Étale cohomology #9

2016-04-06

Last time: fundamental exact sequence

$$U \overset{j}{\underset{\text{open}}{\hookrightarrow}} X \overset{i}{\underset{\text{closed}}{\hookleftarrow}} Z \qquad U = X \setminus Z$$

Given sheaf $\mathcal{F}$ on $X_{\acute{e}t}$, have exact seq:

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

Today: Recollement: describe sheaves on X in terms of sheaves on U and Z.

Observation: Given sheaf $\mathcal{F} \in \mathcal{S}(X)$, we obtain sheaves

$j^*\mathcal{F} \in \mathcal{S}(U)$, $i^*\mathcal{F} \in \mathcal{S}(Z)$ but also adjunction maps (unit maps)

$$\mathcal{F} \xrightarrow{\eta} j_* j^* \mathcal{F}$$
$$\mathcal{F} \to i_* i^* \mathcal{F}$$

Applying i^* to the first gives: $(i^*\mathcal{F}) \xrightarrow{\phi_{\mathcal{F}}} i^* j_* (j^* \mathcal{F})$

Def: $\Pi(X) = (\mathcal{S}(Z), \mathcal{S}(U), i^* j_*)$ is the category with

objects: triples (F_1, F_2, ϕ) $\quad F_1 \in \mathcal{S}(Z)$, $F_2 \in \mathcal{S}(U)$, $\phi : F_1 \to i^* j_* F_2$

morphisms: $(\psi_1 : F_1 \to F_1', \psi_2 : F_2 \to F_2') : (F_1, F_2, \phi) \to (F_1', F_2', \phi')$ s.th.

$$\begin{array}{ccc} F_1 & \xrightarrow{\phi} & i^* j_* F_2 \\ \psi_1 \downarrow & & \downarrow i^* j_* \psi_2 \\ F_1' & \xrightarrow{\phi'} & i^* j_* F_2' \end{array}$$

<u>Theorem</u> (Recollement, Milne 3.10) There is an equiv of categories

$$S(X) \xrightarrow{\ t\ } T(X)$$

$$\mathcal{F} \overset{t}{\longmapsto} (i^*\mathcal{F},\ j^*\mathcal{F},\ \phi_{\mathcal{F}})$$

$$i_*\mathcal{F}_1 \underset{i_*\phi \searrow\, i_*i^*j_*\mathcal{F}_2}{\times} j_*\mathcal{F}_2 \overset{s}{\longleftarrow\!\!\shortmid} (\mathcal{F}_1, \mathcal{F}_2, \phi)$$

<u>pf</u>: t is functorial since i^*, j^*, η functorial.
s ——"—— $i_*, i^*, j_*, \times$ functorial.

The unit maps $\mathcal{F} \to i_*i^*\mathcal{F}$, $\mathcal{F} \to j_*j^*\mathcal{F}$ induce a map

$$\begin{tikzcd}
\boxed{\mathcal{F} \to st\mathcal{F}} = i_*i^*\mathcal{F} \underset{i_*i^*j_*j^*\mathcal{F}}{\times} j_*j^*\mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & j_*j^*\mathcal{F} \arrow[d] \\
i_*i^*\mathcal{F} \arrow[r, "i_*i^*\eta"'] & i_*i^*j_*j^*\mathcal{F}
\end{tikzcd} \qquad (*)$$

Similarly, the counit maps $i^*i_*\mathcal{F} \xrightarrow{\cong} \mathcal{F}$, $j^*j_*\mathcal{F} \xrightarrow{\cong} \mathcal{F}$ induce maps

$$i^*\Big(i_*\mathcal{F}_1 \underset{\cdots}{\times} j_*\mathcal{F}_2\Big) \overset{i^* \text{ left-exact}}{\cong} i^*i_*\mathcal{F}_1 \underset{i^*j_*\mathcal{F}_2}{\times} i^*j_*\mathcal{F}_2 \cong i^*i_*\mathcal{F}_1 \xrightarrow{\cong} \mathcal{F}_1$$

$$j^*\Big(i_*\mathcal{F}_1 \underset{\cdots}{\times} j_*\mathcal{F}_2\Big) \overset{j^* \text{l.e}}{\cong} j^*i_*\mathcal{F}_1 \underset{j^*i_*i^*j_*\mathcal{F}_2}{\times} j^*j_*\mathcal{F}_2 \cong 0 \times j^*j_*\mathcal{F}_2 = j^*j_*\mathcal{F}_2 \xrightarrow{\cong} \mathcal{F}_2$$

