\mathcal{E})$. We let $\mathrm{Chow}_{0,d}\big(\overline{X}\hookrightarrow\mathbb{P}_S(\mathcal{E})\big)$ be the scheme-theoretic image of $\varphi_{\mathcal{B}} : \Gamma^d\big(\mathrm{Proj}(\mathcal{B})/S\big)\to\mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{B}_1)\big)$ or equivalently, the scheme-theoretic image of

$$\varphi_{\overline{X},\mathcal{E}} : \Gamma^d\big(\mathrm{Proj}(\mathcal{B})/S\big)\to\mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{B}_1)\big)\hookrightarrow\mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{E})\big).$$

We let $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\big)$ be the open subscheme of $\mathrm{Chow}_{0,d}\big(\overline{X} \hookrightarrow \mathbb{P}_S(\mathcal{E})\big)$ given by the image of

$$\Gamma^d(X/S) \subseteq \Gamma^d(\overline{X}/S) \to \mathrm{Chow}_{0,d}\big(\overline{X} \hookrightarrow \mathbb{P}_S(\mathcal{E})\big).$$

This is indeed an open subscheme as $\Gamma^d\big(\overline{X}/S\big) \to \mathrm{Chow}_{0,d}\big(\overline{X} \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is a homeomorphism by Corollary (3.4.8).

Remark **(3.4.10).** Let S be any scheme, $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module and $X \hookrightarrow \mathbb{P}(\mathcal{E})$ an immersion. Let $S' \to S$ be any morphism and let $X' = X \times_S S'$ and $\mathcal{E}' = \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$. There is a commutative diagram

$$\begin{tikzcd}
\Gamma^d(X'/S') \arrow[rr, "\varphi_{X',\mathcal{E}'}"] \arrow[d, "\cong"] \arrow[drr, phantom, "\circ"] & & \mathbb{P}_{S'}(\mathcal{E}') \arrow[d, "\cong"] \\
\Gamma^d(X/S) \times_S S' \arrow[rr, "\varphi_{X,\mathcal{E}} \times_S \mathrm{id}_{S'}"] & & \mathbb{P}_S(\mathcal{E}) \times_S S'
\end{tikzcd}$$

i.e. $\varphi_{X',\mathcal{E}'} = \varphi_{X,\mathcal{E}} \times_S \mathrm{id}_{S'}$. As the underlying sets of $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ and $\mathrm{Chow}_{0,d}\big(X' \hookrightarrow \mathbb{P}(\mathcal{E}')\big)$ are the images of $\varphi_{X,\mathcal{E}}$ and $\varphi_{X',\mathcal{E}'}$ it follows that $\big(\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\big) \times_S S'\big)_{\mathrm{red}}$ and $\mathrm{Chow}_{0,d}\big(X' \hookrightarrow \mathbb{P}_{S'}(\mathcal{E}')\big)_{\mathrm{red}}$ are equal. By the universal property of the scheme-theoretic image it thus follows that we have a nil-immersion

$$\text{(3.4.10.1)} \qquad \mathrm{Chow}_{0,d}\big(X' \hookrightarrow \mathbb{P}_{S'}(\mathcal{E}')\big) \hookrightarrow \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\big) \times_S S'.$$

As the scheme-theoretic image commutes with flat base change [$\mathrm{EGA}_{\mathrm{IV}}$, Lem. 2.3.1] the morphism (3.4.10.1) is an isomorphism if $S' \to S$ is flat.

If $Z \hookrightarrow X$ is an immersion (resp. a closed immersion, resp. an open immersion) then there is an immersion (resp. a closed immersion, resp. an open immersion)

$$\mathrm{Chow}_{0,d}\big(Z \hookrightarrow \mathbb{P}_S(\mathcal{E})\big) \hookrightarrow \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\big).$$

Proposition (3.4.11). *Let $S = \mathrm{Spec}(A)$ where A is affine and such that every residue field of S has at least d elements. Let $X = \mathrm{Proj}(B)$ where B is a graded A-algebra finitely generated in degree one. Then $\mathrm{Chow}\big(X \hookrightarrow \mathbb{P}(B_1)\big)$ is covered by open subsets of the form $U_f = \mathrm{Spec}(C_f)$ for $f \in B_1$ where C_f is the subring of $\Gamma^d_A\big(B_{(f)}\big)$ generated by elements of degree one, i.e. elements of the form $\times_{i=1}^n \gamma^{d_i}(b_i/f)$ with $b_i \in B_1$.*

Proof. ... ☐

3.5. **The Gamma-Chow morphism.** Let us first restate the contents of Proposition (3.4.7) taking into account the definition of $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$.

