Étale cohomology #3

2016-02-10

Flatness

Definitions

Topological properties (+ openness of qfin/unram.)

Relative Cartier divisors, transversally reg imm, quasi-sections

Finite flat morphisms, discriminant

Maps b/w Cohen-Macaulay and regular

Étale morphisms

Definitions, std étale ex, ~~quasi-sections~~

Smooth morphisms, quasi-sections

Comparison w/ analytic topology

Summary: unramified, monomorphism, proper, flat

Local structure of étale morphisms

Flatness [Milne I.2, Altman-Kleiman Ch. V]

M A-mod, $(*)$ $0 \to N' \to N \to N'' \to 0$

Def: M flat if $- \otimes_A M$ exact (ie. $(*)$ exact $\Rightarrow$ $(*) \otimes_A M$ exact)
M faithfully flat if $- \otimes_A M$ faithfully exact ($\Leftrightarrow$) $\Leftrightarrow \begin{cases} M \text{ flat} \\ N \otimes_A M = 0 \Rightarrow N = 0 \end{cases}$

Fact (Milne 2.2) B A-alg. TFAE $\varphi : A \to B$

(i) B flat (as an A-mod)

(ii) B_P flat as an $A_{\varphi^{-1}(P)}$-mod $\forall P \in \operatorname{Spec} B$ (or $\forall P \in \operatorname{Spm} B$)

Fact (Milne 2.7) $\varphi : A \to B$. TFAE
(AK 1.9)

(i) B faithfully flat A-mod

(ii) φ injective and $B/\varphi(A)$ flat A-mod

(iii) B flat, φ universally injective

(iv) B flat, $\operatorname{Spec} B \to \operatorname{Spec} A$ surjective.

Def: A morphism of schemes $f : X \to Y$, is

(i) flat at $x \in X$ if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ flat, $y = f(x)$

(ii) flat if flat at x $\forall x \in X$.

(iii) faithfully flat if flat + surjective.

Similar definition for $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module.

Sorités: open immersions are flat ($\mathcal{O}_{Y,y} \xrightarrow{\cong} \mathcal{O}_{X,x}$ $\forall x$)

flatness is stable under basechange and compositions.

flat-local on source: $X \xrightarrow{f} Y \xrightarrow{g} Z$ $\left.\begin{matrix} f \text{ flat at } x \\ g \circ f \text{ —}\|\text{—} \end{matrix}\right\} \Rightarrow g$ flat at $f(x)$.

pf: straight-fwd. (use that flat local homo is faithfully flat)

Topological properties of flat morphisms

$f : X \to Y$, f loc of fin. type + Y loc noeth (or f loc. of fin. pres.)

$\boxed{\text{Thm 1}}$ (AK 5.5) (Milne 2.16) $\mathrm{Flat}(f) = \{x \in X : f \text{ flat at } x\} \subseteq X$ is open.

Thm 2 (Milne 2.16) If Y integral and $X \neq \emptyset$ (or merely reduced), then $\mathrm{Flat}(f) \neq \emptyset$. "generic flatness"

Thm 3 (Milne 2.8) f flat $\Rightarrow$ f generizing, i.e., $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ surjective $\forall x \in X$.

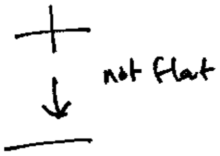
$\boxed{\text{Thm 4}}$ (Milne 2.12) f flat $\Rightarrow$ f universally open.

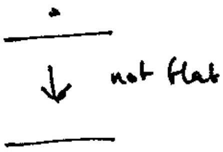
pf: Thm 3 is easy algebra $\Rightarrow$ Thm 2

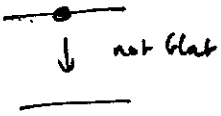
Thm 1 + 4 are proved using Thm 3 and constructible sets (Chevalley's thm).

Consequence: If $f : X \to Y$ flat, then every irreducible comp $E \subset X$ dominates an irreducible comp of Y, i.e., $\overline{f(E)}$ is an irr component.

Ex:







Also holds for embedded components. $f(\mathrm{Ass}(X)) \subset \mathrm{Ass}(Y)$.

When Y regular of dim 1 (e.g. a smooth curve) then this necc. cond is suff.