$\uparrow 0$ b/c $U \cap Z = \emptyset$

which gives

$$\boxed{ts\mathcal{F} \xrightarrow{\cong} \mathcal{F}} \quad (\text{check compatibility w/ } \phi)$$

It remains to verify that $\mathcal{F} \to st\mathcal{F}$ is an isomorphism.

This can be done on <u>stalks</u>.

If $x \in U$, then the diagram $(*)$ becomes:

$$\begin{tikzcd} \mathcal{F}_{\bar{x}} \arrow[r, "\cong"] & 0 \times_0 \mathcal{F}_{\bar{x}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{F}_{\bar{x}} \arrow[d] \\ & 0 \arrow[r] & 0 \end{tikzcd}$$

If $x \in Z$, then the diagram $(*)$ becomes:

$$\begin{tikzcd} \mathcal{F}_{x} \arrow[r, "\cong"] & \mathcal{F}_{\bar{x}} \times_{(j_*j^*\mathcal{F})_{\bar{x}}} (j_*j^*\mathcal{F})_{\bar{x}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (j_*j^*\mathcal{F})_{\bar{x}} \arrow[d, "id"] \\ & \mathcal{F}_{\bar{x}} \arrow[r, "n_{\bar{x}}"'] & (j_*j^*\mathcal{F})_{\bar{x}} \end{tikzcd}$$

$\square$

<u>Def</u>: $\mathcal{F} \in \mathbb{S}(X)$, $\mathrm{Supp}(\mathcal{F}) := \{ x \in X : \mathcal{F}_{\bar{x}} \neq 0 \}$ (independent of choice of $\bar{x}$)

<u>Cor</u> (Milne 3.11) $i_* : \mathbb{S}(Z) \longrightarrow \mathbb{S}(X) \cong \mathbb{T}(X)$ fully faithful

w/ image sheaves w/ $\mathrm{Supp}(\mathcal{F}) \subseteq Z$, or equivalently, triples

$(F_1, 0, 0)$.

<u>Rmk</u> (stalks) $s(F_1, F_2, \phi)_{\bar{x}} = \begin{cases} (F_1)_{\bar{x}} & \text{if } x \in Z \\ (F_2)_{\bar{x}} & \text{if } x \in U \end{cases}$

<u>Thus</u> a sequence

$$(F_1', F_2', \phi') \longrightarrow (F_1, F_2, \phi) \longrightarrow (F_1'', F_2'', \phi'')$$

exact iff

$$F_1' \longrightarrow F_1 \longrightarrow F_1''$$
$$F_2' \longrightarrow F_2 \longrightarrow F_2''$$

are exact.

Adjoint functors $(i^*, i_* = i_!, i^!)$

— " — $(j_!, j^! = j^*, j_*)$

$i^* : F_1 \longleftarrow\!\shortmid (F_1, F_2, \phi)$

$i_* : F_1 \longmapsto (F_1, 0, 0)$

$i^! : \ker\phi \longleftarrow\!\shortmid (F_1, F_2, \phi)$

$i_* i^! \mathcal{F}$ = subsheaf of $\mathcal{F}$ consisting of sections supported on Z

$= \mathcal{H}^0_Z(\mathcal{F})$

$j_! : F_2 \longmapsto (0, F_2, 0)$

$j^* : F_2 \longleftarrow\!\shortmid (F_1, F_2, \phi)$

$j_* : F_2 \longmapsto (i^* j_* F_2, F_2, id)$

$\Gamma(Z, i^!\mathcal{F}) = \ker(\mathcal{F}(X) \longrightarrow \mathcal{F}(U))$

Fundamental exact sequence:

$$0 \longrightarrow j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0$$

$$0 \longrightarrow (0, j^*\mathcal{F}, 0) \longrightarrow (i^*\mathcal{F}, j^*\mathcal{F}, \phi_{\mathcal{F}}) \longrightarrow (i^*\mathcal{F}, 0, 0) \longrightarrow 0$$

$i^*, i_*, j_!, j^!$ are exact but $i^!, j_*$ only left-exact.