Proposition (3.5.1). *Let S be a scheme, $q : X \to S$ quasi-projective and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module of finite type such that there is an immersion $X \hookrightarrow \mathbb{P}(\mathcal{E})$. Let $k \geq 1$ be an integer. Then*

(i) *The canonical map*

$$\mathrm{S}\big(\Gamma^d_{\mathcal{O}_S}(\mathrm{S}^k\mathcal{E})\big) \to \bigoplus_{i\geq 0} \Gamma^d_{\mathcal{O}_S}(\mathrm{S}^{ki}\mathcal{E})$$

induces a morphism

$$\varphi_{\mathcal{E},k} : \Gamma^d(X/S) \hookrightarrow \Gamma^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathrm{S}^k\mathcal{E})\big)$$

which is a universal homeomorphism onto its image. The scheme-theoretical image of $\varphi_{\mathcal{E},k}$ is by definition $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes k})\big)$.

(ii) *Assume that either $\mathcal{E}$ is locally generated by at most $r+1$ elements and $k \geq r(d-1)$ or S has pure characteristic zero, i.e. is a $\mathbb{Q}$-scheme. Then $\varphi_{\mathcal{E},k}$ is a closed immersion and $\Gamma^d(X/S) \to$* $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes k})\big)$ *is an isomorphism.*

Remark **(3.5.2).** As $\Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is a universal homeomorphism, the *topology* of the Chow scheme does not depend on the chosen embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$.

In higher dimension, it is well-known that the Chow variety $\mathrm{Chow}_{r,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ does not depend on the embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$ as a *set*. This follows from the fact that a geometric point corresponds to an r-cycle of degree d [Sam55, §9.4d,h]. The invariance of the topology is also well-known, cf. [Sam55, §9.7]. This implies that the *weak normalization* of the Chow variety does not depend on the embedding in the analytic case cf. [AN67]. This also follows from functorial descriptions of the Chow variety over weakly normal schemes as in [Gue96] over $\mathbb{C}$ or more generally in [Kol96, §1.3][2]. We will now show that the residue fields of $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ do not depend on the embedding.

Proposition (3.5.3). *Let S, $q : X \to S$ and $\mathbb{P}(\mathcal{E})$ as in Proposition* (3.5.1). *The morphism $\varphi_{\mathcal{E}} : \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is a universal homeomorphism with trivial residue field extensions.*

Proof. We have already seen that the morphism $\varphi_{\mathcal{E}}$ is a universal homeomorphism. It is thus enough to show that it has trivial residue field extensions. To show this it is enough to show that for every point $a : \mathrm{Spec}(k) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ with $k = k^{\mathrm{sep}}$ there exists a, necessarily unique, point $b : \mathrm{Spec}(k) \to \Gamma^d(X/S)$ lifting a, i.e. the diagram

$$\begin{tikzcd} \Gamma^d(X/S) \arrow[r, "\varphi_{\mathcal{E}}"] & \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big) \\ & \mathrm{Spec}(k) \arrow[u, "a"'] \arrow[ul, dotted, "b"] \end{tikzcd}$$

has a unique filling. By Theorem (3.2.1) and Remark (3.4.10) the schemes $\Gamma^d(X/S)$ and $\big(\mathrm{Chow}_{0,d}(X \hookrightarrow \mathbb{P}(\mathcal{E}))\big)_{\mathrm{red}}$ commute with base change, i.e.

$$\Gamma^d(X/S) \times_S S' = \Gamma^d(X \times_S S'/S')$$
$$\big(\mathrm{Chow}_{0,d}(X \hookrightarrow \mathbb{P}(\mathcal{E})) \times_S S'\big)_{\mathrm{red}} = \mathrm{Chow}_{0,d}\big(X \times_S S' \hookrightarrow \mathbb{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})\big)_{\mathrm{red}}$$

for any $S' \to S$. We can thus assume that $S = \mathrm{Spec}(k)$ and hence that the image of a is a closed point. Then $\Gamma^d(X/k) = \mathrm{Sym}^d(X/k)$ as X/k is flat.

As $\varphi_{\mathcal{E}}$ is a universal homeomorphism, there is a unique lifting $\overline{b} : \overline{k} \to \Gamma^d(X/k)$ of a corresponding to a closed point of $\mathrm{Sym}^d(X/k)$. A closed point

[2]A suitable reference to Lawson-Friedlander should also be here...

in $\mathrm{Sym}^d(X/k)$ corresponds to a closed orbit in $(X/k)^d$. Let $X' = \coprod_{i=1}^m x_i \hookrightarrow X$ be the projection of the orbit corresponding to $\overline{b}$. Then $\overline{b}$ factors through $\Gamma^d(X'/S) \hookrightarrow \Gamma^d(X/S)$ and

$$\mathrm{im}(a) = \mathrm{im}\left(\varphi_{X,\mathcal{E}} \circ \overline{b}\right) \subseteq \mathrm{im}(\varphi_{X',\mathcal{E}}) = \mathrm{Chow}_{0,d}\big(X' \hookrightarrow \mathbb{P}(\mathcal{E})\big).$$