Algebraic form.

Thm: A Dedekind domain. M A-mod. Then M flat $\Leftrightarrow$ M torsion-free.
(Milne 2.3)

Fact: $f : X \to Y$ $\overset{\text{loc fin type}}{\text{flat}}$ $\Rightarrow$ $y \mapsto \dim f^{-1}(y)$ locally constant.

Ex: $\mathrm{Spec}\, \mathbb{Q} \to \mathrm{Spec}\, \mathbb{Z}$ flat but not open (not of fin. type)

Openness of q-finite and unramified

$f : X \longrightarrow Y$ loc fin. type.

$qf(f) = \{ x \in X : x \text{ isolated in } X_y = f^{-1}(y),\ y = f(x) \} \subseteq X$

$\text{unram}(f) = \{ x \in X : \mathcal{O}_{X_y, x} \text{ separable } \kappa(y)\text{-algebra} \} \subseteq X$

Fact: These subsets are open.

pf for unram(f): $\Omega_{X/Y} \otimes \kappa(x) \cong \Omega_{X_y/\kappa(y)} \otimes \kappa(x) = 0$

$\underset{\text{Nakayama's lemma}}{\Longrightarrow} \quad \Omega_{X/Y} = 0$ in a nbhd of x $\Rightarrow$ f unramified in a nbhd of x.

Ramification locus of f: $X \setminus \text{unram}(f) = X \setminus \text{Supp}(\Omega_{X/Y})$

Flatness: relative Cartier divisors

A, B noeth or φ fin. pres.

Prop (Milne 2.5) $\varphi : A \longrightarrow B$ flat, $b \in B$. TFAE

(i) b not zero-div and $A \longrightarrow B/(b)$ flat.

(ii) $\bar{b} \in B/mB$ not zero-div $\forall m \in \operatorname{Spec} A$.

$\Updownarrow$ X, Y loc noeth or f loc of fin. pres.

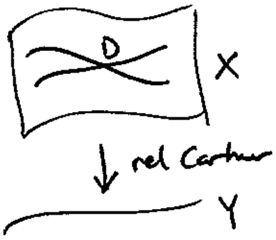
Prop: $f : X \longrightarrow Y$ flat, $D \hookrightarrow X$ (loc.) closed subscheme. TFAE

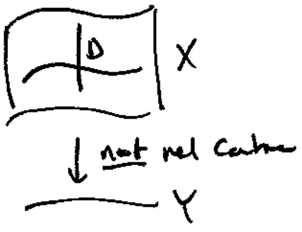
(i) $D \hookrightarrow X$ Cartier divisor and $g : D \hookrightarrow X \longrightarrow Y$ flat

(ii)
$$\begin{array}{ccc} D_y & \hookrightarrow & X_y \\ \| & & \| \\ g^{-1}(y) & & f^{-1}(y) \end{array} \quad \text{Cartier } \forall y \in Y.$$

We say that $D \hookrightarrow X$ is a relative Cartier divisor.

Ex:





Ex: $B = A[z]/(a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0)$

flat $\Leftrightarrow \operatorname{Spec} B \to \operatorname{Spec} A$ q-finite

$\Leftrightarrow (a_1, a_2, \dots, a_n) = A$.

X, Y loc noeth or f loc of fin pres.

Cor: $f : X \longrightarrow Y$ flat, $Z \hookrightarrow X$ (loc.) closed subscheme. TFAE

(i) Locally $Z = \{f_1 = f_2 = \dots = f_r = 0\}$ where $f_1, f_2, \dots, f_r$ reg. seq and $Z \hookrightarrow X \longrightarrow Y$ is flat.

(ii) Locally $Z = \{f_1 = \dots = f_r = 0\}$ where $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_r \in \mathcal{O}_{X_y, x}$ regular seq $\forall x \in X$, $y = f(x)$.

