Homework exercises for lecture #5

TO BE HANDED IN ON MARCH 2.

Either do 1+2 or 3.

1. Show that if X is an irreducible topological space then its topological profinite fundamental group is trivial.
2. Say that two irreducible curves meet at two points. Show that the topological (here it is the Zariski topology) profinite fundamental group is isomorphic to $\hat{\mathbb{Z}}$.
3. (From Lenstra: Open and closed subgroups of profinite groups.)
   Let $\pi = \varprojlim_{i \in I} \pi_i \subset \prod_{i \in I} \pi_i$ be a profinite group of a projective system (I, π_i, f_{ij}), with all π_i finite groups. Let $g_i : \pi \to \pi_i$ be the projection maps for $i \in I$. Let further $\pi' \subset \pi$ be a subgroup.
   (a) Prove: π' is open $\iff$ π' is closed and of finite index $\iff$ $\exists i \in I$ such that $\ker(g_i) \subset \pi'$.
   (b) Prove: π' is closed $\iff$ there is a system of subgroups $\rho_i \subset \pi_i$ for $i \in I$ with $\pi' = \pi \cap \prod_{i \in I} \rho_i$ $\iff$ there is a system of subgroups $\rho_i \subset \pi_i$ for $i \in I$ with $\pi' = \pi \cap \prod_{i \in I} \rho_i$ and for which in addition $f_{ij}(\rho_i) = \rho_j$ for all $i, j \in I$ with $i \geq j$.
   (c) Prove that π' is profinite if it is closed.
   (d) Suppose that π' is a closed normal subgroup of π. Prove that π/π', with the quotient topology, is profinite.