Replacing X with X' we can thus assume that X is a disjoint union of reduced points x_i. By Theorem (3.2.1) we then have that

$$\Gamma^d(X/k) = \Gamma^d\left(\coprod_{i=1}^m x_i/k\right) = \coprod_{\substack{(d_i)\in\mathbb{N}^m\\ \sum_{i=1}^m d_i = d}} \left(\prod_{i=1}^m \Gamma^{d_i}\big(k(x_i)/k\big)\right).$$

As $\overline{b}$ factors through one of these components it is enough to show that any point $k \to \mathrm{Chow}_{0,d_i}\big(k(x_i) \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ lifts uniquely to a point $k \to \Gamma^{d_i}\big(k(x_i)\big)$. We can therefore assume that $X = \mathrm{Spec}(k')$ where $k \hookrightarrow k'$ is an inseparable extension.

Let $s_1, s_2, \dots, s_n$ be generators of the k-vector space $\mathcal{E}$ and let $s_0 \in \mathcal{E}$ be such that $X \hookrightarrow \mathbb{P}(\mathcal{E})$ factors through $\mathrm{D}_+(s_0)$. Let f_i be the image of s_i/s_0 by the homomorphism $k[s_i/s_0] \to k'$ corresponding to the closed immersion $X \hookrightarrow \mathrm{D}_+(s_0)$ and let M be the k-submodule of k' generated by $f_1, f_2, \dots, f_n$. The f_i:s are then a set of generators of k' as a k-algebra and $\mathrm{Chow}_{0,d}\big(k' \hookrightarrow \mathbb{P}_k(\mathcal{E})\big)$ is the spectrum of the subring of $\Gamma^d_k(k')$ generated by $\Gamma^d_k(M)$. In particular $\gamma^d(f_i)$ is in this subring.

By Lemma (**??**) there exists at most one lifting $b : \mathrm{Spec}(k) \to \Gamma^d(k'/k)$ and such a lifting exists if $k'^d \subseteq k$. The lemma also shows that $\overline{b}^*\big(\gamma^d(f)\big) = f^d$ for any $f \in k'$. As $\overline{b}$ lifts a, it thus follows that $f_i^d = \overline{b}^*\big(\gamma^d(f_i)\big) = a^*\big(\gamma^d(f_i)\big) \in k$. In particular $f_i^{p^s} \in k$ where s is the p-order of d which shows that $k'^{p^s} \subseteq k$. Thus $\overline{b}$ is a k-point which concludes the proof. □

Remark **(3.5.4).** Proposition (3.5.3) also follows from the following fact. Let k be a field, E a k-vector space and $X \hookrightarrow \mathbb{P}_k(E)$ a subscheme. Let $\mathcal{Z}$ be an r-cycle on X. The residue field of the point corresponding to $\mathcal{Z}$ in the Chow variety $\mathrm{Chow}_{r,d}\big(X \hookrightarrow \mathbb{P}(E)\big)$, the *Chow field* of $\mathcal{Z}$, does not depend on the embedding $X \hookrightarrow \mathbb{P}^n_k$ [Kol96, Prop-Def I.4.4].

As $\varphi_{\mathcal{E}^{\otimes k}} : \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes k})\big)$ is an isomorphism for sufficiently large k by Proposition (3.5.1) the Chow field coincides with the corresponding residue field of $\Gamma^d(X/S)$.

3.6. **Families of cycles.** Let $\overline{k}$ be an algebraically closed field and let $\alpha : \mathrm{Spec}(\overline{k}) \to \mathrm{Sym}^d(X/S)$ be a geometric point. As $(X/S)^d \to \mathrm{Sym}^d(X/S)$ is integral α lifts (non-uniquely) to a geometric point $\beta : \mathrm{Spec}(\overline{k}) \to (X/S)^d$. Let $\pi_i : (X/S)^d \to X$ be the i^{th} projection and let $x_i = \pi_i \circ \beta$. It is easily seen that the different liftings β of α corresponds to the permutations of the d geometric points $x_i : \mathrm{Spec}(\overline{k}) \to X$. This gives a correspondence between $\overline{k}$-points of $\mathrm{Sym}^d(X/S)$ and zero cycles of degree d on $X \times_S \overline{k}$.

As $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ are universal homeomorphisms, there is a bijection between their geometric points. It is thus reasonable to say that $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ "parameterizes" zero cycles of degree d. Moreover, as $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to$

$\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ have trivial residue field extensions, there is a bijection between k-points for any field k.

Definition (3.6.1). Let $\mathcal{Z}$ be a zero cycle of degree d on $X \times_S \overline{k}$. The residue field of the point in $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ or $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ corresponding to $\mathcal{Z}$ is called the *Chow field* of $\mathcal{Z}$.

