Homework exercises for lecture #14

1. (cf. Milne Exercise V.1.10a, p. 162) Let F be a constructible sheaf (of abelian groups) on $X_{\text{ét}}$. Show that the following are equivalent
   (a) F is locally constant.
   (b) $j_{\overline{x}}^* F$ is a constant sheaf for every geometric point $\overline{x}$, where $j_{\overline{x}}\colon X_{\overline{x}} \to X$ denotes the map from the strict henselization $X_{\overline{x}} = \operatorname{Spec} \mathcal{O}_{X,\overline{x}}$.
   (c) For every specialization $x_1 \rightsquigarrow x_0$, the cospecialization map $F_{\overline{x_0}} \to F_{\overline{x_1}}$ is bijective (for some choice $\mathcal{O}_{X,\overline{x_0}} \to \mathcal{O}_{X,\overline{x_1}}$).
2. Give a counter-example to the previous exercise if F is not constructible. *Hint: take a suitable subsheaf of an infinite direct sum of constant sheaves.*
3. (cf. Milne Exercise V.1.10b, p. 162) Show that every torsion sheaf on $X_{\text{ét}}$ is a direct limit of constructible sheaves of abelian groups.
4. (cf. Milne p. 164) Let X be a noetherian scheme. Let $F = (F_n)$ be a constructible ℓ-adic sheaf on X. Show that X is a finite union of locally closed subschemes Z_i such that $F_n|_{Z_i}$ is locally constant for all n.
5. Give an example of a constructible ℓ-adic sheaf $F = (F_n)$ where F_1 is locally constant but F_2 is not.