Étale cohomology #8

23/3 - 2016

The category of sheaves on $X_{\acute{e}t}$

$\mathrm{PreSh}(X_{\acute{e}t})$ presheaves of abelian groups

Kernels and cokernels are computed pointwise: $0 \to \ker(\alpha) \to P \xrightarrow{\alpha} P'$

$$0 \to \ker(\alpha)(U) \to P(U) \to P'(U)$$

Exactness is pointwise: $P \to P' \to P''$ exact $\Leftrightarrow$ $P(U) \to P'(U) \to P''(U)$ exact $\forall U \to X$

$\mathrm{Sh}(X_{\acute{e}t})$ sheaves of abelian groups

kernels equals presheaf kernels

images / cokernels $\neq$ presheaf images / cokernels

$i: \mathrm{Sh}(X_{\acute{e}t}) \longrightarrow \mathrm{PreSh}(X_{\acute{e}t})$

Sheafification: $\underset{\text{presheaf}}{P} \longrightarrow \underset{\text{associated sheaf}}{aP}$

$\mathrm{Hom}(P, i(\underset{\text{sheaf}}{F})) \xrightarrow{\sim} \mathrm{Hom}(aP, F)$

i.e. a left adjoint to i.

$P_{\bar{x}} \xrightarrow{\sim} (aP)_{\bar{x}}$

Lemma: $\underset{\text{presheaf}}{P} \xrightarrow{i} \underset{\text{sheaf}}{F}$. Then (F, i) is the associated sheaf if

(a) two sections of P are equal in F iff they are locally equal in P.

(b) i is locally surjective (i.e. $\forall s \in F(U \to X)$ $\exists$ covering $\{U_i \to U\}$ s.th. $s|_{U_i}$ in the image of $P(U_i \to U \to X)$ $\forall i$)

Construction of aF:

$P \xrightarrow{i} P^* = \prod_{x \in X} (P_{\bar{x}})^{\bar{x}}$

For every $x \in X$, choose a geom point $\bar{x}: \Omega \to X$ w/ image x

$P_{\bar{x}}$ abelian group (stalk at $\bar{x}$), $(P_{\bar{x}})^{\bar{x}}$ skyscraper sheaf.

This fulfills (a).

$aP :=$ subsheaf of P^* generated by $i(P)$. This also fulfills (b).

<u>Lemma</u>: $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ seq. of sheaves. TFAE

(a) seq is exact

(b) $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ exact $\forall U \xrightarrow{\text{ét}} X$

(c) $0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}}$ exact $\forall \bar{x} \to X$.

<u>pf</u>: Forgetful map $i: Sh \to PreSh$ is right adjoint, hence left exact. so (a) $\Rightarrow$ (b).

<u>Lemma</u>: $\mathcal{F} \xrightarrow{\alpha} \mathcal{F}'$ map of sheaves. TFAE

(a) $\mathcal{F} \to \mathcal{F}' \to 0$ exact.

(b) α locally surjective.

(c) $\mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}}$ surjective $\forall \bar{x} \to X$.

<u>$Sh(X_{ét})$ abelian category</u>

$\ker(\mathcal{F} \to \mathcal{F}')(U) = \ker(\mathcal{F}(U) \to \mathcal{F}'(U))$

$\operatorname{coker}(\mathcal{F} \to \mathcal{F}') = a(\operatorname{coker}(i\mathcal{F} \to i\mathcal{F}'))$

$im = coim$ (check on stalks)

$a: PreSh \to Sh$ is exact (check on stalks)

Direct image

$\pi : Y \to X$ $\quad$ P presheaf on $Y_{\acute{e}t}$

$$(\pi_* P)(U \to X) := P(U \times_X Y \to Y) \qquad \text{(sometimes denoted } \pi_P\text{)}$$

This is a presheaf.