Rmk: In derived category, have Rj_* and $Ri^!$ and Verdier duality gives another exact triangle:

$$i_* Ri^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F}$$

but the sequence

$$0 \longrightarrow i_* i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow j_* j^* \mathcal{F} \longrightarrow 0$$

not exact on the right.

i_*, j_* and $j^!$ are fully faithful.

Rmk: $U=\emptyset \Rightarrow |Z|=|X|$, i.e. $Z\hookrightarrow X$ nil-immersion

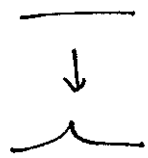
Then recollement says $i_*: \mathcal{S}(Z) \xrightarrow{\simeq} \mathcal{S}(X)$: $i^*=i^!$.

More generally:

Thm (SGA 1, Exp IX, Thm 4.10) If $\pi: Y \longrightarrow X$ universal homeomorphism then $\pi^{-1}: (\acute{e}t/X) \longrightarrow (\acute{e}t/Y)$ is an equivalence of categories, hence $\pi^*: \mathcal{S}(X) \longrightarrow \mathcal{S}(Y)$ is an equivalence of categories.

sketch pf: Descent along π + nil-immersion case.

Ex: Normalization of cuspidal cubic is univ homeo



Rmk: Voevodsky '91 has shown that normal schemes of f.t. $/\mathbb{Q}$ can be recovered from $(\acute{e}t/X)$.

Right-derived functors

$\mathcal{A}$ abelian category (e.g. $\mathcal{S}(X_{\acute{e}t})$)
or $\mathcal{Ab}$ = (ab. grps)

$I \in \mathcal{A}$ is injective if $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to Ab$ exact.

$\mathcal{A}$ has enough injectives if $\forall M \in \mathcal{A} \ \exists \ M \underset{\text{mono}}{\hookrightarrow} \underset{\text{injective}}{I}$.

Thm (Milne III.1, Hartshorne III.1.1A, 1.4)
Let $\mathcal{A}, \mathcal{B}$ abelian categories and $F : \mathcal{A} \to \mathcal{B}$ left-exact covariant functor. If $\mathcal{A}$ has enough injectives, then $\exists$ universal δ-functor $R^\bullet F = (R^i F)_{i \geq 0}$ s.th.

(a) $R^0 F \cong F$

(b) $(R^i F)(I) = 0 \quad \forall I$ injective, $i \geq 1$

Rmk: δ-functor means; $R^i F : \mathcal{A} \to \mathcal{B}$ additive functors; for every exact $0 \to M' \to M \to M'' \to 0$ have $\delta^i : (R^i F)(M'') \to (R^{i+1} F)(M')$ s.th.

(1) $\forall \ 0 \to M' \to M \to M'' \to 0$ SES obtain LES.

(2) δ functorial w.r.t. morphisms b/w SES.

universal δ-functor means: for any δ-functor $(T^i)_{i \geq 0}$ ~~and [illegible]~~ and natural trfm $F \xrightarrow{\varphi^0} T^0$, $\exists$ unique natural trfm $R^i F \xrightarrow{\varphi^i} T^i$ $i \geq 1$ s.th. $\varphi^\bullet : (R^\bullet F) \to (T^\bullet)$ morphism of δ-functors.

<u>Def</u>: $M \in \mathcal{A}$ is <u>F-acyclic</u> if $(R^iF)(M) = 0 \ \forall i \geq 1$.

(so injective $\Rightarrow$ F-acyclic for all F)

<u>Calculating R^iF</u>: Let $M \in \mathcal{A}$ and pick F-acyclic resolution, i.e. (e.g. injective resolution)

$$0 \to M \to N^0 \to N^1 \to N^2 \to \cdots$$

exact w/ N^i F-acyclic $\forall i$. Gives <u>complex</u>:

$$FN^0 \to FN^1 \to FN^2 \to \cdots$$

and $(R^iF)(M) = H^i(FN^\bullet)$.

sketch <u>pf</u>: Split resolution in SES: $\begin{tikzcd}[column sep=small] 0 \arrow[r] & M \arrow[r] & N^0 \arrow[rr] \arrow[dr, two heads] & & N^1 \arrow[rr] \arrow[dr, two heads] & & N^2 \arrow[r] \arrow[dr, two heads] & \cdots \\ & & & Q^0 \arrow[ur, hook] & & Q^1 \arrow[ur, hook] & & {} \end{tikzcd}$ $\square$

Consider resulting LES.