Definition (3.6.2). Let k be a field and X a scheme over k. Let k'/k and k''/k be field extensions of k. Two cycles $\mathcal{Z}'$ and $\mathcal{Z}''$ on $X \times_k k'$ and $X \times_k k''$ respectively, are said to be equivalent if there is a common field extension K/k of k' and k'' such that $\mathcal{Z}' \times_{k'} K = \mathcal{Z}'' \times_{k''} K$. If $\mathcal{Z}'$ is a cycle on $X \times_S k'$ equivalent to a cycle on $X \times_S k''$ then we say that $\mathcal{Z}'$ is *defined over* k''.

Remark **(3.6.3).** If $\mathcal{Z}$ is a cycle on $X \times_S k$ then the corresponding morphism $\mathrm{Spec}(\overline{k}) \to \mathrm{Sym}^d(X/S)$ factors through $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$. Thus if $\mathcal{Z}$ is defined over a field K then the Chow field is contained in K. Conversely $\mathcal{Z}$ is defined over an inseparable extension of the Chow field by (**??**). Thus, in characteristic zero the Chow field of $\mathcal{Z}$ is the unique minimal field of definition of $\mathcal{Z}$. In positive characteristic, it can be shown that the Chow field is the intersection of all minimal field of definitions, cf. [Kol96, Thm. I.4.5].

Let T be any scheme and $f : T \to \mathrm{Sym}^d(X/S)$, $f : T \to \Gamma^d(X/S)$ or $f : T \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ a morphism. A geometric k-point of T then corresponds to a zero cycle of degree d on $X \times_S k$. The following definition is therefore natural.

Definition (3.6.4). A family of cycles parameterized by T is a morphism $f : T \to \mathrm{Sym}^d(X/S)$, $f : T \to \Gamma^d(X/S)$ or $f : T \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$. We use the notation $\mathcal{Z} \to T$ to denote a family of cycles parameterized by T and let $\mathcal{Z}_t$ be the cycle over t, i.e. the cycle corresponding to $\overline{k(t)} \to T \to \mathrm{Sym}^d(X/S)$, etc.

As $\Gamma^d(X/S)$ commutes with base change and has other good properties it is the "correct" parameter scheme and the corresponding morphisms $T \to \Gamma^d(X/S)$ are the "correct" families of cycles.

3.7. **The Hilb-Sym morphism.** ...

4. Outside the degeneracy locus

In this section we will prove that the morphisms

$$\mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$$

are all isomorphisms over an open subset parameterizing "non-degenerated families" of points.

4.1. **Non-degenerated families.**

(4.1.1) *Non-degenerate families of subschemes* — Let k be a field and X be a k-scheme. If $Z \hookrightarrow X$ is a closed subscheme then it is natural say that Z is non-degenerate if $Z_{\overline{k}}$ is reduced, i.e. if $Z \to k$ is geometrically reduced. If Z is of dimension zero then Z is non-degenerate if and only if

$Z \to k$ is étale. Similarily for any scheme S, a finite flat morphism $Z \to S$ of finite presentation is called a non-degenerate family if every fiber is non-degenerate, or equivalently, if $Z \to S$ is étale.

Let $Z \to S$ be a family of zero dimensional subschemes, i.e. a finite flat morphism of finite presentation. The subset of S consisting of $s \in S$ such that the fiber $Z_s \to k(s)$ is non-degenerate is open [EGA_{IV}, Thm. 12.2.1 (viii)]. Thus, there is an open subset $\text{Hilb}^d(X/S)_{\text{nd}}$ of $\text{Hilb}^d(X/S)$ parameterizing non-degenerate families.

(4.1.2) *Non-degenerate families of cycles* — A zero cycle $\mathcal{Z} = \sum_i a_i[z_i]$ on a k-scheme X is called non-degenerate if every point in the support of $\mathcal{Z}_{\overline{k}}$ has multiplicity one. Equivalently the multiplicities a_i are all one and the field extensions $k(z_i)/k$ are separable. It is clear that there is a one-to-one correspondence between non-degenerate zero cycles on X and non-degenerated zero-dimensional subschemes of X.

Given a family of cycles $\mathcal{Z} \to S$, i.e. a morphism $S \to \text{Sym}^d(X/S)$, $S \to \Gamma^d(X/S)$ or $S \to \text{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$, we say that it is non-degenerate family if $\mathcal{Z}_s$ is non-degenerate for every $s \in S$.

(4.1.3) *Degeneracy locus of cycles* — Let $X \to S$ be a morphism of schemes and let $\Delta \hookrightarrow (X/S)^d$ be the big diagonal, i.e. the union of all diagonals $\Delta_{i,j} : (X/S)^{d-1} \to (X/S)^d$. It is clear that the image of Δ by $(X/S)^d \to \text{Sym}^d(X/S)$ parameterizes degenerate cycles and that the open complement parameterizes non-degenerate cycles. We let $\text{Sym}^d(X/S)_{\text{nd}}$, $\Gamma^d(X/S)_{\text{nd}}$ and $\text{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)_{\text{nd}}$ be the open subschemes of $\text{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\text{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$ respectively, parameterizing non-degenerate cycles.

