Étale cohomology #6

2/3 - 2016

Étale fund. grp (cont)

geom point $\mathrm{Spec}(\Omega) \xrightarrow{\bar{x}} X$: $x \in X$, $k(x) \hookrightarrow \Omega$

Thm: X connected, $\bar{x}$ geom pt $\quad \mathrm{F\acute{E}t}_X \xrightarrow{\cong} \pi_{\acute{e}t}(X,\bar{x})$-sets equiv of categories.

$\exists$ pro-finite gp $\pi_{\acute{e}t}(X,\bar{x})$ s.th

Case $X = \mathrm{Spec}\, K$

$K \subset L$ field ext. $\quad L/K$ sep if $\mathrm{irr}_K(\alpha)$ has no mult zeroes $\forall \alpha \in L$

L/K normal if $\mathrm{irr}_K(\alpha)$ factors in linear factors in L

L/K Galois if sep + normal

$I :=$ set of finite Galois extensions E/K contained in fixed Galois ext L/K

$E' \subset E \rightsquigarrow \mathrm{Gal}(E/K) \to \mathrm{Gal}(E'/K)$ i.e. $\{\mathrm{Gal}(E/K)\}_I$ proj system

$\mathrm{Gal}(L/K) := \varprojlim \mathrm{Gal}(E/K)$ topology called Krull topology.

Galois thm:

$$\{E \text{ field} : K \subset E \subset L\} \xleftrightarrow{1-1} \{H < G \text{ closed subgroup}\}$$

$$E \longmapsto \mathrm{Gal}(L/E)$$

$$L^H \longleftarrow H$$

(i) E/K finite $\iff$ H open $\iff$ H closed + finite idx

(ii) E/K Galois $\iff$ H normal $\quad$ Then $G/H \cong \mathrm{Gal}(E/K)$

Def: $\mathrm{Gal}(K^{sep}/K)$ absolute Galois group.

Ex: $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \langle F : a \mapsto a^q \rangle = \mathbb{Z}/n\mathbb{Z}$, $\quad \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$

$X = \mathrm{Spec}\, K$, $\mathrm{F\acute{E}t} = \{ Y \xrightarrow[f]{\acute{e}t} \mathrm{Spec}\, K \}$ $\quad Y = \mathrm{Spec}(\prod_{i=1}^{n} K_i)$ $\quad K_i/K$ separable

The K_i/K gives $\pi_i \subset \pi := \mathrm{Gal}(\bar{K}^{sep}/K)$ open subgroups (up to conjugation, b/c no fixed $K_i \hookrightarrow L$)

E finite π-sets. $\Rightarrow E = \coprod \pi$-orbits, every π-orbit $\Rightarrow \pi \overset{p_i}{\twoheadrightarrow} \pi e_i$

$\pi_i = \ker p_i$.

<u>Correction to last time</u>

$\exists$ pro-finite groups w/ non-open finite index subgroups.

Then $\pi \longrightarrow \hat{\pi}$ is <u>not</u> an iso.

<u>Def</u>: π <u>strongly complete</u> if $\pi \longrightarrow \hat{\pi}$ iso ($\Leftrightarrow$ π profinite and all finite idx subgps are open)

<u>Thm</u> (2003) π profinite, top. f.g. $\Rightarrow$ π strongly complete.

<u>Ex</u> (Milne notes) $\pi = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ not strongly complete

$$\pi = \prod_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \quad \text{not strongly complete}$$

<u>Examples</u>: What is known?

<u>Prop</u>: X normal integral scheme, function field $K = k(X)$.

Take L/K finite sep. Let $Y \longrightarrow X$ be integral closure in L

($Y = \mathrm{Spec}_X \mathcal{A}$ where $\mathcal{A}(U) := \overline{\mathcal{O}_X(U)} \subset L$ int. closure in L)

Say that L/K <u>unramified</u> if $Y \longrightarrow X$ unramified.

Then L/K unramified $\Leftrightarrow$ $Y \longrightarrow X$ finite étale, and

$$F\acute{E}t_{conn}(X) = \{ L/K \text{ unramified} \}$$

<u>Cor.</u> Fix alg closure $K \subset \overline{K}$. Let $M \subset \overline{K}$ be composite of all L/K unramified (i.e. X unramified in L), then $\pi(X, \bar{x}) \cong \mathrm{Gal}(M/K)$

<u>Ex</u>: • $X = \mathrm{Spec}\,\mathbb{Z}$, no unramified ext. of $\mathbb{Q}$ $\Rightarrow$ $\pi(\mathrm{Spec}\,\mathbb{Z}) = \{1\}$

• $X = \mathrm{Spec}\,\mathbb{Z}_p$, $K = \mathbb{Q}_p$, M max unramified ext of $\mathbb{Q}_p \subset \overline{\mathbb{Q}_p}$

$$\mathrm{Gal}(M/K) \cong \mathrm{Gal}(\mathbb{F}_{p^\infty}/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$$

Ex: $X = \mathbb{P}^1_k$, $k = k^{sep}$

$\left(k = \mathbb{C},\ \pi_{top}(X) = \{1\},\ \pi_{Et}(X) = \{1\} \right)$

Riemann-Hurwitz: $Y \xrightarrow{f} X$ finite b/w proper + smooth curves

$$2g_Y - 2 = \deg(f)(2g_X - 2) + R$$

f étale $\Rightarrow R = 0$. But $\underset{X = \mathbb{P}^1}{g_X = 0} \Rightarrow g_Y = 1 - \deg f \Rightarrow \deg f = 1,\ g_Y = 0$.

