Étale cohomology #4

2016 - 02 - 17

Henselian rings and henselization

Local structure of Étale/unramified

Applications: permanence (reg, normal, red.), topological inv.

Henselian pairs, henselian rings

Finite étale covers

Henselization

Strict henselization

Local structure of étale — applications

Thm (Milne I.3.14) $f: X \to Y$ étale $\iff$ locally $A \to (A[z]/p(z))_b = B$
s.th $p(z)$ monic, $p'(z)$ inv in B.

Jacobian variant:

Cor (Milne I.3.16) $f: X \to Y$ étale $\iff$ locally $A \to A[z_1, \dots, z_n]/(P_1, \dots, P_n) = C$
s.th. $\det(\partial P_i/\partial z_j)$ inv in C.

pf: $\Leftarrow$ $\Omega_{X/Y} = \langle dz_1, \dots, dz_n \rangle / \langle dP_1, \dots, dP_n \rangle = 0$ so f unramified.

f unramified $\Rightarrow$ f q-finite $\Rightarrow$ $\bar{P}_1, \dots, \bar{P}_n$ reg. seq in every fiber of f
(b/c $\dim f^{-1}(y) = 0 = n - n$)

$\Rightarrow$ f flat
(a transversally regular immersion)

$\Rightarrow$ WLOG $A \to (A[z]/p(z))_b = B$ as in Thm.

Let $C = A[z,t]/(p(z), tb-1)$. Jacobian $\begin{pmatrix} p'(z) & 0 \\ t\frac{\partial b}{\partial z} & 1 \end{pmatrix}$, $\det = p'$ □

Thm (Milne I.3.15) $f: X \to Y$ unramified $\iff$ locally $A \to (A[z]/p(z))_b / I = B/I$
p monic, p' inv in B.

$$\iff \forall x \in X \, \exists \; \begin{array}{ccccc} X & \overset{\text{open}}{\supset} & U & \overset{\text{closed imm}}{\hookrightarrow} & Z \\ \downarrow & & & \swarrow \text{étale} & \\ Y & \underset{\text{open}}{\supset} & V & & \end{array}$$

(Same proof as I.3.14 but no flatness used at the end)

Prop (Milne I.3.17) $f: X \to Y$ étale, $x \in X$, $y = f(x)$.

(i) $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$

(ii) X regular at x $\Leftrightarrow$ Y regular at y

(iii) X normal at x $\Leftrightarrow$ Y normal at y

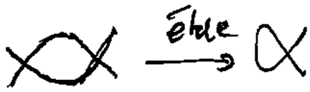
(iv) X reduced at x $\Leftrightarrow$ Y reduced at y

pf: (ii)-(iv) $\Rightarrow$ use that f is flat.

$\Leftrightarrow$ (i) use f quasi-finite + flat: q-finite $\Rightarrow \dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{Y,y})$; flat $\Rightarrow \dim(\mathcal{O}_{X,x}) \geq \dim(\mathcal{O}_{Y,y})$

$\Leftarrow$ use local structure theorem.

Rmk: Not true that $\mathcal{O}_{Y,y}$ int domain $\Rightarrow \mathcal{O}_{X,x}$ integral domain

Ex: 

one says that Y is not unibranch

($\Leftrightarrow$ $Y^{norm} \to Y$ not bijective)

Structure of étale morphisms over normal schemes

Thm (Milne I.3.21) Y connected normal, $f: X \to Y$. TFAE

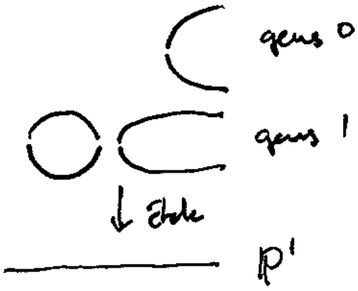
(i) f is étale and separated

(ii) $X = \coprod X_i$, X_i connected normal, $X_i \subset \overline{X_i}$ (open imm), $\overline{X_i} \to Y$ finite normalization

$K(\overline{X_i})/K(Y)$ finite separable

$\mathrm{Supp}(\Omega_{\overline{X_i}/Y}) \cap X_i = \emptyset$.