We will now give an explicit cover of the degeneracy locus of $\text{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\text{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$. Some of the notation is inspired by [ES04, 2.4 and 4.1] and [RS07].

Definition (4.1.4). Let A be a ring and B an A-algebra. Let $\mathbf{x} = (x_1, x_2, \dots, x_d) \in B^d$. We define the symmetrization and anti-symmetrization operators from B^d to $\text{T}^d_A(B)$ as follows

$$\begin{aligned} s(\mathbf{x}) &= \sum_{\sigma \in \mathfrak{S}_d} x_{\sigma(1)} \otimes_A x_{\sigma(2)} \otimes_A \cdots \otimes_A x_{\sigma(d)} \\ a(\mathbf{x}) &= \sum_{\sigma \in \mathfrak{S}_d} (-1)^{|\sigma|} x_{\sigma(1)} \otimes_A x_{\sigma(2)} \otimes_A \cdots \otimes_A x_{\sigma(d)}. \end{aligned}$$

As s and a are A-multilinear, s is symmetric and a is alternating it follows that we get induced homomorphisms, also denoted s and a

$$\begin{aligned} s &: \text{S}^d_A(B) \to \text{TS}^d_A(B) \\ a &: \textstyle\bigwedge^d_A(B) \to \text{T}^d_A(B). \end{aligned}$$

Remark **(4.1.5).** If d is invertible in A, then the symmetrization and anti-symmetrization operators are sometimes defined as $\frac{1}{d}s$ and $\frac{1}{d}a$. We will never use this convention. In [ES04] $a(\mathbf{x})$ is denoted $\nu(\mathbf{x})$ and referred to as a norm vector.

Definition (4.1.6). Let A be a ring and B an A-algebra. Let $\mathbf{x} = (x_1, x_2, \dots, x_d) \in B^d$ and $\mathbf{y} = (y_1, y_2, \dots, y_d) \in B^d$. We define the following element in $\Gamma^d_A(B)$

$$\delta(\mathbf{x}, \mathbf{y}) = \det\bigl(\gamma^1(x_i y_j) \times \gamma^{d-1}(1)\bigr)_{ij}.$$

Following [RS07] we call the ideal $I = I_A = \bigl(\delta(\mathbf{x}, \mathbf{y})\bigr)_{\mathbf{x},\mathbf{y} \in B^d}$, the canonical ideal. As δ is multilinear and alternating in both arguments we extend the definition of δ to a function

$$\delta\,:\,\textstyle\bigwedge^d_A(B) \times \bigwedge^d_A(B) \to \mathrm{S}^2_A\bigl(\bigwedge^d_A(B)\bigr) \to \Gamma^d_A(B).$$

Proposition (4.1.7) ([ES04, Prop 4.4])**.** *Let A be a ring, B an A-algebra and $\mathbf{x}, \mathbf{y} \in B^d$. The image of $\delta(\mathbf{x}, \mathbf{y})$ by $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B) \hookrightarrow \mathrm{T}^d_A(B)$ is $a(\mathbf{x})a(\mathbf{y})$. In particular $a(\mathbf{x})a(\mathbf{y})$ is symmetric.*

Lemma (4.1.8) ([ES04, Lem. 2.5])**.** *Let A be a ring and B and A' be A-algebras. Let $B' = B \otimes_A A'$. Denote by $I_A \subset \Gamma^d_A(B)$ and $I_{A'} \subset \Gamma^d_{A'}(B') = \Gamma^d_A(B) \otimes_A A'$ the canonical ideals corresponding to B and B'. Then $I_A A' = I_{A'}$.*

Lemma (4.1.9). *Let S be a scheme and X and S' be S-schemes. Let $X' = X \times_S S'$. Let $\varphi : \Gamma^d(X'/S') = \Gamma^d(X/S) \times_S S' \to \Gamma^d(X/S)$ be the projection morphism. The inverse image by φ of the degeneracy locus of $\Gamma^d(X/S)$ is the degeneracy locus of $\Gamma^d(X'/S')$.*

Proof. Obvious as we know that a geometric point $\mathrm{Spec}(k) \to \Gamma^d(X/S)$ corresponds to a zero cycle of degree d on $X \times_S \mathrm{Spec}(k)$. □

Lemma (4.1.10). *Let k be a field and B a k-algebra generated as an algebra by the k-vector field $V \subseteq B$. Let k'/k be a field extension and $x_1, x_2, \dots, x_d$ be d distinct k'-points of $\mathrm{Spec}(B \otimes_k k')$. If k has at least $\binom{d}{2}$ elements then there is an element $b \in V$ such that the values of b at $x_1, x_2, \dots, x_d$ are distinct.*