Lemma: $\mathcal{F}$ sheaf, then $\pi_* \mathcal{F}$ is a sheaf.

pf: $U \xrightarrow{\acute{e}t} X$, $\{U_i \to U\}$ covering. Then have

$$\begin{array}{ccccc} (\pi_* \mathcal{F})(U) & \longrightarrow & \prod (\pi_* \mathcal{F})(U_i) & \rightrightarrows & \prod (\pi_* \mathcal{F})(U_i \times_U U_j) \\ \| & & \| & & \| \\ \mathcal{F}(U \times_X Y) & \longrightarrow & \prod \mathcal{F}(U_i \times_X Y) & \rightrightarrows & \prod \mathcal{F}((U_i \times_X Y) \times_{U \times_X Y} (U_j \times_X Y)) \end{array}$$

Bottom row exact since $\mathcal{F}$ sheaf and $\{U_i \times_X Y \to U \times_X Y\}$ covering.
$\Rightarrow$ top row exact $\Rightarrow$ $\pi_* \mathcal{F}$ sheaf. $\square$

Rmk: $\pi_* : \mathrm{PreSh}(Y_{\acute{e}t}) \to \mathrm{PreSh}(X_{\acute{e}t})$ exact: $\quad 0 \to P' \to P \to P'' \to 0$ exact

$$\Rightarrow 0 \to P'(U \times_X Y) \to P(U \times_X Y) \to P''(U \times_X Y) \to 0$$

exact $\forall U \to X$.

$\Rightarrow$ $\pi_* : \mathrm{Sh}(Y_{\acute{e}t}) \to \mathrm{Sh}(X_{\acute{e}t})$ left exact.

Ex: $\pi : X \to \mathrm{Spec}(\bar{k})$. Then $\pi_* = \Gamma$ global sections.

$\mathcal{F} \to \mathcal{F}'$ ~~loc~~ surj. $\not\Rightarrow$ $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}')$ surj. so π_* not right exact.

Stalks: $\bar{y} \to Y \xrightarrow{\pi} X$, $\quad \bar{x} = \pi(\bar{y})$

$$(\pi_* \mathcal{F})_{\bar{x}} = \varinjlim_{(U \times_X Y, \bar{y})} \mathcal{F}(U \times_X Y) \longrightarrow \varinjlim_{(V, \bar{y})} \mathcal{F}(V) = \mathcal{F}_{\bar{y}}$$

in general neither surj nor inj.

<u>Prop</u>: (a) $Z \overset{\pi}{\hookrightarrow} X$ closed imm, $\bar{x} \to X$, $(\pi_* \mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & \text{if } x \in Z \\ 0 & \text{if } x \notin Z \end{cases}$.

(b) $Y \xrightarrow{\pi} X$ finite, $(\pi_* \mathcal{F})_{\bar{x}} = \prod_{y \mapsto x} \mathcal{F}_{\bar{y}}^{d(y)} = \prod_{\substack{\bar{y} \to Y \\ \pi(\bar{y}) = \bar{x}}} \mathcal{F}_{\bar{y}}$

$$d(y) = \overset{\text{sep.}}{\deg}\left(k(y)/k(x)\right)$$

(c) $U \overset{\pi}{\hookrightarrow} X$ open, $(\pi_* \mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & \text{if } x \in U \\ ?? & \text{if } x \notin U \end{cases}$

<u>Extension by zero</u>

$\pi = j : U \hookrightarrow X$ open immersion.

i.e. $V \xrightarrow{\text{ét}} X$ factors via $U \overset{j}{\hookrightarrow} X$

$\mathcal{F}$ presheaf on $U_{\text{ét}}$.

$\mathcal{F}_!(V \xrightarrow{v} X) = \begin{cases} \mathcal{F}(V) & \text{if } \pi(U) \supseteq v(V) \\ 0 & \text{o/w} \end{cases}$ presheaf

$\mathcal{F}$ sheaf on $U_{\text{ét}}$, $\pi_! \mathcal{F} := a(\mathcal{F}_!)$ <u>extension by zero</u>.

<u>Rmk</u>: $(\pi_! \mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$ $\Rightarrow \pi_!$ is exact.