We say that $Z \hookrightarrow X$ is a transversal regular immersion

Thm (Milne 2.25) $f : X \longrightarrow Y$ flat + loc. of fin. pres. + surjective, Y quasi-compact (noetherian)

Then $\exists h : Y' \longrightarrow Y$ flat + quasi-finite + surjective and

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ Y' \arrow[ur, "g"] \arrow[rr, "h"'] \arrow[rr, phantom, "\circ", shift left=3] & & Y \end{tikzcd}$$

"g is a quasi-section"

pf: $\forall y \in |Y|$, $\exists x \in X_y$ s.th. $\mathcal{O}_{X_y, x}$ Cohen-Macaulay. Choose reg seq $f_1, \dots, f_r \in \Gamma(U, \mathcal{O}_X)$ for some nbhd $U \ni x$ s.th. $\mathcal{O}_{X_y, x}/(f_1, \dots, f_r)$ zero-dim. After shrinking U get $Z = \{f_1 = \dots = f_r = 0\} \hookrightarrow U \hookrightarrow X$ transversal reg. imm. Let Y' be disjoint union of a finite number of such Z's. $\square$

Finite flat morphisms

Thm (Milne 2.9) M f.g. A-mod. TFAE

(i) M flat (and f.pres if A not noeth.)

(ii) M projective

(iii) $\tilde{M}$ loc. free, i.e., $\exists f_1, \dots, f_n \in A$: M_{f_i} free A_{f_i}-mod $\forall i$
and $(f_1, \dots, f_n) = (1)$.

(iv) M_p free $\forall p \in \operatorname{Spec} A$ (+ M f.pres if A not noeth)

In particular, if $\mathcal{F}$ coherent sheaf on X, then $\mathcal{F}$ flat $\mathcal{O}_X$-mod $\Leftrightarrow$ $\mathcal{F}$ loc free
and then $x \longmapsto \dim(\mathcal{F} \otimes_{\mathcal{O}_X} k(x))$ locally constant, – the <u>degree of $\mathcal{F}$</u>.

In particular, if $f: X \longrightarrow Y$ <u>finite</u>, then f flat $\Leftrightarrow$ $f_* \mathcal{O}_X$ locally free.
degree of f = $\operatorname{rank}(f_* \mathcal{O}_X)$ is locally constant $= \dim_{k(x)} \Gamma(f^{-1}(x))$.

<u>Discriminant</u> (AK VI 6.6, Milne Exc I.3.9, EGA IV 18.2)

If $f: X \longrightarrow Y$ finite flat, then the <u>branch locus</u> $B \hookrightarrow Y$
$(B = f(\operatorname{Supp}(\Omega_{X/Y})))$ has the structure of a Cartier divisor – the discriminant.

Explicitly: if $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, $A \xrightarrow{\varphi} B$ and B <u>free</u> of rank n,

then $\delta_f = \det(\operatorname{tr}(e_i e_j))$ where $B = A\langle e_1, \dots, e_n \rangle$

δ_f depends on the choice of basis up to an invertible element in A

$\Rightarrow$ ideal (δ_f) well-defined.

Example of flat maps

Thm: $f: X \longrightarrow Y$ quasi-finite, Y regular of equi-dim n. TFAE

(i) f is flat

(ii) X is Cohen-Macaulay of equi-dim n.

pf: (i) $\Rightarrow$ (ii): $\mathcal{O}_{Y,f(x)} \xrightarrow{\varphi} \mathcal{O}_{X,x}$ faithfully flat so a reg seq $g_1, \dots, g_n$ $\Rightarrow$ reg seq $\varphi(g_1), \dots, \varphi(g_n)$.

$\square$

(ii) $\Rightarrow$ (i): see EGA IV 6.1.5.

In particular, if X and Y are smooth varieties, then any quasi-finite map $X \longrightarrow Y$ is flat. More generally:

Thm: $f: X \longrightarrow Y$. X Cohen-Macaulay, Y regular.

If ~~$\dim_x X = \dim_y Y + \dim_x f^{-1}(y)$~~ $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,f(x)} + \dim \mathcal{O}_{f^{-1}(f(x)),x}$

then f is flat.