Thus $\pi_{Et}(X) = \{1\}$.

If $k \neq k^{sep}$, then $\pi_{Et}(\mathbb{P}^1_k) \cong Gal(k^{sep}/k)$

Ex: $X = \mathbb{A}^1_k$, $k = k^{sep}$

$\left(k = \mathbb{C},\ X^{an} \text{ contractible},\ \pi_{top}(X) = \{1\},\ \pi_{Et}(X) = \{1\} \right)$

$char(k) = p > 0$, $\pi(\mathbb{A}^1_k) \longrightarrow \pi(\mathrm{Spec}\, k) = Gal(k^{sep}/k)$ surj but not inj.

kernel contains contribution from Artin-Schreier covers: $Y^p - Y + \alpha = 0$ $\alpha \in k[t]$.

Abhyankar conjectured which groups appears as finite quotients of π(affine curve)

Now proven.

Ex: X_0 proper smooth curve, genus g, over $k = \bar{k}$, char $(k) = p$.

$\left(k = \mathbb{C},\ \pi_{top}(X) = \mathbb{Z}^{*2g} \right)$

If X_0 can be lifted to char 0: $\exists A$ complete dvr, $A/m \cong k$, $char(K(A)) = 0$

$X/\mathrm{Spec}\, A$ smooth proper s.th. $X \otimes_{\mathrm{Spec}\, A} \mathrm{Spec}\, k \cong X_0$. Then $\pi(X_0, \bar{x}) \cong \pi(X, \bar{x})$

$$\underset{\substack{\| \\ \text{profinite completion} \\ \text{of free grp on } 2g \text{ gen.}}}{\pi(X_{\mathrm{Spec}\, K}, \bar{x})} \longrightarrow \pi(X, \bar{x})$$

Grothendieck: over prime-to-p part, this is an iso.

Ex: E elliptic curve $/k = \bar{k}$.

$E \xrightarrow{n\cdot} E$ finite étale, kernel $E[n]$.

$$\pi(E) = \varprojlim E[n](k) = \prod_{\ell \neq p} \mathbb{Z}_\ell^2 \times T_p(E)$$

$T_p(E) = \mathbb{Z}_p^r$ where
$r = 2$ if char $k = 0$
$r = 1$ if ~~char~~ char $k = p$, E ordinary
$r = 0$ if char $k = p$, E supersingular.

Ex: C curve genus g

$$\pi(C)^{ab} = \varprojlim_n \mathrm{Jac}(C)[n] = \prod_{\ell \neq p} \mathbb{Z}_\ell^{2g} \times \mathbb{Z}_p^r \qquad \begin{array}{l} r = 2g \text{ if char } k = 0 \\ 0 \le r \le g \text{ if char } k = p \end{array}$$

Étale topology (following Milne's notes)

$\mathcal{C}$ category w/ fiber products. A Grothendieck topology is:
for each $U \in \mathrm{Ob}(\mathcal{C})$, a set E_U of families of maps $(U_i \to U)_{i \in I}$ called coverings of U s.th.

(a) $\forall V \to U \in \mathrm{Mor}(\mathcal{C})$ and covering $(U_i \to U)_i$ of U
$\Rightarrow (U_i \times_U V \to V)_i$ covering of V

(b) $\forall (U_i \to U)_i$ cov. of U and $(V_{ij} \to U_i)_j$ cov. of U_i
$\Rightarrow (V_{ij} \to U_i \to U)_{i,j}$ cov of U.

(c) $(U \xrightarrow{id} U)$ is a cov. of U.

The pair $T = (\mathcal{C}, E)$ is called a site.

<u>Ex</u>: X top. space

$\mathcal{C}$ = category of open subsets $U \subset X$

E = open coverings $\quad (U \times_V U' = U \cap U'$ in $V)$

$(\mathcal{C}, E)$ is a site

<u>Def</u>: A <u>presheaf</u> of sets/ab. grps on $T = (\mathcal{C}, E)$ is a contravariant functor

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets \ / Ab$$

<u>Notation</u>: $\varphi : U \longrightarrow V$ in $\mathcal{C}$ gives $\mathcal{F}(\varphi) : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ "restriction maps"

$$a \longmapsto a|_U$$