Ex: $Y = \mathrm{Spec}\, k[x]$, (k alg. closed). Every $\overline{X}_i \to Y$ normalization then gives a smooth curve $\overline{X_i}$



Topological invariance of étale morphisms

Thm (Milne I.3.23, EGA IV 18.1.2) $f: X_0 \hookrightarrow X$ nil-immersion (i.e. a closed immersion defined by a (locally) nil-potent ideal). Then

$$f^*: \quad \text{Ét}(X) \longrightarrow \text{Ét}(X_0)$$

$$\text{Ét}(X) = \{Y \to X \text{ étale}\} \qquad \text{Ét}(X_0) = \{Y_0 \to X_0 \text{ étale}\}$$

$$Y \longmapsto Y \times_X X_0$$

is an equivalence of categories.

proof: f^* is fully faithful: $Y_1 \xrightarrow{\text{ét}} X \xleftarrow{\text{ét}} Y_2$

$$\text{Hom}_X(Y_1, Y_2) = \text{Hom}_{Y_1}(Y_1, Y_1 \times_X Y_2)$$

$$\overset{①}{=} \{U \subset Y_1 \times_X Y_2 \text{ open s.th. } U \to Y_1 \times Y_2 \to Y_1 \text{ isomorphism}\}$$

$$\overset{②}{=} \{\ \_\_\_\_ \ \text{"} \ \_\_\_\_\_\_\_\_ \text{ univ. bijective}\}$$

① sections of étale are open

② univ inj + étale $\Leftrightarrow$ open imm

Since $|X_0| = |X|$, $U \to Y_1$ univ bijective

$\Leftrightarrow U \times_X X_0 \to Y_1 \times_X X_0$ univ bijective.

Thus $\text{Hom}_X(Y_1, Y_2) \xrightarrow{\cong} \text{Hom}_{X_0}(f^*Y_1, f^*Y_2)$.

f^* essentially surjective: Let $Y_0 \to X_0$ étale.
Since f^* fully faithful, a lift $Y \to X$ is unique if it $\exists$. Also, local lifts glue automatically. Thus, WLOG, $Y_0 \to X_0$ standard étale:

$$A_0 \longrightarrow (A_0[z]/p_0(z))_{b_0}$$

Lift to:

$$A \longrightarrow (A[z]/p(z))_b$$

with $p(z)$ monic $\Rightarrow$ flat. Auto. unramified since can be checked on fibers and $|X_0| = |X|$. $\square$

Henselian rings (EGA IV 18.5.11)

$\mathrm{ClOp}(X)$ = set of clopen subsets of $|X|$ $\quad (= \mathcal{P}(\pi_0(X))$ if X noetherian$)$

$\mathrm{ClOp}(\mathrm{Spec} A)$ = idempotents of $A = \{a \in A : a^2 = a\}$

Def. A <u>henselian pair</u> is a pair (X, X_0) of a scheme X and a closed subscheme X_0

s.t. $\quad \forall X' \to X$ <u>finite</u>, $\quad \mathrm{ClOp}(X') \longrightarrow \mathrm{ClOp}(X' \times_X X_0)$ is bijective

Def. A local ring $(A, \mathfrak{m})$ is henselian if $(\mathrm{Spec}\, A, \mathrm{Spec}\, A/\mathfrak{m})$ is a henselian pair.

<u>Thm</u> (Milne I. 4.2; EGA IV 18.5.4/18.5.11) A local ring, $X = \mathrm{Spec}\, A$, $X_0 = \{x\} = \mathrm{Spec}\, A/\mathfrak{m}$. TFAE

(a) A henselian

(b) $\forall A \to B$ finite: $B = \prod_{i=1}^{n} B_i$ $\quad B_i$ finite and local.

(c) $\forall X' \xrightarrow{f} X$ q-fin + separated: $X' = X'_1 \amalg X'_2$ $\quad X'_1 \xrightarrow{f_1} X$ finite, $\quad X'_2 \xrightarrow{f_2} X$ q-fin, $f_2^{-1}(x) = \emptyset$.

and $\{X'_1 = X'_{11} \amalg \cdots \amalg X'_{1n} \quad X'_{1j}$ local sch$\}$

(d) $\forall E \xrightarrow{f} X$ étale: $\mathrm{Hom}_X(X, E) \xrightarrow[\varphi]{\sim} \mathrm{Hom}_{X_0}(X_0, E|_{X_0})$

(e) $\forall f(t) \in A[t]$ monic, $\forall \bar{f} = g_0 h_0$, g_0, h_0 monic & strictly coprime $\exists f = gh$, g, h monic & strictly coprime

w/ image $\bar{f}(t) \in (A/\mathfrak{m})[t]$ $\qquad$ s.t. $g_0 = \bar{g}$, $h_0 = \bar{h}$

(e') $\forall f(t) \in A[t]$ monic, (b) holds for $B = A[t]/f$.