(see EGA IV 6.1.5, IV 15.4.2, Hartshorne III Ex 10.9, Milne I. 2.6 b)

Étale morphisms (Milne I.3) (Altman-Kleiman Ch VI)

Def. $f: X \to Y$ is étale if flat + unramified (+ loc. of fin. pres)

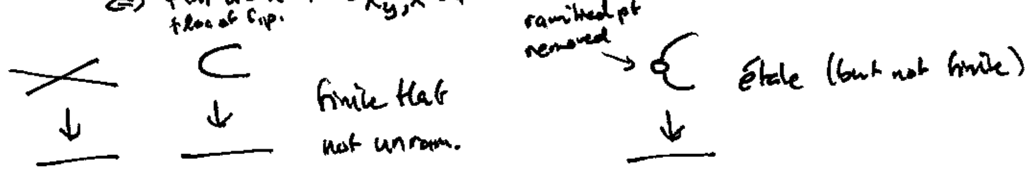
loc. of fin. type $\Uparrow$ (above "unramified")

$\uparrow$ nec. if Y not loc noeth.

f étale at x if étale in a nbhd of x

$\Leftrightarrow$ flat at x + $\mathcal{O}_{X_y,x}$ separable $\kappa(x)$-alg.
+loc of f.p.

Ex:



$k[t,t^{-1}] \longrightarrow k[t,t^{-1},z]/z^n-t$ finite étale of deg n.

Ex (Milne 3.4) $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} A[z]/(g)$ $g = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$

g monic $\Rightarrow f: X \to Y$ finite flat.

$\Omega_f = \langle dz \rangle / (dg)$ $dg = \frac{dg}{dz} dz = g' dz$

So $\Omega_f = 0 \Leftrightarrow g'$ unit in $A[z]/(g)$

or more precisely: f is ramified along $\operatorname{Supp}(\Omega_f) = \{g' = 0\}$.

In particular $\operatorname{Spec} (A[z]/(g))_{g'} \longrightarrow \operatorname{Spec} A$ is étale.

Ex: Artin-Schreier coverings in char p

$z^p - z - a = 0$

finite étale.

Sorités: open immersions are étale

étale stable under base change and composition.

étale-local on source: $X \xrightarrow{f} Y \xrightarrow{g} Z$ $\left.\begin{array}{l} f \text{ étale at } x \\ g\circ f \text{ étale at } x \end{array}\right\} \Rightarrow g$ étale at $f(x)$

Maps b/w étale maps are étale, even stronger:

Prop: (Milne 3.6)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

. $\left.\begin{array}{l} gf \text{ étale} \\ g \text{ unramified} \end{array}\right\} \Rightarrow f$ étale.

pf:

$$\begin{tikzcd} f: X \arrow[r, "\Gamma_f"] & X \times_Z Y \arrow[r, "\text{étale}"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ & X \arrow[r, "gf"'] & Z \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X \times_Z Y \arrow[d, "f \times id"] \\ Y \arrow[r, "\Delta_g", "\text{open imm}"'] & Y \times_Z Y \end{tikzcd}$$

$\square$

Smooth maps (Altman-Kleiman Ch VII, Hartshorne III.10)

Milne 3.24 – 3.26

Def: $f : X \to Y$ is smooth if flat + loc. of fin. pres + geom. fibers are regular.

Fact: $X/\bar{k}$ finite type is regular $\Leftrightarrow$ $\Omega_{X/\bar{k}}$ loc free of rank $n = \dim X$.

Fact: f smooth $\Leftrightarrow$ flat + loc of fin pres + $\Omega_{X/Y}$ loc free of rank n

+ fibers of f are equidim of dim n.

Rmk: f étale $\Leftrightarrow$ f smooth + quasi-finite

Sorités: smoothness stable under base change and composition.

Rmk: $Z = \{f_1 = f_2 = \dots = f_r = 0\} \hookrightarrow X$ ~~smooth~~ $\Rightarrow$ $X \xrightarrow{f} \mathbb{A}^r$, $f^{-1}(0) = Z$.

X smooth, Jacobian criterion for $Z \hookrightarrow X$ $\Leftrightarrow$ smoothness of f.

Thm (Milne 3.26) $f : X \to Y$ smooth, surjective, Y quasi-compact.

Then $\exists\, Y' \xrightarrow{h} Y$ étale + surjective and

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ Y' \arrow[ur, "g"] \arrow[rr, "h"'] & & Y \end{tikzcd}$$

"g is a quasi-section"

pf: Similar as proof for f flat, but pick $f_1, \dots, f_r$ regular s.th.