Proof. For a vector space $V_0 \subseteq V$ we let $B_0 \subseteq B$ be the sub-algebra generated by V_0. There is a finite dimensional vector space $V_0 \subseteq V$ such that the images of $x_1, x_2, \dots, x_d$ in $\mathrm{Spec}(B_0 \otimes_k k')$ are distinct. Replacing V and B with V_0 and B_0 we can thus assume that V is finite dimensional. It is further clear that we can assume that $B = \mathrm{S}(V)$. The points $x_1, x_2, \dots, x_d$ then corresponds to vectors of $V^\vee \otimes_k k'$ and we need to find a k-rational hyperplane which does not contain the $\binom{d}{2}$ difference vectors $x_i - x_j$. A similar counting argument as in the proof of Lemma (3.1.6) shows that if k has at least $\binom{d}{2}$ elements then this is possible. □

Proposition (4.1.11). *Let A be a ring and B an A-algebra. Let $V \subset B$ be an A-submodule such that B is generated by V as an algebra. Consider the following three ideals of $\Gamma^d_A(B)$*

(i) *The canonical ideal $I_1 = \bigl(\delta(\mathbf{x}, \mathbf{y})\bigr)_{\mathbf{x},\mathbf{y} \in B^d}$.*
(ii) $I_2 = \bigl(\delta(\mathbf{x}, \mathbf{x})\bigr)_{\mathbf{x} \in B^d}$.
(iii) $I_3 = \bigl(\delta(\mathbf{x}, \mathbf{x})\bigr)_{\mathbf{x}=(1,b,b^2,\dots,b^{d-1}),\, b \in V}$.

The ideals I_1 and I_2 are ideals of definition for the degeneracy locus of $\Gamma^d\bigl(\operatorname{Spec}(B)/\operatorname{Spec}(A)\bigr) = \operatorname{Spec}\bigl(\Gamma^d_A(B)\bigr)$. If every residue field of A has at least $\binom{d}{2}$ elements then so is I_3.

Proof. The discussion in (4.1.3) shows that it is enough to prove that the image of the ideals I_k by the homomorphism $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B) \hookrightarrow \mathrm{T}^d_A(B)$ set-theoretically defines the big diagonal of $\operatorname{Spec}\bigl(\mathrm{T}^d_A(B)\bigr)$. By Proposition (4.1.7) the image of $\delta(\mathbf{x},\mathbf{y})$ is $a(\mathbf{x})a(\mathbf{y})$. Thus the radicals of the images of I_1 and I_2 equals the radical of $J = \bigl(a(\mathbf{x})\bigr)_{\mathbf{x}\in B^d}$. It is further easily seen that J is contained in the ideal of every diagonal of $\operatorname{Spec}\bigl(\mathrm{T}^d_A(B)\bigr)$. Equivalently, the closed subset corresponding to J contains the big diagonal.

By Lemmas (4.1.8) and (4.1.9) it is enough to show the first part of the proposition after any base change $A \to A'$ such that $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$ is surjective. We can thus assume that every residue field of A has at least $\binom{d}{2}$ elements. Both parts of the proposition then follows if we show that the closed subset corresponding to the ideal

$$K = \bigl(a(1, b, b^2, \dots, b^{d-1})\bigr)_{b\in V} \subseteq \mathrm{T}^d_A(B)$$

is contained in the big diagonal. As the formation of K commutes with base changes $A \to A'$ which are either surjections or localizations we can assume that A is a field with at least $\binom{d}{2}$ elements.

Let $\operatorname{Spec}(k) : x \to \operatorname{Spec}\bigl(\mathrm{T}^d_A(B)\bigr)$ be a point corresponding to d distinct k-points $x_1, x_2, \dots, x_d$ of $\operatorname{Spec}(B \otimes_A k)$. Lemma (4.1.10) shows that there is an element $b \in V$ which takes d distinct values $a_1, a_2, \dots, a_d \in k$ on the d points. The value of $a(1, b, b^2, \dots, b^{d-1})$ at x is then

$$\sum_{\sigma\in\mathfrak{S}_d} (-1)^{|\sigma|} a_1^{\sigma(1)-1} a_2^{\sigma(2)-1} \dots a_d^{\sigma(d)-1} = \det \begin{pmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{d-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_d & a_d^2 & \dots & a_d^{d-1} \end{pmatrix} = \prod_{j<i}(a_i - a_j)$$

which is non-zero. Thus x is not contained in the zero-set of K. This shows that zero-set of K is contained in the big diagonal and hence that zero-set defined by K is the big diagonal. $\square$

4.2. Non-degenerated symmetric tensors and divided powers.

Proposition (4.2.1). *Let A be a ring, B an A-algebra and $x, y \in \bigwedge^d_A(B)$. Then $\Gamma^d_A(B)_{\delta(x,y)} \to \mathrm{TS}^d_A(B)_{\delta(x,y)}$ is an isomorphism.*

Proof. Denote the canonical homomorphism $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ with φ. Let $f \in \mathrm{TS}^d_A(B)$. As the anti-symmetrization operator $a : \mathrm{T}^d_A(B) \to \mathrm{T}^d_A(B)$ is a $\mathrm{TS}^d_A(B)$-module homomorphism we have that $fa(x) = a(fx)$. By Proposition (4.1.7)

$$f\varphi\bigl(\delta(x,y)\bigr) = fa(x)a(y) = a(fx)a(y) = \varphi\bigl(\delta(fx,y)\bigr)$$

which shows that φ is surjective after localization in $\delta(x,y)$.