<u>pf</u>: Note that φ in (d) is always injective: if $s_1, s_2 : X \to E$ sections of f then

$$\begin{array}{ccc} s_1(X) \cap s_2(X) = \mathrm{eq}(s_1, s_2) & \hookrightarrow & E \\ \downarrow & \square & \downarrow \text{open} \\ X & \xrightarrow{(s_1, s_2)} & E \times_X E \end{array}$$

so their equalizer is open in X. If $s_1(x) = s_2(x)$, then $x \in \mathrm{eq}$ $\Rightarrow \mathrm{eq} = X$, b/c X local.

(a) $\Leftrightarrow$ (b) and (e) $\Leftrightarrow$ (e') are easy.

(b) $\Rightarrow$ (c) follows directly from ZMT.

(c) $\Rightarrow$ (d) let $s_0 : X_0 \to E$ (i.e. choose a $\kappa(x)$-rat'l point). We may replace E w/ an open nbhd of $s_0(x)$ $\Rightarrow$ WLOG E separated $\overset{(c)}{\Rightarrow}$ WLOG E finite and $f^{-1}(x) = \{s_0(x)\}$

$\Rightarrow f : E \to X$ finite flat of rank 1 $\Rightarrow f$ iso.

(d) $\Rightarrow$ (e) use that $\mathrm{ClOp}(\mathrm{Spec} B / \mathrm{Spec} A)$ <u>represented</u> by étale map. (e) $\Rightarrow$ (a) easy: use factorization $A \to A[e] \to B$.

Sorites of henselian pairs/rings

- (X, X_{red}) henselian pair
- (X, X_0) henselian, $f: X' \to X$ finite $\Rightarrow$ $(X', X' \times_X X_0)$ henselian (by defn.)

In particular if A henselian local then so is A/I $\forall I$ and all finite local A-algebras.

Prop (Milne I.4.5) A complete local noetherian $\Rightarrow$ A henselian local.

pf: B finite A-alg $\Rightarrow B = \hat{B} = \varprojlim B/m^n B$. Since $B/m^n B$ artinian, it's a product of local artinian: $\mathrm{Spec}(B/mB) = \{m_1, \ldots, m_s\} = \mathrm{Spec}(B/m^n B)$ $\forall n$

$$\varprojlim B/m^n B = \varprojlim \prod_{i=1}^{s} (B/m^n B)_{m_i} = \prod_{i=1}^{s} \varprojlim (B/m^n B)_{m_i} = \prod_{i=1}^{s} B_i$$

B_i complete local ring. So A henselian $\square$

(Milne has pf using formally étale)

Prop: (X, X_0) henselian, $f: X' \to X$ proper, then $(X', X' \times_X X_0)$ henselian

pf:

$$\begin{array}{ccccc} X' & \xrightarrow{g} & Z & \xrightarrow{\text{finite}} & X \\ \uparrow & & \uparrow & & \uparrow \\ X'_0 & \longrightarrow & Z_0 & \longrightarrow & X_0 \end{array}$$

Stein factorization. Then (Z, Z_0) henselian. $Z = \mathrm{Spec}_X f_* \mathcal{O}_{X'}$

Since g has connected fibres and is closed (proper) easily seen that (X', X'_0) henselian. $\square$

Rmk: $f: X \to Y$ étale at $x \Leftrightarrow \hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_{X,x}$ finite étale $\Leftrightarrow \hat{\mathcal{O}}_{Y,y} \xrightarrow{\cong} \hat{\mathcal{O}}_{X,x}$ if $k(x) = k(y)$

Rmk: If X, Y finite type over k and $\exists$ abstract iso $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{Y,y}$ [use Artin approximation]

then $\exists$ $w \in W$ with $W \xrightarrow{\text{ét}} X \ni x$, $w \mapsto x$, and $W \xrightarrow{\text{ét}} Y \ni y$, $w \mapsto y$, inducing $\hat{\mathcal{O}}_{X,x} \xrightarrow{\cong} \hat{\mathcal{O}}_{W,w} \xleftarrow{\cong} \hat{\mathcal{O}}_{Y,y}$

Henselian rings and finite étale covers

<u>Prop</u> (Milne I.4.4, EGA IV 18.5.15) (X, X_0) local henselian, then

$$\text{FÉt}(X) \longrightarrow \text{FÉt}(X_0) \quad \text{equivalence of categories}$$

$$\{E \to X \text{ finite étale}\} \quad \{E_0 \to X_0 \text{ finite étale}\}$$

$$(E \to X) \longmapsto (E \times_X X_0 \to X_0)$$

<u>pf</u>: <u>fully faithfulness</u>: (X, X_0) any henselian pair.

