Étale cohomology #5

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The étale fundamental group

Sources: Milne book
Milne notes
Lenstra "Galois theory of sch"

Background: topological fund grp

X top space, $x_0 \in X$

$\pi(X, x_0)$ = grp of homotopy classes of loops in X based at x_0.

(no good analogy in algebraic geometry of this description)

(*) X (path-connected) semi-locally simply connected ($x \in X$ has nbhd U_x s.th every loop in U_x can be contracted in X)

Def: $f: Y \to X$ covering map if $\forall x \in X$ $\exists U$ open nbhd and $\exists F$ discrete set and $\exists$

$$\begin{tikzcd} U \times F \arrow[rr, "\cong"] \arrow[dr, "\pi_1"'] & \circ & f^{-1}(U) \arrow[dl, "f"] \\ & U & \end{tikzcd}$$

$Cov(X)$ = category of covering maps.

Thm: X with (*), then $x_0 \in X$

$$Cov(X) \xrightarrow[F_{x_0}]{\cong} \pi(X, x_0)\text{-sets} = \begin{cases} \text{Ob: } \pi(X,x_0) \times F \to F & F \text{ discrete set} \\ \text{Mor: } g: F \to F' & \pi(X,x_0)\text{-equi.} \end{cases}$$

$$Y \xrightarrow{f} X \longmapsto f^{-1}(x_0)$$

Thm (1): X connected $\exists$ pro-finite grp $\hat{\pi}(X, x_0)$ and (unique up to top iso)

$$FCov(X) \xrightarrow[F_{x_0}]{\cong} \text{finite } \hat{\pi}(X, x_0)\text{-sets}$$

~~profinite completion.~~

If X satisfies (*) then $\hat{\pi}(X, x_0)$ is the profinite completion of $\pi(X, x_0)$

Thm (2): X connected scheme $\exists$ pro-finite grp $\hat{\pi}_{ét}(X, \bar{x}_0)$ (unique up to top iso)

$$FÉt(X) \xrightarrow[F_{\bar{x}_0}]{\cong} \text{finite } \hat{\pi}_{ét}(X, \bar{x}_0)\text{-sets}$$

$$\operatorname{Spec} \Omega \xrightarrow{\bar{x}_0} X \iff \begin{cases} x_0 \in X \\ \kappa(x_0) \hookrightarrow \Omega \end{cases}$$

Ω alg closed (or sep. closed)

Profinite groups

I directed partially ordered set: $\forall i,j\ \exists h$ s.th. $h \geq i$, $h \geq j$.

$(\pi_i)_{i\in I}$ finite groups.

$(f_{ij} : \pi_i \longrightarrow \pi_j)_{i\geq j}$ s.th. $f_{ii} = id$, $f_{ih} = f_{jh} \circ f_{ij}$ $\forall i \geq j \geq h$

This is a projective system.

Inverse limit $\pi = \varprojlim \pi_i = \{ x_i \in \prod_{i\in I} \pi_i : f_{ij}(x_i) = x_j \ \forall i \geq j \}$

π_i is discrete.

$\prod \pi_i$ has product topology, $\varprojlim \pi_i$ has induced topology.

π is a pro-finite group. (equiv definition π is compact and totally disconnected grp).

Ex: G group, I collection of normal subgroups of finite index (i.e. G/N finite)

$N \geq N'$ if $N \subset N'$ $\rightsquigarrow$ $\pi_N = G/N \longrightarrow \pi_{N'} = G/N'$

$\hat{G} := \varprojlim G/N$ profinite completion of G. Natural map $G \longrightarrow \hat{G}$.

G is profinite $\Leftrightarrow$ $G \rightarrow \hat{G}$ isom.

Ex: $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$ pro-finite ring. $\qquad \hat{\mathbb{Z}} = \prod \hat{\mathbb{Z}}_p$

$\hat{\mathbb{Z}}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ p-adic integers.

Thm (Riemann existence thm) X non-singular variety $/\mathbb{C}$.

$F\acute{E}t_X \xrightarrow{\cong} FCov(X(\mathbb{C})^{an})$

so $\pi_{\acute{e}t}(X, \bar{x}_0) \cong \hat{\pi}_1(X(\mathbb{C}), x_0)$

Galois categories

$\mathcal{C}$ category, $F: \mathcal{C} \longrightarrow$ fin. sets. $\mathcal{C}$ ess. small

G1: $\exists$ terminal obj, $\exists$ fiber products.

G2: $\exists$ finite sums.

G3: $\exists$ epi-mono fact, every mono is direct summand

G4: F commutes w/ terminal obj and fiber products

G5: F commutes w/ finite sums, epis, and quotients by finite grp of autom.

G6: $u \in \mathrm{Mor}(\mathcal{C})$, $F(u)$ iso $\Rightarrow$ u iso (i.e. F is conservative)

$\mathrm{Aut}(F) =$ isomorphisms of functors $F \xrightarrow{\sigma} F$

$\forall f: Y \to Z$ in $\mathcal{C}$

$$\begin{tikzcd} F(Y) \arrow[r, "F(f)"] \arrow[d, "\sigma_Y"', "\cong"] \arrow[dr, phantom, "\circ"] & F(Z) \arrow[d, "\cong"', "\sigma_Z"] \\ F(Y) \arrow[r, "F(f)"'] & F(Z) \end{tikzcd}$$

$$\mathrm{Aut}(F) \subset \prod_{X \in \mathcal{C}} S_{F(X)} \qquad S_{F(X)} = \text{perm grp of } F(X)$$

$$\mathrm{Aut}(F) = \bigcap_{f: Y \to Z} G_f$$

$$G_f = \{ (\sigma_X) \in \prod S_{F(X)} : \sigma_Z F(f) = F(f) \sigma_Y \}$$

closed subgroup of $\prod S_{F(X)}$

$\mathrm{Aut}(F) = \bigcap G_f$ — " —

$\Rightarrow$ $\mathrm{Aut}(F)$ profinite group.

Def: $\pi = \mathrm{Aut}(F)$.

Thm: $F: \mathcal{C} \xrightarrow{H} \pi\text{-sets} \xrightarrow{\text{forget}} \text{sets}$, H is an equivalence of categories.

$X \longmapsto F(X)$

$f \longmapsto F(f)$

π is unique up to (non-unique) iso.

In proof of Thm: One shows that F is pro-representable by $(X_i) \in \mathcal{C}$, i.e.

$$\varinjlim \mathrm{Hom}_{\mathcal{C}}(X_i, Y) \xrightarrow{\sim} F(Y)$$

X_i can be chosen to be connected and Galois $\leftarrow$ $\mathrm{Aut}_{\mathcal{C}}(X_i) \longrightarrow F_{\bar{x}}(X_i)$ bijective.

$\sigma \longmapsto \sigma \circ s$

Then $\pi = \varprojlim \mathrm{Aut}_{\mathcal{C}}(X_i)$

Lemma: $(\mathrm{FCov}_X, F_{\bar{x}})$ and $(\mathrm{FEt}_X, F_{\bar{x}})$ are Galois categories.

Finite étale maps $\Leftrightarrow$ locally a trivial covering in étale topology:

$$(U, \bar{u}) \xrightarrow{g} (X, \bar{x}) \text{ étale nbhd.}$$

Given $f: Y \to X$ finite étale, $\exists\, U \to X$ étale nbhd

s.th.

$$\begin{tikzcd} & U \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ U \times F \arrow[ur, "\cong"] \arrow[r] & U \arrow[r] & X \end{tikzcd}$$