$(\bar{f}_1, \dots, \bar{f}_r) = m_{X_y, x}$ (possible if $\kappa(x)/\kappa(y)$ separable). $\square$

Comparison w/ analytic topology

quasi-finite ⟺ locally finite in analytic topology

unramified ⟺ locally a closed imm —"—

étale ⟺ locally an open imm —"—

smooth ⟺ locally $\mathbb{A}^n$ —"—

Algebraic counterparts

$f : X \to Y$

$$\begin{tikzcd} x & x' \arrow[l, mapsto, bend right] \\ X & X' \arrow[l, "\text{étale}"'] \arrow[d, "f'"] \\ Y \arrow[from=u, "f"'] \arrow[ur, phantom, "\circ"] & Y' \arrow[l, "\text{étale}"'] \end{tikzcd} \qquad (*)$$

Thm:

f q-fin at x ⟺ ∃ comm diagram (*) w/ f' finite

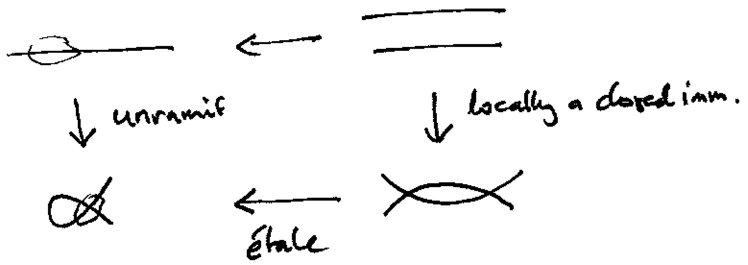
f unramified at x ⟺ —"— f' closed imm

f étale at x ⟺ —"— f' isomorphism (can take $X' \to X$ open imm)