Note that if $E \to X$ finite étale and $E \times_X X_0 \to X_0$ iso then $E \to X$ has rank 1 in a clopen nbhd of X_0 = all of X, so $E \xrightarrow{\cong} X$.

Now $\text{Hom}_X(E_1, E_2) = \{U \subset E_1 \times_X E_2 \text{ } \underline{\text{clopen}} \text{ s.th. } U \to E_1 \text{ iso}\}$

(b/c $E_1 \times_X E_2$ henselian) $= \{U_0 \subset (E_1 \times_X E_2) \times_X X_0 \text{ —''— } U_0 \to E_1 \times_X X_0 \text{ iso}\}$

$= \text{Hom}_{X_0}(E_1 \times_X X_0, E_2 \times_X X_0)$

<u>ess surjective</u>: $(X, X_0) = (\text{Spec } A, \text{Spec } k)$

$E_0 \to X_0$ corr to separable k-algebra $k \to B_0 = k[z]/p_0(z)$. p_0 monic separable

Lift to $A \to B = A[z]/p(z)$ p monic (auto sep.) $\square$

<u>More generally</u> $(X, X_0) \xrightarrow{f} (S, S_0)$ proper, then $\text{FÉt}(X) \xrightarrow{\cong} \text{FÉt}(X_0)$ if

(1) (S, S_0) local complete [use Grothendieck existence, cf EGA IV 18.3.4]

(2) (S, S_0) local henselian [use Artin approximation]

<u>Rmk</u>: Related to <u>proper base change</u>

(X, X_0) henselian $\Leftrightarrow$ $\forall E \to X$ ét $\text{Hom}_X(X, E) \xrightarrow{\sim} \text{Hom}_{X_0}(X_0, E|_{X_0})$ $\longleftrightarrow H^0(X, F) \xrightarrow{\sim} H^0(X_0, F|_{X_0})$

$\text{FÉt}(X) \xrightarrow{\sim} \text{FÉt}(X_0)$ $\longleftrightarrow H^1(X, F) \xrightarrow{\sim} H^1(X_0, F|_{X_0})$

Henselization (Milne I.4.8–4.11, EGA IV 18.6)

Def: An étale neighborhood is an étale map $(X,x) \xrightarrow{f} (Y,y)$, i.e. $f: X \to Y$, $x \mapsto y$, such that $\kappa(x)/\kappa(y)$ is trivial.

Rmk: (X,x) local ring. Then (X,x) henselian $\iff$ every étale nbhd has a section.

Rmk: $\operatorname{Spec} \mathcal{O}_{X,x} = \bigcap_{\substack{x \in U \subset X \\ \text{open nbhd}}} U = \varprojlim U$ $\qquad \mathcal{O}_{X,x} = \varinjlim \Gamma(U, \mathcal{O}_X)$

$$A_p = \varinjlim_{f \notin P} A_f \qquad P \in \operatorname{Spec} A$$

Analogously, we define

$$X^h_x := \varprojlim_{\substack{(X',x') \to (X,x) \\ \text{étale nbhd}}} X' \qquad \mathcal{O}^h_{X,x} := \varinjlim \Gamma(X', \mathcal{O}_{X'}) = \varinjlim \mathcal{O}_{X',x'}$$

Lemma: (i) The étale nbhds is a filtered system.

(ii) The connected of the ét. nbhd are cofinal and form a poset. (noetherian case)

pf: (i) Given $(X_1',x_1') \to (X,x)$, $(X_2',x_2') \to (X,x)$, we have $(X_1' \times_X X_2', y)$ mapping to both.

Given $(X',x') \overset{f_1}{\underset{f_2}{\rightrightarrows}} (X,x)$ we have $(eq(f_1,f_2), x') \xrightarrow[\text{im}]{\text{open}} (X',x') \overset{f_1}{\underset{f_2}{\rightrightarrows}} (X,x)$.

(ii) see Milne (easy).

Thm (EGA IV 18.6.6) A local ring, $A^h := \varinjlim \mathcal{O}_{X',x'}$ $\quad (X',x') \to (\operatorname{Spec} A, \operatorname{Spec} k)$ étale nbhds

(i) A^h is local henselian, $A \to A^h$ is local.

(ii) $A \mapsto A^h$ is a functor, left adjoint to forgetful functor (Local henselian) $\to$ (Local rings)

(iii) $A \to A^h$ is faithfully flat

(iv) A noetherian $\iff$ A^h noetherian

(v) $m_A A^h = m_{A^h}$ and $A/m_A \xrightarrow{\sim} A^h/m_A A^h$. In particular $\hat{A} \xrightarrow{\cong} \widehat{A^h}$.