Choose a surjection $F \twoheadrightarrow B$ with F a flat A-algebra and let I be the kernel of $F \twoheadrightarrow B$. Let J be the kernel of $\mathrm{T}^d_A(F) \twoheadrightarrow \mathrm{T}^d_A(B)$. This is the setting of Notation (3.3.4). As discussed there, the kernel of $\mathrm{TS}^d_A(F) \to \mathrm{TS}^d_A(B)$ is $J^G = J \cap \mathrm{TS}^d_A(F)$.

Let $f \in J^G$. As $f \in J$ we can write f as a sum $f_1 + f_2 + \cdots + f_n$ such that $f_i = f_{i1} \otimes f_{i2} \otimes \cdots \otimes f_{id} \in \mathrm{T}^d_A(F)$ with $f_{ij} \in I$ for some j depending on i. Choose a lifting $z \in \bigwedge^d_A(F)$ of x. Identifying $\Gamma^d_A(F)$ and $\mathrm{TS}^d_A(F)$, we have that $f\delta(z,y) = \delta(fz,y)$. This is a sum of determinants with elements in $\Gamma^d_A(F)$ such that there is a row where every element is in $\gamma^1(I) \times \gamma^{d-1}(1)$. Thus $\delta(fz,y)$ is in the kernel of $\Gamma^d_A(F) \twoheadrightarrow \Gamma^d_A(B)$ by (1.2.7). The image of f in $\Gamma^d_A(B)$ is thus zero after multiplying with $\delta(x,y)$. Consequently φ is injective after localization in $\delta(x,y)$. □

Corollary (4.2.2). *Let S be a scheme and X/S an* AF*-scheme. Then* $\mathrm{Sym}^d(X/S)_{\mathrm{nd}} \to \Gamma^d(X/S)_{\mathrm{nd}}$ *is an isomorphism.*

Proof. By Proposition (3.2.6) we can assume that S and X are affine. The corollary then follows from Propositions (4.1.11) and (4.2.1). □

Definition (4.2.3). Let A be any ring and $B = A[x_1, x_2, \dots, x_r]$. We call the elements $f \in \Gamma^d_A(B)$ of degree one, see Definition (1.5.2), *multilinear* or *elementary multisymmetric functions*. These are elements of the form

$$\gamma^{d_1}(x_1) \times \gamma^{d_2}(x_2) \times \cdots \times \gamma^{d_n}(x_n) \times \gamma^{d-d_1-\cdots-d_n}(1).$$

We let $\Gamma^d_A(A[x_1, x_2, \dots, x_n])_{\text{mult.lin.}}$ denotes the sub-A-algebra of $\Gamma^d_A(A[x_1, x_2, \dots, x_n])$ generated by multi-linear elements.

Remark **(4.2.4).** If the characteristic of A is zero or more generally if $d!$ is invertible in A, then $\Gamma^d_A(A[x_1, x_2, \dots, x_n])_{\text{mult.lin.}} = \Gamma^d_A(A[x_1, x_2, \dots, x_n])$ by Theorem (1.5.4).

Proposition (4.2.5). *Let A be a ring and $B = A[x_1, x_2, \dots, x_n]$. Let $b \in B_1$ and let $\mathbf{x} = (1, b, b^2, \dots, b^{d-1})$. Then $\big(\Gamma^d_A(B)_{\text{mult.lin.}}\big)_{\delta(\mathbf{x},\mathbf{x})} \hookrightarrow \Gamma^d_A(B)_{\delta(\mathbf{x},\mathbf{x})}$ is an isomorphism.*

Proof. Let $f \in \Gamma^d_A(B) = \mathrm{TS}^d_A(B)$. We will show that f is a sum of products of multilinear elements after multiplication by a power of $\delta(\mathbf{x},\mathbf{x})$. As $f\delta(\mathbf{x},\mathbf{x}) = \delta(f\mathbf{x},\mathbf{x})$ and the latter is a sum of products of elements of the type $\gamma^1(c) \times \gamma^{d-1}(1)$ we can assume that f is of this type. As $c \mapsto \gamma^1(c) \times \gamma^{d-1}(1)$ is linear we can further assume that $c = x^\alpha$ for some non-trivial monomial $x^\alpha \in B$. It will be useful to instead assume that $c = x^\alpha b^k$ with $|\alpha| \geq 1$ and $k \in \mathbb{N}$. We will now proceed on induction on $|\alpha|$.