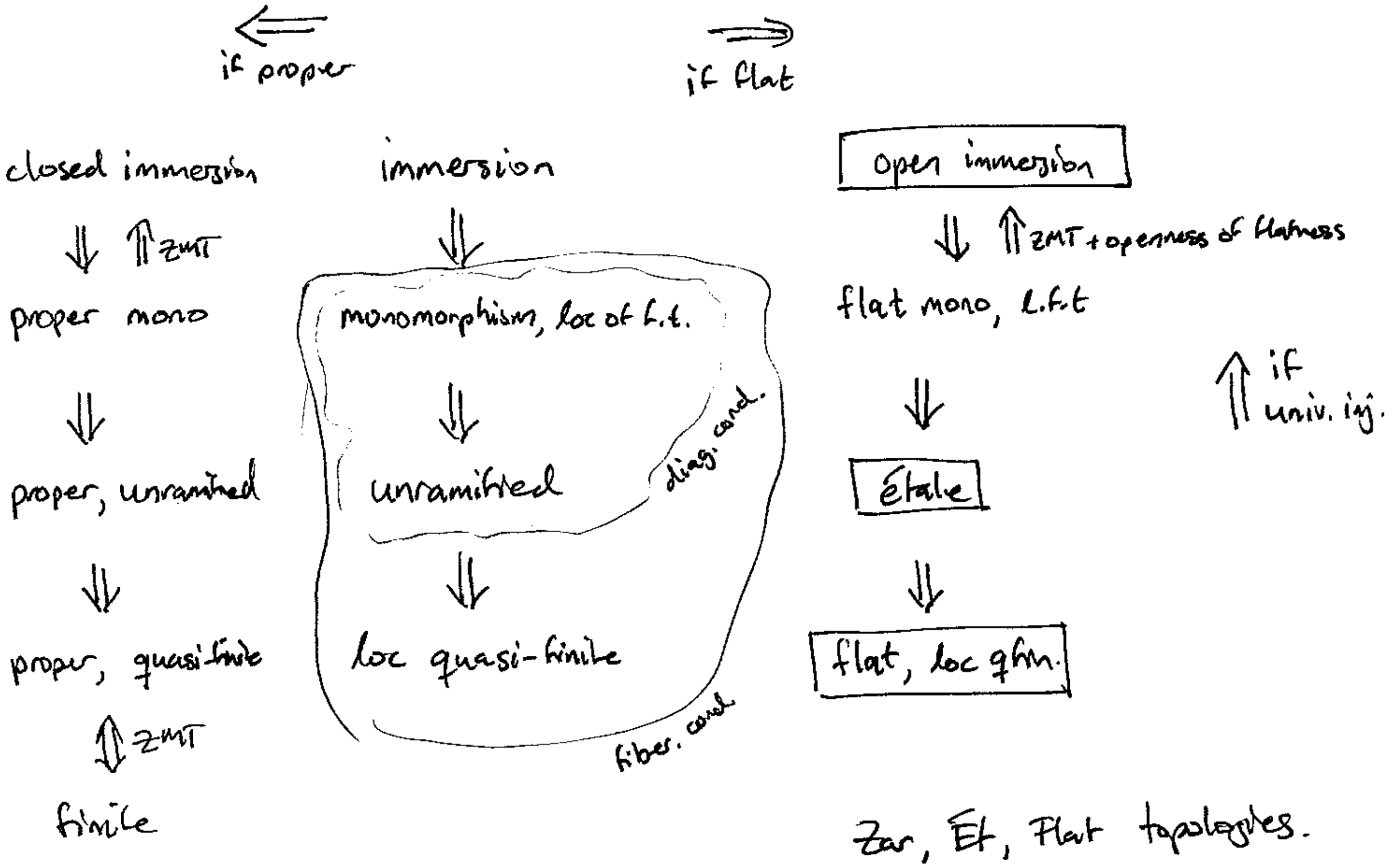
f smooth at x ⟺ ∃

$$\begin{tikzcd} X \arrow[dd, "f"'] & \overset{\overset{x}{\wedge}}{U} \arrow[l, phantom, "\overset{\text{open}}{\supset}"] \arrow[d, "\text{étale}"] \\ & \mathbb{A}^n \times Y \arrow[dl] \\ Y & \end{tikzcd}$$

Ex:



Summarizing diagram



Zar, Et, Flat topologies.

proper + flat: clopen immersion
⇓
finite étale
⇓
finite flat

Local structure of étale morphisms

Thm (Milne 3.14) $f: X \to Y$. TFAE

(i) f is étale at $x \in X$.

(ii) $\exists$ open nbhds $x \in U$, $y = f(x) \in V$ s.th. $f|_U : U \to V$ standard étale

i.e. $V = \operatorname{Spec} A$, $U = \operatorname{Spec} B$, $B = (A[z]/(p(z)))_b$

where $p(z)$ monic and $p'(z)$ invertible in B.

pf: WLOG $X = \operatorname{Spec} C$, $Y = \operatorname{Spec} A$.

WLOG (ZMT) f is finite

WLOG (simple ~~approx~~ arg.) A local w/ max ideal corr to $f(x) = y$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] & C \arrow[d, two heads] & \\
k(y) \arrow[r] & C \underset{A}{\otimes} k(y) \cong k(x) \times C' &
\end{tikzcd}$$

w/ $k(x)/k(y)$ separable b/c f unramified at x.

$$\begin{tikzcd}
\text{Factorize: } k(y) \arrow[r, hook] & k(x) \times k(y) \arrow[r, hook] \arrow[d, equal] & k(x) \times C' \\
& D_0 &
\end{tikzcd}$$

Since D_0 separable $k(y)$-algebra, $D_0 = k(y)[\bar{t}] \subset k(x) \times C'$

Lift to $D = A[t] \subset C$.

$$\begin{tikzcd}
\operatorname{Spec}(C) \arrow[r, "g"] & \operatorname{Spec}(D) \arrow[r] & \operatorname{Spec}(A) \\[-20pt]
x \arrow[r, mapsto] & z \arrow[r, mapsto] & y
\end{tikzcd}$$

$g^{-1}(z) = \{x\}$ by construction.

Since g finite, $C \underset{D}{\otimes} D_{\mathfrak{p}_z} \xrightarrow{\cong} C_{\mathfrak{p}_x}$. Nakayama $\Rightarrow$ $D_{\mathfrak{p}_z} \to C_{\mathfrak{p}_x}$ surj

Injective by flatness $\Rightarrow$ $D_{\mathfrak{p}_z} \cong C_{\mathfrak{p}_x}$.

Thus, in an open nbhd of x, g is an isomorphism. Thus

WLOG $C = A[t]/I$.