We obtain $A \xrightarrow{\text{f.flat}} A^h \xrightarrow{\text{f.flat}} \hat{A}$. $\quad$ (Complete local) $\rightleftarrows$ (local hens) $\rightleftarrows$ (local)

$\hat{A} \leftarrow\!\shortmid A^h \leftarrow\!\shortmid A$

Strict henselization

Recall: (X,x) henselian $\iff$ every étale nbhd has a section

$$\begin{array}{ccc} \operatorname{Spec} k(x) & \hookrightarrow & X' \\ & \searrow & s \uparrow \downarrow \text{étale} \\ & & X \end{array}$$

Def: (X,x) (Spec A, A local) strictly henselian if TFE hold

(i) X henselian and residue field $k(x)$ separably closed

(ii) X henselian and every finite étale $X' \to X$ is trivial (i.e. $X' = X \amalg \dots \amalg X$)

(iii) $\forall$ $\begin{array}{ccc} X' & \ni & x' \\ s \uparrow \downarrow \text{étale} & & \downarrow \\ X & \ni & x \end{array}$ $\exists s$.

(iii') $$\begin{array}{ccc} \operatorname{Spec} k(x)^{sep} & \to & X' \\ & \searrow & s \uparrow \downarrow \text{étale} \\ & & X \end{array}$$

Def: A geometric point of X is a separably closed field k and a map $\bar{x} : \operatorname{Spec} k \to X$

Also written $\bar{x} \to X$. (sometimes k alg closed)

Def: An étale nbhd ~~...~~ of $(X, \bar{x})$ is a commutative diagram

$$\begin{array}{ccc} \bar{x} & \to & X' \\ & \searrow & \downarrow \text{étale} \\ & & X \end{array}$$

Def: $X^{sh}_{\bar{x}} := \varprojlim_{\substack{\bar{x} \to X' \\ \searrow \downarrow \\ X}} X'$ $\qquad \mathcal{O}^{sh}_{X,\bar{x}} := \varinjlim \mathcal{O}_{X',x'}$ $\quad x' =$ image of $\bar{x}$ in X'.

Also written $\mathcal{O}_{X,\bar{x}}$.

Prop: (i) $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}^h_{X,x} = \dim \mathcal{O}^{sh}_{X,x}$

(ii) X regular, normal, reduced, CM, ... at x ($\iff \mathcal{O}_{X,x}$ is ...)

$\iff \mathcal{O}^h_{X,x}$ —— " ——————

$\iff \mathcal{O}^{sh}_{X,x}$ —— " ——————

(Not true for integral domain (unibranch better))

Various constructions of henselization

① As a filtered direct limit of étale extensions $A \to A_\lambda$

② $A \mapsto A^h$ left adjoint to forgetful functor (hens rings) $\to$ (loc rings)

③ (A noetherian) A^h is smallest henselian ring contained in $\hat{A}$: $A^h = \bigcap_{\substack{A' \subset \hat{A} \\ A' \text{ henselian}}} A'$

④ (A normal) $K(A) =: K \subset K^{sep} \subset \overline{K}$

$G = Gal(K^{sep}/K)$

$A^{sep} :=$ integral closure of A in K^{sep}. G acts on A^{sep}.

Pick max ideal $m_{sep} \subset A^{sep}$. $(A^{sep}/m_{sep} = k^{sep})$

$D =$ decomposition group $= \{ \sigma \in G : \sigma(m_{sep}) = m_{sep} \}$

$\cup$

$I =$ inertia group $= \{ \sigma \in D : \sigma \bmod m_{sep} = id \}$

Exact seq: $1 \to I \to D \to Aut(A^{sep}/m_{sep}) \to 1$

$$Aut(A^{sep}/m_{sep}) = Gal(k^{sep}/k)$$

$$\begin{array}{ccccccccc}
K & \subset & K^D & \subset & K^I & \subset & K^{sep} & \subset & \overline{K} \\
\cup & & \cup & & \cup & & \cup & & \\
A & \subset & A^h & \subset & A^{sh} & \subset & A^{sep}_{m_{sep}} & & \leftarrow \text{localization} \\
\| & & \cup & & \cup & & \cup & & \\
A & \subset & (A^{sep})^D & \subset & (A^{sep})^I & \subset & A^{sep} & & \leftarrow \text{integral closure of } A \text{ in top line}
\end{array}$$

⑤ (A finite type over a field?) $A^h \subset \hat{A}$ ring of <u>algebraic power series</u>.