<u>Lemma</u>: (a) a product of injectives is injective.

(b) a functor $F: \mathcal{A} \to \mathcal{B}$ w/ an <u>exact</u> left adjoint $G: \mathcal{B} \to \mathcal{A}$ <u>preserves injectives</u>, i.e. I inj $\Rightarrow$ $F(I)$ injective.

<u>pf</u>: (a) $\mathrm{Hom}(-, \prod_\alpha I_\alpha) = \prod_\alpha \mathrm{Hom}(-, I_\alpha)$ and $\prod_\alpha$ is exact. $\square$

(b) $\mathrm{Hom}(-, F(I)) = \mathrm{Hom}(G(-), I)$ is a comp. of exact functors.

<u>Prop.</u> $\mathcal{S}(X_E)$ has enough injectives.

<u>pf 1</u>: ($E = \acute{e}t$) Let $F \in \mathcal{S}(X_{\acute{e}t})$. For each $x \in X$ choose $\bar{x} \xrightarrow{u_x} X$.

$\mathcal{S}(\bar{x}_{\acute{e}t}) \cong (Ab)$, hence has enough injectives. Choose

$u_x^* F = F_{\bar{x}} \hookrightarrow I_x$ w/ I_x injective.

Consider

$$F \longrightarrow F^* := \prod_x (u_x)_* u_x^* F \longrightarrow F^{**} := \prod_x (u_x)_* I_x$$

Then F^{**} injective by lemma.

$F \longrightarrow F^*$ mono since $F_{\bar{x}} \longrightarrow ((u_x)_* F_{\bar{x}})_{\bar{x}}$ iso.

$F^* \longrightarrow F^{**}$ mono since $(u_x)_*$ left-exact and $\prod$ is left-exact. $\square$

<u>pf 2</u>: Any abelian category satisfying

- AB5 = arb. direct sums exist and are exact
  + filtered direct limits are exact
- AB3* = arb ~~direct~~ products exist (but need not be exact)
- $\exists$ a generator G: for any $u, v : M \to N$, if $u \neq v$, $\exists f : G \to M$ s.th. $uf \neq vf$.

has enough injectives.

(use the generator and Baer's argument)

<u>Lemma</u>: $\mathcal{S}(X_E)$ satisfies AB5 + AB3* + GEN (a <u>Grothendieck abelian cat.</u>)

<u>pf</u>: AB5 + AB3* (see Milne II 2.15)

GEN: take $G = \prod_{\substack{j : U \to X \\ \text{in } \mathcal{C}/X}} j_! \mathbb{Z}$.

Derived functors

(a) $\Gamma(X,-): \mathcal{S}(X_E) \to Ab \rightsquigarrow R^i\Gamma(X,-) = H^i(X,-)$

(b) $\Gamma(U,-): \mathcal{S}(X_E) \to Ab \rightsquigarrow H^i(U,-)$

(c) $i: \mathcal{S}(X_E) \longrightarrow \mathbb{P}(X_E) \rightsquigarrow \underline{H}^i(-)$

(d) Fix $\mathcal{F} \in \mathcal{S}(X_E)$. $\mathrm{Hom}(\mathcal{F},-): \mathcal{S}(X_E) \to Ab \rightsquigarrow \mathrm{Ext}^i(\mathcal{F},-)$

(e) —— ıı —— $\mathcal{H}om(\mathcal{F},-): \mathcal{S}(X_E) \to \mathcal{S}(X_E) \rightsquigarrow \mathcal{E}xt^i(\mathcal{F},-)$

$\mathcal{G} \longmapsto \mathcal{H}om(\mathcal{F},\mathcal{G})$

$U \mapsto \mathrm{Hom}(\mathcal{F}(U),\mathcal{G}(U))$

(f) $\pi: X'_{E'} \longrightarrow X_E \rightsquigarrow \pi_*: \mathcal{S}(X'_{E'}) \to \mathcal{S}(X_E) \rightsquigarrow R^i\pi_*$
continuous

<u>Rmk</u>: Ext^i, $\mathcal{E}xt^i$ also give LES from SES in first variable.