<u>Adjunction</u>: $\operatorname{Hom}_{X_{\text{ét}}}(\mathcal{F}_!, \underset{\text{presheaf}}{Q}) \xrightarrow{\sim} \operatorname{Hom}_{U_{\text{ét}}}(\mathcal{F}, Q|_U)$

$\Rightarrow \operatorname{Hom}_{X_{\text{ét}}}(j_! \mathcal{F}, \underset{\text{sheaf}}{Q}) \xrightarrow{\sim} \operatorname{Hom}_{U_{\text{ét}}}(\mathcal{F}, Q|_U)$

So $j_!$ left adjoint to $j^* = (-)|_U$.

Inverse image

$Y \xrightarrow{\pi} X$, P presheaf on $X_{ét}$

$\pi^p P = P'$ presheaf on $Y_{ét}$ given by

$$P'(V \xrightarrow{v} Y) = \varinjlim_{\substack{V \to U \\ v\downarrow \;\circ\; \downarrow ét \\ Y \xrightarrow{\pi} X}} P(U)$$

not a sheaf in general even if P sheaf.

note that such diagrams $\Rightarrow$

$$\begin{tikzcd} V \arrow[d, "ét"'] \arrow[dr, "ét"] & \\ Y & U \times_X Y \arrow[l, "ét"] \end{tikzcd}$$

$\pi^* \mathcal{F} = a(\pi^p \mathcal{F})$ when $\mathcal{F}$ sheaf

$\pi^* : Sh(X_{ét}) \longrightarrow Sh(Y_{ét})$

Adjunction:

$$\begin{array}{ccc} \mathrm{Hom}_{Y_{ét}}(\underset{\text{presheaf on } Y}{P'}, Q) & \xrightarrow{\sim} & \mathrm{Hom}_{X_{ét}}(\underset{\text{presheaf on } X}{P}, \pi_* Q) \\ \| & & \| \\ \mathrm{Hom}_{Y_{ét}}(\pi^* \mathcal{F}, \underset{\text{sheaf on } Y}{\mathcal{G}}) & \xrightarrow{\sim} & \mathrm{Hom}_{X_{ét}}(\mathcal{F}, \pi_* \mathcal{G}) \end{array}$$

So π^* left adjoint of π_*.

Rmk: $\pi : U \to X$ étale, then $\pi^* \mathcal{F} = \mathcal{F}|_U$.
$\mathcal{F}$ sheaf

Stalks: $\pi : Y \longrightarrow X$, $\bar{y} \to Y$, $\mathcal{F}$ sheaf, $\bar{x} = \pi(\bar{y})$, then $(\pi^* \mathcal{F})_{\bar{y}} = \mathcal{F}_{\bar{x}}$ $\Rightarrow$ π^* <u>is exact</u>.

Moreover: $\bar{x} \longrightarrow X$ then $\bar{x}^* \mathcal{F} = \mathcal{F}_{\bar{x}}$ so stalks are a special case of pull-backs = inverse image.

Fundamental exact sequence of an open immersion

$U \overset{j}{\underset{\text{open}}{\hookrightarrow}} X \overset{i}{\underset{\text{closed}}{\hookleftarrow}} Z \qquad X = U \amalg Z \quad (\text{ie. } U = X \setminus Z)$

$\mathcal{F}$ sheaf on $X_{\acute{e}t}$. Have maps:

$j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \quad$ (counit of adjunction $j_!, j^*$: adjoint of $j^*\mathcal{F} \xrightarrow{id} j^*\mathcal{F}$)

$\mathcal{F} \longrightarrow i_* i^* \mathcal{F} \quad$ (unit of adjunction i^*, i_* : adjoint of $i^*\mathcal{F} \xrightarrow{id} i^*\mathcal{F}$)

<u>Claim:</u> $0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$ exact.

<u>pf:</u> Can verify this on stalks:

$$0 \to (j_! j^* \mathcal{F})_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to (i_* i^* \mathcal{F})_{\bar{x}} \to 0$$

If $\bar{x} \in U$, then this becomes:

$$0 \to \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to 0 \to 0$$

If $\bar{x} \in Z$, then this becomes

$$0 \to 0 \to \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to 0 .$$

□