Assume that $|\alpha| = 1$. If $k = 0$ then $f = \gamma^1(x^\alpha b^k) \times \gamma^{d-1}(1)$ is multilinear. We continue with induction on k to show that $f \in \Gamma^d_A(B)_{\text{mult.lin.}}$. We have that

$$\begin{aligned} f = \gamma^1(x^\alpha b^k) \times \gamma^{d-1}(1) = \big(\gamma^1(x^\alpha b^{k-1}) \times \gamma^{d-1}(1)\big)\big(\gamma^1(b) \times \gamma^{d-1}(1)\big) \\ - \gamma^1(x^\alpha b^{k-1}) \times \gamma^1(b) \times \gamma^{d-2}(1) \end{aligned}$$

and by induction it is enough to show that the last term is in $\Gamma^d_A(B)_{\text{mult.lin.}}$. Similar use of the relation

$$\begin{aligned} \gamma^1(x^\alpha b^{k-\ell}) \times \gamma^\ell(b) \times \gamma^{d-\ell-1}(1) = \big(\gamma^1(x^\alpha b^{k-\ell-1}) \times \gamma^{d-1}(1)\big)\big(\gamma^{\ell+1}(b) \times \gamma^{d-\ell-1}(1)\big) \\ - \gamma^1(x^\alpha b^{k-\ell-1}) \times \gamma^{\ell+1}(b) \times \gamma^{d-\ell-2}(1) \end{aligned}$$

with $1 \leq l \leq d-2$ and $l \leq k-1$ shows that it is enough to consider either $\gamma^1(x^\alpha) \times \gamma^k(b) \times \gamma^{d-k-1}(1)$ if $k \leq d-1$ or $\gamma^1(x^\alpha b^{k-d+1}) \times \gamma^{d-1}(b)$ if $k > d-1$. The first element of these is multilinear and the second is the product of the multilinear element $\gamma^d(b)$ and $\gamma^1(x^\alpha b^{k-d}) \times \gamma^{d-1}(1)$ which by the induction on k is in $\Gamma^d_A(B)_{\mathrm{mult.lin.}}$.

If $|\alpha| > 1$ then $x^\alpha = x^{\alpha'} x^{\alpha''}$ for some α', α'' such that $|\alpha'|, |\alpha''| < |\alpha|$. We have that

$$f = \gamma^1(c) \times \gamma^{d-1}(1) = \big(\gamma^1(x^{\alpha'} b^k) \times \gamma^{d-1}(1)\big)\big(\gamma^1(x^{\alpha''}) \times \gamma^{d-1}(1)\big) \\ - \gamma^1(x^{\alpha'} b^k) \times \gamma^1(x^{\alpha''}) \times \gamma^{d-2}(1).$$

By induction it is enough to show that the last term is a sum of products of multilinear elements, after suitable multiplication by $\delta(\mathbf{x},\mathbf{x})$. Let $g = \gamma^1(x^{\alpha'} b^k) \times \gamma^1(x^{\alpha''}) \times \gamma^{d-2}(1)$. Then $g\delta(\mathbf{x},\mathbf{x}) = \delta(g\mathbf{x},\mathbf{x})$ which is a sum of products of elements of the kind $\gamma^1(x^{\alpha'} b^{t'}) \times \gamma^{d-1}(1)$ and $\gamma^1(x^{\alpha''} b^{t''}) \times \gamma^{d-1}(1)$. By induction on $|\alpha|$ these are in $\big(\Gamma^d_A(B)_{\mathrm{mult.lin.}}\big)_{\delta(\mathbf{x},\mathbf{x})}$. □

Corollary (4.2.6). *Let X/S be quasi-projective morphism of schemes and let $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ be an immersion for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type. Then $\Gamma^d(X/S)_{\mathrm{nd}} \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)_{\mathrm{nd}}$ is an isomorphism.*

Proof. As Γ commutes with arbitrary base change and Chow commutes with flat base change we may assume that S is affine and, using Lemma (1.3.3), that every residue field of S has at least $\binom{d}{2}$ elements. If $\mathcal{E}' \twoheadrightarrow \mathcal{E}$ is a surjection of $\mathcal{O}_S$-modules then $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big) = \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}')\big)$ by Definition (3.4.9) and we may thus assume that $\mathcal{E}$ is free. Further as $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is the schematic image of $\Gamma^d(X/S) \hookrightarrow \Gamma^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ we may assume that $X = \mathbb{P}(\mathcal{E}) = \mathbb{P}^n$.

By Proposition (3.4.11) and the assumption on the residue fields of $S = \mathrm{Spec}(A)$, the scheme $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ is covered by affine open subsets over which the morphism $\Gamma^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ corresponds to the inclusion of rings

$$\Gamma^d_A(A[x_1, x_2, \dots, x_n])_{\mathrm{mult.lin.}} \hookrightarrow \Gamma^d_A(A[x_1, x_2, \dots, x_n]).$$

The corollary now follows from Propositions (4.1.11) and (4.2.5). □

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