<u>Prop</u>: For any $U \to X$ in $\mathbb{C}/X$: $H^i(U,\mathcal{F}) \cong H^i(U,\mathcal{F}|_U)$

<u>pf</u>: $\mathcal{S}(X_E) \xrightarrow[\text{exact}]{|_U} \mathcal{S}(U_E) \xrightarrow{\Gamma} Ab$

$\mathcal{F} \longmapsto \mathcal{F}|_U$

Enough to prove that $|_U$ <u>preserves injectives</u> (so we can calculate $H^i(U,\mathcal{F}|_U)$ by an injective resolution on X). Follows from lemma:

<u>Lemma</u>: For any $j: U \to X$ in $\mathbb{C}/X$ exists left adjoint $j_!$ to j^* and $j_!$ exact.

<u>pf</u>: $(j_!\mathcal{F})(\underbrace{V}_{\varphi: V \to X}) = a\Big(\bigoplus_{\substack{V \xrightarrow{\alpha} U \\ \varphi \searrow \ \swarrow j \\ X}} \mathcal{F}(\underbrace{V}_{\alpha: V \to U})\Big)$ $\quad \mathcal{F} \in \mathcal{S}(U_E)$

<u>Cor</u>: $\underline{H}^i(\mathcal{F})$ is presheaf $U \longmapsto H^i(U,\mathcal{F}) \cong H^i(U,\mathcal{F}|_U)$

<u>Rmk</u>: $\pi : X \longrightarrow Y$ morphism. Then π^* <u>exact</u> so induces

$$H^i(Y, \mathcal{F}) \longrightarrow H^i(X, \pi^*\mathcal{F})$$

(for quasi-coh. cohomology, needs π flat for this)

<u>Rmk</u>: If π <u>univ homeomorphism</u>, then $H^i(Y,\mathcal{F}) \xrightarrow{\cong} H^i(X, \pi^*\mathcal{F})$.

<u>Rmk</u>: If π_* <u>exact</u>, e.g. π <u>finite</u>, then induces

$$H^i(X, \mathcal{F}) \longrightarrow H^i(Y, \pi_*\mathcal{F})$$

Not true for <u>π affine</u>. In general, need Leray spectral sequence.

<u>Ex</u> (Galois cohomology)

$X = \operatorname{Spec} k$, $G = \operatorname{Gal}(k) = \operatorname{Gal}(k^{sep}/k)$.

$\mathcal{S}(X_{\acute{e}t}) \cong$ discrete G-modules (i.e. M abelian group w/ discrete topology w/ continuous action of G w/ profinite topology $\Leftrightarrow$ stabilizers are open subgroups)

$\mathcal{F} \longmapsto \mathcal{F}_{\bar{x}}$

$H^i(\operatorname{Spec} k, \mathcal{F}) \cong \underbrace{H^i(G, \mathcal{F}_{\bar{x}})}_{\text{Galois cohomology}}$ often non-trivial for $i>0$

$H^i(\operatorname{Spec} k^{sep}, \mathcal{F}) = 0 \quad \forall i > 0$.

<u>Rmk</u>: Very different from quasi-coherent cohomology. ~~No affines~~ Affines and even fields have cohomology. But we have analogue of local rings:

$X = \operatorname{Spec} \mathcal{O}^h_{X,\bar{x}}$ then $\mathcal{S}(X_{\acute{e}t}) \cong \mathcal{S}(\bar{x}_{\acute{e}t}) \cong$ ~~[illegible]~~ Ab $\left(\operatorname{Spec} \Omega \xrightarrow{\bar{x}} X\right)$ and $\Gamma(-)$ an iso

so $H^i(\operatorname{Spec} \mathcal{O}^h_{X,\bar{x}}, \mathcal{F}) = 0 \quad \forall i > 0$